EXPANDING DOMAIN LIMIT FOR INCOMPRESSIBLE FLUIDS IN THE PLANE

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ABSTRACT. The general class of problems we consider is the following: Let Ω_1 be a bounded domain in $\mathbb{R}^d$ for $d \geq 2$ and let u^0 be a velocity field on all of $\mathbb{R}^d$. Suppose that for all $R \geq 1$ we have an operator $\mathcal{T}_R$ that projects u^0 restricted to $R\Omega_1$ (Ω_1 scaled by R) into a function space on $R\Omega_1$ for which the solution to some initial value problem is well-posed with $\mathcal{T}_R u^0$ as the initial velocity. Can we show that as $R \to \infty$ the solution to the initial value problem on $R\Omega_1$ converges to a solution in the whole space?

We answer this question when $d = 2$ for weak solutions to the Navier-Stokes and Euler equations. For the Navier-Stokes equations we assume the lowest regularity of u^0 for which one can obtain adequate control on the pressure. For the Euler equations we assume the lowest feasible regularity of u^0 for which uniqueness of solutions to the Euler equations is known (thus, we allow "slightly unbounded" vorticity). In both cases, we obtain strong convergence of the velocity and the vorticity as $R \to \infty$ and, for the Euler equations, the flow. Our approach yields, in principle, a bound on the rates of convergence.

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1. Introduction

The properties of the solutions to the Navier-Stokes equations (which we refer to as (NS)) and to the Euler equations (which we refer to as (E)) are reasonably well understood in two dimensions in the setting of a bounded domain and in the whole space (as well as for periodic domains). It is a natural question to ask whether the solution to (NS) or (E) in a bounded domain approaches the solution to (NS) or (E) in the entire space as we let the size of the bounded domain increase to infinity.

More precisely, let Ω_1 be a bounded domain with a C^2-boundary Γ_1. For simplicity, we assume that Ω_1 is connected and simply connected. Define

$$\Omega_R := R\Omega_1 \text{ and } \Gamma_R := R\Gamma_1 = \partial\Omega_R \text{ for } R \text{ in } [1, \infty). \tag{1.1}$$

We assume that the origin lies in the interior of Ω_1, so that Ω_R fills the whole space as $R \to \infty$. For $R = \infty$, we define Ω_R to be $\mathbb{R}^2$ and Γ_R to be empty.

Let $X(\Omega_R)$ be a function space for which (NS) or (E) is well-posed on Ω_R. Let u^0 lie in $X(\mathbb{R}^2)$ and suppose that $\mathcal{T}_R$ is a "truncation" operator that maps $X(\mathbb{R}^2)$ to $X(\Omega_R)$ in such a way that $\|u^0|_{\Omega_R} - \mathcal{T}_R u^0\|_{X(\Omega_R)} \to 0$ as $R \to \infty$. The question we address is the following: If u_R is the solution (velocity) to (NS) or (E) on Ω_R with initial velocity $\mathcal{T}_R u_0$ and u is the solution to (NS) or (E) on $\mathbb{R}^2$, can we show that $\|u|_{\Omega_R} - u_R\|_{L^2([0,T];X(\Omega_R))} \to 0$ as $R \to \infty$?

We show in Theorem 8.1 that, in fact, such convergence does occur in $X(\Omega_R) = H^1(\Omega_R)$. For solutions to (NS) we need only assume that u^0 lies in $H^1(\mathbb{R}^2)$. For solutions to (E), though, we need a stronger assumption on u^0 to have well-posedness. We will assume that the initial velocity has Yudovich vorticity, described in Section 2. This is a class of vorticities introduced by Yudovich in [19] for which he showed uniqueness of solutions to (E) in a bounded domain in $\mathbb{R}^d$, $d \geq 2$. This class is slightly broader than initial vorticities lying in L^∞, for which Yudovich established the same uniqueness result in [18]. It is the natural class of initial vorticities for us to use because it is ideally suited to the use of energy methods, and is the largest such class for which existence and uniqueness of solutions to (E) has been established. (For the larger class of initial vorticities defined by Misha Vishik in [15] existence is not known. Also, this class is not as readily amenable to the use of energy methods; see, however, [2].)

We will restrict ourselves to solutions in the whole space that have finite energy, though this is a stronger condition than required. For instance, the spaces E_m of [1] which allow infinite energy or spaces that allow even slower decay of the velocity at infinity can be dealt with using our techniques. The assumption of finite energy simplifies the analysis considerably, however, in large part because it does not require us to make significant adaptations to the standard existence and uniqueness results for the Navier-Stokes and Euler equations, and because it simplifies considerably the definition of the truncation operator $\mathcal{T}_R$.

Our results seem to be most closely related to those of [5] and [6], in which the authors consider the limit as $\epsilon \to 0$ of solutions of (E) and (NS) on the domain external to $\Omega_\epsilon = \epsilon\Omega_1$, where Ω_1 is a fixed simply connected domain. In a sense, this is the opposite limit to what we consider. They start with a smooth initial vorticity ω^0 whose support is compact and does not contain the origin. For $\epsilon > 0$, they use as an initial velocity the unique divergence-free vector field in Ω_ϵ^C that is tangent to $\partial\Omega_\epsilon$, has a curl equal to ω^0 in Ω_ϵ^C, and has a given fixed circulation γ. Using a weak vorticity formulation of (E), they find, roughly speaking, that a subsequence of solutions to (E) converges in the limit as $\epsilon \to 0$ to a solution to (E) with an additional forcing term of $\gamma\delta$. (Here, δ is the Dirac delta function.) In contrast, for (NS) they find that a subsequence converges to a solution to (NS) whose initial vorticity is $\omega^0 + \gamma\delta$. (The smoothness of the initial vorticity is not the critical point; their convergence argument for (E) would apply for initial vorticities in L^p for $p > 2$ and even less smoothness is required for (NS), as they note.)

The limits considered here and in [5] and [6] can be viewed as falling into the broad class of limits of singularly perturbed domains, as considered in detail for elliptic problems in [12].

This paper is organized as follows: In Section 2 we define Yudovich vorticity and in Section 3 we define the function spaces we will use. In Section 4 we describe how we adjust the initial velocity to satisfy the boundary conditions. We define a weak solution to (NS) and (E) in Section 5 and give the basic existence, uniqueness, and regularity results for the velocity and pressure in Section 6. We also require a uniform-in-time bound on how fast solutions to (NS) and (E) in all of $\mathbb{R}^2$ vanish at infinity, which we discuss in Section 7. Our main result, in which we establish convergence of solutions to (NS) and (E) as $R \to \infty$, is given in Section 8. We include in the appendix various lemmas we use in the body of the paper.

A few words on notation: We define the vorticity of a vector field u on $\mathbb{R}^2$ by $\omega(u) := \partial_1 u^2 - \partial_2 u^1$. By T, we always mean an arbitrary, but fixed, positive real number representing time. The symbol C stands for a positive constant that can hold different values on either side of an inequality, though always has the same value on each side of an equality. The constant may have dependence on certain parameters, such as viscosity, but will never have any dependence on our scaling factor, R. We use the notation $\int fg$ when we sometimes should more properly write (f, g)—the pairing of f in a function space X with an element g in the dual space of X.

2. Yudovich Vorticity

Definition 2.1. Let $\theta : [p_0, \infty) \to \mathbb{R}^+$ for some p_0 in $(1, 2)$. We say that θ is *admissible* if the function $\beta_M : (0, \infty) \to [0, \infty)$ defined, for some $M > 0$, by[1]

$$\beta_M(x) := 2C_0 \inf\left\{(M^\epsilon x^{1-\epsilon}/\epsilon)\theta(1/\epsilon) : \epsilon \text{ in } (0, (2+\epsilon_0)^{-1}]\right\}, \tag{2.1}$$

where C_0 is a fixed absolute constant and $\epsilon_0 > 0$ is fixed as in Lemma A.5, satisfies

$$\int_0^1 \frac{dx}{\beta_M(x)} = \infty. \tag{2.2}$$

Because

$$\begin{aligned}\beta_M(x) &= 2C_0 M \frac{x}{M} \inf\left\{((x/M)^{-\epsilon}/\epsilon)\theta(1/\epsilon) : \epsilon \text{ in } (0, (2+\epsilon_0)^{-1}]\right\}\\ &= M\beta_1(x/M),\end{aligned}$$

this definition is independent of the value of M. Also, β_M is a monotonically increasing continuous function, with $\lim_{x\to 0^+}\beta_M(x) = 0$.

Yudovich proves in [19] that for a bounded domain in $\mathbb{R}^n$, if $\|\omega^0\|_{L^p} \le \theta(p)$ for some admissible function θ, then at most one solution to the Euler equations exists. Because of this, we call such a vorticity, *Yudovich vorticity*:

Definition 2.2. We say that a vector field v has *Yudovich vorticity* if for some admissible function $\theta : [p_0, \infty) \to \mathbb{R}^+$ with p_0 in $(1, 2)$, $\|\omega(v)\|_{L^p} \le \theta(p)$ for all p in $[p_0, \infty)$.

Examples of admissible bounds on vorticity are

$$\theta_0(p) = 1, \theta_1(p) = \log p, \dots, \theta_m(p) = \log p \cdot \log\log p \cdots \log^m p, \tag{2.3}$$

where $\log^m$ is log composed with itself m times. These admissible bounds are described in [19] (see also [7].) Roughly speaking, the L^p–norm of a Yudovich vorticity can grow in p only slightly faster than $\log p$ and still be admissible. Such growth in the L^p–norm arises, for example, from a point singularity of the type $\log\log(1/|x|)$.

3. Function Spaces

We will use the following function spaces:

$$\begin{aligned}H(\Omega_R) &= \left\{v \in (L^2(\Omega_R))^2 : \operatorname{div} v = 0 \text{ in } \Omega_R \text{ and } v\cdot\mathbf{n} = 0 \text{ on } \Gamma_R\right\},\\ V^{(E)}(\Omega_R) &= \left\{v \in (H^1(\Omega_R))^2 : \operatorname{div} v = 0 \text{ in } \Omega_R \text{ and } v\cdot\mathbf{n} = 0 \text{ on } \Gamma_R\right\},\\ V^{(NS)}(\Omega_R) &= \left\{v \in (H^1(\Omega_R))^2 : \operatorname{div} v = 0 \text{ in } \Omega_R \text{ and } v = 0 \text{ on } \Gamma_R\right\}.\end{aligned} \tag{3.1}$$

We equip $H(\Omega_R)$ with the $L^2(\Omega_R)$-norm and $V^{(E)}(\Omega_R)$ and $V^{(NS)}(\Omega_R)$ with the $H^1(\Omega_R)$-norm.

[1]The definition of β_M in Equation (2.1) differs from that in [7] in that it directly incorporates the factor of p that appears in the Calderón-Zygmund inequality; in [7] this factor is included in the equivalent of Equation (2.2).

Our solutions to (E) at time t will lie in $V^{(E)}(\Omega_R)$, solutions to (NS) in $V^{(NS)}(\Omega_R)$. In general, $V^{(NS)}(\Omega_R) \subsetneq V^{(E)}(\Omega_R) \subsetneq H(\Omega_R)$; however, when $\Omega_R = \mathbb{R}^2$, the first two spaces coincide, and we simply write $V(\mathbb{R}^2)$.

Given a function $\theta : [p_0, \infty) \to \mathbb{R}^+$ admissible in the sense of Definition 2.1 for some p_0 in $(1, 2)$, we define the subspace

$$\mathbb{Y}_\theta(\Omega_R) = \left\{ v \in V^{(E)}(\Omega_R) : \|\omega(v)\|_{L^p} \le C\theta(p) \text{ for all } p \text{ in } [p_0, \infty) \right\}$$

for some constant C. We define a norm on $\mathbb{Y}_\theta$ by

$$\|v\|_{\mathbb{Y}_\theta(\Omega_R)} = \|v\|_{L^2(\Omega_R)} + \sup_{p\in[p_0,\infty)} \|\omega(v)\|_{L^p(\Omega_R)} / \theta(p). \tag{3.2}$$

Finally, we define the space

$$\mathbb{Y}(\Omega_R) = \left\{ v \in Y_\theta^{(E)}(\Omega_R) : \text{ for some admissible } \theta \right\},$$

but place no norm on this space.

4. Truncation of the initial velocity

Definition 4.1 ("Truncation" operator)**.** Let

$$\Sigma_1 = \{x \in \Omega_1 : \operatorname{dist}(x, \Gamma_1) < 1/2\overline{\kappa}\},$$

where $\overline{\kappa}$ is the maximum curvature of Γ_1. Let φ_1 in $C^\infty(\Omega_1)$ taking values in $[0, 1]$ be defined so that $\varphi_1 = 1$ on $\Omega_1 \setminus \Sigma_1$ and $\varphi_1 = 0$ on Γ_1, and let $\varphi_R(\cdot) = \varphi_1(\cdot/R)$ and $\Sigma_R = R\Sigma_1$. Let ψ be a stream function for $u \in H(\mathbb{R}^2)$; that is, $u = \nabla^\perp\psi$ (ψ is unique up to the addition of a constant). Finally, define $\mathcal{T}_R \colon H(\mathbb{R}^2) \to H(\Omega_R)$ by

$$\mathcal{T}_R u := \nabla^\perp(\varphi_R\psi_R), \tag{4.1}$$

where $\psi_R = \psi - |\Sigma_R|^{-1} \int_{\Sigma_R} \psi$, so that $\int_{\Sigma_R} \psi_R = 0$ and $u = \nabla^\perp\psi_R$ on all of $\mathbb{R}^2$.

Lemma 4.2. *$\mathcal{T}_R \colon H(\mathbb{R}^2) \to H(\Omega_R)$ with an operator norm that is independent of R. For any u in $H(\mathbb{R}^2)$,*

$$\|u - \mathcal{T}_R u\|_{H(\Omega_R)} \to 0 \text{ as } R \to \infty. \tag{4.2}$$

$\mathcal{T}_R \colon V(\mathbb{R}^2) \to V^{(E)}(\Omega_R)$ with an operator norm that is independent of R. For any u in $V(\mathbb{R}^2)$,

$$\|u - \mathcal{T}_R u\|_{H^1(\Omega_R)} \to 0 \text{ as } R \to \infty. \tag{4.3}$$

$\mathcal{T}_R \colon \mathbb{Y}_\theta(\mathbb{R}^2) \to \mathbb{Y}_\theta(\Omega_R)$ with an operator norm that is independent of R. For any u in $\mathbb{Y}_\theta(\mathbb{R}^2)$,

$$\|\omega(u) - \omega(\mathcal{T}_R u)\|_{L^p(\Omega_R)} \to 0 \text{ as } R \to \infty \tag{4.4}$$

uniformly over all p in $[p_0, \infty)$, p_0 being as in Definition 2.2.

If in Definition 4.1 we impose the extra condition on the cutoff function φ_1 that $\nabla\varphi_1 = 0$ on Γ_1 then also

$$\mathcal{T}_R \colon V(\mathbb{R}^2) \to V^{(NS)}(\Omega_R) \tag{4.5}$$

with an operator norm that is independent of R, and Equation (4.2) and Equation (4.3) continue to hold.

Proof. Define Σ_R, φ_R, and ψ_R as in Definition 4.1. Observe that

$$\|\nabla\varphi_R\|_{L^\infty(\Sigma_R)} \le C/R, \quad \|\nabla\nabla\varphi_R\|_{L^\infty(\Sigma_R)} \le C/R^2,$$

and by Lemma A.3,

$$\|\psi_R\|_{L^p(\Sigma_R)} \le C_p R \|\nabla\psi_R\|_{L^p(\Sigma_R)} = C_p R\|u\|_{L^p(\Sigma_R)}$$

for all p in $[1, \infty]$ for some constant C_p. Thus,

$$\begin{aligned}
\|u - \mathcal{T}_R u\|_{H(\Omega_R)} &= \|u - \nabla^\perp(\varphi_R\psi_R)\|_{L^2(\Omega_R)} = \|u - \varphi_R\nabla^\perp\psi_R - \psi_R\nabla^\perp\varphi_R\|_{L^2(\Omega_R)} \\
&\le \|1 - \varphi_R\|_{L^\infty(\Sigma_R)}\|u\|_{L^2(\Sigma_R)} + \|\nabla\varphi_R\|_{L^\infty(\Sigma_R)}\|\psi_R\|_{L^2(\Sigma_R)} \\
&\le \|u\|_{L^2(\Sigma_R)} + \frac{C_2}{R}R\|u\|_{L^2(\Sigma_R)} \le C\|u\|_{L^2(\Sigma_R)}.
\end{aligned}$$

This converges to 0 as $R \to \infty$ since u is in $L^2(\mathbb{R}^2)$, giving Equation (4.2).

The same calculation with the first term dropped gives

$$\|\mathcal{T}_R u\|_{H(\Omega_R)} \le \|u\|_{L^2(\Omega_R)} + C_2\|u\|_{L^2(\Sigma_R)} \le C\|u\|_{L^2(\Omega_R)}, \tag{4.6}$$

which bounds the operator norm of $\mathcal{T}_R \colon H(\mathbb{R}^2) \to H(\Omega_R)$ independently of R.

Similarly,

$$\begin{aligned}
\|\nabla u - \nabla\mathcal{T}_R u\|_{L^2(\Omega_R)} &= \|\nabla u - \nabla\nabla^\perp(\varphi_R\psi_R)\|_{L^2(\Omega_R)} \\
&= \|\nabla u - \nabla(\varphi_R\nabla^\perp\psi_R) - \nabla(\psi_R\nabla^\perp\varphi_R)\|_{L^2(\Omega_R)} \\
&= \|\nabla u - \varphi_R\nabla\nabla^\perp\psi_R - \nabla\varphi_R \otimes \nabla^\perp\psi_R - \nabla\psi_R \otimes \nabla^\perp\varphi_R - \psi_R\nabla\nabla^\perp\varphi_R\|_{L^2(\Omega_R)} \\
&= \|(1 - \varphi_R)\nabla u - \nabla\varphi_R \otimes \nabla^\perp\psi_R - \nabla\psi_R \otimes \nabla^\perp\varphi_R - \psi_R\nabla\nabla^\perp\varphi_R\|_{L^2(\Omega_R)} \\
&\le \|\nabla u\|_{L^2(\Sigma_R)} + 2\,\|\nabla\varphi_R\|_{L^\infty(\Sigma_R)}\,\|u\|_{L^2(\Sigma_R)} + \|\nabla\nabla^\perp\varphi_R\|_{L^\infty(\Sigma_R)}\,\|\psi_R\|_{L^2(\Sigma_R)} \\
&\le \|\nabla u\|_{L^2(\Sigma_R)} + \frac{C}{R}\|u\|_{L^2(\Sigma_R)} + \frac{C_2}{R^2}R\|u\|_{L^2(\Sigma_R)} \le C\|u\|_{H^1(\Sigma_R)},
\end{aligned}$$

which converges to zero because u is in $H^1(\mathbb{R}^2)$. This gives Equation (4.3).

The same calculation with the first term dropped gives

$$\|\nabla\mathcal{T}_R u\|_{L^2(\Omega_R)} \le \|\nabla u\|_{L^2(\Omega_R)} + (C/R)\|u\|_{L^2(\Sigma_R)} \le C\|u\|_{H^1(\Omega_R)}.$$

Together with Equation (4.6), this bounds the operator norm of $\mathcal{T}_R \colon V(\mathbb{R}^2) \to V^{(E)}(\Omega_R)$ independently of R.

Requiring that $\nabla\varphi_1 = 0$ on Γ_1 (so $\nabla\varphi_R = 0$ on Γ_R) affects none of the calculations above while ensuring that $\mathcal{T}_R u$ lies in $V^{(NS)}(\Omega_R)$, since then $\mathcal{T}_R u = \varphi_R\nabla^\perp\psi_R + \psi_R\nabla^\perp\varphi_R = 0$ on Γ_R, giving Equation (4.5) and the independence of the operator norm on R.

Now assume that u lies in $\mathbb{Y}_\theta(\mathbb{R}^2)$. Then for all p in the interval $[p_0, \infty)$,

$$\begin{aligned}
&\|\omega(u) - \omega(\mathcal{T}_R u)\|_{L^p(\Omega_R)} \\
&\quad = \|\omega(u) - \omega(\varphi_R \nabla^\perp \psi_R) - \omega(\psi_R \nabla^\perp \varphi_R)\|_{L^p(\Omega_R)} \\
&\quad = \|\omega(u) - \varphi_R \omega(\nabla^\perp \psi_R) + \nabla \varphi_R \cdot (\nabla^\perp \psi_R)^\perp \\
&\qquad\qquad - \psi_R \omega(\nabla^\perp \varphi_R) + \nabla \psi_R \cdot (\nabla^\perp \varphi_R)^\perp \|_{L^p(\Omega_R)} \\
&\quad = \|(1 - \varphi_R)\omega(u) - 2\nabla \varphi_R \cdot \nabla \psi_R - \psi_R \omega(\nabla^\perp \varphi_R)\|_{L^p(\Sigma_R)} \\
&\quad \le \|\omega(u)\|_{L^p(\Sigma_R)} + \frac{C}{R}\|\nabla \psi_R\|_{L^p(\Sigma_R)} + \frac{C}{R^2}\|\psi_R\|_{L^p(\Sigma_R)}.
\end{aligned} \tag{4.7}$$

We wish to obtain a bound on the last term that is independent of p. When $p \ge 2$,

$$\begin{aligned}
&\frac{C}{R^2}\|\psi_R\|_{L^p(\Sigma_R)} \le \frac{C}{R^2}\|\psi_R\|_{L^2 \cap L^\infty(\Sigma_R)} \\
&\qquad \le \max\{C_2, C_\infty\} \frac{C}{R^2} R \|\nabla \psi_R\|_{L^2 \cap L^\infty(\Sigma_R)} \le \frac{C}{R}\|u\|_{L^2 \cap L^\infty(\Sigma_R)},
\end{aligned}$$

which converges to 0 because u is in $L^2(\mathbb{R}^2)$ by assumption and is in $L^\infty(\mathbb{R}^2)$ by Lemma A.4. For p in $[p_0, 2)$, let q and b be such that $1/p = 1/2 + 1/q$ and $1/p_0 = 1/2 + 1/b$. Then

$$\begin{aligned}
&\frac{C}{R^2}\|\psi_R\|_{L^p(\Sigma_R)} \le \frac{C}{R^2}\|\psi_R\|_{L^2(\Sigma_R)}\|1\|_{L^q(\Sigma_R)} \le C R^{2/q - 2} C_2 R \|u\|_{L^2(\Sigma_R)} \\
&\qquad = C R^{2/q - 1}\|u\|_{L^2(\Sigma_R)}.
\end{aligned}$$

Since $q > b > 2$, we have

$$\frac{C}{R^2}\|\psi_R\|_{L^p(\Sigma_R)} \le C R^{2/b - 1}\|u\|_{L^2 \cap L^\infty(\Sigma_R)} \le C R^{2/b - 1}\|u\|_{L^2 \cap L^\infty(\mathbb{R}^2)},$$

an inequality that, in fact, holds for all p in $[p_0, \infty)$. Similarly,

$$\frac{C}{R}\|\nabla \psi_R\|_{L^p(\Sigma_R)} \le C R^{2/b - 1}\|u\|_{L^2 \cap L^\infty(\mathbb{R}^2)}.$$

Then from Equation (4.7), we have

$$\|\omega(u) - \omega(\mathcal{T}_R u)\|_{L^p(\Omega_R)} \le \|\omega(u)\|_{L^p(\Sigma_R)} + C R^{2/b - 1}\|u\|_{L^2 \cap L^\infty(\mathbb{R}^2)}.$$

This converges to 0 as $R \to \infty$ because $\omega(u)$ is in $L^p(\mathbb{R}^2)$, u is in $L^2 \cap L^\infty(\mathbb{R}^2)$, and $2/b - 1 < 0$, giving Equation (4.4).

A similar argument gives

$$\|\omega(\mathcal{T}_R u)\|_{L^p(\Omega_R)} \le \|\omega(u)\|_{L^p(\mathbb{R}^2)} + C R^{2/b - 1}\|u\|_{L^2 \cap L^\infty(\mathbb{R}^2)}.$$

From interpolation of Lebesgue spaces and Lemma A.4,

$$\begin{aligned}
&\|u\|_{L^2 \cap L^\infty(\mathbb{R}^2)} \le \max\left\{\|u\|_{L^2(\mathbb{R}^2)}, \|u\|_{L^\infty(\mathbb{R}^2)}\right\} \\
&\qquad \le C\left(\|u\|_{L^2(\mathbb{R}^2)} + \|\omega(u)\|_{L^4(\mathbb{R}^2)}\right) \le C\,\|u\|_{\mathbb{Y}_\theta(\mathbb{R}^2)}.
\end{aligned}$$

Thus by Equation (3.2),

$$\begin{aligned}\|\mathcal{T}_R u\|_{\mathbb{Y}_\theta(\Omega_R)} &\le \|u\|_{L^2(\mathbb{R}^2)} + \sup_{p\in[p_0,\infty)} \left(\frac{\|\omega(u)\|_{L^p(\mathbb{R}^2)} + CR^{2/b-1}\|u\|_{\mathbb{Y}_\theta(\mathbb{R}^2)}}{\theta(p)} \right) \\ &\le C\|u\|_{\mathbb{Y}_\theta(\mathbb{R}^2)},\end{aligned}$$

showing that $\mathcal{T}_R \colon \mathbb{Y}_\theta(\mathbb{R}^2) \to \mathbb{Y}_\theta(\Omega_R)$ with an operator norm that is independent of R. □

5. Weak Solutions

Definition 5.1 (Weak Navier-Stokes Solutions)**.** Given viscosity $\nu > 0$ and initial velocity u^0 in $H(\Omega_R)$, u in $L^2([0,T]; V^{(NS)})$ with $\partial_t u$ in $L^2([0,T]; (V^{(NS)})')$ is a weak solution to the Navier-Stokes equations (without forcing) if $u(0) = u^0$ and

$$(\mathbf{NS}) \qquad \int_{\Omega_R} \partial_t u \cdot v + \int_{\Omega_R} (u\cdot\nabla u)\cdot v + \nu \int_{\Omega_R} \nabla u \cdot \nabla v = 0$$

for almost all t in $[0,T]$ and for all v in $V^{(NS)}(\Omega_R)$.

For the Euler equations, existence is only known if the L^p-norm of the initial vorticity is finite for some p in $(1,\infty]$, and uniqueness is known only under even stronger assumptions, such as the initial velocity lying in $\mathbb{Y}$ (see also [15]). This is reflected in the following definition of a weak solution to the Euler equations.

Definition 5.2 (Weak Euler Solutions)**.** Given an initial velocity u^0 in $\mathbb{Y}(\Omega_R)$, u in $L^\infty([0,T]; V^{(E)})$ with $\partial_t u$ in $L^2([0,T]; (V^{(NS)})')$ is a weak solution to the Euler equations (without forcing) if $u(0) = u^0$ and

$$(\mathbf{E}) \qquad \int_{\Omega_R} \partial_t u \cdot v + \int_{\Omega_R} (u\cdot\nabla u)\cdot v = 0$$

for almost all t in $[0,T]$ and for all v in $V^{(E)}(\Omega_R)$.

Given a solution to (NS), there exists a distribution p (tempered, if $R = \infty$) such that

$$\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u, \tag{5.1}$$

equality holding in the sense of distributions. This follows from a result of Poincaré and de Rham that any distribution that is a curl-free vector is the gradient of some scalar distribution.

Given a solution to (E), there exists a pressure p such that

$$\partial_t u + u\cdot\nabla u + \nabla p = 0, \tag{5.2}$$

but we can only interpret p as a distribution when $R = \infty$. Otherwise, we must view $\partial_t u + u\cdot\nabla u$ as lying in $H^{-1}(\Omega_R)$ and p as lying in $L^2(\Omega_R)$. (Equation (5.2) follows, for instance, from Remark I.1.9 p. 14 of [14].)

In both Equation (5.1) and Equation (5.2) the pressure is unique up to the addition of a function of time. We resolve this ambiguity for $R < \infty$ by requiring that $\int_{\Omega_R} p(t) = 0$ and for $R = \infty$ by requiring that $p(t)$ lie in $L^2(\Omega_R)$ for almost all t in $[0, T]$.

6. Properties of the Velocity and Pressure

Theorem 6.1. ***(1)*** *Assume that u^0 is in $V(\mathbb{R}^2)$. Then there exists a unique weak solution (u, p) to (NS) with initial velocity u^0 for $R = \infty$ and initial velocity $\mathcal{T}_R u^0$ (see Definition 4.1) for R in $[1, \infty)$, with*

$$\begin{array}{ll} u \in L^\infty([0,T]; H(\Omega_R)), & \nabla u \in L^\infty([0,T]; L^2(\Omega_R)), \\ u \in L^4([0,T]; L^\infty(\Omega_R)), & \Delta u \in L^2([0,T]; L^2(\Omega_R)), \\ \partial_t u \in L^2([0,T]; H(\Omega_R)), & \nabla p \in L^2([0,T]; L^2(\Omega_R)), \\ u \in L^\infty([0,T]; H^1(\Omega_R)), & u \in L^2([0,T]; H^2(\Omega_R)), \end{array}$$

and the norms in these spaces can be bounded independently of R in $[1, \infty]$. If $R < \infty$ then p is in $L^2([0,T]; L^2(\Omega_R))$ and if $R = \infty$ then p is in $L^\infty([0,T]; L^2(\mathbb{R}^2))$ and ∇p is in $L^4([0,T]; L^2(\mathbb{R}^2))$.

(2) *Assume that u^0 is in $\mathbb{Y}_\theta(\mathbb{R}^2)$. Then there exists a unique weak solution (u, p) to (E) in the sense of Definition 5.2 with initial velocity u^0 for $R = \infty$ and initial velocity $\mathcal{T}_R u^0$ for R in $[1, \infty)$. The velocity u lies in $L^\infty([0,T]; \mathbb{Y}_\theta)$ and is unique in that class. We have,*

$$\begin{array}{ll} u \in L^\infty([0,T]; H(\Omega_R)), & \nabla u \in L^\infty([0,T]; L^2(\Omega_R)), \\ u \in L^\infty([0,T] \times \Omega_R), & u \in C([0,T] \times \overline{\Omega_R}) \\ \partial_t u \in L^\infty([0,T]; H(\Omega_R)), & \nabla p \in L^\infty([0,T]; L^2(\Omega_R)), \end{array}$$

and the norms in these spaces and of u in $L^\infty([0,T]; \mathbb{Y}_\theta)$ can be bounded independently of R in $[1, \infty]$. The pressure p is in $L^\infty([0,T]; H^1(\mathbb{R}^2))$. Also,

$$\|\omega(t)\|_{L^q(\Omega_R)} = \|\omega^0\|_{L^q(\Omega_R)} \tag{6.1}$$

for all q in $[p_0, \infty)$ (and for $q = \infty$ if ω^0 is in $L^\infty(\Omega_R)$) and almost all $t \geq 0$, where p_0 is as in Definition 2.2.

Furthermore, there is a bound on the modulus of continuity of $u(t, x)$ in t that is independent of x and a bound on the modulus of continuity of $u(t, x)$ in x that is independent of t, and both of these bounds are independent of R in $[1, \infty]$. There exists a unique flow X associated with u with bounds on the moduli of continuity in time and in space with the same properties just described for u. Finally, the bound, μ, on the modulus of continuity of $u(t, x)$ in x satisfies $\int_0^1 ds/\mu(s) = \infty$.

Proof. The facts regarding solutions to (NS) in (1) are entirely classical except perhaps for the independence of the norms on R. In that regard, we note that no domain-dependent constants enter into the bounds on u in $L^\infty([0,T]; H(\Omega_R))$ or ∇u in $L^2([0,T]; L^2(\Omega_R))$, as these bounds follow from the most basic energy equality derived by multiplying Equation (5.1) by u and integrating over Ω_R. (This is true even with forcing, though then the domain-independent bounds grow with T.) Only the norms of u^0 and ∇u^0 in $L^2(\Omega_R)$ enter into these bounds, and by Lemma 4.2 the truncation operator $\mathcal{T}_R$ is bounded in L^2 and H^1; hence, the bounds can be made independent of R.

In the bounds on ∇u in $L^\infty([0,T];L^2(\Omega_R))$ and Δu in $L^2([0,T];L^2(\Omega_R))$, domain-dependent constants do enter. These bounds follow by an energy inequality derived (formally) by multiplying Equation (5.1) by Au and integrating over Ω_R (see, for instance, the proof of Theorem III.3.10 p. 213-214 of [14] for details). Here, A is the Stokes operator.

The proof of this energy inequality relies on two key inequalities, the first being

$$C \left\|\Delta u\right\|_{L^2(\Omega_R)} \le \left\|Au\right\|_{L^2(\Omega_R)} \le \left\|\Delta u\right\|_{L^2(\Omega_R)}. \tag{6.2}$$

The constant C is independent of R because Au and Δu scale the same way with R. The second key inequality is Equation (A.3) applied to ∇u instead of u, giving

$$\left\|\nabla u\right\|_{L^4(\Omega_R)}^2 \le C \left\|\nabla u\right\|_{L^2(\Omega_R)} \left(\left\|\nabla\nabla u\right\|_{L^2(\Omega_R)} + (1/R)\left\|\nabla u\right\|_{L^2(\Omega_R)}\right).$$

But it follows from basic elliptic regularity theory (see, for instance, Theorem 8.12 p. 176 of [4])) that

$$\left\|\nabla\nabla u\right\|_{L^2(\Omega_R)} \le C \left(\left\|\Delta u\right\|_{L^2(\Omega_R)} + (1/R)\left\|\nabla u\right\|_{L^2(\Omega_R)}\right), \tag{6.3}$$

with a scaling argument to give the factor of $1/R$ and the independence of C on R. Other than the additional term of $(1/R)\left\|\nabla u\right\|_{L^2(\Omega_R)}$, which is easy to accommodate, the derivation of the energy inequality proceeds as usual, giving bounds on ∇u in $L^\infty([0,T];H^1(\Omega_R))$, on u in $L^\infty([0,T];L^2(\Omega_R))$, and on Δu in $L^2([0,T];L^2(\Omega_R))$ that are independent of R (though not of the shape of the domain).

Because u, ∇u, and Δu are each in $L^2([0,T];L^2(\Omega_R))$ with bounds on their norms that are independent of R, it follows from Equation (6.3) that u is in $L^2([0,T];H^2(\Omega_R))$ with a bound on its norm that is independent of R.

The remaining bounds on u, $\partial_t u$, and ∇p follow from these basic bounds, and in that way we obtain independence of all the stated norms on R.

By Lemma 4.2, the operator norm of $\mathcal{T}_R \colon \mathbb{Y}_\theta(\mathbb{R}^2) \to \mathbb{Y}_\theta(\Omega_R)$ is independent of R. So too then are the bounds on the norms in (2), which derive from the energy inequality and the transport of vorticity along the flow lines and so involve no domain-dependent constants.

For solutions to (E) in (2), the existence, uniqueness, and regularity of u for $R < \infty$ were proved in the special case of bounded initial vorticity by Yudovich in [18]. He extended uniqueness to the case of Yudovich initial vorticity in [19] for $R < \infty$; uniqueness for $R = \infty$ is essentially the same (see [7]). For R in $[1, \infty]$, existence in the class $\mathbb{Y}(\Omega_R)$ follows from Theorem 4.1 p. 126 and the comment immediately preceding Remark 4.4 p. 132 of

[10], the comment being that the L^p-norm of vorticity is independent of time for any p for which ω^0 is in L^p. For $R < \infty$, existence can also be established as in [17], [18] (see comment in the introduction to [19]). Uniqueness in the class $\mathbb{Y}(\Omega_R)$ for $R < \infty$ is established by Yudovich in [19], and his argument extends with little change to $R = \infty$.

To establish the facts concerning the moduli of continuity of the velocity and flow in the last paragraph of (2), however, it is much easier to adapt the approach in Majda's proof of existence and uniqueness of solutions to (E) as elucidated on p. 311-319 of [11]. (The proof is worked out in all of $\mathbb{R}^2$ but can be adapted to a bounded domain without difficulty.) The only significant change we need make for the unbounded initial vorticities in $\mathbb{Y}_\theta(\Omega_R)$ is to substitute the potential theory arguments in Lemma 6.2 for those in [11]. □

Lemma 6.2. *Let u lie in the space $L^\infty([0,T]; \mathbb{Y}_\theta(\Omega_R))$ for R in $[1, \infty]$ and assume that u is locally integrable in $[0,T] \times \Omega_R$. Then there exists a unique associated flow $X\colon [0,T] \times \Omega_R \to \Omega_R$. The moduli of continuity of $u(t, \cdot)$ and $X(t, \cdot)$ are each bounded by a function that depends only upon the norm of u in $L^\infty([0,T]; \mathbb{Y}_\theta(\Omega_R))$ and upon the function θ itself (in particular, the bound is independent of t in $[0,T]$.) Furthermore, if μ is the bound on the modulus of continuity of the u in space, then $\int_0^t ds/\mu(s) = \infty$.*

Proof. For $R = \infty$ this result follows from Theorem 3.1 of [15] (or see Chapter 5 of [8]). For $R < \infty$ it follows from Lemma 4.2 and Theorem 2 of [19] except for the independence of the moduli of continuity on R, but this follows from a scaling argument. In both cases, the bound depends only upon the function θ (via the function μ). □

As noted in [19], there is the somewhat surprising relationship between μ and the function β_1 of Equation (2.1) that $\mu(r) = (C/r)\beta_1(r^2/4)$.

7. Tail of the Velocity

For our solutions to (E) and (NS) in all of $\mathbb{R}^2$, at any time $t > 0$ the velocity $u(t)$ and its gradient $\nabla u(t)$ lie in $L^2(\mathbb{R}^2)$ and hence vanish at infinity, though at no specific a priori rate. In the proof of Theorem 8.1, however, we will need the stronger property that $u(t)$ vanishes at infinity in the L^2-norm at a rate that is bounded in $L^\infty([0,T])$ and, for (NS), that $\nabla u(t)$ vanishes in the L^2-norm at a rate that is bounded in $L^2([0,T])$. The rate itself, while unimportant to obtain convergence, will be determined by the rate at which u^0 vanishes at infinity, though will never be faster than C/R.

Lemma 7.1. *Let (u,p) be a solution to (E) in all of $\mathbb{R}^2$ with initial velocity in $\mathbb{Y}(\mathbb{R}^2)$. Then*

$$\|u\|_{L^\infty([0,T];L^2(\Omega_R^C))} \to 0 \text{ as } R \to \infty. \tag{7.1}$$

Let (u, p) be a solution to (NS) in all of $\mathbb{R}^2$ with initial velocity in $H(\mathbb{R}^2)$. Then Equation (7.1) holds and also

$$\|\nabla u\|_{L^2([0,T];L^2(\Omega_R^C))} \to 0 \text{ as } R \to \infty. \tag{7.2}$$

Proof. The lemma follows by a standard energy argument that involves scaling by R a cutoff function defined to be 0 on $\Omega_{1/2}$ and 1 on Ω_1^C. □

8. Main Result: Convergence of Solutions

Theorem 8.1. *Let u^0 be in $V(\mathbb{R}^2)$ and let (u_R, p_R) be the solution to (NS) of Definition 5.1 for R in $[1, \infty)$ with initial velocity $\mathcal{T}_R u^0$ in $V^{(NS)}(\Omega_R)$. ($\mathcal{T}_R$ is defined in Definition 4.1.) Let (u, p) be the solution to (NS) in all of $\mathbb{R}^2$ with initial velocity u^0. Then*

$$\|u_R - u\|_{L^\infty([0,T];L^2(\Omega_R))} \to 0 \text{ as } R \to \infty \tag{8.1}$$

and

$$\|\nabla u_R - \nabla u\|_{L^2([0,T];L^2(\Omega_R))} \to 0 \text{ as } R \to \infty. \tag{8.2}$$

Let u^0 be in $\mathbb{Y}(\mathbb{R}^2)$ and let (u_R, p_R) be the unique solution to (E) of Definition 5.2 for R in $[1, \infty)$ with initial velocity $\mathcal{T}_R u^0$ in $\mathbb{Y}(\Omega_R)$. Let (u, p) be the solution to (E) in all of $\mathbb{R}^2$ with initial velocity u^0. Then

$$\|u_R - u\|_{L^\infty([0,T];L^2 \cap L^\infty(\Omega_R))} \to 0 \text{ as } R \to \infty \tag{8.3}$$

and

$$\|\nabla u_R - \nabla u\|_{L^\infty([0,T];L^p(\Omega_R))} \to 0 \text{ as } R \to \infty \tag{8.4}$$

for all p in $[p_0, \infty)$, where p_0 is as in Definition 2.2. Also, if X_R and X are the flows associated to u_R and u, as given by Theorem 6.1, then

$$\|X_R - X\|_{L^\infty([0,T]\times\Omega_R)} \to 0 \text{ as } R \to \infty. \tag{8.5}$$

Proof. **Basic energy inequality**: For the first part of the proof we will treat (NS) and (E) in a unified manner, since, formally, (E) is simply (NS) with $\nu = 0$. We start with a basic energy argument. Let

$$w = u_R - u$$

and observe that $\|w(0)\|_{H^1(\Omega_R)} = \|u^0 - \mathcal{T}_R u^0\|_{H^1(\Omega_R)} \to 0$ as $R \to \infty$ by Lemma 4.2.

Subtracting Equation (5.1) for (u, p) from Equation (5.1) for (u_R, p_R), we have, on Ω_R,

$$\partial_t w + u_R \cdot \nabla u_R - u_R \cdot \nabla u + u_R \cdot \nabla u - u \cdot \nabla u + \nabla p_R - \nabla p = \nu \Delta w$$

or

$$\partial_t w + u_R \cdot \nabla w + w \cdot \nabla u + \nabla p_R - \nabla p = \nu \Delta w.$$

Multiplying by w and integrating over space, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega_R)} &+ \int_{\Omega_R}(u_R\cdot\nabla w)\cdot w + \int_{\Omega_R}(w\cdot\nabla u)\cdot w\\ &+ \int_{\Omega_R}\nabla(p_R - p)\cdot w = \nu\int_{\Omega_R}\Delta w\cdot w\\ &= -\nu\int_{\Omega_R}\nabla w\cdot\nabla w + \nu\int_{\Gamma_R}(\nabla w\cdot\mathbf{n})\cdot w\\ &= -\nu\int_{\Omega_R}|\nabla w|^2 - \nu\int_{\Gamma_R}(\nabla w\cdot\mathbf{n})\cdot u.\end{aligned}$$

In the last equality we used $\nu = 0$ for (E) and $u_R = 0$ on Γ_R for (NS).

But,

$$\begin{aligned}\int_{\Omega_R}(u_R\cdot\nabla w)\cdot w &= \int_{\Omega_R}u_R^j\partial_j w^i w^i = \frac{1}{2}\int_{\Omega_R}u_R^j\partial_j|w|^2 = \frac{1}{2}\int_{\Omega_R}u_R\cdot\nabla|w|^2\\ &= -\frac{1}{2}\int_{\Omega_R}(\operatorname{div}u_R)|w|^2 + \frac{1}{2}\int_{\Gamma_R}(u_R\cdot\mathbf{n})\cdot|w|^2 = 0,\end{aligned}$$

since $\operatorname{div} u_R = 0$ and $u_R\cdot n = 0$ on Γ_R (in fact, $u_R = 0$ on Γ_R for (NS)). Thus, we have,

$$\begin{aligned}&\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega_R)} + 2\nu\|\nabla w\|^2_{L^2(\Omega_R)}\\ &\quad = -2\int_{\Omega_R}\nabla(p_R - p)\cdot w - 2\nu\int_{\Gamma_R}(\nabla w\cdot\mathbf{n})\cdot u - 2\int_{\Omega_R}(w\cdot\nabla u)\cdot w.\end{aligned}$$

Integrating in time gives

$$\begin{aligned}&\|w(t)\|^2_{L^2(\Omega_R)} + 2\nu\int_0^t\|\nabla w\|^2_{L^2(\Omega_R)}\\ &\quad = \|w(0)\|^2_{L^2(\Omega_R)} - 2\int_0^t\int_{\Omega_R}\nabla(p_R - p)\cdot w\\ &\qquad - 2\nu\int_0^t\int_{\Gamma_R}(\nabla w\cdot\mathbf{n})\cdot u - 2\int_0^t\int_{\Omega_R}(w\cdot\nabla u)\cdot w.\end{aligned}\tag{8.6}$$

Letting $\mathcal{E}$ be the extension operator of Lemma A.1, we have

$$\int_{\Omega_R}\nabla(p_R - p)\cdot w = -\int_{\Omega_R}\nabla(p_R - p)\cdot u = \int_{\Omega_R^C}\nabla(\mathcal{E}p_R - p)\cdot u.$$

The first equality follows from $\int_{\Omega_R}\nabla(p_R - p)\cdot u_R = 0$ and the second from $\int_{\mathbb{R}^2}\nabla(\mathcal{E}p_R - p)\cdot u = 0$. Then,

$$\begin{aligned}\left|\int_0^t\int_{\Omega_R^C}\nabla p\cdot u\right| &\le \|\nabla p\|_{L^2([0,T];L^2(\mathbb{R}^2))}\|u\|_{L^2([0,T];L^2(\Omega_R^C))},\\ \left|\int_0^t\int_{\Omega_R^C}\nabla\mathcal{E}p_R\cdot u\right| &\le \|\nabla\mathcal{E}p_R\|_{L^2([0,T];L^2(\mathbb{R}^2))}\|u\|_{L^2([0,T];L^2(\Omega_R^C))}.\end{aligned}\tag{8.7}$$

The first integral in Equation (8.7) converges to 0 as $R \to \infty$ by Theorem 6.1 and Equation (7.1). Because

$$\|\nabla \mathcal{E} p_R\|_{L^2(\mathbb{R}^2)} \le C \left(\|\nabla p_R\|_{L^2(\Omega_R)} + \frac{1}{R} \|p_R\|_{L^2(\Omega_R)} \right) \le C \|\nabla p_R\|_{L^2(\Omega_R)}$$

by Lemma A.1 and Lemma A.3 (recall that $\int_{\Omega_R} p_R = 0$), the second integral in Equation (8.7) converges to 0 as well.

For solutions to (NS), we extend w to all of $\mathbb{R}^2$ as $w = \mathcal{E} u_R - u$ (we do not need a divergence-free extension). Then

$$\int_{\Gamma_R} (\nabla w \cdot \mathbf{n}) \cdot u = -\int_{\Omega_R^C} \nabla w \cdot \nabla u - \int_{\Omega_R^C} \Delta w \cdot u$$

so

$$\begin{aligned}\left| \int_0^t \int_{\Gamma_R} (\nabla w \cdot \mathbf{n}) \cdot u \right| &\le \|\nabla w\|_{L^2([0,T];L^2(\mathbb{R}^2))} \|\nabla u\|_{L^2([0,T];L^2(\Omega_R^C))} \\ &\qquad + \|\Delta w\|_{L^2([0,T];L^2(\mathbb{R}^2))} \|u\|_{L^2([0,T];L^2(\Omega_R^C))}.\end{aligned}$$

By Theorem 6.1, $\|\nabla u\|_{L^2([0,T];L^2(\mathbb{R}^2))} \le C$. Also,

$$\|\nabla \mathcal{E} u_R\|_{L^2([0,T];L^2(\mathbb{R}^2))} \le C \|u_R\|_{L^2([0,T];H^1(\Omega_R))} \le C$$

by Lemma A.1 and Theorem 6.1 so $\|\nabla w\|_{L^2([0,T];L^2(\mathbb{R}^2))} \le C$. Similar reasoning gives $\|\Delta w\|_{L^2([0,T];L^2(\mathbb{R}^2))} \le C$. Therefore,

$$\left| \int_0^t \int_{\Gamma_R} (\nabla w \cdot \mathbf{n}) \cdot u \right| \to 0$$

as $R \to \infty$ by Equation (7.1) and Equation (7.2). (It is only in this bound that we require that u^0 lie in $V(\mathbb{R}^2)$. For the other bounds, u^0 in $H(\mathbb{R}^2)$ would have sufficed.)

From Equation (8.6) and the estimates above, we have that

$$\|w(t)\|_{L^2(\Omega_R)}^2 + 2\nu \int_0^t \|\nabla w\|_{L^2(\Omega_R)}^2 \le K + 2 \int_0^t \int_{\Omega_R} |\nabla u| \, |w|^2, \tag{8.8}$$

where $K \to 0$ as $R \to \infty$.

Solutions to (NS) with u^0 in V: Assume that (u_R, p_R) and (u, p) are solutions to (NS) with u^0 in $V^{(NS)}(\mathbb{R}^2)$. Applying Lemma A.2, Young's inequality, and the inequality $(A + B)^2 \le 2(A^2 + B^2)$ to Equation (8.8), we have

$$\begin{aligned}\|w(t)\|_{L^2(\Omega_R)}^2 &+ 2\nu \int_0^t \|\nabla w\|_{L^2(\Omega_R)}^2 \le K + 2 \int_0^t \|\nabla u\|_{L^2(\Omega_R)} \|w\|_{L^4(\Omega_R)}^2 \\ &\le K + 2^{3/2} \int_0^t \|\nabla u\|_{L^2(\Omega_R)} \|w\|_{L^2(\Omega_R)} \left(\|\nabla w\|_{L^2(\Omega_R)} + \frac{1}{R} \|w\|_{L^2(\Omega_R)} \right) \\ &\le K + \nu \int_0^t \left(\|\nabla w\|_{L^2(\Omega_R)}^2 + \frac{1}{R^2} \|w\|_{L^2(\Omega_R)}^2 \right) + C \int_0^t \|\nabla u\|_{L^2(\Omega_R)}^2 \|w\|_{L^2(\Omega_R)}^2,\end{aligned}$$

or,

$$\|w(t)\|^2_{L^2(\Omega_R)} + \nu \int_0^t \|\nabla w\|^2_{L^2(\Omega_R)} \le K + \int_0^t \left(C \|\nabla u\|^2_{L^2(\Omega_R)} + \frac{\nu}{R^2} \right) \|w\|^2_{L^2(\Omega_R)}$$
$$\le K + C \int_0^t \|w\|^2_{L^2(\Omega_R)},$$

where we used Theorem 6.1 in the last inequality. Applying Gronwall's lemma gives Equation (8.1) and Equation (8.2).

Solutions to (E): By Lemma 4.2 and Theorem 6.1, there exists a unique solution (u_R, p_R) to (E) for all R in $[1, \infty)$ and both u_R and u lie in $L^\infty(\mathbb{R} \times \Omega_R)$ with a norm that is independent of $\mathbb{R}$. Thus,

$$M = \sup_{R \ge 1} \| \, |w|^2 \, \|_{L^\infty([0,T] \times \Omega_R)} \tag{8.9}$$

is finite and independent of R in $[1, \infty]$.

We now proceed as in [19] or [7]. Let s be in $[0, T]$, and let

$$A = |w(s,x)|^2, \quad B = |\nabla u(s,x)|, \quad L(s) = \|w(s)\|^2_{L^2}.$$

Then for all $1/\epsilon$ in $[2 + \epsilon_0, \infty)$,

$$\int_{\Omega_R} |\nabla u(s,x)| \, |w(s,x)|^2 \, dx = \int_{\Omega_R} AB = \int_{\Omega_R} A^\epsilon A^{1-\epsilon} B \le M^\epsilon \int_{\Omega_R} A^{1-\epsilon} B$$
$$\le M^\epsilon \left\| A^{1-\epsilon} \right\|_{L^{1/(1-\epsilon)}} \|B\|_{L^{1/\epsilon}} = M^\epsilon \|A\|^{1-\epsilon}_{L^1} \|B\|_{L^{1/\epsilon}}$$
$$= M^\epsilon L(s)^{1-\epsilon} \|\nabla u(s)\|_{L^{1/\epsilon}} \le C M^\epsilon L(s)^{1-\epsilon} \frac{1}{\epsilon} \|\omega^0\|_{L^{1/\epsilon}}$$
$$\le C M^\epsilon L(s)^{1-\epsilon} \frac{1}{\epsilon} \theta(1/\epsilon),$$

where θ is as in Definition 2.1. Here we used Lemma A.5 and the bounds on the L^p-norms of the vorticity given by Equation (6.1). Since this inequality holds for all ϵ in $(0, 1/(2 + \epsilon_0)^{-1}]$ it follows that

$$2 \int_{\mathbb{R}^2} |\nabla u(s,x)| \, |w(s,x)|^2 \, dx \le C \beta_M(L(s)),$$

with β_M as in Equation (2.1). From Equation (8.8), then, we have

$$L(t) \le K + C \int_0^t \beta_M(L(r)) \, dr. \tag{8.10}$$

By Lemma A.6,

$$\int_K^{L(t)} \frac{ds}{C \beta_M(s)} \le \int_0^t ds = t. \tag{8.11}$$

It follows that for all t in $(0, T]$,

$$\int_K^1 \frac{ds}{\beta_M(s)} \le CT + \int_{L(t)}^1 \frac{ds}{\beta_M(s)}.$$

Since Equation (2.2) holds, as $R \to \infty$ the left side becomes infinite; hence, so must the right side. But this implies that $L(t) \to 0$ as $R \to \infty$, and that the convergence is uniform over $[0, T]$: this is Equation (8.1). It also follows from Equation (8.11) that

$$\int_K^{L(t)} \frac{dr}{\beta_M(r)} \le Ct,$$

which can be used, in principle, to bound the rate of convergence. Also, Equation (8.3) follows by an application of Corollary 8.4 to u_R and $u|_{\Omega_R}$.

Vorticity for solutions to (E): We have,

$$\begin{aligned}
\|\omega_R(t) - \omega(t)\|_{L^p(\Omega_R)} &= \|\omega^0(\mathcal{T}_R u^0) \circ X_R^{-1}(t) - \omega^0 \circ X^{-1}(t)\|_{L^p(\Omega_R)} \\
&\le \|\omega^0(\mathcal{T}_R u^0) \circ X_R^{-1}(t) - \omega^0 \circ X_R^{-1}(t)\|_{L^p(\Omega_R)} \\
&\qquad + \|\omega^0 \circ X_R^{-1}(t) - \omega^0 \circ X^{-1}(t)\|_{L^p(\Omega_R)} \\
&= \|\omega^0(\mathcal{T}_R u^0) - \omega^0\|_{L^p(\Omega_R)} + \|\omega^0 \circ X_R^{-1}(t) - \omega^0 \circ X^{-1}(t)\|_{L^p(\Omega_R)},
\end{aligned} \tag{8.12}$$

using, in the last step, that $X_R^{-1}(t)$ is measure-preserving and maps Ω_R to itself. The first term on the right-hand side of Equation (8.12) converges to zero as $R \to \infty$ by Lemma 4.2.

This leaves the second term on the right-hand side of Equation (8.12), which converges to zero by Lemma 8.2 if $X_R^{-1} \to X^{-1}$ in $L^\infty([0, T] \times \Omega_R)$, which we now show.

The inverse flow X^{-1} is given by

$$X^{-1}(t, x) = x - \int_0^t u(s, X^{-1}(s, x))\, ds,$$

and similarly for X_R^{-1}. Then,

$$\begin{aligned}
\left|X_R^{-1}(t, x) - X^{-1}(t, x)\right| &= \left|\int_0^t (u_R(s, X_R^{-1}(s, x)) - u(s, X^{-1}(s, x)))\, ds\right| \\
&\le \int_0^t \left|u_R(s, X_R^{-1}(s, x)) - u(s, X_R^{-1}(s, x))\right|\, ds \\
&\qquad + \int_0^t \left|u(s, X_R^{-1}(s, x)) - u(s, X^{-1}(s, x))\right|\, ds.
\end{aligned}$$

But,

$$\left|u(s, X_R^{-1}(s, x)) - u(s, X^{-1}(s, x))\right| \le \mu(\left|X_R^{-1}(s, x) - X^{-1}(s, x)\right|),$$

where μ is the bound on the modulus of continuity in space of u given by Theorem 6.1. Also,

$$\int_0^t \left|u_R(s, X_R^{-1}(s, x)) - u(s, X_R^{-1}(s, x))\right|\, ds \le A(R)T,$$

where $A(R) = \|u_R - u\|_{L^\infty([0,T]\times\Omega_R)}$; this converges to zero as $R \to \infty$ by Equation (8.3). Thus,

$$\left|X_R^{-1}(t,x) - X^{-1}(t,x)\right| \le A(R)T + \int_0^t \mu(\left|X_R^{-1}(s,x) - X^{-1}(s,x)\right|).$$

Letting $L_R(t) = \left|X_R^{-1}(t,x) - X^{-1}(t,x)\right|$, we have

$$L_R(t) \le A(R)T + \int_0^t \frac{ds}{\mu(s)}.$$

Applying Lemma A.6 gives

$$\int_{A(R)T}^{L_R(t)} \frac{ds}{\mu(s)} = t.$$

Because $\int_0^1 \mu(s)\,ds = \infty$, we conclude that $X_R^{-1} \to X^{-1}$ in $L^\infty([0,T]\times\Omega_R)$, thus completing the demonstration of Equation (8.5). Applying Lemma A.5 for $p \ge 2 + \epsilon_0$ and standard elliptic regularity bounds along with Equation (8.3) for p in $[p_0, 2+\epsilon_0)$ gives Equation (8.4). □

We can obtain an upper bound on the rate of convergence of solutions to (NS) in Equation (8.1) and Equation (8.2) by examining the bounds in the proof above, in the proof of Lemma 7.1, and the proof of Lemma 4.2. Similarly, we can obtain a bound on the rate of convergence of solutions to (E) in Equation (8.3). For (NS), the convergence rate is controlled by the rate of decay with R of $\|u^0\|_{L^2(\Omega_R^C)}$ and $\|\nabla u^0\|_{L^2(\Omega_R^C)}$. For solutions to (E), the convergence rate is controlled by the rate of decay with R of $\|u^0\|_{L^2(\Omega_R^C)}$ and by the function β_M of Definition 2.1. (The function β_M enters into these bounds much as in [7] or [9].)

We can also obtain a bound on the rate of convergence in Equation (8.4), but this ultimately relies on measure-theoretic properties of ω^0 that are hard to usefully characterize let alone quantify. The rate of convergence of the flow, however, can be determined much as for the convergence in Equation (8.3).

We used the following lemmas in the proof of Theorem 8.1:

Lemma 8.2. *Let f be in $L^p(\mathbb{R}^d)$, $1 \le p < \infty$, $d \ge 1$ and let (X_n) and (Y_n) be sequences of measure-preserving homeomorphisms from a domain Σ_R of $\mathbb{R}^d$ to all of $\mathbb{R}^d$ with*

$$\|X_n - Y_n\|_{L^\infty(\Sigma_R)} \le M(n)$$

with $M(n) \to 0$ as $n \to \infty$. Then there exists a nondecreasing function $N : (0,\infty) \to \mathbb{Z}^+$ such that for all $\epsilon > 0$ if $n \ge N(\epsilon)$ then

$$\|f \circ X_n - f \circ Y_n\|_{L^p(\Sigma_R)} \le \epsilon.$$

Furthermore, the function N depends only upon the functions f and M.

Proof. Our proof is an adaptation of the proof that translation is continuous in $L^p(\mathbb{R}^d)$ (see, for instance, Theorem 8.19 p. 134-135 of [16]). Approximate f in $L^p(\mathbb{R}^d)$ by a sequence of functions (f_k) that are finite linear combinations of characteristic functions of cubes in $\mathbb{R}^d$. It is easy to see that if g_1 is the characteristic function of a cube, then

$$\|g_1 \circ X_n - g_1 \circ Y_n\|_{L^p(\Sigma_R)} \le \|g_1(\cdot + M(n)\mathbf{e}_j) - g_1(\cdot)\|_{L^p(\Sigma_R)},$$

and that $\|g_1(\cdot + M(n)\mathbf{e}_j) - g_1(\cdot)\|_{L^p(\Sigma_R)} \to 0$ as $n \to \infty$. Here, $\mathbf{e}_j$ is any of the coordinate basis vectors. If g_2 is also the characteristic function of a cube, then

$$\begin{aligned}
&\|(g_1 + g_2) \circ X_n - (g_1 + g_2) \circ Y_n\|_{L^p(\Sigma_R)} \\
&\quad = \|g_1 \circ X_n - g_1 \circ Y_n + g_2 \circ X_n - g_2 \circ Y_n\|_{L^p(\Sigma_R)} \\
&\quad \le \|g_1 \circ X_n - g_1 \circ Y_n\|_{L^p(\Sigma_R)} + \|g_2 \circ X_n - g_2 \circ Y_n\|_{L^p(\Sigma_R)} \\
&\quad \le \|g_1(\cdot + M(n)\mathbf{e}_j) - g_1(\cdot)\|_{L^p(\Sigma_R)} + \|g_2(\cdot + M(n)\mathbf{e}_j) - g_2(\cdot)\|_{L^p(\Sigma_R)},
\end{aligned}$$

so $\|(g_1 + g_2) \circ X_n - (g_1 + g_2) \circ Y_n\|_{L^p(\Sigma_R)} \to 0$ as $n \to \infty$ at a rate that is bounded in terms of $M(n)$. We conclude then that each f_k has the property that $\|f_k \circ X_n - f_k \circ Y_n\|_{L^p(\Sigma_R)} \to 0$ as $n \to \infty$ at a rate that is bounded in terms of $M(n)$.

Now let $\epsilon > 0$ and choose k large enough that $\|f_k - f\|_{L^p(\mathbb{R}^2)} < \epsilon/4$. Then

$$\begin{aligned}
\|f \circ X_n - f \circ Y_n\|_{L^p(\Sigma_R)} &\le \|f \circ X_n - f_k \circ X_n\|_{L^p(\Sigma_R)} \\
&\qquad + \|f_k \circ X_n - f_k \circ Y_n\|_{L^p(\Sigma_R)} + \|f_k \circ Y_n - f \circ Y_n\|_{L^p(\Sigma_R)} \\
&= \|f_k \circ X_n - f_k \circ Y_n\|_{L^p(\Sigma_R)} + \|f_k - f\|_{L^p(X_n^{-1}(\Sigma_R))} + \|f_k - f\|_{L^p(Y_n^{-1}(\Sigma_R))} \\
&\le \|f_k \circ X_n - f_k \circ Y_n\|_{L^p(\Sigma_R)} + 2\|f_k - f\|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

If we choose N large enough that $\|f_k \circ X_n - f_k \circ Y_n\|_{L^p(\Sigma_R)} < \epsilon/2$ for all $n \ge N$, it follows that $\|f \circ X_n - f \circ Y_n\|_{L^p(\Sigma_R)} < \epsilon$ for all $n \ge N$. What we have constructed is the desired map $N = N(\epsilon)$ from the properties only of M and f. $\square$

Lemma 8.3. *Let*

$$\mathcal{F}(\Omega_R) = \left\{u \in (C(\Omega_R))^2 : |u(x) - u(y)| \le \rho(|x - y|)\right\},$$

where ρ is a nondecreasing continuous function with $\rho(0) = 0$. (That is, $\mathcal{F}(\Omega_R)$ consists of all continuous functions on Ω_R with a given common bound on their modulus of continuity.) Then there exists a continuous function $F : [0, \infty) \to [0, \infty)$ with $F(0) = 0$ such that for all u_1, u_2 in $\mathcal{F}(\Omega_R)$,

$$\|u_1 - u_2\|_{L^\infty(\Omega_R)} \le F(\|u_1 - u_2\|_{L^2(\Omega_R)}).$$

Moreover, a choice of F can be made that is independent of R in $[1, \infty]$.

Proof. Assume first that $R = \infty$ and let u_1, u_2 be in $\mathcal{F}(\Omega_R)$. Fix x in Ω_R and let

$$\delta = |u_1(x) - u_2(x)|\,.$$

Now suppose that y is in the ball B of radius a about x, where $a = \rho^{-1}(\delta/4)$. Then

$$|u_1(x) - u_1(y)| \le \rho(|x-y|) \le \rho(a) = \delta/4$$

and also $|u_2(x) - u_2(y)| \le \delta/4$. It follows that

$$|u_1(y) - u_2(y)| \ge \delta/2$$

for all y in B, and thus that

$$\|u_1 - u_2\|_{L^2(\mathbb{R}^2)} \ge \|u_1 - u_2\|_{L^2(B)} \ge \left(\int_B (\delta/2)^2\right)^{1/2} = \frac{\sqrt{\pi}}{2} a\delta.$$

Hence,

$$h(\delta) := \frac{\sqrt{\pi}}{2}\delta\rho^{-1}(\delta/4) \le \|u_1 - u_2\|_{L^2(\mathbb{R}^2)} \tag{8.13}$$

so

$$|u_1(x) - u_2(x)| = \delta \le h^{-1}(\|u_1 - u_2\|_{L^2(\mathbb{R}^2)}).$$

Since this is true for all x in Ω_R,

$$\|u_1 - u_2\|_{L^\infty(\mathbb{R}^2)} \le F(\|u_1 - u_2\|_{L^2(\mathbb{R}^2)}), \tag{8.14}$$

where $F = h^{-1}$, and where we note that $F(0) = 0$.

The only modification required for R in $[1, \infty)$ is that we must replace the ball B with $B \cap \Omega_R$. If B has radius

$$r < 1/(2\overline{\kappa}_R) = R/(2\overline{\kappa}_1) = CR,$$

where $\overline{\kappa}_R$ is the maximum curvature of Γ_R (which is necessarily positive), then it is easy to see that $\operatorname{Area}(B \cap \Omega_R) \ge (1/4) \operatorname{Area} B$. This has the effect of changing the constant $\sqrt{\pi}/2$ in Equation (8.13) to $\sqrt{\pi}/8$ and gives $F(x) = h^{-1}(x)$ for x in the interval $[0, CR]$. For $x > CR$, the constant in Equation (8.13) decreases below $\sqrt{\pi}/8$ resulting in an F that increases more rapidly than h^{-1}. In any case, it follows that the function F that results for $R = 1$ serves as an upper bound on F for all R in $[1, \infty]$. □

Corollary 8.4. *Let $u_j \colon [0,T] \times \Omega_R \to \mathbb{R}^2$, $j = 1, 2$, with $u_j(t)$ in $\mathcal{F}(\Omega_R)$ for almost all t in $[0,T]$, where $\mathcal{F}(\Omega_R)$ is as in Lemma 8.3. Then there exists a continuous function $F : [0, \infty) \to [0, \infty)$ with $F(0) = 0$ such that*

$$\|u_1 - u_2\|_{L^\infty([0,T]\times\Omega_R)} \le F(\|u_1 - u_2\|_{L^\infty([0,T];L^2(\Omega_R))}).$$

Proof. Apply Lemma 8.3 to $u_1(t)$ and $u_2(t)$ for all t in $[0,T]$. □

Appendix A. Various Lemmas

Lemma A.1. *For any R in $[1,\infty)$ there exists a single bounded linear extension operator $\mathcal{E} = \mathcal{E}_R$, $\mathcal{E} : H^{n,p}(\Omega_R) \to H^{n,p}(\mathbb{R}^2)$ for all $n = 0, 1, \dots$ and all p in $[1,\infty]$, with*

$$\|\mathcal{E}f\|_{H^{n,p}(\mathbb{R}^2)} \le C_n \|f\|_{H^{n,p}(\Omega_R)}, \tag{A.1}$$

where the constant C_n is independent of p and R in $[1,\infty]$.

If f is in $H^{1,p}(\Omega_R)$ then

$$\|\nabla \mathcal{E}f\|_{L^p(\mathbb{R}^2)} \le C \left(\|\nabla f\|_{L^p(\Omega_R)} + \frac{1}{R} \|f\|_{L^p(\Omega_R)} \right) \tag{A.2}$$

with a constant C that is independent of p and R in $[1,\infty]$.

Proof. First define the extension operator $\mathcal{E}_1$ on Ω_1. We can use, for instance, a partition of unity and the extension operator of Theorem 5' p. 181 of [13], since we have sufficient smoothness of the boundary. This gives Equation (A.1) for $R = 1$ with independence of C_n on p. (The extension operator of Theorem 5 p. 181 of [13] would suffice, except for the independence of C_n on p.)

Now let R be in $[1,\infty)$ with f in $H^{n,p}(\Omega_R)$, and define f in $H^{n,p}(\Omega_1)$ by $f_1(x) = f(Rx)$. Then define $\mathcal{E}_R$ by $\mathcal{E}_R f(x) = (\mathcal{E}_1 f_1)(x/R)$. The factor of $1/R$ in Equation (A.2) and the independence of C_n on R in $[1,\infty)$ follow by scaling. □

The following is Ladyzhenskaya's inequality and a simple consequence of it.

Lemma A.2. *For u in $H^1_0(\Omega_R)$ with R in $[1,\infty]$,*

$$\|u\|^2_{L^4(\Omega_R)} \le 2^{1/2} \|u\|_{L^2(\Omega_R)} \|\nabla u\|_{L^2(\Omega_R)}.$$

For u in $H^1(\Omega_R)$ with R in $[1,\infty)$,

$$\|u\|^2_{L^4(\Omega_R)} \le C \|u\|_{L^2(\Omega_R)} \left(\|\nabla u\|_{L^2(\Omega_R)} + \frac{1}{R} \|u\|_{L^2(\Omega_R)} \right), \tag{A.3}$$

where C is independent of R in $[1,\infty]$.

Proof. The first inequality is Ladyzhenskaya's inequality (see, for instance, Lemma III.3.3 p. 197 of [14]). The second inequality follows from the first, since $H^1_0(\mathbb{R}^2) = H^1(\mathbb{R}^2)$, and from Lemma A.1:

$$\begin{aligned}\|u\|^2_{L^4(\Omega_R)} &\le \|\mathcal{E}u\|^2_{L^4(\Omega_R)} \le 2^{1/2} \|\mathcal{E}u\|_{L^2(\Omega_R)} \|\nabla \mathcal{E}u\|_{L^2(\Omega_R)} \\ &\le C \|u\|_{L^2(\Omega_R)} \left(\|\nabla u\|_{L^2(\Omega_R)} + \frac{1}{R} \|u\|_{L^2(\Omega_R)} \right).\end{aligned}$$

□

Lemma A.3 (Poincaré's inequality)**.** *Let U be an open bounded connected subset of $\mathbb{R}^2$ with a C^1-boundary, and let $U_R = RU$. Then for all f in $H^{1,p}(U_R)$ with $\int_{U_R} f = 0$,*

$$\|f\|_{L^p(U_R)} \le C_p R \|\nabla f\|_{L^p(U_R)}$$

for all p in $[1, \infty]$, where C_p is independent of R.

Proof. This is classical; see, for instance, Theorem 1 p. 275 of [3]. To verify that the scaling factor is R, assume that

$$\|f\|_{L^p(U_R)} \le C_p(R) \|\nabla f\|_{L^p(U_R)}. \tag{A.4}$$

Let f be in $L^p(U_R)$ and define f_1 in $L^p(U_1)$ by $f_1(x) = f(Rx)$. Then the chain rule and a change of variables gives

$$\|f_1\|_{L^p(U_1)} = R^{-2/p} \|f\|_{L^p(U_R)},$$

while

$$\|\nabla f_1\|_{L^p(U_1)} = R^{1-2/p} \|\nabla f\|_{L^p(U_R)}.$$

Multiplying both sides of Equation (A.4) by $R^{-2/p}$ gives

$$\|f_1\|_{L^p(U_1)} \le C_p(R) R^{-1} \|\nabla f\|_{L^p(U_R)}.$$

Since this is true for all f in $L^p(U_R)$ it follows that $C_p(1) \le C_p(R)R^{-1}$. Interchanging the roles of U_R and U_1 it follows that $C_p(R) = C_p(1)R$. □

Lemma A.4. *Let f be a scalar- or vector-valued function in $L^2(\mathbb{R}^2)$ with ∇f in $L^a(\mathbb{R}^2)$ for some a in $(2, \infty)$. Then f is in $L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, and for all b in $(a, \infty]$,*

$$\|f\|_{L^b(\mathbb{R}^2)} \le C \left(\|f\|_{L^2(\mathbb{R}^2)} + C \|\nabla f\|_{L^a(\mathbb{R}^2)} \right), \tag{A.5}$$

where the constant C depends on a and on b.

Let v be a divergence-free vector field in $L^2(\mathbb{R}^2)$ with vorticity ω lying in $L^a(\mathbb{R}^2)$ for some a in $(2, \infty)$. Then v is in $L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, and for all b in $(a, \infty]$,

$$\|v\|_{L^b(\mathbb{R}^2)} \le C \left(\|v\|_{L^2(\mathbb{R}^2)} + \frac{a^2}{a-1} \|\omega\|_{L^a(\mathbb{R}^2)} \right), \tag{A.6}$$

where the constant C depends on a and on b.

Proof. This can be proven by decomposing v into low and high-frequencies using Littlewood-Paley operators. See, for instance, Lemma 2B.1 p. 23-24 of [8]. □

The following is a result of Yudovich's:

Lemma A.5. *Fixing $\epsilon_0 > 0$, for any p in $[2+\epsilon_0, \infty)$ and any u in $V^{(E)}(\Omega_R)$ (recall that Ω_R is simply connected),*

$$\|\nabla u\|_{L^p(\Omega_R)} \le Cp \|\omega(u)\|_{L^p(\Omega_R)},$$

with a constant C that is independent of p and of R in $[1, \infty]$.

Proof. Let u be in $V^{(E)}(\Omega_R)$. Then ψ, the stream function for u, can be assumed to vanish on Γ_R since Ω_R is simply connected. Applying Corollary 1 of [17] with the operator $L = \Delta$ and $r = 0$ gives

$$\|\nabla u\|_{L^p(\Omega_R)} \le \|\psi\|_{H^{2,p}(\Omega_R)} \le C(\Omega_R)p\,\|\Delta\psi\|_{L^p(\Omega_R)} = C(\Omega_R)p\,\|\omega(u)\|_{L^p(\Omega_R)}.$$

To demonstrate the independence of $C(\Omega_R)$ on R, let u be an arbitrary element of $V^{(E)}(\Omega_R)$. Then $u(\cdot) = u_1(\cdot/R)$ for some u_1 in $V^{(E)}(\Omega_1)$. But, $\|\nabla u\|_{L^p(\Omega_R)} = R^{2/p-1}\,\|\nabla u_1\|_{L^p(\Omega_1)}$ and $\|\omega(u)\|_{L^p(\Omega_R)} = R^{2/p-1}\,\|\omega(u_1)\|_{L^p(\Omega_1)}$, so $C(\Omega_R) \le C(\Omega_1)$; the argument in reverse shows equality of the two constants. □

The following is Osgood's lemma (see, for instance, p. 92 of [1]). The succinct proof is due to M. Tehranchi.

Lemma A.6 (Osgood's lemma)**.** *Let L be a measurable nonnegative function and γ a nonnegative locally integrable function, each defined on the domain $[t_0, t_1]$. Let $\mu\colon [0,\infty) \to [0,\infty)$ be a continuous nondecreasing function, with $\mu(0) = 0$. Let $a \ge 0$, and assume that for all t in $[t_0, t_1]$,*

$$L(t) \le a + \int_{t_0}^t \gamma(s)\mu(L(s))\,ds. \tag{A.7}$$

If $a > 0$, then

$$\int_a^{L(t)} \frac{ds}{\mu(s)} \le \int_{t_0}^t \gamma(s)\,ds.$$

If $a = 0$ and $\int_0^\infty ds/\mu(s) = \infty$, then $L \equiv 0$.

Proof. We have,

$$\begin{aligned}\int_a^{L(t)} \frac{dx}{\mu(x)} &\le \int_a^{a+\int_{t_0}^t \gamma(u)\mu(L(u))\,du} \frac{dx}{\mu(x)} \\ &\le \int_{t_0}^t \frac{\gamma(s)\mu(L(s))\,ds}{\mu(a + \int_{t_0}^s \gamma(u)\mu(L(u))\,du)} \le \int_{t_0}^t \gamma(s)\,ds.\end{aligned}$$

The last inequality follows from Equation (A.7), since μ is nondecreasing. □

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