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SCATTERING FOR THE YANG-MILLS EQUATIONS

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ABSTRACT. We construct wave and scattering operators for the Yang-Mills equations on Minkowski space, $\mathbf{M}_0 \cong \mathbf{R}^4$. Sufficiently regular solutions of the Yang-Mills equations on $\mathbf{M}_0$ are known to extend uniquely to solutions of the corresponding equations on the universal cover of its conformal compactification, $\widetilde{\mathbf{M}} \cong \mathbf{R} \times S^3$. Moreover, the boundary of $\mathbf{M}_0$ as embedded in $\widetilde{\mathbf{M}}$ is the union of "lightcones at future and past infinity", $C_\pm$. We construct wave operators $W_\pm$ as continuous maps from a space $\mathbf{X}$ of time-zero Cauchy data for the Yang-Mills equations to Hilbert spaces $\mathbf{H}(C_\pm)$ of Goursat data on $C_\pm$. The scattering operator is then a homeomorphism $S: \mathbf{X} \to \mathbf{X}$ such that $UW_+ = W_-S$, where $U: \mathbf{H}(C_+) \to \mathbf{H}(C_-)$ is the linear isomorphism arising from a conformal transformation of $\widetilde{\mathbf{M}}$ mapping C_- onto C_+. The maps $W_\pm$ and S are shown to be smooth in a certain sense.

1. INTRODUCTION

The conformal invariance of the Yang-Mills equations in four dimensions greatly facilitates the study of the temporal asymptotic behavior of their solutions. There is a natural conformal embedding $\imath$ of Minkowski space, $\mathbf{M}_0 \cong \mathbf{R}^4$, into the universal cover of its conformal compactification, $\widetilde{\mathbf{M}} \cong \mathbf{R} \times S^3$. Conformally invariant wave equations on $\mathbf{M}_0$ may thereby be extended to corresponding equations on $\widetilde{\mathbf{M}}$ [1, 2]. In particular, if A is a connection satisfying the Yang-Mills equations on $\widetilde{\mathbf{M}}$, $\imath^* A$ satisfies the Yang-Mills equations on $\mathbf{M}_0$. Techniques used to prove global existence for the Yang-Mills Cauchy problem on $\mathbf{M}_0$ extend to prove global existence on $\widetilde{\mathbf{M}}$, allowing the derivation of precise asymptotics for Yang-Mills fields on $\mathbf{M}_0$ [3, 4]. Here we construct wave and scattering operators for the Yang-Mills equations on $\mathbf{M}_0$ in terms of the embedding $\imath: \mathbf{M}_0 \to \widetilde{\mathbf{M}}$.

The boundary of $\mathbf{M}_0$ in $\widetilde{\mathbf{M}}$ is the union of two characteristic cones $C_\pm$, the "lightcones at past and future infinity". Points of $C_\pm$ correspond to limits of null lines in $\mathbf{M}_0$ as the time coordinate in $\mathbf{M}_0$ approaches $\pm\infty$. The scattering theory of conformally invariant wave equations on $\mathbf{M}_0$ is thus closely related to the Goursat problem on $\widetilde{\mathbf{M}}$, in which solutions are determined by data on a characteristic cone [5]. Moreover the surface where the time coordinate of $\mathbf{M}_0$

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is zero is just the Cauchy surface $\{0\}\times S^3 \subset \widetilde{\mathbf{M}}$ with a single point removed. For the equation $\Box f + \lambda f^3 = 0$ this allowed the formulation of wave operators $W_\pm$ as nonlinear maps from a Hilbert space of Cauchy data at time zero to Hilbert spaces of Goursat data on the cones $C_\pm$ [6]. These maps can be correlated with the traditional wave operators mapping sufficiently regular solutions of $\Box f + \lambda f^3 = 0$ to temporally asymptotic solutions of the free wave equation. Moreover, the maps $W_\pm$ are smooth with smooth inverses, so the scattering operator $W_+(W_-)^{-1}$ exists and is a diffeomorphism [7].

A difficulty in extending this approach to the Yang-Mills equations is that global existence has not been proven for finite-energy Cauchy data, but only for data having more derivatives. The techniques used in [6] to solve the Goursat problem and invert the operators $W_\pm$ rely heavily upon the fact that the spaces of Goursat data have energy-type norms. Thus while we construct wave operators for the Yang-Mills equations and show they are smooth in a certain sense, we do not prove them invertible.

Nonetheless, a scattering operator of a different sort can be constructed. The group $SO^{\sim}(2,4)$ of conformal transformations of $\widetilde{\mathbf{M}}$ has a unique central element ζ mapping C_- onto C_+, and the action of this element ζ on solutions of conformally invariant equations on $\widetilde{\mathbf{M}}$ corresponds to scattering [5]. We show that for the Yang-Mills equations this map is smooth, and describe its relationship to the wave operators.

2. The scattering operator

First we recall the basic global existence theorem of [4]. We shall identify the universal cover of conformally compactified Minkowski space, $\widetilde{\mathbf{M}}$, with $\mathbf{R}\times S^3$ given the metric $dt^2 - ds^2$, where dt^2 and ds^2 are the standard Riemannian metrics on $\mathbf{R}$ and S^3, respectively. Let $\mathbf{g}$ be the Lie algebra of a compact Lie group. Given a smooth manifold M, possibly with boundary, let $\Omega^p(M,\mathbf{g})$ denote the $\mathbf{g}$-valued differential p-forms over M. (We use this notation in informal contexts when no particular degree of differentiability need be specified.) Following the notation of [8], the Yang-Mills equations for $A \in \Omega^1(\widetilde{\mathbf{M}},\mathbf{g})$ may be written as:

(1) $$F = dA + \tfrac{1}{2}[A,A]; \quad d*F + [A, *F] = 0 .$$

In temporal gauge, the dt component of A is assumed to vanish, where t is the $\mathbf{R}$-valued coordinate on $\widetilde{\mathbf{M}}$. We shall identify elements $A \in \Omega^1(\widetilde{\mathbf{M}},\mathbf{g})$ with vanishing dt component with functions $A\colon \mathbf{R} \to \Omega^1(S^3,\mathbf{g})$. Let d_s denote exterior differentiation of $\mathbf{g}$-valued forms on S^3, and let $*_s$ denote the Hodge $*$-operator on $\mathbf{g}$-valued forms on S^3 with respect to the metric ds^2. Given $A, B \in \Omega^1(s^3,\mathbf{g})$, we make the following definitions:

$$A\times B = *_s[A,B], \quad [A;B] = *_s[A, *_s B],$$
$$\nabla\cdot A = *_s\, d_s *_s A, \quad \nabla\times A = *_s d_s A,$$

and given $f \in \Omega^0(S^3,\mathbf{g})$ we define ∇f to be $d_s f$. Note that if Δ deno Laplace-Beltrami operator on $\Omega^1(S^3,\mathbf{g})$, we have $\Delta A = \nabla\times(\nabla\times A) - \nabla($ Using $'$ to denote ∂_t, and working in temporal gauge, the equations (equivalent to the evolution equation

(2) $$A'' + \nabla\times(\nabla\times A) + A\times(\nabla\times A) + \tfrac{1}{2}\nabla\times(A\times A) + \tfrac{1}{2}A\times(A\times A) =$$

together with the constraint

(3) $$\nabla\cdot A' + [A;A'] = 0 .$$

Identifying S^3 with $SU(2)$, let V_i, $1 \le i \le 3$, be an orthonormal ba left-invariant vector fields on S^3. Let $\Omega^1_k(S^3,\mathbf{g})$ denote the space of $\mathbf{g}$-v one-forms on S^3 with all components lying in the Sobolev space $H^k(S^3)$, the structure of a real Hilbert space with the following norm:

$$\|A\|_k^2 = \sum_{1\le i\le 3}\int_{S^3} |(\Delta+1)^{k/2}A(V_i)|^2$$

where here Δ denotes the Laplacian as a selfadjoint operator on $L^2(S^3,\mathbf{g})$ $|\cdot|$ denotes any Hilbert norm on $\mathbf{g}$.

The evolution equation (2) may be desingularized by differentiating it respect to t and using the constraint (3) to rewrite the resulting term $\nabla\times(\nabla$ as $\Delta A' - \nabla[A;A']$. Let $\mathbf{H}$ denote the real Hilbert space $\Omega^1_3(S^3,\mathbf{g})\oplus\Omega^1_2(S^3,$ $\Omega^1_1(S^3,\mathbf{g})$; $\mathbf{H}$ will be used as a space of Cauchy data (A, A', A''). Let $\mathbf{X}$ de the set of Cauchy data in $\mathbf{H}$ satisfying (2) and (3), that is, those $(A_1, A_2, A$ $\mathbf{H}$ such that $\nabla\cdot A_2 + [A_1;A_2] = 0$ and

$$A_3 + \nabla\times(\nabla\times A_1) + A_1\times(\nabla\times A_1) + \tfrac{1}{2}\nabla\times(A_1\times A_1) + \tfrac{1}{2}A_1\times(A_1\times A_1) =$$

Note that $\mathbf{X}$ is a closed subset of $\mathbf{H}$; we give $\mathbf{X}$ the metric arising from norm on $\mathbf{H}$. Let L be the unbounded operator on $\mathbf{H}$ with domain $\Omega^1_4(S^3,$ $\Omega^1_3(S^3,\mathbf{g})\oplus\Omega^1_2(S^3,\mathbf{g})$ given by

$$L(A_1, A_2, A_3) = (A_2, A_3, -(\Delta+1)A_2),$$

and let $N\colon \mathbf{H}\to\mathbf{H}$ be the function given by

$$N(A_1, A_2, A_3) = (0, 0, k(A_1, A_2))$$

where

$$k(A_1,A_2) = \nabla[A_1;A_2] - A_1\times(\nabla\times A_2) - A_2\times(\nabla\times A_1) - \nabla\times(A_1\times A_2)$$
$$-A_1\times(A_1\times A_2) - \tfrac{1}{2}A_2\times(A_1\times A_1) + A_2 .$$

It follows from results in [4] that L generates a strongly continuous or parameter semigroup on $\mathbf{H}$, and that the function N is C^∞ from $\mathbf{H}$ to with all derivatives bounded on bounded sets.

The global existence result may be stated as follows:

Lemma 1. *For any Cauchy datum* $u_0 \in \mathbf{X}$ *there is a unique continuous* $u: \mathbf{R} \to \mathbf{X}$ *satisfying*

$$u(t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)}N(u(s))\,ds. \tag{4}$$

Let $(A_1(t), A_2(t), A_3(t)) = u(t)$, *and let* $A \in \Omega^1(\widetilde{\mathbf{M}}, \mathbf{g})$ *denote the* $\mathbf{g}$*-valued one-form corresponding to the continuous function* $A_1: \mathbf{R} \to \Omega^1_3(S^3, \mathbf{g})$. *Then* A *satisfies* (2) *and* (3) *in the distributional sense.*

Proof. This follows from Theorems 3 and 4 of [4]. □

Define the map $T: \mathbf{R} \times \mathbf{X} \to \mathbf{X}$ as follows: given $u_0 \in \mathbf{X}$, let $u: \mathbf{R} \to \mathbf{X}$ be the unique continuous solution of (4), and let $T(t)u_0 = u(t)$. T is clearly a group action of $\mathbf{R}$ on the space $\mathbf{X}$. This group action extends to an action of $\mathbf{R} \times SO(4)$, which is the identity component of the group of isometries of $\widetilde{\mathbf{M}}$:

Lemma 2. *The group* $SO(4)$ *has a strongly continuous orthogonal representation on* $\mathbf{H}$ *given by*:

$$g(A_1, A_2, A_3) = ((g^{-1})^* A_1, (g^{-1})^* A_2, (g^{-1})^* A_3)$$

where $g^*: \Omega^1(S^3, \mathbf{g}) \to \Omega^1(S^3, \mathbf{g})$ *denotes the pullback map induced by the diffeomorphism* $g: S^3 \to S^3$ *corresponding to* $g \in SO(4)$. *If* $g \in SO(4)$ *and* $u \in \mathbf{X}$, *then* $gu \in \mathbf{X}$. *Moreover, there is a group action* $\Gamma: \mathbf{R} \times SO(4)\mathbf{X} \to \mathbf{X}$ *given by*:

$$\Gamma(t, g)u = T(t)(gu), \quad (t, g) \in \mathbf{R} \times SO(4),\ u \in \mathbf{H}.$$

Proof. Since the action of $SO(4)$ on S^3 is isometric the linear transformation of $\mathbf{H}$ given by $u \mapsto gu$ is an isometry. The action of $SO(4)$ on any Sobolev space $H^k(S^3)$ is strongly continuous, so $(g, u) \mapsto gu$ is a continuous orthogonal representation. Moreover, if $u = (A_1, A_2, A_3) \in \mathbf{X}$ and $gu = (B_1, B_2, B_3)$, then simple computations show that $\nabla \cdot B_2 + [B_1; B_2] = 0$ and

$$B_3 + \nabla \times (\nabla \times B_1) + B_1 \times (\nabla \times B_1) + \tfrac{1}{2}\nabla \times (B_1 \times B_1) + \tfrac{1}{2}B_1 \times (B_1 \times B_1) = 0,$$

hence $gu \in \mathbf{X}$. That Γ is a group action follows from the fact that $T(t)gu = gT(t)u$, a consequence of the $SO(4)$-invariance of equation (4). □

As mentioned above, the action of the central element $\zeta \in SO^{\sim}(2,4)$ on the space of solutions of the Yang-Mills equation may be interpreted as the scattering operator [5]. In its action as a conformal transformation of $\widetilde{\mathbf{M}}$ it corresponds to the isometry $(\pi, -I) \in \mathbf{R} \times SO(4)$. Thus we define the scattering operator $S: \mathbf{X} \to \mathbf{X}$ by $S = \Gamma(\pi, -I)$. In a certain sense S is smooth, but since $\mathbf{X}$ is not a submanifold of $\mathbf{H}$ [9], some care is required to make this precise. We will show that the action Γ of $\mathbf{R} \times SO(4)$ on $\mathbf{X}$ extends to a local action $\overline{\Gamma}$ of $\mathbf{R} \times SO(4)$ as C^∞ maps defined on neighborhoods of $\mathbf{X}$.

More precisely:

Theorem 3. *For any* $(t, g) \in \mathbf{R} \times SO(4)$ *there is an open set* $U_t \subseteq \mathbf{H}$ *co* $\mathbf{X}$ *and a function* $\overline{\Gamma}(t, g): U_t \to \mathbf{H}$ *such that*:

(a) *The restriction of* $\overline{\Gamma}(t, g)$ *to* $\mathbf{X}$ *is* $\Gamma(t, g)$.

(b) *When both sides are defined,* $\overline{\Gamma}(t, g)\overline{\Gamma}(s, h)v = \overline{\Gamma}(t + s, gh)v$.

(c) *If* $|s| \le |t|$ *then* $U_t \subseteq U_s$. *If* $t \ge 0$ *and* $v \in U_t$, $\overline{\Gamma}(s, I)v$ *is con as a function of* s *from* $[-t, t]$ *to* $\mathbf{H}$.

(d) *For all* $n \ge 1$ *the function* $\overline{\Gamma}(t, g): U_t \to \mathbf{H}$ *has a continuous nth derivative* $D^n\overline{\Gamma}(t, g): \mathbf{H} \times \mathbf{H}^n \to \mathbf{H}$, *where* $\mathbf{H}^n$ *denotes the n-fold produc For all* $v \in U_t$ *the multilinear map* $D^n\overline{\Gamma}(s, g)v: \mathbf{H}^n \to \mathbf{H}$ *satisfies*

$$\sup\{\|D^n\overline{\Gamma}(s, g)v\| : g \in SO(4), |s| \le |t|\} < \infty.$$

Proof. Suppose that $u_0 \in \mathbf{X}$ and $t \ge 0$. We shall show that for some ε, *i* if $\|v_0 - u_0\| < \varepsilon$, $0 \le t_0 \le t$, and $v: [-t_0, t_0] \to \mathbf{H}$ is a continuous fu such that

$$v(s) = e^{Ls}v_0 + \int_0^s e^{L(s-s')}N(v(s'))\,ds', \tag{5}$$

then $\sup_{s \in [-t_0, t_0]} \|v(s)\| \le M$. It then follows from the theory of no semigroups [10] that for any $v_0 \in \mathbf{H}$ with $\|v_0 - u_0\| < \varepsilon$ there is a continuous function $v: [-t, t] \to \mathbf{H}$ satisfying (5).

Let $F: \mathbf{R}^2 \to \mathbf{R}$ be a continuous function, increasing in each variabl that

$$\|N(x) - N(y)\| \le F(\|x\|, \|y\|)\|x - y\|, \qquad x, y \in \mathbf{H}.$$

Let $u: \mathbf{R} \to \mathbf{X}$ be as in Lemma 1, and let

$$M = \sup_{s \in [-t, t]} \|u(s)\| + 1 \quad \text{and} \quad C = \sup_{s \in [-t, t]} \|e^{Ls}\|.$$

Choose $\varepsilon > 0$ such that $\varepsilon C e^{CF(M,M)t} < 1$. The value of ε depends and t; let us write it as $\varepsilon(u_0, t)$. By the definition of ε and the invariance of the equations (4) and (5), we may choose this function s $\varepsilon(u_0, t) = \varepsilon(gu_0, t)$ for all $g \in SO(4)$. Moreover we may choose this fu so that $0 \le s \le t$ implies $\varepsilon(u_0, t) \le \varepsilon(u_0, s)$.

Let $v_0 \in \mathbf{H}$ have $\|v_0 - u_0\| < \varepsilon(u_0, t)$. Let $0 \le t_0 \le t$, and let $v: [-t_0$ $\mathbf{H}$ be continuous, satisfying (5). The proof now proceeds by contrad: assume that for some $s \in [-t_0, t_0]$ we have $\|v(s) - u(s)\| > 1$. Let

$$\tau = \inf\{|s| : s \in [-t_0, t_0], \|v(s) - u(s)\| > 1\}.$$

By continuity, either $\|v(\tau) - u(\tau)\| = 1$ or $\|v(-\tau) - u(-\tau)\| = 1$. W assume that $\|v(\tau) - u(\tau)\| = 1$, as the other case is analogous. By (4) an for any $s \in [0, \tau]$ we have

$$\|v(s) - u(s)\| \le C\varepsilon(u_0, t) + C\int_0^s F(\|v(s')\|, \|u(s')\|)\|v(s') - u(s')\|\,ds'$$
$$\le C\varepsilon(u_0, t) + CF(M, M)\int_0^s \|v(s') - u(s')\|\,ds'.$$

Thus by Gronwall's inequality and our choice of $\varepsilon(u_0, t)$,

$$\|v(\tau) - u(\tau)\| \leq \varepsilon(u_0, t) C e^{CF(M,M)\tau} < 1,$$

a contradiction. We conclude that for all $s \in [-t_0, t_0]$, $\|v(s) - u(s)\| \leq 1$; hence $\sup_{s\in[-t_0,t_0]} \|v(s)\| \leq M$ as was to be shown.

Now for any $t \in \mathbf{R}$ define the open set $U_t \subseteq \mathbf{H}$ as follows:

$$U_t = \bigcup_{u\in\mathbf{X}} \{v : \|v - u\| \leq \varepsilon(u, |t|)\}.$$

Given $v_0 \in U_t$, let $v: [-t, t] \to \mathbf{H}$ be the unique continuous function satisfying (5). Given s with $|s| \leq t$ let $\overline{T}(s)v_0 = v(s)$, and for $g \in SO(4)$ let

$$\overline{\Gamma}(s, g)v_0 = g\overline{T}(s)v_0 . \tag{7}$$

Statements (a) and (c) of the theorem now follow directly from the definitions. Statement (b) follows from the same arguments used in Lemma 2. To prove the differentiability claimed in (d) it suffices by (7) and Lemma 2 to prove the existence of continuous Fréchet derivatives $D^n\overline{T}(t): \mathbf{H} \times \mathbf{H}^n \to \mathbf{H}$. This follows by induction on n using Theorem B$'$ of [11] and the fact that N is C^∞ with bounded derivatives on bounded sets. This theorem also establishes that for $v \in U_t$ and $|s| \leq |t|$, the Fréchet derivative $D^n\overline{T}(s)v: \mathbf{H}^n \to \mathbf{H}$ is a strongly continuous function of s. By the uniform boundedness theorem we conclude that

$$\sup_{|s|\leq|t|} \|D^n\overline{T}(s)v\| < \infty .$$

Since $SO(4)$ acts as orthogonal transformations of $\mathbf{H}$ and $\overline{\Gamma}(s, g) = g\overline{T}(s)$, we have $\|D^n\overline{\Gamma}(s, g)v\| = \|D^n\overline{T}(s)v\|$, hence

$$\sup\{\|D^n\overline{\Gamma}(s, g)v\| : g \in SO(4), |s| \leq |t|\} < \infty \quad \square$$

Corollary 4. *For some open sets $U, V \subseteq \mathbf{H}$ containing $\mathbf{X}$, the scattering operator $S: \mathbf{X} \to \mathbf{X}$ extends to a diffeomorphism $\overline{S}: U \to V$.*

Proof. Take $U = U_\pi \cap \overline{\Gamma}(-\pi, -I)U_\pi$, let $\overline{S}$ be the restriction of $\overline{\Gamma}(\pi, -I)$ to U, and let $V = \overline{S}U$. Then $\overline{S}: U \to V$ is C^∞ by part (c) of the theorem, and if $v \in U$, $\overline{\Gamma}(-\pi, -I)\overline{S}v = v$ by part (b), so $\overline{\Gamma}(-\pi, -I)|V$ is a C^∞ inverse of $\overline{S}$. $\square$

3. The wave operators

In this section we describe wave operators mapping a Cauchy datum in $\mathbf{X}$ to restrictions of the corresponding solution of the Yang-Mills equation to the "lightcones at infinity", $C_\pm \subset \widetilde{\mathbf{M}}$. Let ρ denote the arclength from the point in S^3 corresponding to the identity of $SU(2)$. Let $C_\pm$ be the subsets of $\widetilde{\mathbf{M}}$ given by the equations $t = \pm(\pi - \rho)$. We define the Sobolev spaces $H^k(C_\pm)$ through the identification of $C_\pm$ with S^3 by means of the one-to-one maps

$$(t, x) \mapsto x, \qquad (t, x) \in C_\pm \subset \mathbf{R} \times S^3 .$$

The Sobolev space $H^1(C_\pm)$ is of particular significance, being the s "finite-energy Goursat data" for the conformally invariant scalar wave e on $\widetilde{\mathbf{M}}$, that is, the space of restrictions to $C_\pm$ of finite-energy solut $(\Box + 1)\varphi = 0$, where $\Box$ denotes the D'Alembertian on functions φ: I [6]. Let $E_\pm$ denote the pullback of the bundle of $\mathbf{g}$-valued one-forms or the subspace $C_\pm$, and let $\mathbf{H}(C_\pm)$ denote the space of sections of $E_\pm$ components lying in the Sobolev space $H^1(C_\pm)$.

We shall formulate the wave operators for the Yang-Mills equations i of the spaces $\mathbf{H}(C_\pm)$, as follows. As Cauchy datum $u \in \mathbf{X}$ determines a s of the Yang-Mills equations in temporal gauge, $A \in \Omega^1(\widetilde{\mathbf{M}}, \mathbf{g})$. Lett denote the lifts to $\widetilde{\mathbf{M}}$ of the previously described vector fields V_i on S letting $X_0 = \partial_t$, the vector fields X_i, $0 \leq i \leq 3$, form an orthonormal b vector fields on $\widetilde{\mathbf{M}}$. We shall show that the restrictions $A|C_\pm$, $(X_iA)|C$ $(X_0X_iA)|C_\pm$ are well-defined as elements of $\mathbf{H}(C_\pm)$. Moreover, the fu $u \mapsto A|C_\pm$, $u \mapsto (X_iA)|C_\pm$, and $u \mapsto (X_0X_iA)|C_\pm$ extend to smooth fu defined on an open neighborhood of $\mathbf{X}$.

We make use of a lemma on the inhomogeneous wave equation for g- one-forms. Let R be the region of $\widetilde{\mathbf{M}}$ defined by $\{-\pi \leq t \leq \pi\}$, and be the space of $\mathbf{g}$-valued one-forms on R with all components in the S space $L^2(R)$, given the structure of a real Hilbert space.

Lemma 5. *If A_1, $A_2 \in \Omega^1(S^3, \mathbf{g})$ and $h \in \Omega^1(\widetilde{\mathbf{M}}, \mathbf{g})$ are C^∞ sections there is a unique C^∞ section $A \in \Omega^1(\widetilde{\mathbf{M}}, \mathbf{g})$ such that*

$$(\Box + 1)A = h, \quad (A, X_0A)|_{t=0} = (A_1, A_2), \tag{6}$$

where $\Box$ denotes the D'Alembertian on $\Omega^1(\widetilde{\mathbf{M}}, \mathbf{g})$. The restrictions $A|C$ functions of $h|R$, A_1, and A_2, and the functions $(h|R, A_1, A_2) \mapsto A|C_\pm$ defined extend uniquely to continuous maps

$$T_\pm: \mathbf{V} \oplus \Omega_1^1(S^3, \mathbf{g}) \oplus \Omega_0^1(S^3, \mathbf{g}) \to \mathbf{H}(C_\pm) .$$

Proof. This is a straightforward consequence of Lemma 6 of [7], which an analogous result for the inhomogeneous scalar wave equation. $\square$

Note that by the theory of the Cauchy problem for the inhomogeneous equation, if $(A_1, A_2) \in \Omega_1^1(S^3, \mathbf{g}) \oplus \Omega_0^1(S^3, \mathbf{g})$ and $h \in \mathbf{V}$ then there a unique $A \in \Omega_1^1(R, \mathbf{g})$ satisfying (6) in the distributional sense. The lemma then allows us to define $A|C_\pm$ as an element of $\mathbf{H}(C_\pm)$.

Theorem 6. *Given $u \in \mathbf{X}$, let $(A_1(t), A_2(t), A_3(t)) = T(t)u$, and let A d the element of $\Omega^1(\widetilde{\mathbf{M}}, \mathbf{g})$ corresponding to $A_1: \mathbf{R} \to \Omega_3^1(S^3, \mathbf{g})$. Then A $(X_iA)|C_\pm$, and $(X_0X_iA)|C_\pm$, where $0 \leq i \leq 3$, are well-defined elemen $\mathbf{H}(C_\pm)$ as in Lemma 5. Moreover, for some open set $U \subseteq \mathbf{H}$ containing $\mathbf{X}$ functions $u \mapsto A|C_\pm$, $u \mapsto (X_iA)|C_\pm$, and $u \mapsto (X_0, X_iA)|C_\pm$ extend to maps from U to $\mathbf{H}(C_\pm)$.*

Proof. By symmetry it suffices to consider the restrictions to C_+. Suppose that $u \in U_\pi$, and for $|t| \leq \pi$ let $(A_1(t), A_2(t), A_3(t)) = \overline{\Gamma}(t, I)u$, where the open set $U_\pi \subseteq \mathbf{H}$ and $\overline{\Gamma}(t, I)$ are as in Theorem 3. Let A denote the element of $\Omega^1(R, \mathbf{g})$ corresponding to $A_1 : [-\pi, \pi] \to \Omega^1_3(S^3, \mathbf{g})$, and let $h(u)$ denote the element of $\mathbf{V}$ corresponding to

$$A_3 + (\Delta + 1)A_1 : [-\pi, \pi] \to \Omega^1_1(S^3, \mathbf{g}).$$

By equation (5) and the definition of L and N, we have $(\Box + 1)A = h(u)$ in the distributional sense. Note that by statements (c), (d) of Theorem 3, the functions $u \mapsto h(u)$ is C^∞ from U_π to $\mathbf{V}$. Let us write $u = (B_1, B_2, B_3)$. By Lemma 5 and the remarks following, we may define $A|C_+ = T_+(h(u), B_1, B_2)$, and conclude that $u \mapsto A|C_+$ is a C^∞ function from U_π to $\mathbf{H}(C_+)$.

Next we treat $X_iA|C_+$. Since the Lie derivations X_i commute with the differential operator $\Box$, we have

$$(\Box + 1)X_iA = X_ih(u)$$

in the distributional sense. For $1 \leq i \leq 3$ the map $u \mapsto X_ih(u)$ is C^∞ from U_π to $\mathbf{V}$, by Theorem 3. Moreover, by (5) we have

$$X_0h(u) = X_0(A_3 + (\Delta + 1)A_1) = k(A_1, A_2) \tag{7}$$

in the distributional sense. Since N is C^∞ with bounded derivatives on bounded sets, k is C^∞ from $\Omega^1_3(S^3, \mathbf{g}) \oplus \Omega^1_2(S^3, \mathbf{g})$ to $\Omega^1_1(S^3, \mathbf{g})$, with bounded derivatives on bounded sets. It follows from Theorem 3 that the map $u \mapsto X_0h(u)$ is C^∞ from U_π to $\mathbf{V}$.

For each i, $X_iA|_{t=0}$ and $X_0X_iA|_{t=0}$ are well-defined distributions by the theory of the Cauchy problem. Then for $1 \leq i \leq 3$ we have

$$(X_iA, X_0X_iA)|_{t=0} = (V_iB_1, V_iB_2),$$

while for $i = 0$ we have

$$(X_0A, X_0^2A)|_{t=0} = (B_2, B_3).$$

For each i it is evident that the map $u \mapsto (X_iA, X_0X_iA)|_{t=0}$ is C^∞ from $\mathbf{H}$ to $\Omega^1_1(S^3, \mathbf{g}) \oplus \Omega^1_0(S^3, \mathbf{g})$. Thus by Lemma 5 and the remarks following, we may define

$$(X_iA)|C_+ = T_+(X_ih(u), (X_iA, X_0X_iA)|_{t=0}),$$

and conclude that $u \mapsto (X_iA)|C_+$ is a C^∞ function from U_π to $\mathbf{H}(C_+)$.

We use a similar argument for $(X_0X_iA)|C_+$. Given $u \in \mathbf{X}$ we have

$$(\Box + 1)X_0X_iA = X_0X_ih(u)$$

in the distributional sense. By (7) we have

$$X_0X_ih(u) = X_ik(A_1, A_2).$$

For $1 \leq i \leq 3$, since X_ik is C^∞ from $\Omega^1_3(S^3, \mathbf{g}) \oplus \Omega^1_2(S^3, \mathbf{g})$ to $\Omega^1_0(S^3,$ [illegible] follows from Theorem 3 that the map $u \mapsto X_0X_ih(u)$ is C^∞ from U_π t[illegible] For $i = 0$ we obtain the same conclusion as follows. By the chain rule we

$$X_0^2h(u) = X_0k(A_1, A_2) = k_1(A_1, A_2)A_2 + k_2(A_1, A_2)A_3$$

where k_i denotes the Fréchet derivative with respect to A_i of the expre[illegible] for $k(A_1, A_2)$. It follows from the Sobolev inequalities that the right si[illegible] C^∞ from $\mathbf{H}$ to $\Omega^1_0(S^3, \mathbf{g})$, with bounded derivatives on bounded sets (see Lemma 2.2 of [4]). Thus by Theorem 3, the map $u \mapsto X_0^2h(u)$ is C^∞ f[illegible] U_π to $\mathbf{V}$.

For each i, $X_0X_iA|_{t=0}$ and $X_0^2X_iA|_{t=0}$ are well-defined distributions, b[illegible] theory of the Cauchy problem. For $1 \leq i \leq 3$ we have

$$(X_0X_iA, X_0^2X_iA)|_{t=0} = (V_iB_2, V_iB_3),$$

and for $i = 0$ we have

$$(X_0^2A, X_0^3A)|_{t=0} = (B_3, -(\Delta + 1)B_2 + k(B_1, B_2)).$$

For each i, it is evident that the map $u \mapsto (X_0X_iA, X_0^2X_iA)|_{t=0}$ is C^∞ fr[illegible] U_π to $\Omega^1_1(S^3, \mathbf{g}) \oplus \Omega^1_0(S^3, \mathbf{g})$. Thus by Lemma 5 we may define

$$(X_0X_iA)|C_+ = T_+(X_0X_ih(u), (X_0X_iA, X_0^2X_iA)|_{t=0}),$$

and note that $u \mapsto (X_0X_iA)|C_+$ is C^∞ from U_π to $\mathbf{H}(C_+)$. □

By the above theorem there exist continuous "wave operators" $W_\pm : \mathbf{X}$ [illegible] $\mathbf{H}(C_\pm)$ such that $W_\pm(u) = A|C_\pm$ where $A \in \Omega^1(\widetilde{\mathbf{M}}, \mathbf{g})$ is as in the theore[illegible] The relation between these wave operators and the scattering operator S [illegible] given as follows:

Theorem 6. *Let $U : \mathbf{H}(C_+) \to \mathbf{H}(C_-)$ be the isomorphism of real Hilbert spa*[illegible] *given by*

$$(UA)(t, x) = A(t + \pi, -x) \qquad (t, x) \in C_+ \subset \mathbf{R} \times S^3.$$

Then we have $W_-S = UW_+$.

Proof. Suppose $u \in \mathbf{X}$, and let $A \in \Omega^1(\widetilde{\mathbf{M}}, \mathbf{g})$ be the corresponding soluti[illegible] of the Yang-Mills equation in temporal gauge, as in the statement of Theore[illegible] 5. Then if $B \in \Omega^1(\widetilde{\mathbf{M}}, \mathbf{g})$ denotes the solution of the Yang-Mills equati[illegible] in temporal gauge corresponding to $Su \in \mathbf{X}$, by the definition of S we ha[illegible] $B(t, x) = A(t + \pi, -x)$. Thus

$$W_-Su = B|C_- = U(A|C_+) = UW_+u. \quad \Box$$

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