R-commutative Geometry and Quantization of Poisson Algebras

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Abstract

An r-commutative algebra is an algebra A equipped with a Yang-Baxter operator $R: A \otimes A \to A \otimes A$ satisfying $m = mR$, where $m: A \otimes A \to A$ is the multiplication map, together with the compatibility conditions $R(a \otimes 1) = 1 \otimes a$, $R(1 \otimes a) = a \otimes 1$, $R(\mathrm{id} \otimes m) = (m \otimes \mathrm{id})R_2 R_1$ and $R(m \otimes \mathrm{id}) = (\mathrm{id} \otimes m)R_1 R_2$. The basic notions of differential geometry extend from commutative (or supercommutative) algebras to r-commutative algebras. Examples of r-commutative algebras obtained by quantization of Poisson algebras include the Weyl algebra, noncommutative tori, quantum groups, and certain quantum vector spaces. In many of these cases the r-commutative de Rham cohomology is stable under quantization.

1 Introduction

Following the initiative of A. Connes [6], there has been a surge of interest in noncommutative geometry, in which one treats a noncommutative algebra as if it consisted of smooth functions on a space and pursues analogs of differential-geometric constructions. As pointed out by Connes, Karoubi [16], and Woronowicz [28, 30], the analog of differential forms for a noncommutative algebra A is a *differential calculus* for A, that is, a differential graded algebra Ω that is generated as such by $\Omega^0 \cong A$. Every differential calculus for A is a quotient of a certain "universal" differential calculus $\Omega_u(A)$. For commutative algebras one usually works with the "classical" differential calculus $\Omega_c(A)$, which is the quotient of $\Omega_u(A)$ by the relations

$$adb = (db)a\ , \quad d1 = 0\ .$$

When A is the algebra of smooth functions on a manifold M, $\Omega_c(A)$ is isomorphic as a differential graded algebra to the differential forms $\Omega(M)$.

The simplest generalization of the classical differential calculus treats $\mathbf{Z}_2$-graded algebras that are "supercommutative," satisfying $ab = (-1)^{\deg a \deg b} ba$. The extension of concepts to supercommutative algebras is based on the rule that the twist map

$$a \otimes b \mapsto b \otimes a$$

should be replaced everywhere by the graded twist map

$$a \otimes b \mapsto (-1)^{\deg a \deg b} b \otimes a \ .$$

Thus one constructs an analog of the classical differential calculus for a supercommutative algebra A as the quotient of $\Omega_u(A)$ by the relations

$$a\, db = (-1)^{\deg a \deg b}\,(db)a\ ,\quad d1 = 0\ .$$

Supercommutative algebras arose in physics from the desire to treat bosons and fermions in an even-handed manner. The interchange of two identical bosons is modelled mathematically by the twist map, while for fermions one uses the map

$$a \otimes b \mapsto -b \otimes a\ .$$

More recently, mathematical investigations of low-dimensional physics have raised the possibility of other particle types, so-called "anyons," for which interchange is modelled by an operator $R\colon A\otimes A \to A\otimes A$ satisfying the Yang-Baxter equations

$$(R\otimes \mathrm{id})(\mathrm{id}\otimes R)(R\otimes \mathrm{id}) = (\mathrm{id}\otimes R)(R\otimes \mathrm{id})(\mathrm{id}\otimes R)\ .$$

Such operators define representations of the braid group. This circle of ideas leads to the concept of an "r-algebra," an algebra equipped with a Yang-Baxter operator

$$a\otimes b \mapsto R(a\otimes b)$$

compatible with the algebra structure in a certain sense. An algebra A equipped with such an "r-structure" is said to be "r-commutative" if $m = mR$, where $m\colon A\otimes A \to A$ is the multiplication map. Many interesting noncommutative analogs of manifolds are r-commutative. In addition to supermanifolds, these include quantum groups, quantum matrix algebras, quantum vector spaces [11, 12, 15, 20, 28, 29, 30], noncommutative tori [6, 8, 25], the Weyl and Clifford algebras [26], and certain universal enveloping algebras.

Generalizing the classical differential calculus to r-commutative algebras is straightforward when the r-structure R is "strong," that is, $R^2 = \mathrm{id}$. One simply forms the quotient of $\Omega_u(A)$ by the relations

$$adb = \sum_i (db^i)a_i\ ,\quad d1 = 0\ ,$$

where $R(a\otimes b) = \sum_i b^i \otimes a_i$. Most of our work concerns this case, which is relevant to the Weyl and Clifford algebras and noncommutative tori. For quantum groups, quantum matrix algebras, and quantum vector spaces one needs r-structures that are not strong. Here there are still many basic open questions.

The plan of the paper is as follows. In section 1 we state the basic definitions concerning r-commutative algebra and differential forms on strong r-commutative algebras. In section 2 we describe two basic r-commutative algebras: the r-symmetric algebras, which generalize the symmetric and exterior algebras, and the r-Weyl algebras, which generalize the Weyl and Clifford algebras. In section 3 we sketch the relation between r-commutative geometry and the quantization of Poisson algebras, and work through the details for the Weyl algebra, giving a new proof of Segal's Poincaré lemma for "quantized differential forms" [26]. In section 4 we give a similar treatment of noncommutative tori. In section 5 we sketch what is known about the r-commutative geometry of quantum groups, quantum matrix algebras and quantum vector spaces.

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2 R-commutative Geometry

We begin by describing "r-commutative algebras," in which, following the ideas of Manin [20], the role of the twist map

$$\tau(a \otimes b) = b \otimes a$$

is replaced by an arbitrary Yang-Baxter operator. We develop only a small piece of this theory. One could phrase some of our work in the language of tensor categories [9, 17, 19], but we take a more pedestrian approach.

Initially we work over an arbitrary field k. By an "algebra" we will always mean a unital associative algebra over k. Let V be a vector space over k. Given $R \in End(V \otimes V)$, define $R_i \in End(V^{\otimes n})$ for $1 \leq i < n$ by

$$R(v_1 \otimes \cdots v_n) = v_1 \otimes \cdots \otimes v_{i-1} \otimes R(v_i \otimes v_{i+1}) \otimes v_{i+1} \otimes \cdots \otimes v_n \ .$$

Given $R \in End(V \otimes V)$, we say R is a *Yang-Baxter operator* on V if R is invertible and

$$R_1 R_2 R_1 = R_2 R_1 R_2$$

on $V \otimes V \otimes V$. We say R is *strong* if also $R^2 = \mathrm{id}$.

Yang-Baxter operators are closely related to the braid group and symmetric group, as follows. Let s_i, $1 \leq i < n$, denote the standard generators of the braid group B_n, which satisfy the relations

$$s_i s_j = s_j s_i \ \ , \ \ |i - j| \geq 2 \ ,$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \ .$$

Let $\pi \colon B_n \to S_n$ denote the homomorphism such that

$$\pi(s_i) = \sigma_i \ ,$$

where σ_i is the ith elementary transposition. Then an element $R \in End(V \otimes V)$ is a Yang-Baxter operator if and only if for all n the map $s_i \mapsto R_i$ extends to a representation ρ of B_n on $V^{\otimes n}$. The Yang-Baxter operator R is strong if and only if for all n the representation ρ factors through S_n, that is, $\rho = \rho' \pi$ for some $\rho' \colon S_n \to End(V^{\otimes n})$.

Define the element $s_{nm} \in B_{n+m}$ by

$$s_{nm} = (s_m \cdots s_1)(s_{m+1} \cdots s_2) \cdots (s_{n+m-1} \cdots s_n) \; .$$

Pictorially, this element can be represented as the braid in Figure 1, which makes it clear that $\pi(s_{nm}) \in S_{n+m}$ is the permutation

$$(1, \ldots, n+m) \mapsto (n+1, \ldots, n+m, 1, \ldots, n) \; .$$

Let $m \colon A \otimes A \to A$ be the multiplication map. We define an *r-algebra* to be an algebra A equipped with a Yang-Baxter operator such that

$$R(1 \otimes a) = a \otimes 1 \; , \quad R(a \otimes 1) = 1 \otimes a$$

for all $a \in A$, and the following diagram commutes:

$$\begin{array}{ccc} A^{\otimes 4} & \xrightarrow{\rho(s_{22})} & A^{\otimes 4} \\ {\scriptstyle m \otimes m} \downarrow & & \downarrow {\scriptstyle m \otimes m} \\ A^{\otimes 2} & \xrightarrow{R} & A^{\otimes 2} \end{array} \qquad (1)$$

A Yang-Baxter operator on an algebra A satisfying these conditions will be called an *r-structure* for A. If in addition the Yang-Baxter operator is strong, we call it a *strong* r-structure for A, and say that A is a *strong* r-algebra. We say that A is *r-commutative* if $m = mR$. The reader may verify that every $\mathbf{Z}_2$-graded algebra is a strong r-algebra with r-structure given by

$$R(a \otimes b) = (-1)^{\deg a \, \deg b} \, b \otimes a \; ,$$

and that in this case r-commutativity is equivalent to graded commutativity.

The commutative diagram (1) deserves some comment. Roughly speaking, it describes how to move a product $ab \in A$ to the right of $cd \in A$ if we know how to move the factors a and b to the right of c and d. It has a pictorial interpretation given in Figure 2, where the joining of two strands denotes multiplication. The following lemma gives an alternate formulation in terms of equations (3), which are dual to the axioms for a quasi-triangular Hopf algebra [11]. Figure 3 gives a pictorial interpretation of these equations.

Lemma 1. *Suppose that A is an algebra, and that $R\colon A\otimes A\to A\otimes A$ is a Yang-Baxter operator. Then R is an r-structure for A if and only if*

$$R(1\otimes a)=a\otimes 1\ ,\quad R(a\otimes 1)=1\otimes a\ , \tag{2}$$

and

$$R(m\otimes \mathrm{id})=(\mathrm{id}\otimes m)R_1R_2\ ,\quad R(\mathrm{id}\otimes m)=(m\otimes \mathrm{id})R_2R_1\ , \tag{3}$$

as maps from $A^{\otimes 3}$ to $A^{\otimes 2}$.

Proof - Note that $R_1R_2=\rho(s_{21})$ and $R_2R_1=\rho(s_{12})$. Let id_n denote the identity on $A^{\otimes n}$, and note that

$$\begin{aligned}\rho(s_{22}) &= (\mathrm{id}_1\otimes\rho(s_{21}))(\rho(s_{21})\otimes \mathrm{id}_1)\\ &= (\rho(s_{12})\otimes \mathrm{id}_1)(\mathrm{id}_1\otimes\rho(s_{12}))\ .\end{aligned}$$

Given the identities (1) and (2) we have

$$\begin{aligned}(\mathrm{id}_1\otimes m)\rho(s_{21})(a\otimes b\otimes c) &= (m\otimes m)(\mathrm{id}_1\otimes\rho(s_{21}))(1\otimes a\otimes b\otimes c)\\ &= (m\otimes m)(\mathrm{id}_1\otimes\rho(s_{21}))(\rho(s_{21})\otimes \mathrm{id}_1)(a\otimes b\otimes 1\otimes c)\\ &= (m\otimes m)\rho(s_{22})(a\otimes b\otimes 1\otimes c)\\ &= R(m\otimes m)(a\otimes b\otimes 1\otimes c)\\ &= R(m\otimes \mathrm{id}_1)(a\otimes b\otimes c)\ .\end{aligned}$$

The other part of (3) follows similarly.

Conversely, given equations (3) we have

$$\begin{aligned}(m\otimes m)\rho(s_{22}) &= (\mathrm{id}_1\otimes m)(m\otimes \mathrm{id}_2)(\rho(s_{12})\otimes \mathrm{id}_1)(\mathrm{id}_1\otimes\rho(s_{12}))\\ &= (\mathrm{id}_1\otimes m)(R\otimes \mathrm{id}_1)(\mathrm{id}_1\otimes m\otimes \mathrm{id}_1)(\mathrm{id}_1\otimes\rho(s_{12}))\\ &= (\mathrm{id}_1\otimes m)(R\otimes \mathrm{id}_1)(\mathrm{id}_1\otimes R)(\mathrm{id}_2\otimes m)\\ &= (\mathrm{id}_1\otimes m)\rho(s_{21})(\mathrm{id}_2\otimes m)\\ &= R(m\otimes \mathrm{id}_1)(\mathrm{id}_2\otimes m)\\ &= R(m\otimes m)\ . \qquad \square\end{aligned}$$

There is a simple formula for moving the product $a_1\cdots a_j$ to the right of the product $b_1\cdots b_k$ in an r-algebra.

Lemma 2. *Suppose A is an algebra with r-structure R. For any $j\geq 1$, let $m_j\colon A^{\otimes j}\to A$ denote the map given by $m_j(a_1\otimes\cdots\otimes a_j)=a_1a_2\cdots a_j$. Then given $j,k\geq 1$ and $a_1,\dots a_j,b_1,\dots b_k\in A$, we have*

$$R(a_1\cdots a_j\otimes b_1\cdots b_k)=(m_k\otimes m_j)\rho(s_{jk})(a_1\otimes\cdots\otimes a_j\otimes b_1\otimes\cdots\otimes b_k)\ ,$$

where ρ denotes the representation of B_{j+k} on $A^{\otimes(j+k)}$ determined by R.

Proof — We prove that

$$R(m_j \otimes m_k) = (m_k \otimes m_j)\rho(s_{jk})$$

for all $j, k \geq 1$ by induction. The case $j, k = 1$ simply says that $R = R$. Assume as an inductive hypothesis that $R(m_j \otimes m_k) = (m_k \otimes m_j)\rho(s_{jk})$ for all $j \leq J$, $k \leq K$. It suffices to prove that for such j, k we have

$$R(m_{j+1} \otimes m_k) = (m_k \otimes m_{j+1})\rho(s_{j+1,k})$$

and

$$R(m_j \otimes m_{k+1}) = (m_{k+1} \otimes m_j)\rho(s_{j,k+1}) .$$

We only prove the first, as the second is analogous. By Lemma 1 we have

$$\begin{aligned} R(m_j \otimes m_k) &= R(m \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes m_j \otimes m_k) \\ &= (\mathrm{id}_1 \otimes m)\rho(s_{21})(\mathrm{id}_1 \otimes m_j \otimes m_k) \\ &= (\mathrm{id}_1 \otimes m)(R \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes R)(\mathrm{id}_1 \otimes m_j \otimes m_k) , \end{aligned}$$

so by the inductive hypothesis

$$R(m_j \otimes m_k) = (\mathrm{id}_1 \otimes m)(R \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes m_k \otimes m_j)(\mathrm{id}_1 \otimes \rho(s_{jk})) ,$$

and the inductive hypothesis also implies that $R(\mathrm{id}_1 \otimes m_k) = (m_k \otimes \mathrm{id}_1)\rho(s_{1k})$, so

$$\begin{aligned} R(m_j \otimes m_k) &= (\mathrm{id}_1 \otimes m)(m_k \otimes m_j)(\rho(s_{1k}) \otimes \mathrm{id}_j)(\mathrm{id}_1 \otimes \rho(s_{jk})) \\ &= (m_k \otimes m_{j+1})\rho(s_{j+1,k}) \end{aligned}$$

as desired. □

As a consequence, an r-structure R on A is determined by its action on any subspace $V \subseteq A$ generating A with $R(V \otimes V) \subseteq V \otimes V$.

Lemma 3. *Suppose an algebra A is generated by a subspace $V \subseteq A$. Given a Yang-Baxter operator R on V, there exists at most one r-structure on A, say $\widetilde{R} \in End_k(A \otimes A)$, extending $R \in End_k(V \otimes V)$. If R is strong then $\widetilde{R}$ is strong if it exists.*

Proof - Uniqueness is an immediate corollary of Lemma 2. If R is strong then $s_{kj}s_{jk}$ acts as the identity on $V^{\otimes(j+k)}$ for all $j, k \geq 1$, so Lemma 2 implies

$$\begin{aligned} \widetilde{R}^2(v_1 \cdots v_j \otimes w_1 \cdots w_k) &= \widetilde{R}(m_k \otimes m_j)\rho(s_{jk})(v_1 \cdots v_j \otimes w_1 \cdots w_k) \\ &= (m_j \otimes m_k)\rho(s_{jk})\rho(s_{kj})(v_1 \cdots v_j \otimes w_1 \cdots w_k) \\ &= v_1 \cdots v_j \otimes w_1 \cdots w_k \end{aligned}$$

for all $v_1, \ldots, v_j, w_1, \ldots, w_k \in V$, so so $\widetilde{R}$ is strong. □

We define an *r-ideal* of an r-algebra A to be a two-sided ideal $I \subseteq A$ such that R preserves $I \otimes A + A \otimes I$. Given algebras A and B with r-structures R_A and R_B, respectively, we say $f \colon A \to B$ is an *r-morphism* if f is a homomorphism and $(f \otimes f)R_A = R_B(f \otimes f)$.

Lemma 4. *Let A be an r-algebra with an r-ideal I. Then there is a unique r-structure on A/I such that the quotient map $j: A \to A/I$ is an r-morphism. Conversely, the kernel of any r-morphism $f: A \to B$ is an r-ideal in A.*

Proof - Suppose I is an r-ideal of A. If R' is a strong Yang-Baxter operator on A/I such that j is an r-morphism, we must have

$$R'(j \otimes j)x = (j \otimes j)Rx$$

for all $x \in A \otimes A$. To show that this formula defines an element $R' \in End_k(A/I \otimes A/I)$ it suffices to note that the kernel of $j \otimes j$ is $I \otimes A + A \otimes I$, which is preserved by R, so that $\ker(j \otimes j) \subseteq \ker(j \otimes j)R$. One may easily check that R' is an r-structure for A/I, and that j becomes an r-morphism relative to this r-structure.

Conversely, suppose $f: A \to B$ is an r-morphism. Since $R_B(f \otimes f) = (f \otimes f)R_A$, it follows that $\ker(f \otimes f)$ is preserved by R_A. Since $\ker(f \otimes f) = \ker f \otimes A + A \otimes \ker f$, it follows that $\ker f$ is an r-ideal. $\square$

Note that the quotient of a strong r-algebra by an r-ideal is strong, and the quotient of an r-commutative algebra by an r-ideal is r-commutative. In parallel to Lemma 3, we also have:

Lemma 5. *Let A and B be r-algebras with r-structure R_A and R_B, respectively. A homomorphism $f: A \to B$ is an r-morphism if $(f \otimes f)R_A(v \otimes w) = R_B(f \otimes f)(v \otimes w)$ for all $v, w \in V$, where $V \subseteq A$ is a subspace generating A with $R(V \otimes V) \subseteq V \otimes V$.*

Proof - It suffices to show that $(f \otimes f)R_A(a \otimes b) = R_B(f \otimes f)(a \otimes b)$ for $a = v_1 \cdots v_j$, $b = w_1 \cdots w_k$, where $v_1, \ldots, v_j, w_1, \ldots, w_k \in V$. Let ρ_A and ρ_B denote the representations of the braid group determined by R_A and R_B. By Lemma 2 we have

$$\begin{aligned} (f \otimes f)R_A(a \otimes b) &= (f \otimes f)(m_k \otimes m_j)\rho_A(s_{jk})(v_1 \otimes \cdots \otimes w_k) \\ &= (m_k \otimes m_j)f^{\otimes(j+k)}\rho_A(s_{jk})(v_1 \otimes \cdots \otimes w_k) \\ &= (m_k \otimes m_j)\rho_B(s_{jk})f^{\otimes(j+k)}(v_1 \otimes \cdots \otimes w_k) \\ &= R_B(m_j \otimes m_k)f^{\otimes(j+k)}(v_1 \otimes \cdots \otimes w_k) \\ &= R_B(f \otimes f)(a \otimes b) \ . \qquad \square \end{aligned}$$

We now turn to an analog of the classical differential calculus $\Omega_c(A)$ for strong r-commutative r-algebras. Let A be an r-algebra, and let $\Omega_u(A)$ be the universal differential calculus over A [6, 16]. Recall that this may be defined by the property that for any differential graded algebra Ω and any homomorphism $f: A \to \Omega^0$, there exists a unique differential graded algebra morphism $\widetilde{f}: \Omega_u(A) \to \Omega$ extending f. We define the algebra of *differential forms* over A, $\Omega_R(A)$, to be the quotient of $\Omega_u(A)$ by the differential ideal generated by $d1$ together with all elements of the form

$$adb - \sum_i (db^i)a_i \ ,$$

where $a, b \in A$ and $R(a \otimes b) = \sum_i b^i \otimes a_i$. Clearly $\Omega_R(A)$ is a differential calculus for A. Just as a smooth mapping of manifolds induces a homomorphism of differential forms, we have:

Lemma 6. *Let A and B be r-algebras and $f: A \to B$ an r-morphism. Then there exists a unique morphism of differential graded algebras $f_*: \Omega_R(A) \to \Omega_R(B)$ extending f. Given r-morphisms $f: A \to B$ and $g: B \to C$, we have $(gf)_* = g_* f_*$.*

Proof - By the universal property of $\Omega_u(A)$, there is a unique morphism of differential graded algebras $\widetilde{f}: \Omega_u(A) \to \Omega_R(B)$ extending f, so it suffices for the first claim to show that $\ker \widetilde{f}$ contains $d1$ and the elements $adb - \sum db^i\, a_i$ for all $a, b \in A$. This follows directly from the definition of $\Omega_R(B)$. For the second claim, suppose $f: A \to B$ and $g: B \to C$ are r-morphisms. Since $\Omega_R(A)$ is generated as a differential graded algebra by A, and since f_* and g_* are differential graded algebra morphisms, we must have $(gf)_* = g_* f_*$. □

We emphasize that while $\Omega_R(A)$ is well-defined for any r-algebra A, it is the correct generalization of the classical differential calculus only if A is r-commutative and strong. We defer remarks on the non-strong case to section 6.

3 R-symmetric and R-Weyl algebras

We now introduce two fundamental examples of r-commutative algebras. The first, variously called the "Yang-Baxter" or "Zamolodchikov" algebra [19, 20], is a kind of universal r-commutative algebra. As it generalizes the symmetric algebra, we prefer to call it the "r-symmetric algebra."

Lemma 7. *Let R be a strong Yang-Baxter operator on a vector space V. Then TV has a unique r-algebra structure $\widetilde{R} \in End(TV \otimes TV)$ extending $R \in End(V \otimes V)$. If R is strong then $\widetilde{R}$ is strong.*

Proof - Define $\widetilde{R}: TV \otimes TV \to TV \otimes TV$ such that for all $a \in V^{\otimes n}$ and $b \in V^{\otimes m}$,

$$\widetilde{R}(a \otimes b) = \rho(s_{nm})(a \otimes b) \in V^{\otimes m} \otimes V^{\otimes n} .$$

Note that here we are first identifying $a \otimes b \in V^{\otimes n} \otimes V^{\otimes m}$ with an element of $V^{\otimes(n+m)}$, and then identifying $\rho(s_{nm})(a \otimes b) \in V^{\otimes(n+m)}$ with an element of $V^{\otimes m} \otimes V^{\otimes n}$, so that $\widetilde{R}: V^{\otimes n} \otimes V^{\otimes m} \to V^{\otimes m} \otimes V^{\otimes n}$. As maps from $V^{\otimes \ell} \otimes V^{\otimes m} \otimes V^{\otimes n}$ to $V^{\otimes n} \otimes V^{\otimes m} \otimes V^{\otimes \ell}$, we have

$$\widetilde{R}_1 \widetilde{R}_2 \widetilde{R}_1 = (\rho(s_{mn}) \otimes \mathrm{id})(\mathrm{id} \otimes \rho(s_{\ell n})(\rho(s_{\ell m}) \otimes \mathrm{id})$$

while

$$\widetilde{R}_2 \widetilde{R}_1 \widetilde{R}_2 = (\mathrm{id} \otimes \rho(s_{mn}))(\rho(s_{\ell n}) \otimes \mathrm{id})(\mathrm{id} \otimes \rho(s_{\ell m}) ,$$

and one may check that these are equal using either the braid group relations or the pictorial representation of Figure 1. It is straightforward to check that $\widetilde{R}$ makes TV into an r-algebra. Uniqueness follows from Lemma 3, as does the fact that $\widetilde{R}$ is strong if R is strong. $\square$

Lemma 8. *Let R be a Yang-Baxter operator on the vector space V. Let TV be given the r-algebra structure $\widetilde{R}$ as in Lemma 1. Let $I \subseteq TV$ be the ideal generated by all elements of the form $v \otimes w - R(v \otimes w)$ for $v, w \in V$. Then I is an r-ideal and TV/I is an r-commutative algebra.*

Proof - Any element of $I \otimes A$ is a linear combination of those of the form $a \otimes b$, where

$$a = v_1 \otimes \cdots \otimes (1 - R)(v_i \otimes v_{i+1}) \otimes \cdots \otimes v_n \ ,$$

$$b = v_{n+1} \otimes \cdots\cdots \otimes v_{n+m} \ .$$

We have

$$a \otimes b = (1 - \rho(s_i))(v_1 \otimes \cdots \otimes v_{n+m}) \ ,$$

so

$$\begin{aligned} \widetilde{R}(a \otimes b) &= \rho(s_{nm})(1 - \rho(s_i))(v_1 \otimes \cdots \otimes v_{n+m}) \\ &= (1 - \rho(s_{m+i}))\rho(s_{nm})(v_1 \otimes \cdots \otimes v_{n+m}) \ . \end{aligned}$$

It follows that $\widetilde{R}(a \otimes b)$ is a linear combination of elements of the form

$$(1 - \rho(s_{m+i}))(w_1 \otimes \cdots \otimes w_{n+m}) \ ,$$

which lie in $A \otimes I$. Thus $R(I \otimes A) \subseteq A \otimes I$; a similar argument shows that $R(A \otimes I) \subseteq I \otimes A$, so that I is an r-ideal.

Next let us show that TV/I is r-commutative. The algebra TV/I is spanned by elements a, b of the form

$$a = [v_1 \otimes \cdots \otimes v_n] \ , \quad b = [v_{n+1} \otimes \cdots \otimes v_{n+m}] \ .$$

We have

$$m\widetilde{R}(a \otimes b) = [\rho(s_{nm})(v_1 \otimes \cdots \otimes v_{n+m})] \ .$$

Since s_{nm} is a product of generators $s_i \in B_{n+m}$, and since

$$\begin{aligned} [\rho(s_i)(w_1 \otimes \cdots \otimes w_{n+m})] &= [w_1 \otimes \cdots R(w_i \otimes w_{i+1}) \otimes \cdots \otimes w_{n+m}] \\ &= [w_1 \otimes \cdots w_i \otimes w_{i+1} \otimes \cdots \otimes w_{n+m}] \end{aligned}$$

for all $w_1, \ldots, w_{n+m} \in V$, we have

$$m\widetilde{R}(a \otimes b) = [v_1 \otimes \cdots \otimes v_{n+m}] = m(a \otimes b) \ . \ \square$$

Given a vector space V equipped with a Yang-Baxter operator, we denote the r-commutative algebra TV/I constructed in the above lemma by $S_R V$, the *r-symmetric algebra* over V. As the quotient of TV by a homogeneous ideal, $S_R V$ is naturally a graded algebra. When R is the twist map $\tau(v \otimes w) = w \otimes v$, we have $S_R V = SV$, while if $R = -\tau$, $S_R V$ is the exterior algebra ΛV. Just as the symmetric algebra SV may be identified with the coordinate ring of a vector space, the algebra $S_R V$ may be regarded as the coordinate ring of a "quantum vector space" [20]. Similarly, elements of $\Omega_R(S_R(V))$ may be regarded as differential forms on a quantum vector space.

Just as the exterior and symmetric algebras are special cases of r-symmetric algebras, both the Clifford and Weyl algebras are "r-Weyl algebras." Let V be a vector space equipped with a Yang-Baxter operator R. We say that a bilinear form $\omega: V \otimes V \to k$ is a *skew form* on V if $\omega \circ R = -\omega$. Given a skew form ω on V, we define the *r-Weyl algebra* $W_R(V, \omega)$ to be the quotient of the tensor algebra TV by the ideal I generated by the elements

$$v \otimes w - w \otimes v - \omega(v \otimes w)1$$

for all $v, w \in V$, where 1 denotes the identity of TV.

In addition to the Clifford, Weyl, and unified Clifford-Weyl algebras, a few other r-Weyl algebras have already been studied. For example, Arik and Coon [1], and more recently Goodearl [14], Morikawa [21], and Gelfand and Fairlie [13] have considered a "q-deformed" Weyl algebra which is a special case of our r-Weyl algebra. Namely, if one takes $V = k \oplus k$ with the basis $\{v, w\}$, and defines $R \in End(V \otimes V)$ to be the strong Yang-Baxter operator such that

$$R(x \otimes x) = x \otimes x \ , \quad R(y \otimes y) = y \otimes y \ , \quad R(x \otimes y) = qy \otimes x \ ,$$

where $q \neq 0$, there is a unique skew map $\omega: V \otimes V \to k$ with $\omega(x, y) = 1$, and the r-Weyl algebra $W_R(V, \omega)$ is isomorphic to $k[x, y]/\langle xy - qxy - 1\rangle$.

Note that if $\omega = 0$ the r-Weyl algebra $W_R(V, \omega)$ is just the r-symmetric algebra $S_R V$. Interestingly, the r-Weyl algebra admits an r-commutative r-structure that reduces to the standard one on $S_R V$ in the special case $\omega = 0$.

Theorem 1. *Let V be a vector space, R a Yang-Baxter operator on V, and ω a skew form on V. Then there is a unique r-structure $\widetilde{R}$ on $W_R(V, \omega)$ such that*

$$\widetilde{R}(v \otimes w) = R(v \otimes w) + \omega(v \otimes w)\,(1 \otimes 1) \ .$$

Moreover, the r-structure $\widetilde{R}$ is r-commutative, and $\widetilde{R}$ is strong if R is strong.

Proof - The uniqueness of $\widetilde{R}$ follows from Lemma 3, since the span of V and 1 in $W_R(V, \omega)$ is a subspace U generating $W_R(V, \omega)$, with $\widetilde{R}(U \otimes U) = U \otimes U$. For existence, it is useful to construct an r-symmetric algebra containing a formal variable x, which specializes to $W_R(V, \omega)$ when we set x equal to $1 \in k$. Let $V' = V \oplus k$, and let $x = (0,1) \in V'$. One easily checks that there is a unique Yang-Baxter operator R' on V' given by

$$R'((v + \alpha x) \otimes (w + \beta x)) =$$

$$R(v \otimes w) + w \otimes \alpha x + \beta x \otimes v + (\alpha\beta + \omega(v \otimes w))(x \otimes x)$$

for all $v, w \in V$, $\alpha, \beta \in k$. The r-symmetric algebra $S_{R'}(V')$ is automatically r-commutative, and is strong if R, hence R', is strong.

Let I be the ideal in $S_{R'}(V')$ generated by $x - 1$. We claim that I is an r-ideal. The relations above imply that $x-1$ is central in $S_{R'}(V')$, so I is spanned by elements of the form $(x-1)v_2 \cdots v_j$, where $v_2, \ldots, v_j \in V'$. Taking $w_1, \ldots, w_k \in V'$, we have

$$R'((x-1)v_2 \cdots v_j \otimes w_1 \cdots w_k) =$$

$$(m_k \otimes m_k)\rho(s_{jk})((x-1) \otimes v_2 \otimes \cdots \otimes v_j \otimes w_1 \otimes \cdots \otimes w_k)$$

by Lemma 2. Since $R'((x-1) \otimes v) = v \otimes (x-1)$ for all $v \in V'$, the right-hand side above is a linear combination of elements of the form

$$y_1 \cdots y_k \otimes (x-1)x_2 \cdots x_j$$

for $x_i, y_i \in V'$. This implies that $R(A \otimes I) \subseteq I \otimes A$. A similar argument implies that $R(I \otimes A) \subseteq A \otimes I$, so I is an r-ideal. Note that $S_{R'}(V')$ is the quotient of TV' by the relations

$$v \otimes w - R(v \otimes w) = \omega(v \otimes w)(x \otimes x) \ , \quad v \otimes x = x \otimes v$$

for all $v, w \in V$, hence $S_{R'}(V')/I$ is the quotient of TV' by the above relations together with the relation $x = 1$. Thus there is a unique isomorphism $S_{R'}(V')/I \cong W_R(V, \omega)$ such that the following diagram commutes:

$$\begin{array}{ccc} TV' & \xrightarrow{x \mapsto 1} & TV \\ \downarrow & & \downarrow \\ S_{R'}(V')/I & \xrightarrow{\sim} & W_R(V, \omega) \end{array}$$

We use this isomorphism to transfer the quotient r-structure on $S_{R'}(V')/I$ to $W_R(V, \omega)$. Denoting this r-structure on $W_R(V, \omega)$ by $\widetilde{R}$, it is clear that this r-structure is r-commutative, is strong if R is strong, and satisfies

$$\widetilde{R}(v \otimes w) = R(v \otimes w) + \omega(v \otimes w)\,(1 \otimes 1)\ . \qquad \square$$

4 Quantized Differential Forms

In classical mechanics the observables of a physical system are typically represented by a *Poisson algebra*, a commutative algebra A equipped with a Lie bracket $\{\cdot,\cdot\}$ such that $\{a,\cdot\}$ is a derivation of A for any $a \in A$. In quantum theory the observables of physical system are typically modeled by a noncommutative algebra. (In physics these are algebras over $\mathbf{C}$, but we work over an arbitrary field k.) One way of obtaining such algebras is by quantizing Poisson algebras. There are many variations on this theme, such as formal quantization [3, 18] and deformation quantization [24], but perhaps the simplest is algebraic quantization. An *algebraic quantization* of the Poisson algebra A is an associative product $*$ on $A[x]$ such that

$$a * b \equiv ab \bmod x$$

and

$$a * b - b * a \equiv x\{a, b\} \bmod x^2$$

for all $a, b \in A$, and

$$x * a = a * x = xa$$

for all $a \in A[x]$. We write $\widetilde{A}$ for $A[x]$ equipped with the product $*$. One may effectively assign the variable x any value $\hbar \in k$ by forming the quotient $A_\hbar = \widetilde{A}/\langle x - \hbar\rangle$. Note that $A_0 = A$ is commutative, while $A_\hbar$ is generally noncommutative for $\hbar \neq 0$.

Since the noncommutativity of the quantization $\widetilde{A}$ is "controlled" by the Poisson bracket in A, the algebras $\widetilde{A}$ and $A_\hbar$ may well be r-commutative with the quotient map $j\colon \widetilde{A} \to A_\hbar$ an r-morphism. This allows the systematic study of the geometry of the commutative algebra A, its algebraic quantization $\widetilde{A}$ and the specializations $A_\hbar$. In the spirit of algebraic geometry, one may regard a particular value of $\hbar$ as a point in the "line" $k[x]$, and picture $\widetilde{A}$ as a fiber bundle over the line, with fiber over $x = \hbar$ equal to $A_\hbar$. Since the line is contractible, one may expect the map $j_*\colon \Omega_R(\widetilde{A}) \to \Omega_R(A_\hbar)$ to induce an isomorphism on cohomology for any $\hbar \in k$. If this is the case, the cohomology $H(\Omega_R(A_\hbar))$ of any fiber will equal that of the "classical fiber" $A = A_0$. In this section we use these ideas to give a new proof of Segal's "Poincaré lemma" for differential forms on the Weyl algebra [26].

Now we turn to the Weyl algebra. Let V be a vector space and ω an antisymmetric bilinear form on V. We write x for $(0, 1) \in V \oplus k$. The space $V \oplus k$ is a Lie algebra, essentially the Lie algebra of the Heisenberg group, with bracket given by

$$[v + \alpha x, w + \beta x] = \omega(v, w)x$$

for all $v, w \in V$, $\alpha, \beta \in k$. Let the *Heisenberg algebra* $\widetilde{A}$ over V denote the universal enveloping algebra of $V \oplus k$. Given $\hbar \in k$, let *Weyl algebra* $A_\hbar$ over V denote the quotient of $\widetilde{A}$ by the ideal generated by $x - \hbar$. Let $j\colon \widetilde{A} \to A_\hbar$ denote the quotient map. We write simply A for A_0; note that A is the symmetric algebra SV.

We now give the Heisenberg and Weyl algebras r-structures such that $j\colon \widetilde{A} \to A_\hbar$ is an r-morphism. By Theorem 1, $A_\hbar$ has a unique r-structure R such that

$$R(v \otimes w) = w \otimes v + \hbar\omega(v,w)(1 \otimes 1)$$

for all $v, w \in V$, and this r-structure is strong and r-commutative. By the following theorem, if k is not of characteristic 2 then there is a unique r-structure $\widetilde{R}$ on $\widetilde{A}$ such that

$$\widetilde{R}((v + \alpha x) \otimes (w + \beta x)) = (w + \beta x) \otimes (v + \alpha x) + \frac{1}{2}\omega(v,w)(x \otimes 1 + 1 \otimes x)$$

for all $v, w \in V$ and $\alpha, \beta \in k$, and $\widetilde{R}$ is strong and r-commutative.

Theorem 2. *Let $\underline{\mathbf{g}}$ be a Lie algebra over a field k not of characteristic 2, with $[\underline{\mathbf{g}}, [\underline{\mathbf{g}}, \underline{\mathbf{g}}]] = 0$. Then there exists a unique r-structure R on the universal enveloping algebra $U\underline{\mathbf{g}}$ such that*

$$R(v \otimes w) = w \otimes v + \frac{1}{2}([v,w] \otimes 1 + 1 \otimes [v,w]) \ .$$

Moreover, this r-structure is strong and r-commutative.

Proof - Uniqueness follows from Lemma 2, since the span of $\underline{\mathbf{g}}$ and 1 in $U\underline{\mathbf{g}}$ is a subspace generating $U\underline{\mathbf{g}}$ whose tensor product with itself is preserved by R. For existence, let $L = \underline{\mathbf{g}} \oplus k$, and write e for $(0,1) \in L$. One may verify by explicit calculation that there is a strong Yang-Baxter operator R' on L given by

$$R'((v + \alpha e) \otimes (w + \beta e)) = (w + \beta e) \otimes (v + \alpha e) + \frac{1}{2}([v,w] \otimes e + e \otimes [v,w])$$

for all $v, w \in V$ and $\alpha, \beta \in k$. This calculation uses the fact that $[\underline{\mathbf{g}}, [\underline{\mathbf{g}}, \underline{\mathbf{g}}]] = 0$. Let us also use R' to denote the r-structure in $S_{R'}(L)$. Noting that $R'((e-1) \otimes u) = u \otimes (e-1)$ for all $u \in L$, it follows as in the proof of Theorem 1 that the ideal $I \subseteq S_{R'}(L)$ generated by $e - 1$ is an r-ideal. Note that $S_{R'}(L)$ is the quotient of TL by the ideal generated by the elements

$$v \otimes w - w \otimes v - \frac{1}{2}([v,w] \otimes e + e \otimes [v,w]) \ , \quad v \otimes e - e \otimes v$$

for all $v, w \in V$. It follows that $S_{R'}(L)/I$ is isomorphic to the quotient of $T\underline{\mathbf{g}}$ by the ideal generated by the elements

$$v \otimes w - w \otimes v - [v,w]$$

for all $v, w \in \underline{\mathbf{g}}$. This gives a natural isomorphism $S_R(L)/I \cong U\underline{\mathbf{g}}$, which we may use to endow $U\underline{\mathbf{g}}$ with an r-structure with the desired properties. $\square$

Corollary 1. *Let V be a vector space over a field k not of characteristic 2, and let ω be an anti-symmetric bilinear form on V. Then for any $\hbar \in k$ the quotient map $j: \widetilde{A} \to A_\hbar$ from the Heisenberg algebra to the Weyl algebra over V is an r-morphism.*

Proof - This follows from Lemma 5 and the calculation

$$\begin{aligned}(j \otimes j)\widetilde{R}((v + \alpha x) \otimes (w + \beta x)) &= (j \otimes j)((w + \beta x) \otimes (v + \alpha x) + \frac{1}{2}\omega(v, w)(x \otimes 1 + 1 \otimes x)) \\ &= (w + \beta\hbar) \otimes (v + \alpha\hbar) + \hbar\omega(v, w)(1 \otimes 1) \\ &= R((v + \alpha\hbar) \otimes (w + \beta\hbar)) \\ &= R(j \otimes j)((v + \alpha x) \otimes (w + \beta x)) \ . \qquad \square\end{aligned}$$

We now consider differential forms on the Heisenberg and Weyl algebras. We write simply $\Omega(\widetilde{A})$ and $\Omega(A_\hbar)$ for the differential forms on these algebras, suppressing reference to the r-structures involved. One easily verifies that $\Omega(\widetilde{A})$ is the quotient of $\Omega_u(\widetilde{A})$ by the differential ideal generated by the relations

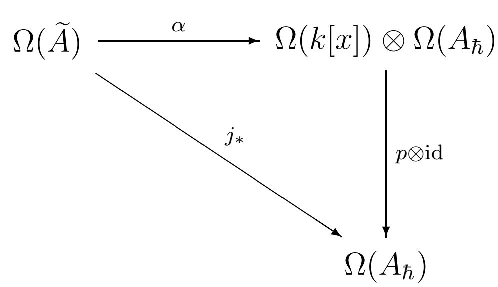
$$vdw - wdv = \frac{1}{2}\omega(v, w)dx \ , \quad xdv = (dv)x \ , \quad vdx = (dx)v \ , \quad xdx = (dx)x$$

for all $v, w \in V$. The quotient map $j: \widetilde{A} \to A_\hbar$ induces a surjection $j_*: \Omega(\widetilde{A}) \to \Omega(A_\hbar)$ with kernel generated as an ideal by $x - \hbar$ and dx. Moreover, $\Omega(A_\hbar)$ is the quotient of $\Omega_u(A_\hbar)$ by the differential ideal generated by the relations

$$vdw = (dw)v$$

for all $v, w \in V$. Taking $\hbar = 1$, elements of $\Omega(A_\hbar)$ are precisely Segal's "quantized differential forms." Taking $\hbar = 0$, $\Omega(A_0) = \Omega(A)$ is isomorphic to the algebra of algebraic differential forms on V^*.

Our structure theorem for the differential forms on $\widetilde{A}$ and $A_\hbar$ is motivated by the fiber bundle picture described above. If the "total space" $\widetilde{A}$ were simply a product of the "fiber" $A_\hbar$ and the line $k[x]$, one would expect there to be a differential graded algebra isomorphism $\alpha: \Omega(\widetilde{A}) \to \Omega(k[x]) \otimes \Omega(A_\hbar)$ such that the following diagram commutes:



Here $\Omega(k[x])$ denotes the differential forms on $k[x]$ equipped with the twist map as its r-structure, and $p: \Omega(k[x]) \to k$ is the homomorphism determined by $p(x) = \hbar$ and

$p(dx) = 0$. This conjecture turns out to be only sightly over-optimistic. Note that when $k[x]$ is equipped with the twist map as an r-structure, the natural inclusion $k[x] \hookrightarrow \widetilde{A}$ is an r-morphism, so it induces a differential graded algebra morphism $\Omega(k[x]) \to \Omega(\widetilde{A})$. Thus $\Omega(\widetilde{A})$ becomes an $\Omega(k[x])$-bimodule.

Theorem 3. *For any $\hbar \in k$, there is a map $\alpha\colon \Omega(\widetilde{A}) \to \Omega(k[x]) \otimes \Omega(A_\hbar)$, an isomorphism of differential complexes and of $\Omega(k[x])$-bimodules, such that the following diagram commutes:*

$$\begin{array}{ccc} \Omega(\widetilde{A}) & \xrightarrow{\ \alpha\ } & \Omega(k[x]) \otimes \Omega(A_\hbar) \\ & \searrow^{j_*} & \downarrow{\scriptstyle p\otimes \mathrm{id}} \\ & & \Omega(A_\hbar) \end{array}$$

Proof - It follows from the Diamond Lemma [5] that elements of the form

$$x^{i_0}(dx)^{j_0}\, e_1^{i_1}(de_1)^{j_1} \cdots e_n^{i_n}(de_n)^{j_n}$$

are a basis for $\Omega(\widetilde{A})$. Let $\alpha\colon \Omega(\widetilde{A}) \to \Omega(k[x]) \otimes \Omega(A_\hbar)$ be defined by

$$\alpha\left(x^{i_0}(dx)^{j_0}\, e_1^{i_1}(de_1)^{j_1} \cdots e_n^{i_n}(de_n)^{j_n}\right) = x^{i_0}(dx)^{j_0} \otimes e_1^{i_1}(de_1)^{j_1} \cdots e_n^{i_n}(de_n)^{j_n} \ .$$

One may check that α is a morphism of differential complexes and $\Omega(k[x])$-bimodules by explicit calculation. It is also easy to check that $(p \otimes \mathrm{id})\alpha = j_*$. To show that α is one-to-one and onto, it suffices to note, again using the Diamond Lemma, that elements of the form

$$x^{i_0}(dx)^{j_0} \otimes e_1^{i_1}(de_1)^{j_1} \cdots e_n^{i_n}(de_n)^{j_n}$$

are a basis for $\Omega(k[x]) \otimes \Omega(A_\hbar)$. □

We emphasize that α is not natural, as it depends on the choice of ordered basis e_i, nor is it an algebra homomorphism. Theorem 3 has as a corollary a "Poincaré lemma" for quantized differential forms. In the case $k = \mathbf{R}$, the part of this corollary concerning $\Omega(A_\hbar)$ was proved by Segal [26].

Corollary 2. *If the field k is of characteristic zero, the r-commutative de Rham cohomology $H^p(\Omega(\widetilde{A}))$ vanishes for $p > 0$, and equals k for $p = 0$. The same holds for $H^p(\Omega(A_\hbar))$ for any $\hbar \in k$.*

Proof - By Theorem 3 and the Künneth product formula we have $H(\Omega(\widetilde{A})) \cong H(\Omega(k[x]) \otimes H(\Omega(A_\hbar))$ for any value of $\hbar \in k$. By the Poincaré lemma for algebraic differential forms, it follows that $H^p(\Omega(k[x])$ equals k for $p=0$ and vanishes otherwise, and taking $\hbar = 0$ the same holds for $H^p(\Omega(A))$. Thus $H(\Omega(\widetilde{A}))$ equals k for $p=0$ and vanishes otherwise. Again using the Künneth formula, it follows that the same must hold for $H(\Omega(A_\hbar))$ for all $\hbar \in k$. $\Box$

We should make clear the sense in which the Heisenberg algebra $\widetilde{A}$ is an algebraic quantization of the symmetric algebra $A = SV$. Restricting $\alpha\colon \Omega(\widetilde{A}) \to \Omega(k[x]) \otimes \Omega(A)$ to elements of degree zero, we obtain a $k[x]$-module isomorphism of $\widetilde{A}$ and $A[x]$. We can use this to transfer the product in $\widetilde{A}$ to $A[x]$; call this product $*$. Then we have

$$a * b \equiv ab \bmod x \ , \quad a * b - b * a \equiv x\{a,b\} \bmod x^2$$

for all $a, b \in A$, and

$$x * a = a * x = xa$$

for all $a \in A[x]$.

It is worth noting that $\Omega(\widetilde{A})$ also arises as a cochain complex for the cohomology of the Lie algebra $V \oplus k$ with coefficients in $\widetilde{A}$. Since $V \oplus k$ is nilpotent this gives an alternate proof of Corollary 2. But, as we shall see in the next section, the approach using r-commutative geometry also works in cases which do not arise through Lie algebra cohomology.

5 Noncommutative Tori

Our definition of algebraic quantization in the previous section is not really sufficiently general. At the very least, the deformation parameter space should be allowed to be an arbitrary algebraic variety. Sometimes it is the punctured plane, that is, the Laurent polynomials $k(x)$. Here one specializes by setting x equal to any nonzero $q \in k$, regarding q as the equivalent of $e^\hbar$, so that $q = 1$ corresponds to the classical (commutative) case. This occurs in the theory of quantum groups. Starting with a Cartan subalgebra of its Lie algebra, there is a canonical way to make any complex semisimple Lie group G into a Poisson manifold [11, 12]. The algebraic functions A on G thus become a Poisson algebra, and there is a product $*$ on $A(x)$ such that

$$a * b \equiv ab \bmod (x-1) \ ,$$

$$a * b - b * a \equiv x\{a,b\} \bmod (x-1)^2$$

for all $a, b \in A$, and

$$x * a = a * x = xa$$

for all $a \in A(x)$. Writing $\widetilde{A}$ for $A(x)$ equipped with the product $*$, it turns out that the "quantum group" $\widetilde{A}$ is naturally an r-commutative algebra, as are all its specializations $A_q = \widetilde{A}/\langle x - q\rangle$.

Noncommutative tori also have as their deformation parameter space the punctured plane, or a product of copies thereof. They arise from quantizing the algebra of functions on T^n, which has translation-invariant Poisson structures of the form

$$\pi = \sum_{1 \le i < j \le n} a_{ij} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial \theta_j} \ .$$

They have been intensively studied from the C*-algebraic viewpoint [25]. Our goal here is to describe the r-commutative de Rham theory of noncommutative tori in a purely algebraic setting, making use of a "universal noncommutative torus" which has all the noncommutative n-tori as quotients.

We work over an arbitrary field k, and fix $n \ge 0$ and a collection $q = \{q_{ij}\}$, $1 \le i < j \le n$, of nonzero elements of k. We define $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$. The *noncommutative torus* T_q is the algebra generated by elements $u_1, \dots, u_n$ and their inverses, with the relations

$$u_i u_j = q_{ij}^2 u_j u_i$$

for $1 \le i < j \le n$. (The appearance of q_{ij}^2 here rather than q_{ij} is a purely technical matter.) The *universal noncommutative torus* $\widetilde{T}$ is the algebra with generators u_i, x_{ij}, and their inverses, where $1 \le i < j \le n$, with the relations

$$u_i u_j = x_{ij}^2 u_j u_i \ , \ \ x_{ij} u_k = u_k x_{ij} \ , \ \ x_{ij} x_{kl} = x_{kl} x_{ij} \ .$$

Note that the quotient of $\widetilde{T}$ by the ideal generated by all the elements $x_{ij} - q_{ij}$ is isomorphic to T_q. Let $j\colon \widetilde{T} \to T_q$ denote the quotient map. We now give $\widetilde{T}$ and T_q r-structures making this quotient map an r-morphism:

Theorem 4. *The universal noncommutative torus $\widetilde{T}$ has a unique strong r-structure $\widetilde{R}$ such that*

$$\widetilde{R}(u_i \otimes u_j) = x_{ij} u_j \otimes x_{ij} u_i \ ,$$

$$\widetilde{R}(u_i \otimes x_{jk}) = x_{jk} \otimes u_i \ , \ \ \widetilde{R}(x_{ij} \otimes x_{kl}) = x_{kl} \otimes x_{ij} \ .$$

This r-structure is r-commutative.

Proof - Let a *multi-index* be an n-tuple of integers, $I = (i_1, \dots i_n)$, and let a *double multi-index* be a family of integers $B = \{b_{ij}\}_{1 \le i < j \le n}$. Given any multi-index I, let

$$u^I = u_1^{i_1} \cdots u_n^{i_n} \ ,$$

and given a double multi-index B, let

$$x^B = \prod_{1 \le i < j \le n} x_{ij}^{b_{ij}} \ .$$

For any multi-indices I and J, there is a unique double multi-index (IJ) such that

$$u^I u^J = x^{(IJ)} u^{I+J} \ .$$

Similarly

$$u^I u^J = x^{2(I,J)} u^J u^I$$

where $2(I,J) = (IJ) - (JI)$. By the Diamond Lemma, elements of the form $x^B u^I$ form a basis of $\widetilde{T}$.

To prove the uniqueness of $\widetilde{R}$, we first determine what it does to elements of the form u_i^{-1} and x_{ij}^{-1}. Write $R(x_{ij}^{-1} \otimes x_{kl}) = \sum_{\alpha,\beta} c_{\alpha\beta} e_\alpha \otimes e_\beta$, where $c_{\alpha\beta} \in k$ and $\{e_\alpha\}$ is a basis of $\widetilde{T}$ consisting of elements of the form $x^B u^I$. Then on the one hand, equation (3) implies

$$(\mathrm{id} \otimes m)\rho(s_{21})(x_{ij}^{-1} \otimes x_{ij} \otimes x_{kl}) = R(1 \otimes x_{kl}) = x_{kl} \otimes 1 \ .$$

On the other hand,

$$\begin{aligned} (\mathrm{id} \otimes m)\rho(s_{21})(x_{ij}^{-1} \otimes x_{ij} \otimes x_{kl}) &= (\mathrm{id} \otimes m)R_1(x_{ij}^{-1} \otimes x_{kl} \otimes x_{ij}) \\ &= \sum_{\alpha,\beta} c_{\alpha,\beta} e_\alpha \otimes e_\beta x_{ij} \ . \end{aligned}$$

Comparing these, we conclude that the only nonzero term $c_{\alpha\beta} e_\alpha \otimes e_\beta$ is $x_{kl} \otimes x_{ij}^{-1}$. In other words,

$$\widetilde{R}(x_{ij}^{-1} \otimes x_{kl}) = x_{kl} \otimes x_{ij}^{-1} \ .$$

Analogous arguments imply the following:

$$\begin{aligned} \widetilde{R}(x_{ij}^{-1} \otimes x_{kl}^{-1}) &= x_{kl}^{-1} \otimes x_{ij}^{-1} \\ \widetilde{R}(u_i \otimes x_{jk}^{-1}) &= x_{jk}^{-1} \otimes u_i \\ \widetilde{R}(u_i^{-1} \otimes x_{jk}) &= x_{jk} \otimes u_i^{-1} \\ \widetilde{R}(u_i^{-1} \otimes x_{jk}^{-1}) &= x_{jk}^{-1} \otimes u_i^{-1} \ . \end{aligned}$$

Writing $R(u_i^{-1} \otimes u_j) = \sum_{\alpha,\beta} c_{\alpha\beta} e_\alpha \otimes e_\beta$, we may calculate using the above results that

$$\begin{aligned} (\mathrm{id} \otimes m)\rho(s_{21})(u_i^{-1} \otimes x_{ij}^{-1} u_i \otimes x_{ij}^{-1} u_j) &= (\mathrm{id} \otimes m)R_1(u_i^{-1} \otimes u_j \otimes u_i) \\ &= \sum_{\alpha,\beta} c_{\alpha\beta} e_\alpha \otimes e_\beta u_i \end{aligned}$$

while by (3),

$$\begin{aligned} (\mathrm{id} \otimes m)\rho(s_{21})(u_i^{-1} \otimes x_{ij}^{-1} u_i \otimes x_{ij}^{-1} u_j) &= R(x_{ij}^{-1} \otimes x_{ij}^{-1} u_j) \\ &= x_{ij}^{-1} u_j \otimes x_{ij}^{-1} \ , \end{aligned}$$

so that

$$R(u_i^{-1} \otimes u_j) = x_{ij}^{-1} u_j \otimes x_{ij}^{-1} u_i^{-1} \ .$$

Similarly, we can show that

$$R(u_i^{-1} \otimes u_j^{-1}) = x_{ij} u_j^{-1} \otimes x_{ij} u_i^{-1} \ .$$

With these determinations of the action of $\widetilde{R}$ on tensor products of elements of the form u_i, u_i^{-1}, x_{ij}, and x_{ij}^{-1}, we may calculate the action of $\widetilde{R}$ on all of $\widetilde{T} \otimes \widetilde{T}$ using Lemma 2. Thus there is a unique strong r-structure $\widetilde{R}$ on $\widetilde{T}$ meeting the hypotheses of the theorem.

To prove existence, define $\widetilde{R} \in End_k(\widetilde{T} \otimes \widetilde{T})$ by

$$\widetilde{R}(x^B u^I \otimes x^C u^J) = x^{C+(I,J)} u^J \otimes x^{B+(I,J)} u^I \ .$$

Straightforward calculations show that $\widetilde{R}$ is a strong Yang-Baxter operator. Since $(I,0) = (0,I)$ for all multi-indices I,

$$\widetilde{R}(1 \otimes x^B u^I) = x^B u^I \otimes 1 \ , \ \ \widetilde{R}(x^B u^I \otimes 1) = 1 \otimes x^B u^I \ .$$

We complete the proof that $\widetilde{R}$ is an r-structure for $\widetilde{T}$ using Lemma 2. To show that $\widetilde{R}(\mathrm{id} \otimes m) = (m \otimes \mathrm{id})\rho(s_{12})$, one notes that

$$\begin{aligned} \widetilde{R}(\mathrm{id} \otimes m)(x^B u^I \otimes x^C u^J \otimes x^D u^K) &= \widetilde{R}(x^B u^I \otimes x^{C+D+(JK)} u^{J+K}) \\ &= x^{C+D+(JK)+(I,J+K)} u^{J+K} \otimes x^{B+(I,J+K)} u^I \end{aligned}$$

while

$$\begin{aligned} (m \otimes \mathrm{id})\rho(s_{12})(x^B u^I \otimes x^C u^J \otimes x^D u^K) &= (m \otimes \mathrm{id})\widetilde{R}_2(x^{C+(I,J)} u^J \otimes x^{B+(I,J)} u^I \otimes x^D u^K) \\ &= (m \otimes \mathrm{id})(x^{C+(I,J)} u^J \otimes x^{D+(I,K)} u^K \otimes x^{B+(I,J)+(I,K)} u^I) \\ &= x^{C+D+(I,J)+(I,K)+(JK)} u^{J+K} \otimes x^{B+(I,J)+(I,K)} u^I \ . \end{aligned}$$

It then suffices to note that $(I, J+K) = (I,J) + (I,K)$ for all multi-indices I, J, K. The proof that $\widetilde{R}(m \otimes \mathrm{id}) = (\mathrm{id} \otimes m)\rho(s_{21})$ is similar.

Lastly, to show that $\widetilde{R}$ is r-commutative one notes that

$$\begin{aligned} m\widetilde{R}(x^B u^I \otimes x^C u^J) &= x^{B+C+2(I,J)+(JI)} u^{I+J} \\ &= x^{B+C+(IJ)} u^{I+J} \\ &= m(x^B u^I \otimes x^C u^J) \ . \ \ \square \end{aligned}$$

Corollary 3. *There is a unique r-structure R on T_q such that*

$$R(u_i \otimes u_j) = q_{ij}^2 u_j \otimes u_i \ .$$

This r-structure is strong and r-commutative. The quotient map $j\colon \widetilde{T} \to T_q$ is an r-morphism.

Proof - Using the notation of the proof of Theorem 3, define $u^I \in T_q$ for any multi-index I by

$$u^I = u_1^{i_1} \cdots u_n^{i_n} \ ,$$

and define q^B for any double-index B by

$$q^B = \prod_{1 \le i < j \le n} q_{ij}^{b_{ij}} \ .$$

For any $u^I, u^J \in T_q$ we have $u^I u^J = q^{(IJ)} u^{I+J}$ and $u^I u^J = q^{2(I,J)} u^J u^I$. By the Diamond Lemma, $\{u^I\}$ is a basis of T_q. Arguing as in the proof of Theorem 3, one can show that the r-structure R must satisfy

$$\begin{array}{rcl} R(u_i^{-1} \otimes u_j) & = & q_{ij}^{-2} u_j \otimes u_i^1 \\ R(u_i \otimes u_j^{-1}) & = & q_{ij}^{-2} u_j^{-1} \otimes u_i \\ R(u_i^{-1} \otimes u_j^{-1}) & = & q_{ij}^{2} u_j^{-1} \otimes u_i^{-1} \, . \end{array}$$

Lemma 3 thus implies that R is unique.

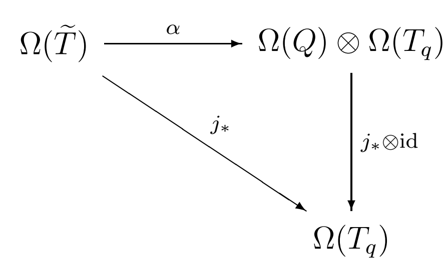
For existence, note using the technique in the proof of Theorem 1 that the ideal generated by the elements $x_{ij} - q_{ij}$ is an r-ideal. It follows that there is a strong r-commutative r-structure R on T_q such that $j \colon \widetilde{T} \to T_q$ is an r-morphism. By the r-morphism property, this r-structure satisfies

$$R(u_i \otimes u_j) = (j \otimes j)(x_{ij} u_j \otimes x_{ij} u_i) = q_{ij}^2 u_j \otimes u_i \, . \qquad \square$$

We now develop an analog of Theorem 3 for noncommutative tori. Equipping $\widetilde{T}$ and T_q with the r-structures given in the theorem and corollary above, we write simply $\Omega(\widetilde{T})$ and $\Omega(T_q)$ for the differential forms on $\widetilde{T}$ and T_g. The r-morphism $j \colon \widetilde{T} \to T_q$ induces a differential graded algebra morphism $j_* \colon \Omega(\widetilde{T}) \to \Omega(T_q)$.

Define Q to be the algebra of Laurent polynomials in the $\frac{1}{2} n(n-1)$ variables $\{x_{ij}\}$. This algebra plays the role of a moduli space for noncommutative n-tori. There is a natural algebra inclusion $Q \hookrightarrow \widetilde{T}$. Giving Q the twist map as an r-structure, this inclusion is an r-morphism, so it induces a differential graded algebra morphism $\Omega(Q) \to \Omega(\widetilde{T})$, making $\Omega(\widetilde{T})$ into a bimodule over $\Omega(Q)$.

Theorem 5. *There is a map $\alpha \colon \Omega(\widetilde{T}) \to \Omega(Q) \otimes \Omega(T_q)$, an isomorphism of $\Omega(Q)$-modules and differential complexes, such that the following diagram commutes:*



Proof - We define α by

$$\alpha \left(\Big(\prod_{1 \le i < j \le n} x_{ij}^{a_{ij}} dx_{ij}^{b_{ij}} \Big) u_1^{k_1} (du_1)^{\ell_1} \cdots u_n^{k_n} (du_n)^{\ell_n} \right) =$$

$$\prod_{1 \le i < j \le n} x_{ij}^{a_{ij}} dx_{ij}^{b_{ij}} \otimes u_1^{k_1} (du_1)^{\ell_1} \cdots u_n^{k_n} (du_n)^{\ell_n} \, .$$

That α is well-defined, one-to-one, and onto follows from the Diamond Lemma. One may verify by explicit computation that α is a morphism of $\Omega(Q)$-modules and differential complexes. To prove that the diagram commutes, note that

$$\begin{aligned} j_*(\omega u_1^{i_1} du_1^{j_1} \cdots u_n^{i_n} du_n^{j_n}) &= j_*(\omega)(u_1^{i_1} du_1^{j_1} \cdots u_n^{i_n} du_n^{j_n}) \\ &= (j_* \otimes \mathrm{id})\alpha(\omega u_1^{i_1} du_1^{j_1} \cdots u_n^{i_n} du_n^{j_n}) \end{aligned}$$

where $\omega \in \Omega(Q)$. □

As in Theorem 3, the isomorphism α of the above theorem is neither natural, nor an algebra homomorphism. Interestingly, however, we have:

Corollary 4. *The isomorphism $\alpha: \Omega(\widetilde{T}) \to \Omega(Q) \otimes \Omega(T_q)$ induces an isomorphism of cohomology rings $\alpha_*: H(\Omega(\widetilde{T})) \to H(\Omega(Q)) \otimes H(T_q)$.*

Proof - The only point not following immediately from the theorem above is that α_* is an algebra homomorphism. Define $\omega_{ij}, \mu_i \in \Omega(\widetilde{T})$ by

$$\omega_{ij} = x_{ij}^{-1} dx_{ij} \ , \quad \mu_i = u_i^{-1} du_i \ .$$

Using the basis for $\Omega(\widetilde{T})$ in the proof of Theorem 5, one can see that these anticommuting elements of degree 1 anticommute generate a subalgebra of $\Omega(\widetilde{T})$ isomorphic to an exterior algebra. Restricted to this subalgebra, α is a homomorphism. Noting that ω_{ij} and μ_i are closed and that $[\alpha(\omega_{ij})]$ and $[\alpha(\mu_i)]$ generate $H(\Omega(Q)) \otimes H(T_q)$, the corollary follows. □

Corollary 5. *The r-commutative de Rham cohomology $H(\Omega(\widetilde{T}))$ is isomorphic as an algebra to the exterior algebra on $n + \frac{1}{2}n(n-1)$ generators of degree 1. For any q, the r-commutative de Rham cohomology $H(\Omega(T_q))$ is isomorphic as an algebra to the exterior algebra on n generators of degree 1.*

Proof - A consequence of the proof of Corollary 4. □

It is interesting to compare work on the K-theory and cyclic cohomology of noncommutative tori [6]. These behave similarly to r-commutative de Rham cohomology in that, at least in a C*-algebraic setting, they are independent of the deformation parameters q_{ij}. More refined K-theoretic invariants vary with the q_{ij}, however. It seems natural to study these phenomena using a C*-algebraic version of the universal noncommutative torus (which may in this case be simplified by taking the universal cover of the "base space" Q). To relate K-theory and cyclic cohomology more precisely to r-commutative geometry, it would seem useful to develop a theory of characteristic classes in the framework of r-commutative geometry.

To conclude this section we briefly discuss the relation between noncommutative tori and certain r-symmetric algebras, the "quasipolynomial algebras." As in the previous section, let k be any field, fix $n \geq 0$ and a collection $q = \{q_{ij}\}$, $1 \leq i < j \leq n$, of nonzero elements of k, and let $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$. Let V be an n-dimensional vector space with basis $\{x_i\}$, where $1 \leq i \leq n$. There is a unique strong Yang-Baxter operator R on V such that

$$R(x_i \otimes x_j) = q_{ij} x_j \otimes x_i \ .$$

Following De Concini and Kac [10], we call the r-symmetric algebra $S_R V$ is a *quasipolynomial algebra.*

Let us describe $\Omega_R(S_R V)$ for such algebras. We say that a multi-index $I = (i_1, \ldots, i_n)$ is *non-negative* if all its components are non-negative, and *short* if all its components equal 0 or 1. Given any multi-index I and short multi-index J, let ω_{IJ} denote the element

$$\omega_{IJ} = x_1^{i_1} dx_1^{j_1} \cdots x_n^{i_n} dx_n^{j_n}$$

in $\Omega_R(S_R V)$. Let $\Omega(SV)$ denote the differential forms on SV equipped with the twist map as its r-structure.

Theorem 6. *Let $S_R V$ be a quasipolynomial algebra. Then there is an isomorphism of differential complexes $\alpha \colon \Omega_R(S_R V) \to \Omega(SV)$ given by*

$$\alpha(\omega_{IJ}) = x_1^{i_1} dx_1^{j_1} \cdots x_n^{i_n} dx_n^{j_n} \ .$$

Proof - Note that $\Omega_R(S_R V)$ may be defined as the algebra generated by the elements x_i, dx_i, with the relations

$$x_i x_j = q_{ij} x_j x_i \ , \quad x_i dx_j = q_{ij} dx_j x_i \ , \quad dx_i dx_j = -q_{ij} dx_j dx_i \ .$$

It follows that the elements ω_{IJ}, where I is arbitrary and J is short, span $\Omega_R(S_R V)$. Moreover, the Diamond Lemma implies that they form a basis. It follows that α is an isomorphism of vector spaces. It is straightforward that α is a morphism of differential complexes. □

As a corollary we have the following Poincaré lemma:

Corollary 6. *Let $S_R V$ be a quasipolynomial algebra over a field of characteristic zero. Then $H^p(\Omega_R(S_R V)) = 0$ for $p \geq 1$, while $H^0(\Omega_R(S_R V)) = k$.*

Proof - A consequence of the usual Poincaré lemma and the above theorem. □

The isomorphism of Theorem 6 is not canonical, as it depends on the ordering of the basis x_i. Note that if R is a Yang-Baxter operator on a vector space V, so is $-R$. One may define $\Lambda_R V$, the *r-exterior algebra* over V, to be the r-commutative algebra $S_{-R} V$ [20]. When R is the twist map, $\Lambda_R V = \Lambda V$. By the same techniques used to prove the above theorem, one may show that there is a canonical isomorphism of vector spaces (not of algebras) $\beta\colon \Omega_R(S_R V) \to S_R V \otimes \Lambda_R V$ given by

$$\beta(v_1 \cdots v_p dw_1 \cdots dw_q) = v_1 \cdots v_p \otimes w_1 \cdots w_q \ .$$

Many of the usual formulas in the differential geometry of vector spaces extend to quasipolynomial algebras. For example, there are operations $\partial_k\colon \omega_R(S_R V) \to \Omega_R(S_R V)$, $1 \leq k \leq n$, such that $d\omega = \sum dx_k\, \partial_k \omega$, and the fact that $d^2 = 0$ follows from the fact that $\partial_j \partial_k = q_{kj} \partial_k \partial_j$.

The relation of quasipolynomial algebras to noncommutative tori is two-fold. First, there is an algebra inclusion $S_R V \hookrightarrow T_q$ given by

$$x_i \mapsto u_i \ .$$

By Lemma 5 this inclusion is an r-morphism. Second, there is a way to obtain noncommutative tori as quotients of certain quasipolynomial algebras. This construction is especially interesting when $k = \mathbf{C}$ and the q_{ij} are of unit modulus. Equip $V \oplus V$ with the basis $\{z_i, \overline{z}_i\}$, where

$$z_i = (x_i, 0) \ , \ \ \overline{z}_i = (0, x_i) \ .$$

Then there is a unique strong Yang-Baxter operator R on $V \oplus V$ given by

$$R(z_i \otimes z_j) = q_{ij} z_j \otimes z_i \ , \ \ R(z_i \otimes \overline{z}_j) = q_{ji} \overline{z}_j \otimes z_i \ , \ \ R(\overline{z}_i \otimes \overline{z}_j) = q_{ij} \overline{z}_j \otimes \overline{z}_i \ .$$

The second relation together with $R^2 = \mathrm{id}$ implies

$$R(\overline{z}_i \otimes z_j) = q_{ji} z_j \otimes \overline{z}_i \ .$$

The algebra $A = S_R(V \oplus V)$ is a quasipolynomial algebra. We may define operators $\partial, \overline{\partial}\colon \Omega_R(A) \to \Omega_R(A)$ by

$$\partial \omega = \sum_i dz_i\, \partial_i \omega \ , \ \ \overline{\partial} \omega = \sum_i d\overline{z}_i\, \overline{\partial}_i \omega \ ,$$

where ∂_i and $\overline{\partial}_i$ denote the partial derivatives with respect to z_i and $\overline{z}_i$, defined as above. One may easily verify that, as in classical complex geometry,

$$d = \partial + \overline{\partial} \ , \ \ \partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0 \ .$$

The algebra A is a noncommutative analog of the real-algebraic coordinate ring of $\mathbf{C}^n$. Just as $\mathbf{C}^n$ contains an embedded torus given by the equations $z_i \overline{z}_i = 1$, the quotient of A by the ideal I generated by the elements $z_i \overline{z}_i - 1$ is isomorphic to T_q via the map

$$z_i \mapsto u_i \ , \ \overline{z}_i \to u_i^{-1} \ .$$

One may check using Lemma 5 that this map is an r-morphism.

6 Quantum Groups

There are many directions one could take in the further study of r-commutative geometry. The most immediately fruitful may be those providing insight into the representation theory of quantum groups. Just as all representations of $SL_q(2)$ may be constructed as "line bundles" (projective modules) over the quantum projective plane [27], we may expect interesting representations of quantum groups to arise as sections of homogeneous vector bundles satisfying invariant differential equations. The principal difficulty is that the Yang-Baxter operators involved are not strong. Here we sketch what is known and raise some open questions, referring the reader to our forthcoming paper [2] for further details.

The differential forms $\Omega_R(A)$ as we have defined them are suited to the case when A is r-commutative and R is strong, but must be generalized in the non-strong case. Here we only treat quantum vector spaces (r-symmetric algebras) satisfying certain Hecke-type identities. Suppose that the Yang-Baxter operator R on the vector space V is of *type* q, that is,

$$R^2 = (1-q)R + q$$

for some nonzero $q \in k$. Letting $A = S_R(V)$, one may define the *front* differential calculus Ω_f for A to be the quotient of $\Omega_u(A)$ by the differential ideal generated by $d1$ together with all the elements of the form

$$qv\,dw - \sum_i (dw^i)v_i \ ,$$

where $R(v \otimes w) = \sum w^i \otimes v_i$. The extra factor of q prevents the differential from being over-determined, in the following sense. The modified Hecke identity implies

$$\sum_i R(w^i \otimes v_i) = qv \otimes w + (1-q)\sum_i w^i \otimes v_i \ ,$$

so that in Ω one has

$$qv\,dw = \sum_i (dw^i)v_i$$

and

$$q\sum_i w^i dv_i = q(dv)w + (1-q)\sum_i (dw^i)v_i \ ,$$

a linear combination of which gives the relation

$$(dv)w + v\,dw = \sum_i (dw^i)v_i + w^i dv_i \ ,$$

that must hold in any differential calculus for A.

One may equally well work with the *back* differential calculus Ω_b for A, the quotient of $\Omega_u(A)$ by the differential ideal generated by $d1$ together with all the elements of the form

$$q(dv)w - \sum_i w^i\,dv_i \ .$$

The front and back differential calculi are not in general isomorphic (unless $q = 1$). Both these differential calculi have been considered by Pusz and Woronowicz [22, 23], and Wess and Zumino [31, 32], in the special case where V has the basis e_i, $1 \leq i \leq n$, and

$$R(e_i \otimes e_j) = \begin{cases} e_j \otimes e_i & \text{if } i = j \\ q^{\frac{1}{2}} e_j \otimes e_i & \text{if } i < j \\ q^{\frac{1}{2}} e_j \otimes e_i + (1-q) e_i \otimes e_j & \text{if } i > j \ . \end{cases}$$

Up to various conventional normalizations this R is the R-matrix for the quantum group $GL_q(n)$. In this case the algebra $A = S_R V$ is often regarded as a "q-analog" of an n-dimensional vector space.

One motivation for the study of supercommutative algebras is the supercommutativity of the algebra of differential forms on a manifold (with exterior product), relative to the $\mathbf{Z}_2$-grading

$$\Omega_{even}(M) = \bigoplus_{p \geq 0} \Omega^{2p}(M) \ , \ \ \Omega_{odd}(M) = \bigoplus_{p \geq 0} \Omega^{2p+1}(M) \ .$$

As a generalization, we can construct an r-commutative r-structure for the front differential calculus Ω_f. Let $i\colon V \to \Omega_f$ be the natural inclusion map, and let $d\colon V \to \Omega_f$ be inclusion composed with the differential in Ω_f. Then Ω_f has a unique r-structure $\widetilde{R}$ such that

$$\begin{aligned} \widetilde{R}(i \otimes i) &= (i \otimes i) R \\ \widetilde{R}(i \otimes d) &= q^{-1} (d \otimes i) R \\ \widetilde{R}(d \otimes i) &= (i \otimes d + (1 - q^{-1}) d \otimes i) R \\ \widetilde{R}(d \otimes d) &= -q^{-1} (d \otimes d) R \ . \end{aligned}$$

This r-structure is r-commutative. Taking $q = 1$, and using the fact that any strong r-commutative algebra A is the quotient of the r-symmetric algebra $S_R A$ by an r-ideal, one can show that the r-structure R on a strong r-commutative algebra A extends uniquely to a strong r-commutative r-structure on $\Omega_R(A)$ such that

$$\widetilde{R}(a \otimes db) = (d \otimes i) \widetilde{R}(a \otimes b)$$

and

$$\widetilde{R}(da \otimes db) = -(d \otimes d) \widetilde{R}(a \otimes b)$$

for all $a, b \in A$.

The r-commutativity of quantum groups may be shown either using the definition of quasitriangular Hopf algebras, together with Lemma 1, or via quantum matrix algebras. Let V be a vector space equipped with a Yang-Baxter operator R. One may construct a Yang-Baxter operator $\widetilde{R}$ on $End(V)$ using the natural isomorphism $End(V) \otimes End(V) \cong End(V \otimes V)$, as follows:

$$\widetilde{R}(S \otimes T) = R \circ (S \otimes T) \circ R^{-1} \ .$$

The r-symmetric algebra $M_R V = S_{\widetilde{R}}(End(V))$ is called a quantum matrix algebra [12]. Quantum groups typically inherit the structure of r-commutative algebras from quantum matrix algebras. For example, taking V and R as in the example above, the quantum matrix algebra $M_R V$ is often denoted $M_q(n)$. The quantum determinant $\det_q \in M_q(n)$ is an element with

$$\begin{aligned} \widetilde{R}(\det_q \otimes a) &= a \otimes \det_q , \\ \widetilde{R}(a \otimes \det_q) &= \det_q \otimes a , \end{aligned}$$

for all $a \in M_R V$. It follows that the quantum group $SL_q(n)$, which is defined as the quotient $M_q(n)/\langle \det_q - 1 \rangle$, has a unique r-structure such that the quotient map $M_q(n) \to SL_q(n)$ is an r-morphism.

Returning to the general case, it is known that $M_R V$ is a bialgebra and has a coaction on $S_R V$. If e_i is a basis for V and $e^i_j \in End(V)$ are matrix units with $e^i_j e_k = \delta^i_k e_j$, where δ is the Kronecker delta, the coproduct $\Delta \colon M_R V \to M_R V \otimes M_R V$ is determined by

$$\Delta(e^i_j) = \sum_k e^k_j \otimes e^i_k ,$$

while the coaction $\Phi \colon S_R V \to M_R V \otimes S_R V$ is determined by

$$\Phi(e_j) = \sum_i e^i_j \otimes e_i .$$

Woronowicz [28, 30] has initiated the study of differential calculi invariant under the coaction of a bialgebra. If R is of type q, the front differential calculus is covariant for the coaction Φ of $M_R V$ on $S_R V$. Namely, Φ extends uniquely to a coaction $\Phi_* \colon \Omega_f \to M_R V \otimes \Omega_f$ satisfying

$$(\mathrm{id} \otimes d)\Phi_* = \Phi_* d .$$

It is natural to attempt to construct covariant differential calculi for quantum groups and quantum matrix algebras using r-commutative geometry. One could hope by this approach to generalize the differential calculus for $SU_q(2)$ constructed by Woronowicz [28]. In particular, one should seek to explain the mysterious fact that this differential calculus is left-covariant but not right-covariant for $q \neq 1$. One could also hope to give a new proof of the fact that the cohomology of this differential calculus is independent of q, using the ideas by which we treated noncommutative tori. The difficulty is that, apart from the front and back differential calculi, the right generalization of differential forms to r-commutative algebras that are not strong is not known.

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