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The Irreducible Representations of the Lie Algebra $\mathfrak{sl}(2)$ and of the Weyl Algebra

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Contents.

1. INTRODUCTION

This paper presents what is in a sense a solution of—or at least decisive progress on—two longstanding problems on infinite-dimensional representations of algebras, namely, the problems of determining the irreducible representations of the Lie algebra $\mathfrak{sl}(2)$ and of the Weyl algebra $\mathfrak{A}_1$.

These problems had been regarded as hopeless. Indeed, let $\mathfrak{g}$ be any (finite-dimensional) nonabelian Lie algebra over the complex numbers $\mathbf{C}$, and consider (algebraically) irreducible representations of $\mathfrak{g}$ (or equivalently of the enveloping algebra $U\mathfrak{g}$) acting on a vector space which is allowed to be infinite dimensional. The subject of enveloping algebras is largely concerned with these representations, but even in the simplest nonabelian case, with $\mathfrak{g} = \mathfrak{h}$ the three-dimensional (nilpotent) Heisenberg algebra, as Dixmier remarks in the preface of [15], "a deeper study reveals the existence of an

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enormous number of irreducible representations of $\mathfrak{h}$ .... It seems that these representations defy classification. A similar phenomenon exists for $\mathfrak{g} = \mathfrak{sl}(2)$, and most certainly for all nonabelian Lie algebras." The extensive work in the subject has concentrated on the classification of the kernels, that is, the primitive ideals of $U\mathfrak{g}$, and the construction of interesting series of the representations.

The present paper considers the problem of the classification of the irreducible representations of $\mathfrak{h}$ and of $\mathfrak{sl}(2)$ and also of the two-dimensional nonabelian Lie algebra $\mathfrak{b}$, and thus of the prototypes respectively of nilpotent, semisimple and solvable Lie algebras. Our results for $\mathfrak{sl}(2)$ are roughly this: the simple modules are (up to isomorphism) the highest weight modules, the Whittaker modules, and a family of (mutually nonisomorphic) modules (mostly new) which we construct. The modules in the latter family are in bijective correspondence to the set of all pairs $(\gamma, [b])$, where γ is a scalar (given by the action of the Casimir element c) and $[b]$ is a similarity class (containing b) of irreducible elements of the algebra $\mathfrak{B}$ of differential operators with rational function coefficients. The simple module for a given $(\gamma, [b])$ is constructed by giving, in the simple $\mathfrak{B}$-module $\mathfrak{B}/\mathfrak{B}b$, a generator for a simple submodule over a certain copy of $U\mathfrak{sl}(2)/(c-\gamma)$ in $\mathfrak{B}$; actually, we give a sufficient condition for an element of $\mathfrak{B}/\mathfrak{B}b$ to be in this submodule, and show how to find (nonzero) elements satisfying this condition. In the case of elements (necessarily irreducible) of $\mathfrak{B}$ of degree (that is, order) one, where the set of similarity classes has a very simple description, we carry out explicitly the construction of the corresponding simple $\mathfrak{sl}(2)$-modules (as specific subsets of $\mathbf{C}(q)$).

The same classification problem is also considered for the (first) Weyl algebra $\mathfrak{A} = \mathfrak{A}_1(\mathbf{C})$—this is the associative algebra $\mathbf{C}[q, p]$ with two generators p, q subject to the relation $pq - qp = 1$, which may be realized as the algebra of differential operators with polynomial coefficients. As is well known, the classification problems for $\mathfrak{h}$ and for $\mathfrak{A}$ are equivalent, since the primitive quotients of $U\mathfrak{h}$ are $U\mathfrak{h}/U\mathfrak{h}(z-\alpha)$ (where $0 \neq z \in$ center of $\mathfrak{h}$ and $\alpha \in \mathbf{C}$), and $U\mathfrak{h}/U\mathfrak{h}(z-\alpha) \cong \mathfrak{A}$ if $\alpha \neq 0$. Our result for $\mathfrak{A}$ is similar to that indicated above for $\mathfrak{sl}(2)$. Our result for $\mathfrak{b}$ is somewhat more complicated in that only a subset (which we determine) of the set of similarity classes of irreducible elements of $\mathfrak{B}$ is used in the parameter set. The results of previous studies of the irreducible modules over $\mathfrak{A}$ (including [5, 6, 12–14, 24]) and over $\mathfrak{sl}(2)$ (including [2–4, 23, 25, 26]) are fragmentary, and the study of the irreducible modules over $\mathfrak{b}$ does not seem to have been undertaken before.

The algebra $\mathfrak{B}$ mentioned above may be written as $\mathfrak{B} = \mathfrak{B}(\mathbf{C}) = \mathbf{C}(q)[p]$, the associative algebra of polynomials in p $(= d/dz)$ with coefficients in $\mathbf{C}(q)$ (where $q =$ multiplication by z) subject again to the relation $pq - qp = 1$; thus $\mathfrak{B}$ is the localization of $\mathfrak{A}$ at $S = \mathbf{C}[q] - \{0\}$. While both $\mathfrak{B}$ and its subalgebra $\mathfrak{A}$ are simple rings, $\mathfrak{B}$ has the added property that it is a principal left (or right) ideal domain (in fact Euclidean). By a classical theory the representations of $\mathfrak{B}$ can be described in terms of factorization of elements of $\mathfrak{B}$. We recall [18] that a $\mathfrak{B}$-module M is simple if and only if $M \cong \mathfrak{B}/\mathfrak{B}b$ for some $b \in \mathfrak{B}$ which is irreducible (in the usual sense: $b = ac$ implies a or c is a unit); and $\mathfrak{B}/\mathfrak{B}b \cong \mathfrak{B}/\mathfrak{B}a$ if and only if a and b are similar, that is, there exists $c \in \mathfrak{B}$ such that 1 is a greatest common right divisor of b and c and ac is a least common left multiple of b and c (similar is a noncommutative generalization of associate).

For any algebra A we denote by $A^\wedge$ the set of isomorphism classes of simple A-modules. Thus we may identify $\mathfrak{B}^\wedge$ with the set of similarity classes of irreducible elements of $\mathfrak{B}$, with the isomorphism class $[N]$ of a simple module N identified with the similarity class $[b]$ where for some $0 \neq n \in N$, b is a minimal annihilator of n (= annihilator of n of lowest degree (in p), such b necessarily being irreducible).

We now describe in more detail our principal results for $A^\wedge$ for the three main cases mentioned above, namely, for $A = \mathfrak{A}$, $A = U\mathfrak{b}$ and $A = U\mathfrak{sl}(2)$. In each case we shall give a parameter set (described in terms of $\mathfrak{B}^\wedge$) in bijective correspondence with $A^\wedge$, and for each element of the parameter set we shall construct a simple module in the corresponding isomorphism class. In this introduction we state our main results over $\mathbf{C}$; in the body of the paper the results are done over an arbitrary base field of characteristic 0. For $\mathfrak{A}^\wedge$, we take $\mathfrak{B}^\wedge \cup \mathbf{C}$ as the parameter set. If N is a simple $\mathfrak{B}$-module then the socle $\operatorname{Soc}_{\mathfrak{A}} N$ (= sum of the simple $\mathfrak{A}$-submodules of N) is simple (see Section 2.2), and to $[N]$ we make correspond $[\operatorname{Soc}_{\mathfrak{A}} N]$. To $\alpha \in \mathbf{C}$ we make correspond the $\mathfrak{A}$-module $\mathbf{C}[p]$ with p acting as multiplication and q as $\alpha - d/dp$; we denote this module by $(\mathbf{C}[p], \alpha)$. The modules $(\mathbf{C}[p], \alpha)$ are (up to isomorphism) precisely the simple $\mathfrak{A}$-modules which are S-torsion ($S = \mathbf{C}[q] - \{0\}$), or equivalently, for which q has an eigenvector.

Returning to the S-torsionfree simple $\mathfrak{A}$-modules $\operatorname{Soc}_{\mathfrak{A}} N$, we now have to construct these modules, that is, actually find $\operatorname{Soc}_{\mathfrak{A}} \mathfrak{B}/\mathfrak{B}b$ for $b \in \mathfrak{B}$ irreducible, say by determining a generator. To this end we regard b as a differential operator. For each (finite) singular point α (if any) of b we consider the indicial equation $Q_{\alpha,b}(\xi) = 0$ relative to α (the same polynomial equation used in discussing expansions about α of solutions of the differential equation $bw = 0$). Let $\theta_\alpha(b)$ be the least integral indicial root (relative to α), with $\theta_\alpha(b) = +\infty$ if there is no integral root. Note that given b and α, $\theta_\alpha(b)$ is computable. For $s \in S$ and $\beta \in \mathbf{C}$ we denote by $v_\beta(s)$ the order of s at β. We now give our recipe for constructing S-torsionfree simple $\mathfrak{A}$-modules, which gives every such module.

THEOREM 1. *Let $b \in \mathfrak{B}$ be irreducible and pick $s \in S$ such that $v_\alpha(s) \geqslant -\theta_\alpha(b)$ for every singularity α of b. Then $(\mathfrak{A}s + \mathfrak{B}b)/\mathfrak{B}b$ is a simple $\mathfrak{A}$-module*

($= \mathrm{Soc}_{\mathfrak{A}} \mathfrak{B}/\mathfrak{B}b$). *Up to isomorphism every S-torsionfree simple $\mathfrak{A}$-module arises in this way, and from a b which is unique up to similarity.*

This theorem appears as a special case of Theorem 4.4 below. As remarked above, the remaining simple $\mathfrak{A}$-modules are easily described: $A^{\wedge}(S\text{-torsion}) = \{[(\mathbf{C}[p], \alpha)] \mid \alpha \in \mathbf{C}\}$. (Here and throughout the paper we use the following notation: if A is an algebra and $\mathscr{P}$ is an isomorphism-invariant property of simple A-modules, then $A^{\wedge}(\mathscr{P}) = \{[M] \in A^{\wedge} \mid M$ has property $\mathscr{P}\}$).

This result is illustrated in Section 7.1 by giving the complete list of simple $\mathfrak{A}$-modules of degree one, that is, for which some nonzero element is annihilated by some element of the form $b_1 p + b_0$ where $b_1, b_0 \in \mathbf{C}[q]$ (but q has no eigenvector). The modules are the subrings $\mathbf{C}[q, (q-\alpha_1)^{-1} \cdots (q-\alpha_j)^{-1}]$ of $\mathbf{C}(q)$ with $j \geqslant 0$, $\alpha_1, \ldots, \alpha_j \in \mathbf{C}$, q acting as multiplication and p as $t + d/dq$, where $t \in K(q)$ with poles $\alpha_1, \ldots, \alpha_j$ and with no simple pole having an integral residue; two such modules are isomorphic if and only if the respective t differ by the logarithmic derivative of some $f \in \mathbf{C}(q) - \{0\}$ (thus the modules are parameterized by $\mathbf{C}(q)$ modulo the additive subgroup generated by $\{(q-\alpha)^{-1} \mid \alpha \in \mathbf{C}\}$).

Let $\mathfrak{b}$ be the two-dimensional nonabelian Lie algebra, realized as the Borel subalgebra $\mathbf{C}h + \mathbf{C}e$ of $\mathfrak{sl}(2)$, so that $[h, e] = 2e$. We denote by ρ the homomorphism of $U\mathfrak{b}$ to $\mathfrak{A}$ (or $\mathfrak{B}$) with $\rho h = 2qp$ and $\rho e = q$. Then ρ is injective, and so $U\mathfrak{b}$ may be identified with $\rho U\mathfrak{b}$; in particular then $S \subseteq U\mathfrak{b}$. Again the study of $U\mathfrak{b}$ splits into two parts: $U\mathfrak{b}^{\wedge}$ (S-torsion) (which is easy) and $U\mathfrak{b}^{\wedge}$ (S-torsionfree). An S-torsionfree simple $U\mathfrak{b}$-module M localizes to a simple $\mathfrak{B}$-module, but unlike the case of $\mathfrak{A}$, it is not true that every simple $\mathfrak{B}$-module contains a simple $U\mathfrak{b}$-submodule. We must determine for precisely which $[b]$ is $\mathrm{Soc}_{U\mathfrak{b}} \mathfrak{B}/\mathfrak{B}b$ nonzero (it is then simple), and for such $[b]$ we must describe the socle. In doing this we shall again make use of the indicial equations relative to the singular points of b. The condition on the coefficients of b in the following theorem is equivalent to the condition that 0 is a singularity of b at which the indicial polynomial has degree zero.

THEOREM 2. *Let* $b = \sum_j b_j(q) p^j \in \mathfrak{B}$ *be irreducible. Then* $\mathrm{Soc}_{U\mathfrak{b}} \mathfrak{B}/\mathfrak{B}b \neq 0$ *if and only if* $b_0(q) \neq 0$ *and, for all* $j > 0$, *the rational function* $b_j(q)/b_0(q)$ *has a zero at* 0 *of order at least* $j+1$. *Suppose this condition holds, and pick* $s \in S$ *such that* $v_\alpha(s) \geqslant -\theta_\alpha(b)$ *for every singularity* $\alpha \neq 0$ *of b. Then* $(U\mathfrak{b}s + \mathfrak{B}b)/\mathfrak{B}b$ *is a simple* $U\mathfrak{b}$*-module* ($= \mathrm{Soc}_{U\mathfrak{b}} \mathfrak{B}/\mathfrak{B}b$). *Up to isomorphism every S-torsionfree simple* $U\mathfrak{b}$*-module arises in this way, and from a b which is unique up to similarity.*

This theorem is included in Theorem 6.2. The theorem gives $U\mathfrak{b}^{\wedge}$ (e has no eigenvector). Again it is easy to describe the remaining simple $U\mathfrak{b}$-modules (this is included in Proposition 6.1): they are the modules induced from one-dimensional nontrivial modules for the subalgebra $\mathbf{C}e$, and the one-dimensional modules. Alternatively these modules may be described as the restrictions to $\mathfrak{b}$ of the simple Whittaker modules for $\mathfrak{sl}(2)$ (see below), together with the one-dimensional modules, realized, say, as $\mathfrak{b}$-submodules of the highest weight modules over $\mathfrak{sl}(2)$.

The Weyl algebra $\mathfrak{A}$ and $U\mathfrak{b}$ may each be regarded as a skew-polynomial ring. Thus each consists of polynomials in x with coefficients in $\mathbf{C}[q]$ and with $xf-fx=\partial f$ for $f\in \mathbf{C}[q]$, where ∂ is the derivation d/dq (and $x=p$) for $\mathfrak{A}$, and $\partial = q(d/dq)$ (and $x=qp$) for $U\mathfrak{b}$. Theorems 1 and 2 suggest that there should be a common generalization, which at the other extreme should include the commutative case—where $\partial=0$ and there are no S-torsionfree simple modules. We give such a result in Theorem 4.4, in fact with $\mathbf{C}[q]$ replaced by an arbitrary Dedekind domain R and with ∂ any derivation of R. The ∂-invariant prime ideals of R play a role analogous to that of the prime q in Theorem 2, and for the remaining prime ideals we use a generalization of indicial polynomials defined in Section 3. The commutative case, where S-torsionfree simple modules correspond to maximal ideals contracting to 0, is given in Corollary 4.5.

We turn now to the case of $U\mathfrak{sl}(2)$, using a family of embeddings (due to Conze [11]) of $\mathfrak{sl}(2)$ in $\mathfrak{A}$. (For $\mathfrak{sl}(2,\mathbf{R})$ we shall need another family of embeddings σ_γ in $\mathfrak{B}$, introduced in Section 5.2). Let $\mathfrak{s}=\mathfrak{sl}(2,\mathbf{C})$ with the usual basis e, f, h, and for $\lambda\in\mathbf{C}$, let ρ_λ be the homomorphism of $U\mathfrak{s}$ to $\mathfrak{A}$ (or $\mathfrak{B}$) determined by $\rho_\lambda e=q$, $\rho_\lambda h=2qp+\lambda+1$, $\rho_\lambda f=-(qp+\lambda+1)p$. Also let c denote the Casimir element $4fe+h^2+2h$ of $U\mathfrak{s}$. For every simple $U\mathfrak{s}$-module, c acts as a scalar. The kernel of ρ_λ is $U\mathfrak{s}(c-\lambda^2+1)$, and the ideals $U\mathfrak{s}(c-\gamma)$ $(\gamma\in\mathbf{C})$ are precisely the minimal primitive ideals of $U\mathfrak{s}$ (the only other primitive ideals of $U\mathfrak{s}$ being those of finite codimension n^2, $n=1,2,\ldots$).

THEOREM 3. *Suppose $\lambda\in\mathbf{C}$ and $u\in U\mathfrak{s}$ such that $\rho_\lambda u$ is irreducible in $\mathfrak{B}$. Pick $s\in S$ such that $v_\alpha(s)\geqslant -\theta_\alpha(\rho_\lambda u)$ for every singularity α of $\rho_\lambda u$ and $v_0(s)\geqslant -\theta_0(\rho_{-\lambda}u)$ if 0 is a singularity of $\rho_{-\lambda}u$. Then $((\rho_\lambda U\mathfrak{s})s+\mathfrak{B}\rho_\lambda u)/\mathfrak{B}\rho_\lambda u$ is a simple $\rho_\lambda U\mathfrak{s}$-module $(=\mathrm{Soc}_{\rho_\lambda U\mathfrak{s}}\mathfrak{B}/\mathfrak{B}\rho_\lambda u)$. Let its pullback along ρ_λ be denoted by $M(u,\lambda^2-1)$ (which is thus a simple $U\mathfrak{s}$-module with c acting as λ^2-1). Every simple $U\mathfrak{s}$-module for which e has no eigenvector is isomorphic to some $M(u,\lambda^2-1)$, and $M(u,\lambda^2-1)\cong M(u_1,\lambda_1^2-1)$ if and only if $\lambda^2=\lambda_1^2$ and $\rho_\lambda u$ is similar to $\rho_\lambda u_1$.*

This theorem is included in Theorem 5.5. Again the remaining simple $U\mathfrak{s}$-modules (that is, those for which e has an eigenvector) are easily described: they are the well-known highest weight modules, and the simple Whittaker modules of Kostant [21] (these being modules for which e has a nonzero eigenvalue); this is included in Proposition 5.3. (It is interesting that these modules, like the $\mathbf{C}[e]$-torsionfree $U\mathfrak{s}$-modules, arise by pullback along an appropriate ρ_λ from the $\rho_\lambda U\mathfrak{s}$-socle of a simple $\mathfrak{A}$-module; specifically, for the Whittaker modules, from the $(\mathbf{C}[p], \alpha)$, $\alpha \neq 0$, and for the highest weight modules, from $(\mathbf{C}[p], 0)$). A more intrinsic (not involving ρ_λ) description of the modules $M(u, \lambda^2 - 1)$ is given in Section 5.4.

As is the case for $\mathfrak{A}$, the result of Theorem 3 is also illustrated in Section 7 by giving the complete list of simple $U\mathfrak{s}$-modules of degree one, that is, for which some nonzero element is annihilated by some $b_1(e)h + b_0(e)$, where $b_1, b_2 \in K[e]$ (but e has no eigenvector). The modules are explicitly constructed there as certain specific subsets of $\mathbf{C}(q)$. In particular this gives a new description of the simple Harish–Chandra modules over $\mathfrak{s}$ (with respect to $\mathbf{C}h$).

THEOREM 4. *Suppose M is a simple $U\mathfrak{s}$-module but not a highest weight module. Then the following are equivalent*: (i) *M contains a simple $U\mathfrak{b}$-submodule*; (ii) *M is simple as a $U\mathfrak{b}$-module*; (iii) *some* (*respectively*, (iv) *every*) *$m \in M - \{0\}$ is annihilated by some element of the form $eu + 1$, $u \in U\mathfrak{b}$. Moreover, every simple $U\mathfrak{b}$-module of dimension >1 arises in this way.*

This result is contained in Theorem 6.4. It may be regarded as a beginning of an internal analysis of the simple $U\mathfrak{s}$-modules having no eigenvector for e.

Irreducibility and similarity of elements of $\mathfrak{B}$ are concepts in differential algebra, going back to Frobenius and Poincaré. Kolchin [20] proved that $b \in \mathfrak{B}$ is irreducible if and only if its differential Galois group (an algebraic group) is irreducible. Picard [29, Chap. 17], gave a procedure for determining whether or not a given $b \in \mathfrak{B}$ is irreducible. More recently, some examples of irreducible elements were given in [5, 6, 22, 24]. The ring $\mathfrak{B}$ is a unique factorization domain, where the uniqueness of the irreducible factors is up to (order and) similarity [18, 28]. There remains the problem of obtaining a better description of the similarity classes of irreducible elements of $\mathfrak{B}$ of degree $\geqslant 2$. The methods of the present paper have some bearing on this question, as will be shown elsewhere.

Throughout the paper K denotes a field of characteristic 0, and $\bar{K}$ its algebraic closure. The reader primarily interested in Lie algebras is advised, on first reading, to make the following assumptions: the base field K is algebraically closed, $B = \mathfrak{B}(K)$, $S = K[q] - \{0\}$, and A is one of $\mathfrak{A}(K)$, $\rho U\mathfrak{b}$, $\rho_\lambda U\mathfrak{s}$ $(\lambda \in K)$ or $\sigma_\gamma U\mathfrak{s}$ $(\gamma \in K)$ where $\mathfrak{s} = \mathfrak{sl}(2, K)$ and σ_γ is defined in Section 5.2. Thus B is the localization $S^{-1}A$.

In [8, 9] some of the results of the present paper (for the case K algebraically closed) were announced, without proof (with one exception in [9]). I would like to thank Anthony Joseph for contributing a major simplification to the proof for $\mathfrak{A}$.

2. Localizations

2.1. We begin by recalling some definitions and facts about localizations of noncommutative rings (see [10, pp. 13–16]). Let A be a ring with 1 and S a multiplicative subset of A containing 1. By a (left) localization of A at S is meant a ring $B = S^{-1}A$ containing A as a subring such that every $s \in S$ is invertible in B and $B = \{s^{-1}a \mid s \in S, a \in A\}$. A localization of A at S exists if and only if S contains no zero divisor of A and satisfies the (left) Ore condition: for each pair $(s, a) \in S \times A$, $As \cap Sa \neq \varnothing$. The ring B is unique up to isomorphism. If A also has a right localization at S then the two localizations can be identified with each other, giving a two-sided localization at S. Suppose $B = S^{-1}A$ exists and M is an A-module. A localization $S^{-1}M$ can be constructed analogously to the construction of $S^{-1}A$; $S^{-1}M$ is a B-module which is canonically isomorphic to the induced module $B \otimes_A M$. The canonical map $\varphi: M \to S^{-1}M$ ($\varphi m = 1 \otimes m$ if $S^{-1}M$ is identified with $B \otimes_A M$) is injective if and only if M is S-torsionfree (that is, $sm = 0$ implies $m = 0$). The mapping of B-submodules of $S^{-1}M$ to submodules of M given by $N \mapsto \varphi^{-1}(N)$ is injective and has as its image the set of submodules L of M for which M/L is S-torsionfree. Suppose M is a left ideal of A, considered as an A-module $M = {}_AM$; then the localization $S^{-1}M$ is canonically identified with $\{s^{-1}m \mid s \in S, m \in M\}$, a left ideal of B. In particular if $M = {}_AA$ then $S^{-1}M = B$, φ is the inclusion map, and if N is a left ideal of B then $\varphi^{-1}N = A \cap N$ and $S^{-1}(A \cap N) = N$. Thus the map $N \mapsto A \cap N$ of the set of left ideals of B to the set of left ideals of A is injective, and its image consists of those J for which A/J is S-torsionfree. In particular, for any left ideal I of A there is a unique smallest left ideal J of A such that J contains I and A/J is S-torsionfree; moreover $J = A \cap S^{-1}J$. For future reference the following lemma singles out a special case of this.

Lemma 2.1. *Suppose that B is a localization of A at S, J is a maximal left ideal of A, and A/J is S-torsionfree. Then J is the intersection with A of a maximal left ideal (namely, $S^{-1}J$) of B.*

If A has a localization at S then it follows from the Ore condition that the set $\{m \in M \mid sm = 0 \text{ for some } s \in S\}$ of S-torsion elements of an A-module M is a submodule, the S-torsion submodule. In particular, if M is simple, then M is either S-torsionfree or an S-torsion module.

2.2. Suppose A has a localization $B = S^{-1}A$ at S. If M is an S-torsionfree simple A-module, then $S^{-1}M$ is a simple B-module. Hence we have a canonical map, which we also denote by S^{-1},

$$S^{-1}: [M] \mapsto [S^{-1}M]$$

from $A^{\wedge}$ (S-torsionfree) to $B^{\wedge}$.

If M is S-torsionfree we shall identify M with its image in $S^{-1}M$. Then M is an *essential* submodule of ${}_A(S^{-1}M)$, that is, every nonzero submodule intersects it.

LEMMA 2.2.1. *Suppose* $B = S^{-1}A$. *The canonical map* $S^{-1}: A^{\wedge}(S$-*torsionfree*) $\to B^{\wedge}$ *is injective and has as its image the set of classes* $[N]$ *for which* $\operatorname{Soc}_A N \neq 0$. *In particular, if* N *is a simple* B-*module and* ${}_A N$ *contains a simple submodule* N' *then* N' *is essential* (*hence unique*) *and* $N \cong S^{-1}N'$.

Proof. Suppose M_1 and M_2 are S-torsionfree simple A-modules and $S^{-1}M_1 \cong S^{-1}M_2$. Then M_i is an essential simple submodule of ${}_A(S^{-1}M_i)$ $(i = 1, 2)$. Such a submodule is unique and so is preserved by the isomorphism $S^{-1}M_1 \to S^{-1}M_2$. Hence $M_1 \cong M_2$ and therefore S^{-1} is injective. Now suppose N is a simple B-module and ${}_A N$ contains a simple submodule N'. Then $S^{-1}N'$ is a simple B-module. By a universal mapping property the inclusion map $N' \to N$ extends to a B-map of $S^{-1}N'$ to N. This latter map is an isomorphism since both N and $S^{-1}N'$ are simple. Hence $[N] = S^{-1}[N']$ and N', being essential in $S^{-1}N'$, is also essential in N.

Q.E.D.

As we shall see in Theorems 4.4 and 5.5, if $A = \mathfrak{A}$ or $\rho_\lambda U\mathfrak{sl}(2)$ and $B = \mathfrak{B}$ then the above canonical injection is surjective, that is, any simple B-module N contains a simple A-submodule $(= \operatorname{Soc}_A N)$; our approach will yield much more, namely, we shall determine elements of $\operatorname{Soc}_A N$, and thus a construction of it. This close relationship of $B^{\wedge}$ and $A^{\wedge}$(S-torsionfree) was unexpected; however, it is possible to give an easy, but nonconstructive, proof of the surjectivity, that is, of the mere existence of the simple A-submodule of N, as follows. For this purpose we use some invariants $\mathscr{A}(L, \infty)$, $\mathscr{B}(L, \infty)$, $d(L)$, defined in [4], of a left ideal $L \neq 0$ of a primitive quotient $A = U\mathfrak{sl}(2)/(c - \gamma)$ of $U\mathfrak{sl}(2)$. The $\mathscr{A}(L, \infty)$, $\mathscr{B}(L, \infty)$ are certain nonzero ideals of polynomials in one indeterminate, and $d(L) \in \mathbf{Z}$. They satisfy the property that if M is another left ideal with $L \subseteq M$ then $\mathscr{A}(L, \infty) \subseteq \mathscr{A}(M, \infty)$, $\mathscr{B}(L, \infty) \subseteq \mathscr{B}(M, \infty)$, and $d(L) \geqslant d(M)$; in [4, Prop. 2] it is proved that if, moreover, $\mathscr{A}(L, \infty) = \mathscr{A}(M, \infty)$, $\mathscr{B}(L, \infty) = \mathscr{B}(M, \infty)$, $d(L) = d(M)$, and A is simple then $L = M$. If A is not simple then the proof of [4, Prop. 2] gives instead the conclusion that $\dim_K M/L < \infty$.

LEMMA 2.2.2. *Suppose* $A = \mathfrak{A}$ *or a primitive quotient of* $U\mathfrak{sl}(2)$. *If* $0 \neq I$ *is a left ideal of* A *then* A/I *has finite length.*

Proof. If $A = \mathfrak{A}$ (respectively, a simple quotient of $U\mathfrak{sl}(2)$) this was proved in [27, 14] (resp. [4]). Now let A be a nonsimple primitive quotient of $U\mathfrak{sl}(2)$. Thus A contains a unique proper ideal J of finite codimension. As A in Noetherian it suffices to prove that A/I contains a minimal submodule. Suppose it does not. Then there exists a strictly descending chain $\{L_\omega\}$ of left ideals $\neq I$ and having intersection I, the chain being indexed by the ordinals less than some given ordinal. The $\mathscr{A}(L_\omega, \infty)$, $\mathscr{B}(L_\omega, \infty)$, and $d(L_\omega)$ become stationary since they are bounded respectively below by $\mathscr{A}(I, \infty)$, $\mathscr{B}(I, \infty)$ and above by $d(I)$. Hence there is an ω such that $\dim_K L_\omega/L_{\omega'} < \infty$ and $\operatorname{ann} L_\omega/L_{\omega'} = J$ for all $\omega' > \omega$. Therefore $JL_\omega \subseteq \bigcap_{\omega' > \omega} L_{\omega'} = I$. But then any cyclic submodule of L_ω/I is annihilated by J and hence is finite dimensional, and so contains a minimal submodule, a contradiction. Q.E.D.

COROLLARY 2.2. *Suppose $A = \mathfrak{A}$ or $\rho_\lambda U\mathfrak{sl}(2)$. Then every simple $\mathfrak{B}$-module N and every simple $\mathfrak{A}$-module M contain a simple A-submodule. In particular, for this A (with $S = K[q] - \{0\}$) the canonical injection S^{-1}: $A\hat{}\,(S$-torsionfree$) \to B\hat{}$ is surjective.*

Proof. If $0 \neq n \in N$ then $\operatorname{ann}_{\mathfrak{B}} n \neq 0$ and hence $I = \operatorname{ann}_A n \neq 0$. Therefore the A-module $An \cong A/I$ contains a simple submodule. A similar argument works for M, using $S^{-1}A = S^{-1}\mathfrak{A}$ when M is S-torsionfree, and using $0 \neq \operatorname{ann}_{\mathfrak{A}} m) \cap S \subseteq A$ when M is S-torsion. Q.E.D.

The algebra $A = K[q, qp] \subseteq \mathfrak{A}$ (thus $A \cong \rho Ub$) is an example for which the above conclusions fail (this follows from Corollary 4.4.1 below).

2.3. Suppose $B = S^{-1}A$. By Lemma 2.1, any S-torsionfree simple A-module is isomorphic to $A/A \cap L$ for some maximal left ideal L of B. Given a maximal left ideal L of B, $A/A \cap L$ is S-torsionfree. When is it simple? The following lemma gives a sufficient condition.

LEMMA 2.3. *Suppose $B = S^{-1}A$, L is a maximal left ideal of B, and for every simple S-torsion A-module M there exists an element of $A \cap L$ acting injectively on M. Then the left ideal $A \cap L$ of A is maximal.*

Proof. Let J be a maximal left ideal of A containing $A \cap L$. If A/J is S-torsionfree then, by Lemma 2.1, $S^{-1}J$ is a maximal left ideal of B, $J = A \cap S^{-1}J \supseteq A \cap L$, $S^{-1}J \supseteq L$, and hence $S^{-1}J = L$, and $J = A \cap L$. Now suppose A/J is not S-torsionfree. Then a acts injectively on A/J for some $a \in A \cap L$. But for the coset $1 + J \in A/J$, $a(1 + J) = 0$ since $A \cap L \subseteq J$, a contradiction. Q.E.D.

2.4. Suppose B is a principal left ideal domain which is not a division ring. Then the maximal left ideals of B are precisely those left ideals

Bb for which b is irreducible. If N is a simple B-module and $0 \neq n \in N$, then the annihilator $\operatorname{ann}_B n$ of n in B is a maximal left ideal of B, hence of the form Bb, b irreducible. We call b a *minimal annihilator* of n; it is uniquely determined up to left multiplication by a unit of B. Thus an irreducible element of B is an annihilator of n if and only if it is a minimal annihilator of n. If also $0 \neq n' \in N$ then $N \cong B/\operatorname{ann}_B n \cong B/\operatorname{ann}_B n'$, and so any minimal annihilator of n' is similar to b. Conversely, if $a \in B$ is similar to b, say with ad a least common left multiple (l.c.l.m.) of d and b and with 1 a g.c.r.d. of d and b, then $d \notin Bb$, $dn \neq 0$, and a is a minimal annihilator of dn. We write $\min \operatorname{ann} N = \{b \in B \mid \exists\, n \in N - \{0\}$ such that b is a minimal annihilator of $n\}$ and summarize as follows.

LEMMA 2.4.1. *Suppose B is a principal left ideal domain which is not a division ring, and N is a simple B-module. Then* $\min \operatorname{ann} N$ *is a similarity class of irreducible elements of B, and the map of $B^{\wedge}$ to the set of similarity classes of irreducible elements of B, $[N] \mapsto \min \operatorname{ann} N$, is a bijection (with inverse $[b] \mapsto [B/Bb]$).*

In certain cases when $B = S^{-1}A$ we shall be able to classify $A^{\wedge}(S$-torsionfree). The following criterion allows us to recognize which isomorphism class a given A-module belongs to.

LEMMA 2.4.2. *Suppose $B = S^{-1}A$ is a principal left ideal domain, M is a simple S-torsionfree A-module, $a \in A$ is irreducible in B and annihilates some nonzero element of M. Then $M \cong \operatorname{Soc}_A B/Ba$.*

Proof. We have $M = \operatorname{Soc}_A S^{-1}M$. Since a is irreducible and an annihilator, $a \in \min \operatorname{ann} S^{-1}M$. Therefore $S^{-1}M \cong B/Ba$. Q.E.D.

3. INDICIAL POLYNOMIALS

3.1. In this chapter we develop some machinery which will be used later in applying Lemma 2.3. The special cases of principal concern are those for which $A = \mathfrak{A}$ (the Weyl algebra) or its subalgebra $K[q, qp]$, and B is their common localization $\mathfrak{B} = K(q)[p]$.

Let R be a commutative domain, ∂ a derivation of R, and $A = R[x]_\partial$ the corresponding skew-polynomial ring (or differential operator ring) in one variable x, that is, $A = R \otimes_{\mathbf{Z}} \mathbf{Z}[x]$ (as a $\mathbf{Z}$-module) with multiplication determined by $xr = rx + \partial r$ $(r \in R)$, where r, ∂r and x also denote the images of these elements under the canonical maps of R and $\mathbf{Z}[x]$ into A (this is a special case of the smash product or semidirect product construction [31]). Thus $A = \mathfrak{A}(K)$ when $R = K[q]$, $\partial = d/dq$, and $p = x$. We also write S for $R - \{0\}$ and let T be the quotient field of R, that is, $T = S^{-1}R$. The derivation ∂ extends uniquely to a derivation of T, also denoted ∂. We shall denote by B the ring $T[x]_\partial$; thus $B = S^{-1}A$, and if $R = K[q]$, $\partial = d/dq$ and $p = x$ then $B = \mathfrak{B}(K)$.

Throughout this and the next chapter (except in Section 4.5) we make the following assumptions: prime ideal means nonzero prime ideal, and

$$R \text{ is a Dedekind domain, } \partial \text{ is a derivation of } R \text{ (hence also of the quotient field } T\text{), } R/P \text{ has characteristic 0 for every prime ideal } P \text{ of } R \text{ such that } \partial P \not\subseteq P,\ A = R[x]_\partial,\ B = T[x]_\partial. \tag{3.1.1}$$

(In our principal applications the hypothesis on characteristic 0 could be strengthened to: R contains $\mathbf{Q}$).

Suppose P is a prime ideal of R; let v_P denote the valuation of T corresponding to P, that is, if $r \in P^i - P^{i+1}$ then $v_P r = i$. The localization R_P is the valuation ring $\{t \in T \mid v_P t \geqslant 0\}$ of v_P; its unique maximal ideal is PR_P. Trusting that there will be no confusion with localization, we denote by K_P the residue class field R_P/PR_P (which may be identified with R/P) and by η_P the canonical map of R_P to K_P (that is, with the above identification, η_P is the canonical map of R to R/P). If a base field K is given then K is identified with a subfield of K_P. In particular, if $R = K[q]$ and $P = (g)$, where g is monic irreducible then we write $v_P = v_g$, $K_P = K_g$ and $\eta_P = \eta_g$. In this case K_g, identified with $K[q]/(g)$, equals $K(\alpha)$ where α, the image of q, is a root of g. If g is linear, say $g = q - \alpha$, then $\eta_g \colon K[q] \to K$ is evaluation at α.

We extend v_P to a function on B, also denoted by v_P, as follows: if $b = \sum b_i x^i \in B$ then

$$v_P b = \min\{v_P(b_i) - i \mid i \geqslant 0\} \qquad \text{if} \quad \partial P \not\subseteq P, \tag{3.1.2}$$

$$v_P b = \min\{v_P(b_i) \mid i \geqslant 0\} \qquad \text{if} \quad \partial P \subseteq P. \tag{3.1.3}$$

LEMMA 3.1. *For any prime ideal P of R, v_P is a valuation on B.*

Proof. We must show (i) $v(a+b) \geqslant \min\{va, vb\}$, (ii) $v(ab) = va + vb$ (where v denotes v_P). We omit the (routine) proof of (i). For (ii) we have

$$ab = \sum a_j x^j \sum b_k x^k = \sum_{j,k} a_j \sum_{0 \leqslant l \leqslant j} \binom{j}{l} (\partial^l b_k)\, x^{j+k-l}$$

$$= \sum_i \sum_j \sum_{0 \leqslant m \leqslant i,j} \binom{j}{m} a_j (\partial^{j-m} b_{i-m})\, x^i. \tag{3.1.4}$$

Write $\delta = 0$ or 1 accordingly as $\partial P \subseteq P$ or not. Then $\nu(\partial t) \geqslant -\delta + \nu t$ for $t \in R$ and hence also for $t \in T$. Suppose that v, w are the largest integers such that $\nu a = \nu a_v - \delta v$, $\nu b = \nu b_w - \delta w$. We have

$$\nu a_j \partial^{j-m} b_{i-m} \geqslant \nu a + \delta j + \nu b + \delta(i-m) - \delta(j-m) = \nu a + \nu b + \delta i.$$

Consider the coefficient of x^i in (3.1.4) when $i = v + w$. Then the inequality just above is strict in each of the following cases: $j > v$; $j < v$ (then $i - m \geqslant i - j > i - v = w$); and $m < j = v$ (then $i - m > i - j = i - v = w$). In the remaining case, $j = v = m$, the inequality becomes an equality. Therefore $\nu(ab)_i = \nu a + \nu b + \delta i$ when $i = v + w$. Q.E.D.

Actually the definition of ν_P in (3.1.2) gives a valuation even if $\partial P \subseteq P$, but we shall not use this.

3.2. Suppose P is a non-∂-invariant prime ideal of R. Then there exists $g \in P - P^2$ such that $\partial g \notin P$. If also $g' \in P - P^2$ then $g' = ug$, where u is a unit of R_P, and $\nu_P \, \partial g' = \nu_P(u \, \partial g + (\partial u) g) = 0$ since $0 = \nu_P u \, \partial g < \nu_P(\partial u) g$. Therefore for every $g \in P - P^2$, $\partial g \notin P$. Now pick $g \in P - P^2$. For $b = \sum b_j x^j \in B$ we define a polynomial $Q_b = Q_b(\xi) = Q_{P,g,b}(\xi) \in K_P[\xi]$ by

$$Q_{P,g,b}(\xi) = \sum_{j \geqslant 0} \eta_P\{g^{-\nu_P b - j} b_j (\partial g)^j\} \, \xi(\xi - 1) \cdots (\xi - j + 1) \tag{3.2.1}$$

($= 0$ if $b = 0$). We call this the *indicial polynomial of b relative to P, g* (or at P, g). Note that the evaluation of η_P in the jth term of (3.2.1) makes sense since

$$\nu_P(b_j(\partial g)^j) = \nu_P b_j + j\nu_P(\partial g) = \nu_P b_j \geqslant \nu_P b + j. \tag{3.2.2}$$

There is an index j for which (3.2.2) is an equality; if k is the highest such j then $Q_b(\xi)$ has degree k. In particular, if $b \neq 0$ then $Q_b \neq 0$.

If $B = \mathfrak{B}(K)$ and $g = q - \alpha$ ($\alpha \in K$) we may, without danger of confusion, write $Q_{\alpha,b}$ in place of $Q_{(g),g,b}$ and call $Q_{\alpha,b}$ the indicial polynomial of b relative to α (or at α); recall that in this case η_P is evaluation at α. In particular suppose $K = \mathbf{C}$ and $\alpha \in \mathbf{C}$; then α is a singular point of b (regarded as the differential operator $\sum_{j=0}^k b_j(q) d/dq$ of order k) if and only if α is a pole of some $b_j(q)/b_k(q)$. In this case the equation $Q_{\alpha,b}(\xi) = 0$ can be seen to coincide with the indicial equation relative to the singularity α considered classically in expanding solutions of the differential equation $bu = 0$ about α; the case of a regular singular point α is precisely that for which degree $Q_{\alpha,b}(\xi) = k$. (Classically one does not consider the indicial equation at an ordinary point; our definition makes the indicial polynomial there a nonzero constant times $\xi(\xi-1)\cdots(\xi-(k-1))$, which has the appropriate roots.)

We define $\bar{Q}_{P,b}$ to be the monic polynomial obtained by dividing $Q_{P,g,b}$ by its leading coefficient ($\bar{Q}_{P,b}=0$ if $Q_{P,g,b}=0$). We call $\bar{Q}_{P,b}$ the *normalized indicial polynomial* of b relative to P.

LEMMA 3.2. *The polynomial $\bar{Q}_{P,b}$ is independent of the choice of g. If R' is a Dedekind domain containing R with derivation extending ∂, P' a prime ideal of R', $P'\cap R=P$, and $v_{P'}|_R=v_P$ (that is, the ramification index is 1), then $\bar{Q}_{P,b}=\bar{Q}_{P',b}$. In particular, if $B=\mathfrak{B}(K)$, $g\in K[q]$ is irreducible and α is a root of g in $\bar{K}$, then $\bar{Q}_{(g),b}=\bar{Q}_{(q-\alpha),b}$ where on the right side b is regarded as an element of $\mathfrak{B}(\bar{K})$.*

Proof. Suppose $g'\in P-P^2$. As before, $g'=ug$, where $u\in R_P$ (and so also $\partial u\in R_P$) and $v_P u=0$. Hence

$$(\partial(ug))^j=(u(\partial g)+(\partial u)g)^j\equiv u^j(\partial g)^j \bmod PR_P.$$

It follows from (3.2.1) that

$$Q_{P,ug,b}=\eta_P(u^{-v_Pb})\,Q_{P,g,b},$$

which gives the first statement. For the second statement we may use the same g, since $g\in P'-P'^2$; the statement then follows from the fact that $\eta_{P'}$ extends η_P, where K_P is canonically identified as a subfield of $K_{P'}$. Q.E.D.

3.3. In this section we shall use the hypothesis

$$P \text{ is a prime ideal of } R,\ \partial P\not\subseteq P,\ g\in P-P^2. \tag{3.3.1}$$

We can regard T as a B-module, with action written $b\cdot t$, where $x\cdot t=\partial t$ and $t_1\cdot t=t_1t$ ($t,t_1\in T$).

LEMMA 3.3.1. *Suppose* (3.3.1) *holds*, $0\neq b\in B$, *and* $0\neq t\in T$. *Then, for the B-action on T, there exists $t_1\in R_P$ such that*

$$g^{-v_Pb}b\cdot t=t_1t,\qquad \eta_Pt_1=Q_{P,g,b}(v_Pt).$$

Proof. Write $b=\sum_j b_jx^j$ ($b_j\in T$), $vb=v_Pb$, $k=v_Pt$. Then $t=g^kt_2$ where $v_Pt_2=0$ (and so in particular $\partial^jt_2\in R_P$ for all $j\geqslant 0$). We have

$$x^j \cdot g^k t_2 \equiv k(k-1)\cdots(k-j+1)\, g^{k-j}(\partial g)^j\, t_2 \bmod g^{k-j+1}R_P,$$

$$g^{-\nu b}b \cdot t = \left\{ g t_3 + \sum_j k(k-1)\cdots(k-j+1)(\partial g)^j\, b_j g^{-\nu b-j} \right\} t,$$

where

$$g t_3 t \in \sum_j g^{-\nu b} b_j g^{k-j+1} R_P \subseteq g^{k+1} R_P$$

and so $t_3 \in R_P$. Denoting by t_1 the coefficient of t on the right side above, we have $t_1 \in R_P$ and $\eta_P t_1 = Q_{P,g,b}(k)$. Q.E.D.

LEMMA 3.3.2. *Suppose* (3.3.1) *holds and* $a, b \in B$. *Then*

$$Q_{P,g,ab}(\xi) = Q_{P,g,a}(\xi + \nu_P b)\, Q_{P,g,b}(\xi). \tag{3.3.2}$$

Proof. We may assume $0 \neq a, b$. There exist infinitely many negative integers k such that $k \leqslant -\nu b$ ($\nu = \nu_P$) and $0 \neq Q_b(k)$ $(= Q_{P,g,b}(k))$. Since the Q's are polynomials, it suffices to prove the formula with ξ specialized to such k. By Lemma 3.3.1, $g^{-\nu b}b \cdot g^k = t_b g^k$, where $\nu t_b \geqslant 0$ and $\eta_P t_b = Q_b(k) \neq 0$ and hence $\nu t_b = 0$. Again by Lemma 3.3.1, $g^{-\nu a}a \cdot g^{k+\nu b} t_b = t_a g^{k+\nu b} t_b$, where $\nu t_a \geqslant 0$ and $\eta_P t_a = Q_a(k+\nu b)$, and similarly for $g^{-\nu(ab)}ab \cdot g^k = t_{ab} g^k$. Then

$$\begin{aligned} t_{ab} g^k &= g^{-\nu(ab)} a g^{\nu b} g^{-\nu b} b \cdot g^k = g^{-\nu b} g^{-\nu a} a \cdot g^{k+\nu b} t_b \\ &= g^{-\nu b} g^{k+\nu b} t_a t_b. \end{aligned}$$

Therefore $t_{ab} = t_a t_b$ and

$$Q_{ab}(k) = \eta_P t_{ab} = (\eta_P t_a)(\eta_P t_b) = Q_a(k+\nu b)\, Q_b(k). \qquad \text{Q.E.D.}$$

3.4. We shall call the roots of $Q_{P,g,b}(\xi)$ the *indicial roots of* b *relative to* P (this being independent of g); if $B = \mathfrak{B}(K)$ and $P = (q - \alpha)$ we shall also call them the indicial roots of b relative to α (in differential equations these are also called exponents).

Suppose $b \in B$. If P is a non-∂-invariant prime ideal of R, we shall say that b is *preserving relative to* P (or at α, if $P = (q - \alpha)$) if there is no negative integer indicial root relative to P. We shall also say that b is *preserving* if it is preserving relative to P for every non-∂-invariant prime P. What it is that b preserves will be seen in Lemma 4.2, below.

When $B = \mathfrak{B}(\mathbf{C})$, the property of b being preserving at α automatically holds at any ordinary point (since our definition implies that the indicial roots there are $0, 1, \ldots, k-1$), while at one of the (finitely many) singular points the property is computable (that is, only finitely many negative integers need to be checked as potential roots).

In the general case, if $b = \sum b_i x^i \in B$ of degree $k > 0$, we define the *special primes* of b to be the non-∂-invariant prime ideals P for which $v_P b = v_P b_i - i$ for some $i < k$ (that is, $v_P b_k - k \geqslant v_P b_i - i$ for some $i < k$). For such a P, $tb_k \in P$ for every $t \in T$ such that $tb \in A$, since if $tb \in \underset{\smile}{A}$ then

$$v_P tb_k = v_P t + v_P b_k \geqslant v_P t + v_P b_i - i + k > 0.$$

Hence b *has only finitely many special primes.* If P is a non-∂-invariant prime which is not a *special prime* of b then b is preserving relative to P, since in this case $v_P b + j = v_P b_k - k + j < v_P b_j$ for all $j < k$ and so

$$Q_{P,g,b} = \eta_P\{g^{-v_P b - k} b_k (\partial g)^k\}\, \xi(\xi - 1) \cdots (\xi - k + 1)$$

has no root in $\mathbf{Z}^-$. Hence *the property of b being preserving depends on only finitely many primes.* Also it obvious (either from the definition of Q_{tb} or Lemma 3.3.2) that if $0 \neq t \in T$ then b is preserving relative to P if and only tb is.

LEMMA 3.4. *Suppose $0 \neq d_1,\ldots, d_j \in B$. Then there exists $s \in S$ such that, for $i = 1,\ldots, j$, $d_i s^{-1}$ is preserving. More precisely, let $\{P_1,\ldots, P_k\}$ be those prime ideals which are a special prime of at least one of $d_1,\ldots, d_j$, and let $g_l \in P_l - P_l^2$ $(l = 1,\ldots, k)$. Also take $v \in \mathbf{N}$ with $v \geqslant v_1$ for every $v_1 \in \mathbf{Z}^+$ (if any) such that $-v_1$ is a root of some $\bar{Q}(P_l, d_i)$. Then for $i = 1,\ldots, j$, $d_i(g_1 \cdots g_k)^{-v}$ is preserving.*

Proof. Set $s = (g_1 \cdots g_k)^v$. Suppose $w \in \mathbf{Z}^-$. For each non-∂-invariant prime ideal P and $g \in P - P^2$, by Lemma 3.3.2 we have

$$Q(P, g, d_i s^{-1})(w) = Q(P, g, d_i)(w + v_P s^{-1})\, Q(P, g, s^{-1})(w),$$

which we must show is nonzero. We have $Q(P, g, s^{-1})$ is a nonzero scalar. If $P \in \{P_1,\ldots, P_k\}$ then $w + v_P s^{-1} = w - v_P s \leqslant w - v < -v$ and so $Q(P, g, d_i)(w + v_P s^{-1}) \neq 0$. If $P \notin \{P_1,\ldots, P_k\}$ then $w + v_P s^{-1} \in \mathbf{Z}^-$ so again $Q(P, g, d_i)(w + v_P s^{-1}) \neq 0$ since in this case, by the remark above, d_i is preserving relative to the *nonspecial prime P*. Q.E.D.

3.5. The ring $\mathfrak{B} = \mathfrak{B}(K)$ (as with more general B) can be written as $T[x]_\partial$ in more than one way, even with the same $T = K(q)$. For example, $\mathfrak{B} = K(q)[p]_{d/dq} = K(q)[qp]_{q(d/dq)}$. Suppose $g \in K[q]$ is monic irreducible. The definitions of v_g, $Q_{g,b}$ $(= Q_{(g),g,b})$ and preserving (relative to g) depend on the particular expression of $\mathfrak{B}$. For example, the prime (q) of $K[q]$ is ∂-invariant with respect to $\mathfrak{B} = K(q)[qp]_{q(d/dq)}$ but not with respect to $\mathfrak{B} = K(q)[p]_{d/dq}$. When necessary for clarity in a statement we shall designate the expression of $\mathfrak{B}$ with respect to which the definitions are taken.

LEMMA 3.5. *Suppose* $\beta \in K$, $b = \sum a_i(qp+\beta)^i \in \mathfrak{B}$ *where* $a_i \in K(q)$, $g \in K[q]$ *is monic irreducible, and* v_g *and* $Q_{g,b}$ *are defined with respect to* $\mathfrak{B} = K(q)[p]_{d/dq}$. *Then if* $g = q$,

$$v_q b = \min\{v_q a_i\}, \tag{3.5.1}$$

$$Q_{q,b}(\xi) = \sum_j \eta_q(q^{-v_q b} a_j)(\xi+\beta)^j, \tag{3.5.2}$$

while if $g \neq q$ *then*

$$v_g b = \min\{v_g a_i - i\}, \tag{3.5.3}$$

$$Q_{g,b}(\xi) = \sum_j \eta_g\{g^{-v_g b - j}(dg/dq)^j q^j a_j\}\, \xi(\xi-1)\cdots(\xi-j+1). \tag{3.5.4}$$

Proof. For any $k, j \in \mathbf{N}$, $j \leqslant k$, there are scalars α_{kj}, with $\alpha_{k0} = 0$ if $k > 0$ and with $\alpha_{kk} = 1$, such that

$$(qp)^k = \sum_{j=1}^{k} \alpha_{kj} q^j p^j;$$

this can be seen by induction (in fact, the α_{kj} are the Stirling numbers of the second kind). Hence

$$b = \sum_i \sum_{k=0}^{i} \sum_{j=0}^{k} \binom{i}{k} \beta^{i-k} \alpha_{kj} a_i q^j p^j.$$

We thus have $b = \sum_j b_j p^j$, where

$$b_j = \sum_{i \geqslant j} \sum_{k=j}^{i} \binom{i}{k} \beta^{i-k} \alpha_{kj} a_i q^j. \tag{3.5.5}$$

Write $v = v_q$ and let l be the largest index such that $va_l = \min\{va_i\}$. Then b_l equals $a_l q^l$ plus terms on which v has higher value, that is, $vb_l = va_l + l$, and also $vb_j \geqslant va_l + j$ for all j. Hence $vb = va_l$, giving (3.5.1). With $K(q)$ regarded as a $\mathfrak{B}$-module with p acting as d/dq, Lemma 3.3.1 implies that for $k \in \mathbf{N}$, there exists $t \in K(q)$ such that $vt \geqslant 0$,

$$q^{-vb} b \cdot q^{-k} = \sum q^{-vb} a_i (qp)^i \cdot q^{-k} = tq^{-k},$$

and $\eta_q t = Q_{q,b}(-k)$. But $(qp+\beta) \cdot q^{-k} = (-k+\beta)\, q^{-k}$ and hence

$$t = \sum_i q^{-vb} a_i(-k+\beta)^i, \qquad \eta_q \left(\sum q^{-vb} a_i\right)(-k+\beta)^i = Q_{q,b}(-k),$$

and (3.5.2) holds for infinitely many values of ξ.

Next write $\nu = \nu_g$ and let l be the largest index such that $\nu a_l - l = \min\{\nu a_i - i\}$. Then for all j,

$$\nu b_j \geqslant \min\{\nu a_i \mid i \geqslant j\} \geqslant \nu a_l - l + j,$$

and $\nu b_l = \nu a_l$ since $\nu a_i \geqslant \nu a_l - l + i > \nu a_l$ if $i > l$. Hence $\min\{\nu b_j - j\} = \min\{\nu a_i - i\} = \nu b$. Also if $i > j$ then $\nu a_i \geqslant \nu b + i > \nu b + j$, and (3.5.5) gives

$$\eta_g(g^{-\nu b - j} b_j) = \eta_g(g^{-\nu b - j} a_j q^j).$$

Formula (3.5.4) follows from this and (3.2.1). Q.E.D.

COROLLARY 3.5. *The functions ν_g, as defined respectively with respect to $\mathfrak{B} = K(q)[p]_{d/dq}$ and $\mathfrak{B} = K(q)[qp + \beta]_{q(d/dq)}$, coincide; and when $g \neq q$ the same holds for Q_g $(= Q_{g,-})$.*

Proof. This follows from a comparison of (3.5.3), (3.5.1) and (3.5.4) with (3.1.2), (3.1.3) and (3.2.1), since if $\partial = qd/dq$ in (3.2.1) then the factor $(\partial g)^j = (dg/dq)^j q^j$. Q.E.D.

4. THE SIMPLE MODULES OVER CERTAIN SKEW-POLYNOMIAL RINGS

4.1. Throughout Sections 4.1–4.4, we continue assuming (3.1.1). For P a prime ideal of R, K_P becomes an R-module by setting $r \cdot \omega = (\eta_P r)\omega$ $(r \in R,\ \omega \in K_P)$. Then we may form the induced A-module $A \otimes_R K_P$. For $r \in R$ we have

$$rx^j = \sum_{i=0}^{j} \binom{j}{i} (-1)^i x^{j-i} (\partial^i r). \tag{4.1.1}$$

Hence A is a free right R-module with basis $\{x^i \mid i \geqslant 0\}$, every element of $A \otimes_R K_P$ can be uniquely expressed in the form $\sum_{j>0} x^j \otimes \omega_j$ where $\omega_j \in K_P$ $(\omega_j = 0$ except for finitely many $j)$, and

$$rx^k \cdot (x^j \otimes \omega) = \sum_{i=0}^{k+j} x^{k+j-i} \otimes \binom{k+j}{i} (-1)^i \eta_P(\partial^i r)\omega. \tag{4.1.2}$$

If $\partial P \not\subseteq P$ we also write, for the A-module $A \otimes_R K_P$,

$$A \otimes_R K_P = V(P).$$

In the Weyl algebra case, $P = (g)$ for some monic irreducible g in $K[q]$, and we write $V(P) = V(g)$. Recall that $K_P = K_g = K(\alpha)$, where $\alpha = \eta_g q$ (η_g the map $K[q] \to K[q]/(g) = K_g$). In this case we identify $A \otimes_R K_P = V(g)$ with the polynomial ring $K_g[p]$ under the correspondence $\sum x^j \otimes \omega_j \mapsto$

$\sum \omega_j p^j$; by (4.1.2), p acts on $K_g[p]$ by multiplication and q acts as $\alpha - d/dp$.

If P is a ∂-invariant maximal ideal of R, we write $\bar{\partial}$ for the derivation of $R/P = K_P$, $r + P \mapsto \partial r + P$, induced by ∂. In this case $\sum Px^i$ is an ideal of A and $A/\sum Px^i \cong K_P[x]_{\bar{\partial}}$ canonically. The simple $K_P[x]_{\bar{\partial}}$ modules are just the quotients $K_P[x]_{\bar{\partial}}/K_P[x]_{\bar{\partial}}b$, where b is an irreducible element of $K_P[x]_{\bar{\partial}}$. We regard these quotients as A-modules by pulling back along the canonical map of A to $K_P[x]_{\bar{\partial}}$. These A-modules are simple, and two of them (for the same P) are isomorphic if and only if the corresponding elements b are similar. Given a similarity class C of irreducible elements of $K_P[x]_{\bar{\partial}}$, we pick an element $b \in C$ (so that $C = [b]$) and write $V(P, [b])$ for the simple A-module just described.

PROPOSITION 4.1. *Suppose* (3.1.1) *holds. Then any simple S-torsion A-module is isomorphic to exactly one of the following, all of which are simple S-torsion*: $V(P)$ (*P prime*, $\partial P \not\subseteq P$) *and* $V(P, [b])$ (*P maximal*, $\partial P \subseteq P$, $[b]$ *a similarity class of irreducible elements of* $K_P[x]_{\bar{\partial}}$).

Proof. If P is prime and $\partial P \not\subseteq P$, suppose M is a nonzero submodule of $V(P)$ and let j be the smallest index such that M contains an element $m = \sum_{i=0}^{j} x^i \otimes \omega_i$ with $\omega_i \in K_P$, $\omega_j \neq 0$. Pick $g \in P - P^2$. If $j > 0$ then by (4.1.2) (and since $\eta_P g = 0$)

$$gm = x^{j-1} \otimes -j\eta_P(\partial g)\, \omega_j + \text{lower terms};$$

but $\partial g \notin P$ and K_P has characteristic 0, and so $j\eta_P(\partial g) \neq 0$, contradicting the minimality of j. Hence $j = 0$, $1 \otimes K_P \subseteq M$, $M = V(P)$, and $V(P)$ is simple. The simplicity of $V(P, [b])$ has already been discussed.

Now suppose M is a simple A-module and there exists $0 \neq m \in M$ such that $I = (\operatorname{ann} m) \cap R \neq 0$. Since R is noetherian of Krull dimension $\leqslant 1$, R/I contains a minimal ideal J/I. Thus the annihilator in R of J/I is a maximal ideal P. Moreover $R/I \cong Rm$ and so there exists $0 \neq n \in Rm$ such that $Pn = 0$. Let ζ be the R-map $R \to M$ with $\zeta r = rn$. Then $\ker \zeta = P$, and we get an R-map ζ' of $K_P = R/P$ to M. By the universal mapping property of an induced module, there exists a unique A-map $\zeta'': A \otimes_R K_P \to M$ such that $\zeta'' \iota = \zeta'$, where ι is the canonical map $K_P \to A \otimes_R K_P$. If $\partial P \not\subseteq P$ then $A \otimes_R K_P = V(P)$, and since $\zeta'' \neq 0$ and both $V(P)$ and M are simple, ζ'' is an isomorphism. On the other hand, if $\partial P \subseteq P$ then the ideal $\sum Px^i \subseteq \sum x^i P$ by (4.1.1), and $(\sum Px^i)M = (\sum Px^i)An \subseteq (\sum x^i P)n = 0$. Hence M may be regarded as a module over $A/\sum Px^i \cong K_P[x]_{\bar{\partial}}$, and $M \cong V(P, [b])$ for some $[b]$ as indicated. In $V(P)$, since $\partial P \not\subseteq P$, (4.1.2) implies that elements of $R - P$ act injectively on $V(P)$, that is $V(P)$ uniquely determines P. Similarly, since $V(P, [b])$ can be regarded as a K_P-module, again elements in $R - P$ act injectively. This gives the desired uniqueness. Q.E.D.

COROLLARY 4.1. $A \otimes_R K_P \cong A/AP$.

Proof. The composite R-module map $R \to A \to A/AP$ has kernel P, hence gives an R-map $K_P = R/P \to A/AP$. By the universal mapping property this R-map extends to an A-map $\pi: A \otimes_R K_P \to A/AP$ which is obviously surjective. If $\partial P \not\subseteq P$ then $A \otimes_R K_P$ is simple and π is an isomorphism. Suppose $\partial P \subseteq P$. Every element of A is uniquely expressible as $\sum x^i r_i$. Writing $\bar{r}$ for the image in $K_P = R/P$ of $r \in R$, we have

$$\ker \pi = \left\{ \sum x^i \otimes \bar{r}_i \;\middle|\; \sum x^i r_i \in AP \right\}.$$

But $AP = \sum x^i P$ and so $\sum x^i r_i \in AP$ if and only if $r_i \in P$ for all i. Therefore $\ker \pi = 0$ again. Q.E.D.

4.2. Suppose $r \in R$, $a \in A$, $[a, r] \in R$ and $j \in \mathbf{N}$. Then

$$r^j a^j = \prod_{i=0}^{j-1} (ra - i[a, r]). \tag{4.2}$$

Indeed, if the formula holds when $j = k$ then

$$\begin{aligned}(ra - k[a, r])\, r^k a^k &= rar^k a^k - k[a, r]\, r^k a^k \\ &= rr^k aa^k + krr^{k-1}[a, r]\, a^k - k[a, r]\, r^k a^k = r^{k+1} a^{k+1},\end{aligned}$$

giving it for $j = k + 1$.

Observe that $A \otimes_R K_P$ is not only an A-module but actually an $R_P[x]_\partial$-module (where $R_P[x]_\partial$ equals the subring of B generated by R_P and x); here the action is given by (4.1.2) for $r \in R_P$. Indeed the induced $R_P[x]_\partial$-module $R_P[x]_\partial \otimes_{R_P} K_P$ can be identified with $A \otimes_R K_P$ since the elements of each are uniquely expressible in the form $\sum x^j \otimes \omega_j$ ($\omega_j \in K_P$). Also observe that if $b \in B$ then the coefficients in $g^{-\nu b} b$ (where $\nu = \nu_P$) are in R_P, and hence the action of $g^{-\nu b} b$ on $A \otimes_R K_P$ is defined.

LEMMA 4.2. *Suppose* (3.1.1) *and* (3.3.1) *hold, $a \in B$, and $k \in \mathbf{Z}^+$. Then, for the action on $A \otimes_R K_P$,*

$$g^{-\nu_P a} a \cdot (x^{k-1} \otimes 1) = x^{k-1} \otimes Q_{P,g,a}(-k) + m,$$

where $m \in x^{k-2} \otimes K_P + \cdots + 1 \otimes K_P$.

Proof. Write $\nu = \nu_P$, $a = \sum_j a_j x^j$. Then by (4.1.2),

$$gx \cdot (x^{k-1} \otimes \eta_P(\partial g) j) = x^{k-1} \otimes (-k)\, \eta_P(\partial g)^{j+1} + \text{lower terms}.$$

Also $g^{-\nu a-j}a_j \in R_P$ for each j. Hence, by (4.2),

$$\begin{aligned}
&g^{-\nu a}a \cdot (x^{k-1} \otimes 1)\\
&\quad = \sum_j g^{-\nu a-j}a_j g^j x^j \cdot (x^{k-1} \otimes 1)\\
&\quad = \sum_j g^{-\nu a-j}a_j (gx-(j-1)\,\partial g) \cdots (gx-\partial g)(gx) \cdot (x^{k-1} \otimes 1)\\
&\quad = \sum_j g^{-\nu a-j}a_j \cdot \{x^{k-1} \otimes (-k-(j-1)) \cdots (-k-1)(-k)\,\eta_P(\partial g)^j\\
&\qquad + \text{lower terms}\}\\
&\quad = x^{k-1} \otimes Q_{P,g,a}(-k) + \text{lower terms.} \qquad \text{Q.E.D.}
\end{aligned}$$

COROLLARY 4.2. *The element a is preserving relative to P if and only if $g^{-\nu_P a}a$ acts injectively on $V(P)$.*

4.3. We now exhibit some nonmaximal left ideals of certain A, and then some maximal left ideals.

LEMMA 4.3. *Suppose* (3.1.1) *holds, P is prime, $\partial P \subseteq P$, $b = \sum b_j x^j \in B$ with $b_0 \neq 0$, and $b_j/b_0 \notin PR_P$ for some $j > 0$. Then $(A \cap Bb) + \sum Px^i \neq A$, and in particular the left ideal $A \cap Bb$ of A is not maximal.*

Proof. It suffices to show $1 \notin (A \cap Bb) + \sum Px^i$; suppose not. Then $1 = ab + d$, where $d \in \sum Px^i$ $(= \sum x^i P)$. Write $\nu = \nu_P$, which is defined by (3.1.3) in this case. Without loss of generality we may assume that $\nu b = 0$. Since $\nu d > 0$ and $\nu 1 = 0$, we have $0 = \nu(ab) = \nu a$. There exists a largest index k such that $\nu a_k = 0$ and a largest index l such that $\nu b_l = 0$. By the hypothesis on b_j/b_0, $l > 0$. By (3.1.4) the coefficient of x^{k+l} in ab is

$$(ab)_{k+l} = \sum_j \sum_{0 \leqslant i \leqslant k+l,j} \binom{j}{i} a_j \partial^{j-i} b_{k+l-i}.$$

We have

$$\nu(a_j \partial^{j-i} b_{k+l-i}) \geqslant \nu a_j + \nu b_{k+l-i} \geqslant 0.$$

If $i < j \leqslant k$ or $i = j < k$ then $\nu b_{k+l-i} > 0$; if $j > k$ then $\nu a_j > 0$; while in the remaining case, $i = j = k$, the inequality becomes an equality. Hence $\nu(ab)_{k+l} = 0$. But $k + l > 0$ since $l > 0$, and $\nu d_{k+l} > 0$, contradicting $1 = ab + d$. Q.E.D.

We single out the contrary of the hypothesis of Lemma 4.3 on the coefficients b_j of an irreducible element $b = \sum b_j x^j$ of B:

If there exists a ∂-invariant prime, then (coefficient of $x^0) \neq 0$ and for every $j > 0$ and every ∂-invariant prime P (coefficient of x^j)/(coefficient of $x^0) \in PR_P$. (4.3)

THEOREM 4.3. *Suppose* (3.1.1) *holds and that* $b = \sum b_j x^j \in B$ *is irreducible and preserving. Then the* (*S-torsionfree*) *A-module* $A/A \cap Bb$ *is simple if and only if b satisfies* (4.3).

Proof. Obviously $A/A \cap Bb$ is S-torsionfree. We will apply Lemma 2.3. We may assume R is not a field, since otherwise $A = B$ and (4.3) is vacuous. Suppose M is an S-torsion simple A-module. By Proposition 4.1, $M \cong V(P)$ for some non-∂-invariant P or $M \cong V(P, [a])$ for some ∂-invariant P and some $[a]$. Suppose first that $M \cong V(P)$, take $g \in P - P^2$ and write $\nu = \nu_P$ and $d = g^{-\nu b} b$. Since b is preserving relative to P, d_M is injective, by Corollary 4.2. Also $d \in Bb$ and $\nu d = 0$; in particular, $\nu d_j \geqslant j$ for all j, and so $d \in R_P[x]_\partial$. Clearing denominators by left multiplying by a suitable $r \in R - P$, we have $rd \in A \cap Bb$ and $(rd)_M$ is injective since r_M is injective. Next suppose $M \cong V(P, [a])$ and $b_j/b_0 \in PR_P$ for all $j > 0$. Again take $g \in P - P^2$, write $\nu = \nu_P$, and set $d = g^{-\nu b} b$. Then $\nu d_0 = 0$, that is, $d_0 \in R_P - PR_P$, and $\nu d_j > 0$, that is, $d_j \in PR_P$, for all $j > 0$. The module $V(P, [a])$ may be considered as an $R_P[x]_\partial$-module, since the canonical homomorphism $A \to K_P[x]_\partial$, $\sum c_j x^j \mapsto \sum \eta_P(c_j)\, x^j$, is actually defined on $R_P[x]_\partial$. Then $d_j x^j V(P, [a]) = 0$ for all $j > 0$, d_0 acts as the nonzero scalar $\eta_P d_0$ on $V(P, [a])$, and hence d_M is injective. Again clearing denominators we get an $rd \in A \cap Bb$ acting injectively on M. It now follows from Lemma 2.3 that if (4.3) holds then $A/A \cap Bb$ is simple. On the other hand, if (4.3) does not hold then, by Lemma 4.3, $A/A \cap Bb$ is not simple. Q.E.D.

4.4. We now give the main result on simple A-modules for A as in (3.1.1).

THEOREM 4.4. *Suppose* (3.1.1) *holds and N is a simple B-module. The following three conditions* (4.4) *are equivalent*:

$$\mathrm{Soc}_A N \neq 0; \tag{4.4.1}$$

$$\textit{for some } b \in \min \operatorname{ann} N, \text{ (4.3) } \textit{holds}; \tag{4.4.2}$$

$$\textit{for every } b \in \min \operatorname{ann} N, \text{ (4.3) } \textit{holds}. \tag{4.4.3}$$

Suppose that $0 \neq n \in N$, *b is a minimal annihilator of n, and s, which may be calculated as in Lemma* 3.4, *is such that* $s \in S$ *and* bs^{-1} *is preserving. If* (4.4) *holds then* Asn ($\cong (As + Bb/Bb)$ *is a simple A-module* (*in particular,*

$sn \in \mathrm{Soc}_A N = Asn$). Every S-torsionfree simple A-module arises in this way, and from an N which is unique up to isomorphism; that is,

$$A\hat{}\ (S\text{-torsionfree}) \to B\hat{}((4.4)), \qquad [M] \mapsto [S^{-1}M],$$

is a bijection, with inverse $[N] \mapsto [\mathrm{Soc}_A N]$ (for N satisfying (4.4)).

Proof. Suppose $a \in \min \operatorname{ann} N$. By Lemma 3.4 there exists $s \in S$ such that $b = as^{-1}$ is preserving. Then b is similar to a and so $B/Bb \cong N$. If (4.4.3) holds then by Theorem 4.3 the A-submodule $(A + Bb)/Bb \cong A/A \cap Bb$ is simple and so equals $\mathrm{Soc}_A B/Bb$ by Lemma 2.2.1. Hence (4.4.3) implies (4.4.1). Now suppose (4.4.1) holds and let M be a simple submodule of ${}_A N$. Then M is S-torsionfree, and it follows from Lemma 2.1 that there exists an irreducible $d \in B$ such that $M \cong A/A \cap Bd$. Hence $A \cap Bd = \mathrm{ann}_A m$ for some $m \in M$, and so $Bd = \mathrm{ann}_B m$ and $d \in \min \operatorname{ann} N$. By Lemma 4.3, d satisfies (4.3). Hence (4.4.1) implies (4.4.2). To show that (4.4.2) implies (4.4.3) we claim first that if $0 \neq t \in T$ and $a = \sum a_j x^j$ satisfies (4.3), then at satisfies (4.3). Thus, by (3.1.4),

$$at = \sum_i \sum_{j \geqslant i} \binom{j}{i} a_j (\partial^{j-i} t)\, x^i = (at)_i\, x^i.$$

Suppose $\partial P \subseteq P$ and write $\nu = \nu_P$. Then $\nu(\partial^j t) \geqslant \nu t$, $(at)_0 = a_0 t$ plus terms with higher ν-value, and hence $\nu(at)_0 = \nu a_0 + \nu t$, while every term of $(at)_i$, for $i > 0$, has higher ν-value than $\nu a_0 + \nu t$. This proves the claim. Now suppose $a \in \min \operatorname{ann} N$ satisfies (4.3). By Lemma 3.4 and the claim just proved, we may suppose that a is preserving. By Theorem 4.3, $A/A \cap Ba$ is simple. Suppose b is similar to a. Thus there exists $d \in B - Ba$ such that bd is an l.c.l.m. of a and d. We may write $d = s^{-1}d'$ where $s \in S$, $d' \in A$. Then $bs^{-1}d'$ is an l.c.l.m. of a and d' and $d' \notin Ba$, that is, bs^{-1} is similar to a by transforming by d'. Also bs^{-1} satisfies (4.3) if and only if b does. Hence in proving that b satisfies (4.3) we may assume that $d \in A$. We have $A/A \cap Ba$ is S-torsionfree simple with canonical generator $m = 1 + A$, and $\mathrm{ann}_A m = A \cap Ba$. Then $dm \neq 0$, $bdm = 0$ and $A \cap Bb \subseteq \mathrm{ann}_A dm$. Also if $b' \in \mathrm{ann}_A dm$ then $b'd \in \mathrm{ann}_A m = A \cap Ba$, $b'd$ is a left multiple of a and d and so of bd, and hence $b' \in A \cap Bb$. Therefore the left ideal $A \cap Bb = \mathrm{ann}_A dm$ is maximal, and so, by Lemma 4.3, b satisfies (4.3). Thus (4.4.2) implies (4.4.3). The next statement follows from Lemma 3.4, Theorem 4.3, and Lemma 2.2.1. The statement about the bijection of $A\hat{}$ (S-torsionfree) to $B\hat{}((4.4))$ follows from Lemma 2.2.1. Q.E.D.

We recall that R is called ∂-simple if R has no ∂-invariant ideal $\neq 0, R$.

LEMMA 4.4. *Suppose* (3.1.1) *holds. Then R has no ∂-invariant prime ideal if and only if R is ∂-simple.*

Proof. Suppose R has no ∂-invariant prime ideal and let I be a maximal ∂-invariant ideal. Then R/I is $\bar{\partial}$-simple (where $\bar{\partial}$ is the induced derivation on R/I). Since R is noetherian of Krull dimension $\leqslant 1$, if $I \neq 0$ then R/I has a minimal ideal. If P is a prime ideal containing I then R/P has characteristic 0. Hence R/I has characteristic 0 and, by [7] or [30], R/I is simple, that is, I is prime, a contradiction. Q.E.D.

COROLLARY 4.4.1. *Suppose* (3.1.1) *holds. A necessary and sufficient condition for the canonical injection* $S^{-1}: A^{\wedge}$ *(S-torsionfree)* $\to B^{\wedge}$, $[M] \mapsto [S^{-1}M]$, *to be bijective is that R be ∂-simple. If this condition holds (example: $A = \mathfrak{A}$, the Weyl algebra) then every simple S-torsionfree A-module is isomorphic to some $A/A \cap Ba$ with a irreducible and preserving, two of these A-modules being isomorphic if and only if the corresponding a are similar.*

Proof. If R is ∂-simple then (4.3) holds vacuously. Conversely, if there exists a ∂-invariant P then $B/B(x+1)$ is simple and $\mathrm{Soc}_A\, B/B(x+1) = 0$. Q.E.D.

The other extreme, when the canonical injection has empty image, can also be characterized.

COROLLARY 4.4.2. *Suppose* (3.1.1) *holds. Then* $A^{\wedge}$ *(S-torsionfree) is empty if and only if there are infinitely many ∂-invariant primes P.*

Proof. If there are infinitely many ∂-invariant P then the intersection $\bigcap PR_P$ over all such P is zero, and (4.3) never holds for an irreducible b. Conversely, if there are only finitely many ∂-invariant P, then their intersection contains a nonzero element d, $dx+1$ satisfies (4.3), and $N = B/B(dx+1)$ is simple with $\mathrm{Soc}_A N \neq 0$. Q.E.D.

4.5. We remark on the commutative case, that is, when $\partial = 0$. Then $A = R[x]$ (commutative polynomials), and (3.1.1) reduces to the hypothesis that R is Dedekind. Simple A-modules correspond to their annihilators, that is, to the maximal ideals of A, and S-torsionfree simple modules correspond to the maximal ideals I such that $I \cap R = 0$. Thus in this case $A^{\wedge}$ (S-torsionfree) is nonempty if and only if $I \cap R = 0$ for some maximal ideal of $R[x]$; this latter condition is equivalent to R being a G-domain (see [19]). Lemma 4.3 above remains valid if, instead of assuming R Dedeind, one merely assumes that R is a Krull domain and P is a height one prime (so that v_P is defined). Thus we obtain the following classification of the maximal ideals of $R[x]$ contracting to 0.[1] Here $\bigcap P$ denotes the intersection

[1] The fact that they are all principal is covered (in a quite different way) by [19, Ex. 11, p. 42]; the finer details appear to be new. By contrast, in $\mathfrak{A}$ the maximal left ideals contracting to 0 in $K[q]$ are not all principal.

of all nonzero prime ideals of R. Actually the generalization from R being Dedekind to R being Krull evaporates since R is a Krull G-domain if and only if R is a principal ideal domain with only finitely many prime ideals.

COROLLARY 4.5. *Suppose R is a Krull domain but not a field. An ideal I of $R[x]$ is maximal and satisfies $R \cap I = 0$ if and only if $I = (b)$ for some b which is irreducible (in $T[x]$) and of the form $b = 1 + \sum_{i>0} r_i x^i$ where for all $i > 0$, $r_i \in \bigcap P$.*

Proof. Suppose I is maximal and $I \cap R = 0$. We have $I = R[x] \cap T[x]b$ where $b = \sum b_i x^i$ is irreducible. Then $b_0 \neq 0$ (otherwise $b = b_1 x$, $I = \sum_{i>0} Rx^i$ and R is a field). By Lemma 4.3, $b_i/b_0 \in \bigcap PR_P = \bigcap P$ for all $i > 0$. Dividing by b_0 we may assume that $b_0 = 1$. If $(\sum t_i x^i)b \in R[x]$, where $t_i \in T$, and if $t_0,..., t_{j-1} \in R$ (for some $j \geqslant 0$) then $t_j + t_{j-1}b_1 + \cdots + t_0 b_j \in R$, and hence $t_j \in R$. Therefore $R[x] \cap T[x]b = R[x]b$ is principal. Conversely, if I has the given form then R has only finitely many primes. The argument just given shows that $I = R[x] \cap T[x]b$, and the latter ideal is maximal (say, by Theorem 4.3 if one wants to stick to the methods of the present paper).

Q.E.D.

5. THE SIMPLE MODULES OVER $\mathfrak{sl}(2, K)$

5.1. Let $\mathfrak{s}$ be the Lie algebra $\mathfrak{sl}(2, K)$ over the field K (of characteristic 0). We take the usual basis e, f, h where $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$, the usual Cartan subalgebra Kh, positive root vector e, and Borel subalgebra $\mathfrak{b} = Kh + Ke$. We write c for the Casimir element

$$c = 4fe + h^2 + 2h = 4ef + h^2 - 2h = 2ef + 2fe + h^2 \in U\mathfrak{s}.$$

It is well known that the center of $U\mathfrak{s}$ is $K[c]$. By Quillen's lemma, if M is a simple $U\mathfrak{s}$-module then c_M is algebraic, hence a scalar if K is algebraically closed.

We begin with a simple lemma which, in particular, expresses $\mathfrak{sl}(2, \mathbf{R})\hat{}$ in terms of $\mathfrak{sl}(2, \mathbf{C})\hat{}(c_M \in \mathbf{C} - \mathbf{R})$ and $\mathfrak{sl}(2, \mathbf{R})\hat{}(c_M \in \mathbf{R})$. If K' is an extension of the base field K and M' is an $\mathfrak{sl}(2, K')$-module, ${}_{\mathfrak{s}}M'$ denotes here the pullback along the inclusion map $\mathfrak{s} = \mathfrak{sl}(2, K) \to \mathfrak{sl}(2, K')$.

LEMMA 5.1. *If M' is a simple module over $\mathfrak{sl}(2, K')$ where $K \subseteq K' \subseteq \bar{K}$, $c_{M'} \in K'$ and $K(c_{M'}) = K'$, then the restriction ${}_{\mathfrak{s}}M'$ is a simple $\mathfrak{s}$-module, and every simple $\mathfrak{s}$-module arises in this way from exactly n such M', say, $M_1,..., M_n$, with $K' = K(\gamma_1),..., K(\gamma_n)$, respectively, where $\gamma_1,..., \gamma_n$ are the roots of the minimal polynomial of c_M and $c_{M_i} = \gamma_i$ (and no two M_i are isomorphic).*

Proof. Clearly ${}_{\mathfrak{s}}M'$ is simple, since $c \in U\mathfrak{s}$. Conversely, suppose M is a simple $\mathfrak{s}$-module. By Quillen's lemma, c_M is algebraic, and by Schur's lemma, the minimal polynomial g of c_M is irreducible. Let $\gamma_1,..., \gamma_n$ be the roots of g in $\bar{K}$. Picking a γ_i and setting $\gamma_i m = cm$ $(m \in M)$, we get a $K(\gamma_i)$-structure on M, and hence an $\mathfrak{sl}(2, K(\gamma_i))$-module M_i with ${}_{\mathfrak{s}}M_i = M$. Now suppose $M = {}_{\mathfrak{s}}M'$ where M' is a simple $\mathfrak{sl}(2, K(\gamma))$-module and $c_{M'} = \gamma$. Then γ is a root of g, say, $\gamma = \gamma_i$. Since $\mathfrak{s}$ and γ_i generate $U\mathfrak{sl}(2, K(\gamma_i))$, $M' = M_i$. Q.E.D.

A similar phenomenon holds for the three-dimensional Heisenberg Lie algebra $\mathfrak{h}$, with z in place of c where $0 \neq z \in \text{Center}\ \mathfrak{h}$.

5.2. Suppose $\lambda \in \bar{K}$. Let $K' = K(\lambda)$, $\mathfrak{s}' = \mathfrak{sl}(2, K')$, and $\mathfrak{b}' = K'h + K'e$. We identify λ with the linear functional on the Cartan subalgebra $K'h$ of $\mathfrak{s}'$ whose value on h is λ. Then $M(\lambda)$ denotes the corresponding Verma module, that is, the induced $U\mathfrak{s}'$-module $U\mathfrak{s}' \otimes_{U\mathfrak{b}'} K$ where here K' is regarded as a $\mathfrak{b}'$-module with $e \cdot K' = 0$ and $h|_{K'} = \lambda - 1$. Also $L(\lambda)$ denotes the unique (absolutely) simple quotient module of $M(\lambda)$. Thus $L(\lambda) = M(\lambda)$ if $\lambda \notin \mathbf{Z}^+$ while if $\lambda \in \mathbf{Z}^+$ then $L(\lambda)$ is the module of dimension λ. We have $c_{L(\lambda)} = \lambda^2 - 1$.

Associated to $M(\lambda)$ we get a realization of $\mathfrak{s}'$ (and hence of $\mathfrak{s}$) by differential operators on the polynomial algebra $K[\xi]$, as follows (this is a special case, for s', of the Conze mappings [11]). The mapping

$$\sum \omega_i \xi^i \mapsto \sum f^i \otimes \omega_i \in U\mathfrak{s}' \otimes_{U\mathfrak{b}'} K'$$

is a linear bijection of $K'[\xi]$ to $M(\lambda)$; let φ_λ be the representation of $\mathfrak{s}$ (or $U\mathfrak{s}$) in $K[\xi]$ obtained via this bijection from the representation of $\mathfrak{s}$ (or $U\mathfrak{s}$) in $M(\lambda)$. Thus, writing $p = d/d\xi$ and $q =$ multiplication by ξ, we have $\varphi_\lambda f = q$, $\varphi_\lambda h = -2qp + \lambda - 1$, $\varphi_\lambda e = -(qp - \lambda + 1)p$. We shall actually use $\rho_{-\lambda} = \varphi_\lambda \tau$ rather than φ_λ, where τ is the automorphism of $\mathfrak{s}$ given by $\tau e = f$, $\tau f = e$, $\tau h = -h$, and we shall also denote by $\rho_{-\lambda}$ the extension of $\rho_{-\lambda}$ to $U\mathfrak{s}$. Thus ρ_λ is the algebra homomorphism of $U\mathfrak{s}$ to the Weyl algebra $\mathfrak{A}(K')$ (or to $\mathfrak{B}(K')$) with

$$\rho_\lambda e = q, \quad \rho_\lambda h = 2qp + \lambda + 1, \quad \rho_\lambda f = -(qp + \lambda + 1)p. \tag{5.2.1}$$

We now define another family of mappings of $U\mathfrak{s}$, this time to $\mathfrak{B} = \mathfrak{B}(K)$. Given $\gamma \in K$, we define a linear map σ_γ of $\mathfrak{s}$ to $\mathfrak{B}$ by

$$\sigma_\gamma e = q, \qquad \sigma_\gamma h = 2qp, \qquad \sigma_\gamma f = ((1/4)\gamma - (qp)^2 - qp)\, q^{-1}$$
$$(= (1/4)\,\gamma q^{-1} - qp^2).$$

Then $[\sigma_\gamma h, \sigma_\gamma e] = 2\sigma_\gamma e$, $[\sigma_\gamma h, \sigma_\gamma f] = -2\sigma_\gamma f$, and $[\sigma_\gamma e, \sigma_\gamma f] = [q, -qp^2] = 2qp = \sigma_\gamma h$. Hence σ_γ extends to a homomorphism (also denoted σ_γ) of $U\mathfrak{s}$ to

$\mathfrak{B}$. We have $U\mathfrak{s} \subseteq K[q, q^{-1}, p] = K[q, q^{-1}][p]_{d/dq}$. Also $\sigma_0 = \rho_{-1}$. We write $\sigma(\gamma)$ for σ_γ and $\rho(\lambda)$ for ρ_λ when using these symbols as subscripts (indicating pullback).

Suppose $\gamma \in K$. The algebra $U\mathfrak{s}/U\mathfrak{s}(c-\gamma)$ will be denoted by $U\mathfrak{s}_\gamma$ (this was called B_γ in [16]) and the canonical map $U\mathfrak{s} \to U\mathfrak{s}_\gamma$ by π_γ. If $u \in U\mathfrak{s}$, $\pi_\gamma u$ will also be denoted by $\bar{u}$. The multiplicative subset $K[\bar{e}] - \{0\}$ of $U\mathfrak{s}_\gamma$ will be denoted by $\bar{S}$.

Straightforward calculations show that

$$\rho_\gamma c = \gamma, \qquad \rho_\lambda c = \lambda^2 - 1.$$

Hence $U\mathfrak{s}(c-\gamma) \subseteq \ker \sigma_\gamma$. But (see [16]) $U\mathfrak{s}_\gamma$ is simple except for $\gamma = n^2 + 2n$ ($n \in \mathbf{N}$), in which case $U\mathfrak{s}_\gamma$ contains a unique ideal $\neq 0$, $U\mathfrak{s}_\gamma$, this ideal being of codimension $(n+1)^2$. Since $\sigma_\gamma U\mathfrak{s}$ is infinite dimensional (containing all q^i), it follows that $\ker \sigma_\gamma = U\mathfrak{s}(c-\gamma)$. We denote by $\bar{\sigma}_\gamma$ the induced map of $U\mathfrak{s}_\gamma$ to $\mathfrak{B}$; thus $\bar{\sigma}_\gamma \pi_\gamma = \sigma_\gamma$. In particular (restricting the co-domain), $\bar{\sigma}_\gamma$ may be regarded as an isomorphism of $U\mathfrak{s}_\gamma$ to $\sigma_\gamma U\mathfrak{s}$.

Suppose $\lambda \in K$. Taking $\gamma = \lambda^2 - 1$, we have $U\mathfrak{s}(c-\gamma) \subseteq \ker \rho_\lambda$, and hence, as before, $U\mathfrak{s}(c-\gamma) = \ker \rho_\lambda$. We denote by $\bar{\rho}_\lambda$ the induced map of $U\mathfrak{s}_\gamma$ to $\mathfrak{B}$. Thus $\bar{\rho}_\lambda \pi_\gamma = \rho_\lambda$, and $\bar{\rho}_\lambda$ may be regarded as an isomorphism of $U\mathfrak{s}_\gamma$ to $\rho_\lambda U\mathfrak{s}$.

The ring $\mathfrak{B}$ is a localization at $S = K[q] - \{0\}$ of $\sigma_\gamma U\mathfrak{s}$ since it is a localization at S of $K[q, qp] \subseteq \sigma_\gamma U\mathfrak{s}$. Similarly, if $\lambda \in K$ then $\mathfrak{B}$ is a localization at S of $\rho_\lambda U\mathfrak{s}$. Since $\bar{\sigma}_\gamma \bar{S} = S$, we have the following result.

LEMMA 5.2. *For $\gamma \in K$, $U\mathfrak{s}_\gamma$ has a (two-sided) localization at $\bar{S} = K[\bar{e}] - \{0\}$, and $\bar{\sigma}_\gamma$ (respectively $\bar{\rho}_\lambda$ if $\lambda \in K$ satisfies $\lambda^2 = \gamma + 1$) extends to an isomorphism, also denoted $\bar{\sigma}_\gamma$ (respectively $\bar{\rho}_\lambda$), of $\bar{S}^{-1}U\mathfrak{s}_\gamma$ to $\mathfrak{B}$.*

5.3. Suppose $d \in K[\xi]$ is monic irreducible, $\lambda \in \bar{K}$ is a root of d and $K' = K(\lambda)$. We form the highest weight module $L(\lambda)$ over $\mathfrak{s}' = \mathfrak{sl}(2, K')$ and regard it as a module, denoted $L(d)$ (or also $L(\lambda)$), over $\mathfrak{s} \subseteq \mathfrak{s}'$. It is easily seen that $L(\lambda)$ remains simple under the restriction to $\mathfrak{s}$ of the operating algebra ($L(\lambda) = M(\lambda)$ if $\lambda \notin K$). Also $L(\lambda)$ as an $\mathfrak{s}$-module is uniquely determined up to isomorphism by d, justifying the notation $L(d)$.

If $d = \xi - \lambda$ then $L(d) = L(\lambda)$ is just the usual highest weight module over $U\mathfrak{s}$. If $\lambda \notin K$, the $U\mathfrak{s}$-module $L(d)$ is an induced module. Indeed we can regard K' as a $U\mathfrak{b}$-module with $eK' = 0$ and h acting by multiplication by $\lambda - 1$, and form the induced $U\mathfrak{s}$-module $U\mathfrak{s} \otimes_{U\mathfrak{b}} K'$. The latter is simple, by the usual argument. If v is a maximal vector of the $\mathfrak{s}'$-module $L(\lambda)$, then the map $K' \to K'v$, $\omega \mapsto \omega v$, is a $U\mathfrak{b}$-isomorphism which by the universal mapping property extends to a $U\mathfrak{s}$ map of $U\mathfrak{s} \otimes_{U\mathfrak{b}} K'$ to $L(d)$. Since both $U\mathfrak{s} \otimes_{U\mathfrak{b}} K'$ and $L(d)$ are simple, this $U\mathfrak{s}$-map is an isomorphism.

Now suppose $g \in K[q]$ is monic irreducible. Recall that $K_g = K[q]/(g)$,

$\alpha = \eta_g q$ (so that $K_g = K(\alpha)$ and $g(\alpha) = 0$), and $V(g)$ is the induced $\mathfrak{A}$-module $\mathfrak{A} \otimes_{K[q]} K_g$. As in Section 4.1, $V(g)$ is identified with $K_g[p]$ with p acting by multiplication and q acting as $\alpha - d/dp$. If $\lambda \in K$ then we consider the $U\mathfrak{s}$-module $V(g)_{\rho(\lambda)}$ obtained by pullback along ρ_λ. Writing $V(g)_{\rho(\lambda)} = K_g[p]$ we have, by (5.2.1),

$$e \cdot (\omega p^j) = \alpha\omega p^j - j\omega p^{j-1}, \tag{5.3.1}$$

$$h \cdot (\omega p^j) = 2\alpha\omega p^{j+1} + (-2(j+1) + \lambda + 1)\,\omega p^j, \tag{5.3.2}$$

$$f \cdot (\omega p^j) = -\alpha\omega p^{j+2} + (j + 2 - \lambda - 1)\,\omega p^{j+1} \tag{5.3.3}$$

for all $\omega \in K_g$, $j \in \mathbf{N}$.

Suppose $g \neq q$, that is, $\alpha \neq 0$. Then q acts bijectively on $V(g)$, and $V(g)$ can be considered as a $K[q, q^{-1}, p]$-module. Hence by pullback along σ_γ we have the $U\mathfrak{s}$-module $V(g)_{\sigma(\gamma)}$. We have $e \cdot p^i = \alpha p^i - ip^{i-1}$ and $h \cdot p^i = 2\alpha p^{i+1} - 2(i+1)\,p^i$. The basis $\{p^i\}$ being rather inconvenient for expressing the operation by f, we indicate another basis. Since $qp_{V(g)} = (\alpha p - (d/dp)p)_{V(g)}$, it is clear that $\{(qp)^i \cdot 1 \mid i \in \mathbf{N}\}$ is also a K_g-basis of $V(g)$. We write (here) $(qp)^i \cdot 1 = y^i$ and thus identify $V(g)$ (as a vector space over K_g) with the polynomial ring $K_g[y]$. Then for $a(y) \in K_g[y]$ we have (using $q(qp) = (qp - 1)q$ and $q^{-1}(qp) = (qp + 1)\,q^{-1}$)

$$h \cdot a(y) = 2ya(y), \qquad e \cdot a(y) = a(y-1)\alpha,$$

$$f \cdot a(y) = ((1/4)\gamma - y^2 - y)\,a(y+1)\,\alpha^{-1}.$$

The special case $\alpha = 1$ gives modules introduced by Arnal and Pinczon [2].

We can regard K_g as a $K[e]$-module with $e \cdot \omega = \alpha\omega$ ($\omega \in K_g$), where $\alpha = \eta_g q$, and form the induced $U\mathfrak{b}$-module $U\mathfrak{b} \otimes_{K[e]} K_g$. We shall also consider the $U\mathfrak{s}$- (or $U\mathfrak{b}$)-module

$$U\mathfrak{s}_\gamma / U\mathfrak{s}_\gamma g(\bar{e}) = U\mathfrak{s}/(U\mathfrak{s}(c - \gamma) + U\mathfrak{s}g(e)),$$

which we denote by $U\mathfrak{s}_{\gamma,g}$.

LEMMA 5.3. *Suppose $g \in K[q]$ is monic irreducible, $g \neq q$, $\gamma \in K$. Then, as $U\mathfrak{b}$-modules, the following are simple and isomorphic*: $U\mathfrak{b} \otimes_{K[e]} K_g$, $U\mathfrak{s}_{\gamma,g}$, $V(g)_{\sigma(\gamma)}$, *and* (*provided $\lambda \in K$ and $\lambda^2 = \gamma + 1$*) $V(g)_{\rho(\lambda)}$.

Proof. Every element of $U\mathfrak{b} \otimes_{K[e]} K_g$ has a unique expression of the form $\sum h^i \otimes \omega_i$, $\omega_i \in K_g$. Since $g(e)\,K_g = 0$ and $eh = (h-2)e$, we have $e^i h^j = (h - 2i)^j e^i$, and

$$g(e)\,h^j \otimes \omega = h^{j-1} \otimes (-2j)\,g'(e)\,e\omega + \text{lower terms} \quad (\omega \in K_g), \tag{5.3.4}$$

from which it follows that $U\mathfrak{b} \otimes_{K[e]} K_g$ is simple. The map $K_g \to U\mathfrak{s}_{\gamma,g}$,

$d(q) + (g) \mapsto d(\bar{e}) + U\mathfrak{s}_\gamma g(\bar{e})$, is a $K[e]$-module map, hence extends by the universal mapping property to a $U\mathfrak{b}$-map π of $U\mathfrak{b} \otimes_{K[e]} K_g$ to $U\mathfrak{s}_{\gamma,g}$. Since $U\mathfrak{b} \otimes_{K[e]} K_g$ is simple, π is injective. To show that π is surjective, it suffices to show that $\bar{f}^i + U\mathfrak{s}_\gamma g(\bar{e}) \in \operatorname{im} \pi$ for all i. Suppose this holds for $i = 1, \ldots, j$. It follows that $\bar{f}^j \bar{h}^k \bar{e}^l + U\mathfrak{s}_\gamma g(\bar{e}) \in \operatorname{im} \pi$ for all k, l. There exist $s, t \in K[q]$ such that $sq + tg = 1$. Taking congruences in $U\mathfrak{s}_\gamma$ modulo $U\mathfrak{s}_\gamma g(\bar{e})$, we have $\bar{e}s(\bar{e}) \equiv \bar{1}$ and

$$\bar{f}^{j+1} \equiv \bar{f}^j \bar{f} \bar{e} s(\bar{e}) \equiv \bar{f}^j (\gamma - \bar{h}^2 - 2\bar{h}) \tfrac{1}{4} s(\bar{e}) \in \operatorname{im} \pi,$$

and so π is surjective and hence a $U\mathfrak{b}$-isomorphism. Also we see that

$$e \cdot U\mathfrak{s}_{\gamma,g} = U\mathfrak{s}_{\gamma,g}. \tag{5.3.5}$$

We next consider $V(g)_{\rho(\lambda)}$ when $\lambda \in K$ and $\lambda^2 = \gamma + 1$. We have

$$\begin{aligned} g(e) \cdot \sum_{i=0}^{j} \omega_i p^i &= g(\alpha - d/dp) \cdot \sum_{i=0}^{j} \omega_i p^i \\ &= -jg'(\alpha)\, \omega_j p^{j-1} + \text{lower terms}, \end{aligned}$$

from which it follows that $V(g)_{\rho(\lambda)}$ is simple as a $U\mathfrak{b}$-module. The identity map of K_g, regarded as a map of the subspace $1 \otimes K_g$ of $U\mathfrak{b} \otimes_{K[e]} K_g$ to the subspace K_g of $K_g[p] = V(g)_{\rho(\lambda)}$, is a $K[e]$-map. This extends to a $U\mathfrak{b}$-map of $U\mathfrak{b} \otimes_{K[e]} K_g$ to $V(g)_{\rho(\lambda)}$, which is an isomorphism since both $U\mathfrak{b} \otimes_{K[e]} K_g$ and $V(g)_{\rho(\lambda)}$ are simple as $U\mathfrak{b}$-modules. Finally, for any $\gamma \in K$, σ_γ coincides on $U\mathfrak{b}$ with ρ_{-1} and hence $V(g)_{\sigma(\gamma)} = V(g)_{\rho(-1)} \cong U\mathfrak{b} \otimes_{K[e]} K_g$ as $U\mathfrak{b}$-modules. Q.E.D.

PROPOSITION 5.3. *Suppose $\gamma \in K$. Then*

$$\begin{aligned} &U\mathfrak{s}^\wedge(K[e]\text{-torsion}, c_M = \gamma) \\ &\quad = \{[U\mathfrak{s}_{\gamma,g}] \mid g \in K[q] \text{ monic irreducible}, g \neq q\} \\ &\quad\quad \cup \{[L(d)] \mid d = \xi - \lambda, \lambda \in K, \lambda^2 = \gamma + 1 \\ &\quad\quad\quad \text{or } d = \xi^2 - \gamma - 1 \text{ irreducible}\}, \end{aligned}$$

the isomorphism classes indicated on the right side all being distinct.

Proof. It follows from the above remarks and Lemma 5.3 that the classes on the right side are in the left side.

Suppose M is a simple $K[e]$-torsion $U\mathfrak{s}$-module with $c_M = \gamma$. There exists a monic irreducible $g \in K[q]$ and $0 \neq m \in M$ such that $g(e)m = 0$. Suppose first that $g \neq q$. Since $g(e)m = 0$, we have a $K[e]$-map $K_g \to K[e]m$, $a(\alpha) = a(q) + (g) \mapsto a(e)m$. This extends to a $U\mathfrak{b}$-map of $U\mathfrak{b} \otimes_{K[e]} K_g$ to M.

Therefore by Lemma 5.3 there is a nonzero $U\mathfrak{b}$-map, say π, of $M' = U\mathfrak{s}_{\gamma,g}$ to M. In both M' and M, $4fe + h^2 + 2h$ acts as γ. Hence, for $v \in M'$, $\pi(f \cdot (e \cdot v)) = \pi((fe) \cdot v) = (fe) \cdot \pi v = f \cdot \pi(e \cdot v)$. But $e \cdot M' = M'$ by (5.3.5), and so π is a $U\mathfrak{s}$-map. Since both M' and M are simple, π is an isomorphism.

Suppose next that $g = q$, that is, $em = 0$. The null space N of e is h-invariant and $h^2 + 2h - \gamma = 0$ on N. If h has an eigenvector in N, say $hn = (\lambda - 1)n$ $(\lambda \in K)$, then $0 = (h^2 + 2h - \gamma)n = (\lambda^2 - 1 - \gamma)n$ and so $\lambda^2 = \gamma + 1$, n is a maximal vector, and $M \cong L(\lambda) = L(d)$, where $d = \xi - \lambda$. Suppose finally that h has no eigenvector in N. Let $d = \xi^2 - \gamma - 1$ and $K' = K(\lambda)$, where λ is a root of d. On N, $d(h+1) = h^2 + 2h - \gamma = 0$. If λ were in K, we would have $(h + 1 - \lambda)(h + 1 + \lambda) = 0$ on N and h would have an eigenvector in N, a contradiction. Hence $\lambda \notin K$ and d is irreducible. Regarding K' as a $U\mathfrak{b}$-module with $e \cdot K' = 0$ and h acting by multiplication by $\lambda - 1$, we have a K-linear map $\pi\colon K' \to Km + Khm$ with $\pi 1 = m$ and $\pi\lambda = (h+1)m$. Then π is a $U\mathfrak{b}$-map since $\pi(e \cdot K') = e\pi(K') = 0$ and

$$\pi(h \cdot \lambda) = \pi(\lambda^2 - \lambda) = \pi(\gamma + 1 - \lambda) = (\gamma - h)m = (h^2 + h)m = h \cdot \pi\lambda.$$

Hence π extends to a $U\mathfrak{s}$-map π' of $L(d) \cong U\mathfrak{s} \otimes_{U\mathfrak{b}} K'$ to M, and since $L(d)$ and M are simple, π' is an isomorphism.

In $U\mathfrak{b} \otimes_{K[e]} K_g$, and hence also in $U\mathfrak{s}_{\gamma,g}$, g is the unique monic irreducible polynomial for which $g(e)$ annihilates a nonzero element, as follows from a formula like (5.3.4). Also, in $L(d)$, d is the unique monic irreducible polynomial for which $d(h+1)$ annihilates the null vectors of e. This proves the uniqueness statement. Q.E.D.

COROLLARY 5.3.1. *Suppose $\gamma \in K$ and $g \neq q$ is monic irreducible. Then $U\mathfrak{s}_{\gamma,g} \cong V(g)_{\sigma(\gamma)}$ (as $U\mathfrak{s}$-modules). If $\lambda \in K$ and $\lambda^2 = \gamma + 1$ then also $U\mathfrak{s}_{\gamma,g} \cong V(g)_{\rho(\lambda)}$.*

If K is algebraically closed, $\lambda \in K$, $g = q - \alpha \neq q$, and we identify α with the linear functional on Ke with $\alpha e = \alpha$, then, in the terminology of Kostant [21] (specialized here to be case of $\mathfrak{s}$), the coset containing $\bar{1}$ is a cyclic Whittaker vector of $U\mathfrak{s}_{\gamma,g}$ with respect to α. Moreover $U\mathfrak{s}_{\gamma,g}$ is the unique (up to isomorphism) Whittaker module V with respect to α whose corresponding central ideal is

$$(\text{center } U\mathfrak{s}) \cap \text{ann } V = K[c](c - \lambda^2 + 1)$$

[21, Theorem 3.1]. The fact that $U\mathfrak{s}_{\gamma,g}$ is simple reflects the fact that the ideal $K[c](c - \lambda^2 + 1)$ is maximal in $K[c]$ [21, Theorem 3.6.1].

Thus in the algebraically closed case, Proposition 5.3 reduces to the known fact that the simple $U\mathfrak{s}$-modules for which e has an eigenvector are the highest weight modules and the simple Whittaker modules. It is interesting that each of these modules occurs as an essential submodule of an appropriate $V(g)_{\rho(\lambda)}$, as the following result shows.

COROLLARY 5.3.2. *Suppose $\lambda \in K$ and $g \in K[q]$ is monic irreducible. Then the $U\mathfrak{s}$-module $V(g)_{\rho(\lambda)}$ contains an essential simple $K[e]$-torsion submodule $V(g)'_{\rho(\lambda)}$, and*

$$\begin{aligned} V(g)'_{\rho(\lambda)} &= V(g)_{\rho(\lambda)} && \text{if } g \neq q; \\ &= V(q)_{\rho(\lambda)} \cong M(\lambda) = L(\lambda) && \text{if } \lambda \notin \mathbf{Z}^+; \\ &= \sum_{j=0}^{\lambda-1} Kp^j \cong L(\lambda) && \text{if } \lambda \in \mathbf{Z}^+. \end{aligned}$$

Proof. If $g \neq q$ the result follows from Lemma 5.3. Suppose $g = q$, that is, $\alpha = 0$ and $K_g = K$. Then, by (5.3.1) and (5.3.2), $1 = p^0$ is a maximal vector and has weight $\lambda - 1$. If $\lambda \notin \mathbf{Z}^+$ then the coefficient $j + 2 - \lambda - 1$ in (5.3.3) never vanishes, 1 generates $V(q)_{\rho(\lambda)}$, and the canonical map of the Verma module $M(\lambda)$ to $V(q)_{\rho(\lambda)}$ (obtained by the universal property of the induced module) is an isomorphism (see [15, Prop. 7.1.8]). If $\lambda \in \mathbf{Z}^+$ then $\sum_{j=0}^{\lambda-1} Kp^j$ is a simple submodule which intersects any nonzero submodule, by (5.3.1). Q.E.D.

5.4. We turn now to the simple $K[e]$-torsionfree $U\mathfrak{s}$-modules for which c acts as a scalar, say γ. For $u \in U\mathfrak{s}$, the isomorphism $\bar{\sigma}_\gamma$: $U\mathfrak{s}_\gamma \to \sigma_\gamma U\mathfrak{s}$ induces a $U\mathfrak{s}_\gamma$-module isomorphism

$$U\mathfrak{s}_\gamma / U\mathfrak{s}_\gamma \cap \bar{S}^{-1} U\mathfrak{s}_\gamma \bar{u} \cong (\sigma_\gamma U\mathfrak{s} / \sigma_\gamma U\mathfrak{s} \cap \mathfrak{B} \sigma_\gamma u)_{\bar{\sigma}(\gamma)}$$

and of course a corresponding isomorphism for the $U\mathfrak{s}$-module obtained by pullback along π_γ. Thus

$$M(u, \gamma) \cong (\sigma_\gamma U\mathfrak{s} / \sigma_\gamma U\mathfrak{s} \cap \mathfrak{B} \sigma_\gamma u)_{\sigma(\gamma)}, \tag{5.4.1}$$

where we write

$$M(u, \gamma) = (U\mathfrak{s}_\gamma / U\mathfrak{s}_\gamma \cap \bar{S}^{-1} U\mathfrak{s}_\gamma \bar{u})_{\pi(\gamma)}.$$

Similarly, if $\lambda \in K$ and $\lambda^2 = \gamma + 1$ then $\bar{\rho}_\lambda$ induces a $U\mathfrak{s}_\gamma$-module isomorphism

$$U\mathfrak{s}_\gamma / U\mathfrak{s}_\gamma \cap \bar{S}^{-1} U\mathfrak{s}_\gamma \bar{u} \cong (\rho_\lambda U\mathfrak{s} / \rho_\lambda U\mathfrak{s} \cap \mathfrak{B} \rho_\lambda u)_{\bar{\rho}(\lambda)}$$

and a corresponding $U\mathfrak{s}$-module isomorphism.

LEMMA 5.4.1. *Suppose $\lambda \in K$ and $\gamma = \lambda^2 - 1$. The map $\bar{\rho}_{-\lambda}\bar{\rho}_\lambda^{-1}: \rho_\lambda U\mathfrak{s} \to \rho_{-\lambda} U\mathfrak{s}$ (respectively, $\bar{\rho}_\lambda \bar{\sigma}_\gamma^{-1}: \sigma_\gamma U\mathfrak{s} \to \rho_\lambda U\mathfrak{s}$) extends uniquely to an automorphism (denoted by the same symbol) of $\mathfrak{B}$, fixing q and sending p to $p - \lambda q^{-1}$ (respectively, to $p + \frac{1}{2}(\lambda + 1) q^{-1}$).*

Proof. Any $b \in \mathfrak{B}$ can be uniquely expressed as

$$b = \sum a_i(2qp + \lambda + 1)^i, \qquad a_i = a_i(q) \in K(q).$$

Pick $s \in S$ such that $sa_i \in K[q]$ for all i. Then

$$\bar{\rho}_{-\lambda}\bar{\rho}_\lambda^{-1} sb = \bar{\rho}_{-\lambda}\bar{\rho}_\lambda^{-1} \rho_\lambda \sum (sa_i)(e)\, h^i = s \sum a_i(2qp - \lambda + 1)^i,$$

giving uniqueness, and the result for the first map follows since there does exist an automorphism fixing q and sending p to $p - \lambda q^{-1}$. The result for the second map is obtained similarly. Q.E.D.

As a consequence of the lemma, if $\lambda \in K$ and $u \in U\mathfrak{s}$ then $\rho_\lambda u$ and $\rho_{-\lambda} u$ have the same degree, and $\rho_\lambda u$ is irreducible in $\mathfrak{B}$ if and only if $\rho_{-\lambda} u$ is (which by Lemma 5.2 is also equivalent to u being irreducible in $\bar{S}^{-1} U\mathfrak{s}_\gamma$).

LEMMA 5.4.2. *Suppose $u \in U\mathfrak{s}$, $\gamma \in K$, $\lambda \in \bar{K}$, and $\lambda^2 = \gamma + 1$. Let G denote the set of monic irreducible polynomials in $K[q]$. The following conditions* (5.4.2) *are equivalent:*

$\rho_\lambda u, \rho_{-\lambda} u$ are preserving at q, and $\sigma_\gamma u$ is preserving at every $g \in G - \{q\}$; (5.4.2.1)

$\rho_\lambda u$ is preserving and $\rho_{-\lambda} u$ is preserving at q; (5.4.2.2)

$\rho_\lambda u$ and $\rho_{-\lambda} u$ are preserving. (5.4.2.3)

For $a \in \mathfrak{B}$ the following conditions (5.4.3) *are equivalent:*

$\bar{\rho}_\lambda \bar{\sigma}_\gamma^{-1} a, \bar{\rho}_{-\lambda}\bar{\sigma}_\gamma^{-1} a$ are preserving at q and a is preserving at every $g \in G - \{q\}$; (5.4.3.1)

a is preserving and $\bar{\rho}_{-\lambda}\bar{\rho}_\lambda^{-1} a$ is preserving at q; (5.4.3.2)

a and $\bar{\rho}_{-\lambda}\bar{\rho}_\lambda^{-1} a$ are preserving. (5.4.3.3)

Proof. Left multiplying u by a suitable power of e we may assume that $\sigma_\gamma u = \sum a_i(qp)^i$, where $a_i = a_i(q) \in K[q]$. Hence

$$\bar{u} = \sum a_i(\bar{e})(\bar{h}/2)^i, \qquad \rho_{\pm\lambda} u = \sum a_i(qp + \tfrac{1}{2}(\pm\lambda + 1))^i.$$

If $g \in G - \{q\}$, then by (3.5.3) and (3.5.4), the indicial polynomials at g of $\sigma_\gamma u$, $\rho_\lambda u$ and $\rho_{-\lambda} u$ are the same. The last part of the lemma is a restatement of the first; we are regarding the automorphisms as maps of $\mathfrak{B}(\bar{K})$ and using the fact (Lemma 3.2) that the notion of preserving does not depend on the base field. Q.E.D.

THEOREM 5.4. *Suppose* $\gamma \in K$, $u \in U\mathfrak{s}$, $\bar{u}$ *is irreducible in* $\bar{S}^{-1}U\mathfrak{s}_\gamma$ (*equivalently*, $\sigma_\gamma u$ *is irreducible in* $\mathfrak{B}$) *and* u *satisfies* (5.4.2). *Then the* $U\mathfrak{s}$-*module* $M(u, \gamma)$ *is simple* (*and* $K[e]$-*torsionfree, with* $c_{M(u,\gamma)} = \gamma$).

Proof. We write $A = \sigma_\gamma U\mathfrak{s}$, $B = \mathfrak{B}$, $S = K[q] - \{0\}$. Thus $B = S^{-1}A$. We also write $a = \sigma_\gamma u$, and have $a = \sum a_i(qp)^i$, where $a_i = a_i(q) \in K[q, q^{-1}]$. We shall apply Lemma 2.3 with $L = Ba$. Let M be a simple S-torsion A-module. Then $M' = M_{\sigma(\gamma)}$ is simple $K[e]$-torsion and $c_{M'} = \gamma$. Hence $M' \cong V(g)_{\sigma(\gamma)} \cong U\mathfrak{s}_{\gamma,g}$ or $L(d)$ as in Proposition 5.3. Suppose first that $M' \cong V(g)_{\sigma(\gamma)}$ (in particular $g \neq q$). Since a is preserving relative to g, Lemma 3.2 implies that a is also preserving at α, where α is a root of g (in $\bar{K}$). Let $K' = K(\alpha)$ and apply Lemma 4.2 (with $\mathfrak{B}(K')$, $\mathfrak{A}(K')$, $R' = K'[q]$, $P' = (q - \alpha)$ in place of B, A, R, P), recalling the canonical identifications $K_g = K[q]/(g) = K' = K'[q]/(q - \alpha) = K_{P'}$ and $V(g) = K_g[p] = K_{P'}[p] = \mathfrak{A}(K') \otimes_{R'} K_{P'}$. Note that $g \in P' - P'^2$, and $v_g a = v_{q-\alpha} a$ since g is separable. Write $va = v_g a$. By Lemma 3.5, $va_i \geqslant va$ for all i. Therefore $g^{-va} a_i \in K[q, q^{-1}]$ for all i. Let k be the smallest element of $\mathbf{N}$ such that $q^k g^{-va} a_i \in K[q]$ for all i. With $(q^k g^{-va} a_i)(e)$ denoting the element of $K[e]$ obtained by substituting e for q, we have

$$q^k g^{-va} a = \sigma_\gamma \sum_i (q^k g^{-va} a_i)(e)(h/2)^i \in A \cap \mathfrak{B}a.$$

By Corollary 4.2, $g^{-va} a\,|_{V(g)}$ is injective. Also $q\,|_{V(g)}$ is injective since $q \neq g$. Hence $q^k g^{-va} a\,|_M$ is injective. This gives the hypothesis of Lemma 2.3 in this case.

Next suppose $M' \cong L(d) = L(\lambda)$ as in Section 5.3, identify M' with $L(d)$, let v be a maximal vector (of $L(\lambda)$ as an $\mathfrak{sl}(2, K')$-module, $K' = K(\lambda)$), and write $\nu = \nu_q$. For all $k \in \mathbf{N}$ and $s = s(q) \in K[q]$ we have

$$s(e) \cdot f^k v = s(0) f^k v + \text{lower terms} = (\eta_q s) f^k v + \text{lower terms}.$$

Since $\sigma_\gamma u = \sigma_\gamma \sum a_i(e)(h/2)^i$, we have

$$u \equiv \sum a_i(e)(h/2)^i \bmod U\mathfrak{s}(c - \gamma).$$

Therefore

$$
\begin{aligned}
e^{-\nu a}u \cdot f^k v &= \sum_i e^{-\nu a} a_i(e)(h/2)^i \cdot f^k v \\
&= \sum_i \eta_q(q^{-\nu a} a_i) \left(\frac{\lambda - 1}{2} - k\right)^i f^k v + \text{lower terms} \\
&= Q_{q,a}\left(\frac{\lambda - 1}{2} - k\right) f^k v + \text{lower terms} \qquad (5.4.4)
\end{aligned}
$$

by (3.5.2). We have

$$\rho_\lambda u = \sum a_i(q)(qp + \tfrac{1}{2}(\lambda + 1))^i,$$

$\nu a = \nu\rho_\lambda u$ by (3.5.1), and

$$
\begin{aligned}
Q_{q,\rho(\lambda)u}(\xi) &= \sum \eta_q(q^{-\nu a} a_i)(\xi + \tfrac{1}{2}(\lambda + 1))^i \\
&= Q_{q,a}(\xi + \tfrac{1}{2}(\lambda + 1))
\end{aligned}
$$

by (3.5.2). Therefore, for every $k \in \mathbf{N}$,

$$Q_{q,a}(\tfrac{1}{2}(\lambda - 1) - k) = Q_{q,\rho(\lambda)u}(-1 - k) \neq 0$$

since $\rho_\lambda u$ is preserving relative to q. It follows from (5.4.4) that $e^{-\nu a}u\,|_{M'}$ is injective. Therefore $q^{-\nu a}a\,|_M$ is injective. Also $q^{-\nu a}a \in A \cap Ba$.

By Lemma 2.3, $A/A \cap Ba$ is a simple A-module. The result then follows from (5.4.1). Q.E.D.

The above proof does not really involve $\rho_\lambda b$ and $\rho_{-\lambda} b$ except as a notational device in discussing the indicial roots of $\sigma_\gamma b = a$ itself. Also if $\xi^2 - \gamma - 1$ has a root in K, then the hypothesis that both $\rho_\lambda b$ and $\rho_{-\lambda} b$ are preserving is used, while if $\xi^2 - \gamma - 1$ is irreducible then $L(\lambda) \cong L(-\lambda)$ as $U\mathfrak{s}$-modules, the roots of $Q_{q,\rho(\lambda)b}$ are sent into those of $Q_{q,\rho(-\lambda)b}$ under the automorphism of K' over K sending λ to $-\lambda$, and $\rho_\lambda b$ is preserving relative to q if and only if $\rho_{-\lambda} b$ is.

5.5. We can now state our main result on $U\mathfrak{s}$-modules. If M is a $U\mathfrak{s}$-module with $c_M = \gamma \in K$ and if $A = \sigma_\gamma U\mathfrak{s}$, then M may be regarded, uniquely, as an A-module ${}_AM$ such that $({}_AM)_{\sigma(\gamma)} = M$, and similarly with ρ_λ in place of σ_γ where $\lambda \in K$ and $\lambda^2 = \gamma + 1$.

THEOREM 5.5. *Suppose $\gamma \in K$, and write $A = \sigma_\gamma U\mathfrak{s}$. Suppose N is a simple $\mathfrak{B}$-module, $0 \neq n \in N$, and let b be a minimal annihilator of n. Then* *there is an s, which may be calculated as in Lemma* 3.4, *such that $s \in S$ and bs^{-1} satisfies* (5.4.3); *for any such s, Asn ($\cong (As + \mathfrak{B}b)/\mathfrak{B}b$) is a simple A-module (in particular $sn \in \mathrm{Soc}_A N = Asn$), and so the pullback of Asn along σ_γ is a simple $U\mathfrak{s}$-module. If M is any $K[e]$-torsionfree simple $U\mathfrak{s}$-module with $c_M = \gamma$, then M arises in this way, and from an N which is unique up to isomorphism; that is*

$$U\mathfrak{s}\hat{\ }(K[e]\text{-}torsionfree, c_M = \gamma) \to \mathfrak{B}\hat{\ }, \qquad [M] \mapsto [S^{-1}{}_A M]$$

is a bijection, with inverse $[N] \mapsto [\mathrm{Soc}_A N]_{\sigma(\gamma)}$. If $\lambda \in K$ and $\lambda^2 = \gamma + 1$ then all the above holds when σ_γ is replaced by ρ_λ.

Proof. Pick $\lambda \in \bar{K}$ such that $\lambda^2 = \gamma + 1$, and set $b^+ = \bar{\rho}_\lambda \bar{\sigma}_\gamma^{-1} b$ and $b^- = \bar{\rho}_{-\lambda} \bar{\sigma}_\gamma^{-1} b$ (hence $b^+, b^- \in \mathfrak{B}(\bar{K})$). By Lemma 3.4 there exists $s_1 \in \bar{K}[q] - \{0\}$ such that $b^+ s_1^{-1}$ and $b^- s_1^{-1}$ are both preserving. Pick $s \in S$ such that $s_1 \mid s$ in $\bar{K}[q]$. It follows from Lemmas 3.3.2 and 3.2 that $b^+ s^{-1}$ and $b^- s^{-1}$ are also preserving. There exists $s_2 \in S$ such that $s_2 b s^{-1} = a \in A$, with say $a = \sigma_\gamma u$ where $u \in U\mathfrak{s}$. Then $\rho_{\pm\lambda} u = \bar{\rho}_{\pm\lambda} \bar{\sigma}_\gamma^{-1} a = s_2 b^{\pm} s^{-1}$ is preserving, and so $A/A \cap \mathfrak{B}a = A/A \cap \mathfrak{B}\sigma_\gamma u$ is simple by Theorem 5.4. But a is a minimal annihilator of sn and so $Asn \cong A/A \cap \mathfrak{B}a$ is simple. It now follows from Lemma 2.2.1 that the map

$$A\hat{\ }(S\text{-torsionfree}) \to \mathfrak{B}\hat{\ }, \qquad [M] \mapsto [S^{-1}M]$$

is a bijection. This gives the bijection of the theorem since $A\hat{\ }(S$-torsionfree) can be identified (via pullback) with $U\mathfrak{s}\hat{\ }(S$-torsionfree, $c_M = \gamma)$.

Given the proviso on λ, the same proof works when σ_γ is replaced by ρ_λ.

Q.E.D.

COROLLARY 5.5. *The simple $U\mathfrak{s}$-modules on which c acts as a scalar are, up to isomorphism, the following: the simple $K[e]$-torsion modules, given in Proposition* 5.3, *and the modules $M(u, \gamma)$, where $\gamma \in K$ and $u \in U\mathfrak{s}$ satisfies the hypotheses of Theorem* 5.4. *Two of the latter modules, say $M(u_1, \gamma_1)$, $M(u_2, \gamma_2)$ (with u_1, u_2 satisfying the hypotheses of Theorem* 5.4) *are isomorphic if and only if $\gamma_1 = \gamma_2$ and $\sigma(\gamma_1) u_1$ is similar to $\sigma(\gamma_2) u_2$, or equivalently—provided $\lambda \in K$ and $\lambda^2 = \gamma_1 + 1 = \gamma_2 + 1$—$\rho_\lambda u_1$ is similar to $\rho_\lambda u_2$.*

Proof. Only the last sentence remains to be proved. If $M(u_1, \gamma_1) \cong M(u_2, \gamma_2)$ then clearly $\gamma_1 = \gamma_2$. Write $A = \sigma_\gamma U\mathfrak{s}$, and suppose $\gamma_1 = \gamma_2 = \gamma$. We apply Lemma 2.4.2 to ${}_A M(u_i, \gamma) \cong A/A \cap \mathfrak{B}\sigma_\gamma u_i$ (by 5.4.1), getting ${}_A M(u_i, \gamma) \cong \mathrm{Soc}_A \mathfrak{B}/\mathfrak{B}\sigma_\gamma u_i$; then $\sigma_\gamma u_1$ and $\sigma_\gamma u_2$ are similar if and only if $\mathfrak{B}/\mathfrak{B}\sigma_\gamma u_1 \cong \mathfrak{B}/\mathfrak{B}\sigma_\gamma u_2$ if and only if $\mathrm{Soc}_A \mathfrak{B}/\mathfrak{B}\sigma_\gamma u_1 \cong \mathrm{Soc}_A \mathfrak{B}/\mathfrak{B}\sigma_\gamma u_2$ if and only if ${}_A M(u_1, \gamma) \cong {}_A M(u_2, \gamma)$ if and only if $M(u_1, \gamma) \cong M(u_2, \gamma)$. Given the proviso on λ, the same proof works with ρ_λ in place of σ_γ. Q.E.D.

6. The Simple Modules over $\mathfrak{b}$

6.1. In this chapter we apply Sections 4.1, 4.4 to obtain the classification of the simple $U\mathfrak{b}$-modules and relate them to simple $\mathfrak{A}$-modules and $U\mathfrak{s}$-modules. Here $\mathfrak{b}$ is the two-dimensional nonabelian Lie algebra over K, a field of characteristic 0. We take a basis h, e of $\mathfrak{b}$ with $[h, e] = 2e$, so that $\mathfrak{b}$ is realized as a Borel subalgebra of $\mathfrak{s} = \mathfrak{sl}(2, K)$. For any $\gamma \in K$, the restriction to $U\mathfrak{b}$ of the map σ_γ of Section 5.2 can be regarded as a homomorphism of $U\mathfrak{b}$ to $\mathfrak{A}$ (or to $\mathfrak{B}$). We denote this map by ρ; thus $\rho e = q$ and $\rho h = 2qp$. The set $\{q^i(2qp)^j \mid i, j \geqslant 0\}$ being linearly independent, ρ is injective.

Take $R = K[q]$ and $\partial = q(d/dq)$, and identify $A = R[x]_\partial$ with $K[q, qp] \subseteq \mathfrak{A}$ by identifying x with qp. Thus $\rho U\mathfrak{b} = A$. Also $B = S^{-1}A = \mathfrak{B}$ since every element in $\mathfrak{B}$ is of the form $\sum a_i x^i$, $a_i \in K(q)$. The primes P are all principal, $P = (g)$ with monic irreducible $g \in K[q]$. The only ∂-invariant prime is (q).

We now analyze the simple $K[e]$-torsion $U\mathfrak{b}$-modules, or equivalently, the simple S-torsion A-modules. Suppose $g \in K[q]$ is monic irreducible. Recall that $K_g = K(\alpha)$ where $\alpha = \eta_g q$ is a root of g. If $g \neq q$ then $V((g)) = A \otimes_{K[q]} K_g$ is a simple A-module. In order to avoid confusion with the $\mathfrak{A}$-module $V(g) = \mathfrak{A} \otimes_{K[q]} K_g \cong \mathfrak{A}/\mathfrak{A}g$ (defined using a different ∂), we shall not use the notation $V(g)$ for A-modules, but rather (using Corollary 4.1) write $A \otimes_{K[q]} K_g = A/Ag$. By pullback along ρ we obtain a simple $U\mathfrak{b}$-module $(A/Ag)_\rho$, which is identified with $U\mathfrak{b}/U\mathfrak{b}g(e)$. This module may also be identified with the space $K_g[x]$ of polynomials over K_g, identifying $x^i \otimes \omega \in A \otimes_{K[q]} K_g$ with ωx^i ($\omega \in K_g$). Since $\partial^i q = q$ for all i, it follows from (4.1.2) that the action of $U\mathfrak{b}$ on $K_g[x]$ is given by

$$h \cdot \omega x^j = 2\omega x^{j+1}, \qquad e \cdot \omega x^j = \sum_{i=0}^{j} \binom{j}{i} (-1)^i \alpha\omega x^{j-i},$$

that is, h acts as multiplication by $2x$ and e acts as

$$\alpha \sum_{i=0}^{\infty} (-1)^i \frac{1}{i!} \left(\frac{d}{dx}\right)^i = \alpha \exp(-d/dx).$$

If $q = g$ then $\alpha = 0$, K_q is identified with K, $\bar{\partial} K_q = 0$, and $K_q[x]_{\bar{\partial}} = K[x]$ (commutative). Thus similarity in $K_q[x]_{\bar{\partial}}$ is the same as associateness in $K[x]$, and the monic irreducible polynomials form a set of representatives of the similarity classes. If $d \in K[x]$ is monic irreducible, then $V((q), [d])_\rho = K[x]/(d)$ with e acting as 0 and h acting as $2\bar{x}$, where $\bar{x}$ is the coset of x modulo d; that is, $M = V((q), [d])_\rho$ is a finite extension field $K(\beta)$ of K with $eM = 0$ and $h_M = 2\beta$.

We now translate Proposition 4.1 to this special case.

Proposition 6.1. *With the above notation, the $U\mathfrak{b}$-modules $U\mathfrak{b}/U\mathfrak{b}g(e)$ ($g \in K[q]$ monic irreducible, $g \neq q$) and $K[x]/(d)$ ($d \in K[x]$ monic irreducible) are simple (and $K[e]$-torsion). Conversely, any simple $K[e]$-torsion $U\mathfrak{b}$-module is isomorphic to exactly one of these.*

6.2. With $\partial = q(d/dq)$, (q) is the unique ∂-invariant prime of $R = K[q]$, $K_{(q)}$ $(= K[q]/(q)) = K$, and (4.3) becomes

$$b = \sum a_j x^j, \qquad a_i \in K(q), \qquad a_0 \neq 0, a_j/a_0 \in qK[q]_{(q)} \qquad \text{for all } j > 0. \tag{6.2.1}$$

We note that (6.2.1) is equivalent to each of the following (where the product of sets denotes the set of products, not sums of products):

$$sb \in S(qA+1) \text{ for every } s \in S \text{ such that } sb \in A; \tag{6.2.2}$$

$$b \in (K(q) - \{0\})(qA+1). \tag{6.2.3}$$

Indeed suppose (6.2.1) holds and $sb \in A$. Then $a_j/a_0 = qs_j/t$ for all $j > 0$ where $s_j, t \in S$ and $q \nmid t$. Since $sa_0 \in K[q]$, we have $sa_0 = q^k(qa_0' + \beta)$ for some $k \in \mathbf{N}$, $a_0' \in K[q]$ and $0 \neq \beta \in K$. Then $t \mid s_j(qa_0' + \beta)$ and

$$sb = \beta q^k \left\{ q \left(\sum_{j>0} \beta^{-1} t^{-1} s_j (qa_0' + \beta) x^j \right) + \beta^{-1} q a_0' + 1 \right\},$$

giving (6.2.2). It is immediate that (6.2.2) implies (6.2.3) and (6.2.3) implies (6.2.1). By (3.5.2), (6.2.1) is also equivalent to

$$Q_{q,b} \in K \tag{6.2.4}$$

(where of course $Q_{q,b}$ is defined with respect to $\mathfrak{B} = K(q)[p]_{d/dq}$). In turn (6.2.4) is equivalent to

$$b = \sum b_j p^j, \text{ where } b_j \in K(q) \text{ and } b_j/b_0 \in q^{j+1}K[q]_{(q)} \text{ for all } j > 0. \tag{6.2.5}$$

We denote by (6.2) the equivalent conditions (6.2.1)–(6.2.5).

We remark that another proof of the necessity of the condition that every minimal annihilator satisfies (6.2) can be given based on the fact that if M is a simple A-module (where $A = \rho U\mathfrak{b}$), $m \in M$ and $qm \neq 0$ then $bqm = -m$ for some $b \in A$, $bq = qa$ for some $a \in A$, and $(qa+1)m = 0$.

We now translate Theorem 4.4 to the present case.

Theorem 6.2. *The $K[e]$-torsionfree simple modules over $U\mathfrak{b}$ (or over $\rho U\mathfrak{b}$) are given by the statement of Theorem 4.4 with the following changes:*

$A = \rho U\mathfrak{b}$ (with $R = K[q]$, $x = qp$ and $\partial = q(d/dq)$ and $B = \mathfrak{B}$ (and so (3.1.1) *holds automatically)), and* (4.3) *is replaced by* (6.2.5).

Proof. The only thing that needs to be noted in this specialization of Theorem 4.4 is that it makes no difference whether preserving of bs^{-1} is defined with respect to $\mathfrak{B} = K(q)[p]_{d/dq}$ or $\mathfrak{B} = K(q)[qp]_{q(d/dq)}$, which follows, in the presence of (6.2) for bs^{-1}, from Corollary 3.5 and (6.2.4).

Q.E.D.

COROLLARY 6.2. *Every simple $K[e]$-torsionfree $U\mathfrak{b}$-module is isomorphic to some $(\rho U\mathfrak{b}/\rho U\mathfrak{b} \cap \mathfrak{B}\rho a)_\rho$, where ρa is irreducible and preserving, and $a \in (eU\mathfrak{b} + 1)$. Two of these $U\mathfrak{b}$-modules are isomorphic if and only if the corresponding two ρa are similar.*

6.3. Given a simple $\mathfrak{B}$-module N we have $\mathrm{Soc}_{\rho U\mathfrak{b}} N \subseteq \mathrm{Soc}_{\mathfrak{A}} N$. It turns out that only the two extremes are possible: equality or $\mathrm{Soc}_{\rho U\mathfrak{b}} N = 0$.

PROPOSITION 6.3. *If M is a simple S-torsionfree $\mathfrak{A}$-module, then the $K[e]$-torsionfree $U\mathfrak{b}$-module M_ρ is simple if it contains a simple submodule. Moreover every simple $K[e]$-torsionfree $U\mathfrak{b}$-module arises in this way from a unique (up to isomorphism) such M.*

Proof. Write $A = \rho U\mathfrak{b}$ and $M' = \mathrm{Soc}_A M$. Thus $M' \subseteq M \subseteq S^{-1}M$. Suppose $M' \neq 0$. Then M' is simple, qM' is a submodule of M' and hence $qM' = M'$. Therefore $q^{-1}M' \subseteq M'$ and so $pM' = q^{-1}qpM' \subseteq M'$, M' is $\mathfrak{A}$-invariant, and $M' = M$. The first statement follows from this. For the second statement, if M_1 is a simple S-torsionfree A-module, then $M_1 = \mathrm{Soc}_A S^{-1}M_1 = \mathrm{Soc}_{\mathfrak{A}} S^{-1}M_1$. The argument above shows that the action of p, hence of $\mathfrak{A}$, on M_1 is uniquely determined. Q.E.D.

6.4. We now consider simple $U\mathfrak{s}$-modules on which c acts as a scalar, and determine which of these contain a simple $U\mathfrak{b}$-submodule. We exclude consideration of the highest weight modules, that is, those for which e has a null vector, as the situation for these is obvious.

THEOREM 6.4. *Suppose M is a simple e-torsionfree $U\mathfrak{s}$-module and $c_M \in K$. The following conditions are equivalent*:

$$\mathrm{Soc}_{U\mathfrak{b}} M \neq 0; \tag{6.4.1}$$

$$\mathrm{Soc}_{U\mathfrak{b}} M = M; \tag{6.4.2}$$

$$\exists\, m \in M - \{0\} \text{ and } \exists\, u \in U\mathfrak{b} \text{ such that } (eu + 1)m = 0; \tag{6.4.3}$$

$$\forall\, m \in M - \{0\},\ \exists\, u \in U\mathfrak{b} \text{ such that } (eu + 1)m = 0. \tag{6.4.4}$$

Moreover, for each $\gamma \in K$ and simple e-torsionfree $U\mathfrak{b}$-module N, there exists a unique (simple) $U\mathfrak{s}$-module M with ${}_{U\mathfrak{b}}M = N$ and $c_M = \gamma$.

Proof. Let N be a simple e-torsionfree $U\mathfrak{b}$-submodule. Then eN is $U\mathfrak{b}$-invariant (since $he = e(h+2)$) and hence $eN = N$. Given $\gamma \in K$, define

$$f(em) = (1/4)(\gamma - h^2 - 2h)m \qquad (m \in N); \tag{6.4.5}$$

this is well defined since N is e-torsionfree. Then straightforward computations, using $he = e(h+2)$, show that $hf(em) - fh(em) = -2f(em)$ and $ef(em) - fe(em) = hem$. Hence N becomes a $U\mathfrak{s}$-module. Clearly $c_N = \gamma$, and this condition makes the above action of f uniquely determined. This proves the theorem's last statement. Also (6.4.1) implies (6.4.2), since if N is a simple $U\mathfrak{b}$-submodule of M and $c_M = \gamma$ then (6.4.5) holds and shows that N is $U\mathfrak{s}$-invariant. If M is $K[e]$-torsion then, by Proposition 5.3 and Lemma 5.3, (6.4.2) holds and $M \cong U\mathfrak{s}_{\gamma,g} \cong U\mathfrak{b} \otimes_{K[e]} K_g$ for some monic irreducible $g \in K[q]$, $g \neq q$. Then by (5.3.4) every $m \in M$ is annihilated by some power of $g(e)$; since powers of $g(e)$ have nonzero constant term, it follows that (6.4.4) holds in this case. Hence we may assume that M is $K[e]$-torsionfree.

Suppose $c_M = \gamma$. By Theorem 5.5, ${}_{\sigma(\gamma)U\mathfrak{s}}M$ is the $\sigma_\gamma U\mathfrak{s}$-socle of some simple $\mathfrak{B}$-module M'. Suppose (6.4.1) holds. Since $\rho = \sigma_\gamma|_{U\mathfrak{b}}$, $\mathrm{Soc}_{\rho U\mathfrak{b}} M' \neq 0$. Hence, by Theorem 6.2 and (6.2.2), for every element m of M' (and a fortiori for every element of ${}_{\sigma(\gamma)U\mathfrak{s}}M$) there is an $a \in \rho U\mathfrak{b}$ such that $(qa+1)m = 0$; with $a = \rho u$, it follows that (6.4.4) holds. Conversely, suppose (6.4.3) holds. Then $(qa+1)m' = 0$ for some $a \in \rho U\mathfrak{b}$ and $m' \in M' - \{0\}$. By (3.5.2), $Q_{q,qa+1} \in K$. We may choose a minimal annihilator b of m' with $b \in \rho U\mathfrak{b}$. Then $qa + 1 \in \mathfrak{B}b$, and, by Lemma 3.3.2, $Q_{q,b} \in K$. By (3.5.2) again, b has the form $q^k(qd+\beta)$ with $k \in \mathbf{N}$, $d \in \rho U\mathfrak{b}$, $0 \neq \beta \in K$. Thus we may pick b so that b has the form $qd+1$, $d \in \rho U\mathfrak{b}$. By Theorem 6.2, M' contains an essential simple $\rho U\mathfrak{b}$-submodule, which must be contained in $\mathrm{Soc}_{\sigma(\gamma)U\mathfrak{s}} M'$. Hence (6.4.1) holds. Q.E.D.

7. Examples

7.1. If N is a simple $\mathfrak{B}$-module, $0 \neq n \in N$ and b is a minimal annihilator of n then $n, pn, \ldots, p^{k-1}n$ form a basis of N over $K(q)$, where $k = \deg b$ (that is, the order of the differential operator b). We define the *degree* $\deg N$ of N to be the dimension of N over $K(q)$. Thus $\deg N$ equals the degree of any minimal annihilator. Now suppose A is a subring of $\mathfrak{B}$ containing $S = K[q] - \{0\}$ and such that $\mathfrak{B} = S^{-1}A$. If M is a simple A-module we define the *degree* $\deg M$ of M to be 0 if M is S-torsion and to be the degree $\deg S^{-1}M$ of the simple B-module $S^{-1}M$ if M is S-torsionfree.

Suppose $0 \neq m \in M$ and $0 \neq a \in A$ is of smallest degree such that $am = 0$. If M is S-torsion then $a \in S$ and $0 = \deg a = \deg M$; if M is S-torsionfree then a is a minimal annihilator of m as an element of $S^{-1}M$, a is irreducible, and $\deg a = \deg M$.

If $\gamma \in K$, the notion of degree may be transferred from $\mathfrak{B}$ to $\bar{S}^{-1}U\mathfrak{s}_\gamma$ by means of $\bar{\sigma}_\gamma$, or equivalently, if $\lambda \in K$ and $\lambda^2 - 1 = \gamma$, by means of $\bar{\rho}_\lambda$. Thus if M is a simple $U\mathfrak{s}$-module and $c_M = \gamma$, we may define $\deg M$ as $\deg_{\sigma(\gamma)U\mathfrak{s}} M$ ($= \deg_{\rho(\lambda)U\mathfrak{s}} M$ if $\lambda \in K$ and $\lambda^2 - 1 = \gamma$).

We now analyze the degree one modules over the rings $\mathfrak{B}$, $\mathfrak{A}$, and $\rho_\lambda U\mathfrak{s}$ ($\lambda \in K$), assuming for convenience that K is algebraically closed. In the degree one case it is easy to describe not only irreducible elements but also their similarity classes. Suppose N is a simple $\mathfrak{B}$-module of degree one and $0 \neq n \in N$. Then there exists $t \in K(q)$ such that $(p - t)n = 0$, the isomorphism class $[N]$ corresponds to the similarity class $[p - t]$, and we may identify N with $K(q)$ (identifying n with 1), where elements of $K(q)$ act by multiplication and p acts as $t + d/dq$. We have $K(q)n = N$, and changing n to

$$n \prod_l (q - \alpha_l)^{j_l}$$

changes t to $t + \sum_l j_l (q - \alpha_l)^{-1}$. In particular, it follows that $p - t$ and $p - t_1$ are similar if and only if $t - t_1$ is a logarithmic derivative of an element of $K(q) - \{0\}$ ([1, 24] contain different proofs of this classical fact). For $\alpha \in K$ we shall make use of the residue $\operatorname{Res}_\alpha t$ at α, that is, the coefficient of $(q - \alpha)^{-1}$ in the partial fraction decomposition of t. Then the above change in t adds j_l to the residue at α_l. Therefore we may choose $0 \neq n \in N$ so that

$$\alpha \in K, v_{q-\alpha} t = -1 \Rightarrow \operatorname{Res}_\alpha t \notin \mathbf{Z} \tag{7.1.1}$$

and, simultaneously, if $\lambda \in K$ is given,

$$v_q t = -1 \Rightarrow \lambda + \operatorname{Res}_0 t \notin \mathbf{Z} - \{0\}. \tag{7.1.2}$$

THEOREM 7.1. *Suppose N is a simple $\mathfrak{B}$-module of degree one and $\lambda \in K$. Identify N as above with $K(q)$, p acting as $t + d/dq$, where $t \in K(q)$ is normalized so that* (7.1.1) *and* (7.1.2) *hold. Then*

$$\operatorname{Soc}_{\mathfrak{A}} N = K[q, (q - \alpha_1)^{-1}, \ldots, (q - \alpha_l)^{-1}],$$

where $\{\alpha_1, \ldots, \alpha_l\} = \{\alpha \in K \mid v_{q-\alpha} t < 0\}$. *Moreover* $\operatorname{Soc}_{\rho(\lambda)U\mathfrak{s}} N = \operatorname{Soc}_{\mathfrak{A}} N$ *unless one of*

$$v_q t \geqslant 0 \qquad \textit{and} \qquad \lambda \in \mathbf{Z}^-, \tag{7.1.3}$$

$$v_q t = -1 \qquad \textit{and} \qquad \operatorname{Res}_0 t = -\lambda \tag{7.1.4}$$

holds. If (7.1.3) *holds then* $\mathrm{Soc}_{\rho(\lambda)U\mathfrak{s}} N = q^{-\lambda}\,\mathrm{Soc}_{\mathfrak{A}} N$. *If* (7.1.4) *holds* (*so that* $\alpha_i = 0$ *for some* i, *say* $i = l$), *then*

$$\mathrm{Soc}_{\rho(\lambda)U\mathfrak{s}} N = K[q, (q-\alpha_1)^{-1}, \ldots, (q-\alpha_{l-1})^{-1}]. \tag{7.1.5}$$

Proof. Write $K[q, (q-\alpha_1)^{-1}, \ldots, (q-\alpha_l)^{-1}] = M_t$. Pick $k \in \mathbf{N}$, $i \in \{1, \ldots, l\}$, and write $a = p - t + k(q-\alpha_i)^{-1}(= p - t$ if $l = 0)$. Then a is preserving; it suffices to prove this relative to $q - \alpha$, where $\alpha = \alpha_j$ with $1 \leqslant j \leqslant l$. Denoting $v_{q-\alpha}$ by v, we have vt, $va \leqslant -1$. If $va < -1$ then $Q_{q-\alpha,a} \in K$, while if $va = -1$ then $vt = -1$ and $Q_{q-\alpha,a} = \xi - \mathrm{Res}_\alpha t + k\delta_{ij}$, which has no root in $\mathbf{Z}^-$ by (7.1.1). Since $a \cdot (q-\alpha_i)^{-k} = 0$ and a is irreducible and preserving, it follows from Theorem 4.4 that $(q-\alpha_i)^{-k} \in \mathrm{Soc}_{\mathfrak{A}} N$. Hence, using partial fraction decompositions, we have $M_t \subseteq \mathrm{Soc}_{\mathfrak{A}} N$. But also $\mathrm{Soc}_{\mathfrak{A}} N = \mathfrak{A} \cdot 1 \subseteq M_t$, giving the result for $\mathfrak{A}$.

Next, in order to check (5.4.3), we consider $d = \bar{\rho}_{-\lambda}\bar{\rho}_\lambda^{-1} a$, which equals $a - \lambda q^{-1}$ by Lemma 5.4.1. Since a is preserving, d is preserving if and only if it is preserving relative to q. Write $v = v_q$. If $vt \geqslant 0$ then $q \neq q - \alpha_i$, $Q_{q,d} = \xi - \lambda$, and d is preserving unless $\lambda \in \mathbf{Z}^-$. If $vt < 0$ then, say, $\alpha_l = 0$; again if $vt < -1$ then $Q_{q,d} \in K$ while if $vt = -1$ then $Q_{q,d} = \xi - \mathrm{Res}_0 t + k\delta_{il} - \lambda$, and, by (7.1.2), d is preserving unless $\lambda = -\mathrm{Res}_0 t$ and $i = l$. Therefore, by the same reasoning as for $\mathfrak{A}$, using Theorem 5.5 in place of Theorem 4.4, $M_t = \mathrm{Soc}_{\rho(\lambda)U\mathfrak{s}} N$ unless either (7.1.3) or (7.1.4) holds.

Suppose (7.1.3) holds and replace $p - t$ by $p - t + \lambda q^{-1}$, which is a minimal annihilator of $q^{-\lambda}$. With $b = a + \lambda q^{-1}$, we have $Q_{q,b} = \xi + \lambda$, and so b and $b - \lambda q^{-1} = a$ are preserving. It follows (as before, but with $q^{-\lambda}$ in place of 1) that $\mathrm{Soc}_{\rho(\lambda)U\mathfrak{s}} N = M_t q^{-\lambda}$ in this case.

Finally, suppose (7.1.4) holds, $\alpha_l = 0$, and let M'_t denote the right-hand side of (7.1.5). We observed above that a and $d = a - \lambda q^{-1}$ are both preserving in this case unless $i = l$, which implies that $(q-\alpha_i)^{-k} \in \mathrm{Soc}_{\rho(\lambda)U\mathfrak{s}} N$ if $i \neq l$, and hence $M'_t \subseteq \mathrm{Soc}_{\rho(\lambda)U\mathfrak{s}} N$. The space M'_t is clearly invariant under $\rho_\lambda e$ and $\rho_\lambda h$. If $u \in M'_t$ then

$$\begin{aligned}\rho_\lambda f \cdot u &= -\left(q\frac{d}{dq} + qt + \lambda + 1\right)\left(\frac{d}{dq} + t\right) u \\ &\equiv -\lambda q^{-1}u - \lambda^2 q^{-1}u + (\lambda + 1)\lambda q^{-1}u \equiv 0 \qquad \text{modulo } M'_t.\end{aligned}$$

Therefore M'_t is invariant under $\rho_\lambda U\mathfrak{s}$ so equals $\mathrm{Soc}_{\rho(\lambda)U\mathfrak{s}} N$. Q.E.D.

7.2. For $\mathfrak{s} = \mathfrak{sl}(2, \mathbf{C})$, the subalgebra $\mathbf{C}h$ is a symmetric algebra, and the simple Harish–Chandra modules are those for which h has an eigenvector or, equivalently, for which h acts diagonally. These modules are included among the degree zero and degree one modules over $\mathfrak{s}$. Indeed at degree zero they comprise the highest weight modules (the other degree zero modules being the Whittaker modules, where h has no eigenvector). The degree one Harish–Chandra modules are those for which h, but not e, has an eigenvector. The condition that $\rho_\lambda h$ has an eigenvector in N (say for eigenvalue β) but $\rho_\lambda e$ has no eigenvector is equivalent to $2qp + \lambda + 1 - \beta \in \min \operatorname{ann} N$, that is, $p - t \in \min \operatorname{ann} N$, where $t = (\beta - \lambda - 1)/2q$. Thus the simple degree one Harish–Chandra modules with $c_M = \gamma$ comprise the modules of Theorem 7.1, pulled back along ρ_λ, where we choose a particular root λ of $\xi^2 = \gamma + 1$, for which $t \in \mathbf{C}q^{-1}$. Two such modules (for the same λ) are isomorphic if and only if the corresponding t differ by an element of $\mathbf{Z}q^{-1}$.

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