Qualifying Examination: Complex Analysis

26 September 2023

Instructions: Answer any 8—and only 8—of the following 9 problems. Circle the problem numbers to be graded. Provide as many details as you can.

Use only the paper provided. Number and initial each page. **Do not staple.**

(1) Determine all entire functions $f(z)$ with the properties that $|f(z)| \leq |e^z|$ whenever $|z| > 1$.

(2) Let a, b, and c be any three distinct points on the complex plane.

(a) Choose a domain of definition for

$$f(z) = \ln\left(\frac{z-a}{z-b}\right)$$

such that it is analytic on an open set containing the point c.

(b) Determine the radius of convergence for the power series representation of $f(z)$ centered at the point c.

(Hint: Your choice of domain depends on the relative positions of the three points. Please exhaust all possibilities.)

(3) Let γ be a rectifiable curve, and φ a function continuous on $\{\gamma\}$. Define a function $f(z)$ for $z \in \mathbb{C} \setminus \{\gamma\}$ with

$$f(z) = \int_\gamma \frac{\varphi(w)}{w-z}\, dw.$$

Prove that $f(z)$ is a continuous function on $\mathbb{C} \setminus \{\gamma\}$.

(4) Evaluate

$$\oint_C \frac{e^{\pi z}}{z(z+2i)}\, dz,$$

where C is the circle of radius 3 centered at the origin and oriented counterclockwise.

(5) Let $f\colon D \to D$ be analytic with $f(0) = f'(0) = 0$. Show that $|f''(0)| \leq 2$ and that if $|f''(0)| = 2$ then $f(z) = cz^2$ for some $|c| = 1$.

(6) Show that a non-constant harmonic function on a domain (open connected subset) in $\mathbb{R}^2$ is an open map.

(7) Let X = a compact Riemann surface. Let $\{p_1, p_2, \ldots, p_n\}$ be n distinct points on X. Prove that there exists no non-constant bounded holomorphic function on $X \setminus \{p_1, p_2, \ldots, p_n\}$.

(8) Prove that any compact Riemann surface can be represented as a branched cover over the Riemann sphere. Find a formula for the total ramification index in terms of the number of sheets of the cover and the genus of the Riemann surface

(9) Let

$$\begin{aligned} X &= \text{ a compact Riemann surface,} \\ \mathbb{C} &= \text{ the complex number field.} \end{aligned}$$

Prove that $H_X^{1,1} \cong \mathbb{C}$, where

$$\begin{aligned} H_X^{1,1} &= \Omega^{1,1}/\overline{\partial}(\Omega^{1,0}), \\ \Omega^{1,1} &= \text{smooth}(1,1) \text{ forms on } X, \\ \Omega^{1,0} &= \text{smooth}(1,0) \text{ forms on } X. \end{aligned}$$

Qualifying Examination: Complex Analysis

21 September 2022

Instructions: Answer any 8—and only 8—of the following 9 problems. Circle the problem numbers to be graded. Provide as many details as you can.

Use only the paper provided. Number and initial each page. **Do not staple.**

(1) Count the number of zeros of the equation $z^4+3z^3+6=0$ inside the circle $\{|z|=2\}$. Give reasons and show detail of your computation.

(2) State and prove the Maximum Modulus Theorem for analytic functions.

(3) Let $f(z)$ be an entire function such that $|f(z)| < M|z|^5$ for all $|z| > R$, where M and R are two positive constants. Prove that $f(z)$ is a polynomial of degree not greater than 5.

(4) Prove that other than the identity function, no analytic function from D to D can have more than one fixed point. That is, suppose that $f\colon D \to D$ is analytic with $f(a) = a$ and $f(b) = b$ for some distinct $a, b \in D$. Show that $f(z) = z$ for all $z \in D$.

(5) Let (f_n) be a sequence of injective analytic functions on the domain Ω which converges to f in $H(\Omega)$, the space of analytic functions on Ω. Prove that either f is injective or f is a constant function.

(6) For what values of $p > 0$ does the infinite product $\prod_{n=1}^{\infty} \frac{1}{n^p}$ converge? For what values of $p > 0$ does it converge absolutely?

(7) Let X be a compact Riemann surface, and $f : X \to \mathbb{C} \cup \infty$ be a meromorphic function such that $f|_{f^{-1}(\mathbb{C})}$ is a local biholomorphism, and f has 3 double poles, 4 simple poles, and no other poles.
(a) Compute the number of zeros of f and the multiplicity of each zero.
(b) Compute the genus of X

(8) Let X be a compact Riemann surface of genus g. For $p \in X$, let $\mathcal{M}_p$ denote the vector space of functions $f : X \to \mathbb{C}$ which are holomorphic on $X \setminus \{p\}$ and have at worst a simple pole at p.
(a) For $g = 0$, prove $\dim(\mathcal{M}_p) > 1$.
(b) For $g > 0$, prove $f \in \mathcal{M}_p$ implies f is constant.
(c) For $g > 0$, compute the dimension of the vector space of holomorphic 1-forms on X which vanish at p.

(9) Let X be a compact Riemann surface of genus 2. Recall that the the fundamental group of X is

$$\pi_1(X) \cong \{a_1, b_1, a_2, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = 1\},$$

and that $\pi_1(X)$ acts freely on its universal covering.
(a) Carfully state the uniformization theorem.
(b) Determine the universal cover of X.

Qualifying Examination: Complex Analysis
September 20, 2021

Instruction: Answer any eight questions from the following nine problems. Circle the problem numbers to be graded. Provide as many details as you can.

Question 1. State and prove the Liouville's Theorem.

Question 2. Find the Laurent series of $f(z) = \frac{1}{z(z-1)(z-2)}$ about the origin in the annulus $Ann(0;1,2) = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

Question 3. Consider each of the following three domains: (a) the annulus $Ann(0;1,2) = \{z \in \mathbb{C} : 1 < |z| < 2\}$; (b) the complex plane $\mathbb{C}$; and (c) the region $S = \mathbb{C} \backslash \{z \in \mathbb{C} : \operatorname{Im} z = 0, \operatorname{Re} z > 1\}$. Determine whether it is possible to find a one-to-one analytic function mapping the given domains onto the unit disc $B(0;1) = \{z \in \mathbb{C} : |z| < 1\}$. If it is not possible, provide reasons. If it is possible, produce the function.

Question 4. Let G be a region and $a \in G$. Suppose that f is continuous on G and analytic on $G \backslash \{a\}$. Provide that f is analytic at a.

Question 5. Let G be a bounded region. Suppose that f is continuous on $\overline{G}$ and analytic on G. Show that if there exitsts a real constant $C \geq 0$ such that $|f(z)| = C$ for all z on the boundary of G, then either f is a constant function or f has a zero in G.

Question 6. Let G be a region and u a non-constant harmonic function on G. Show that u is an open map from G to $\mathbb{R}$.

Question 7. (a) State the Riemann-Roch formula for a compact Riemann surface. (b) Use the Riemann-Roch formula to show that there exists a meromorphic function on a torus with simple poles at two distinct points.

Question 8. State and prove the Riemann-Hurwitz formula for a holomorphic mapping $f : \mathbb{X} \to \mathbb{Y}$, where $\mathbb{X}$ and $\mathbb{Y}$ are compact Riemann surfaces.

Question 9. Let $\mathbb{X}$ be a compact Riemann surface and $\Omega^{p,q}(\mathbb{X})$ the space of (p,q)-forms on $\mathbb{X}$ so that $\Omega^{0,0}(\mathbb{X})$ is the space of smooth functions. Define

$$H^{1,0} = \ker \overline{\partial} : \Omega^{1,0}(\mathbb{X}) \longrightarrow \Omega^{1,1}(\mathbb{X}),$$

$$H^{0,1} = \operatorname{coker} \overline{\partial} : \Omega^{0,0}(\mathbb{X}) \longrightarrow \Omega^{0,1}(\mathbb{X}), \quad \text{i.e.} \quad H^{0,1} = \frac{\Omega^{0,1}(\mathbb{X})}{\overline{\partial}(\Omega^{0,0}(\mathbb{X}))}.$$

Let $\sigma : H^{1,0} \to H^{0,1}$ be the map induced by the complex conjugation $\alpha \longmapsto \overline{\alpha}$ for $\alpha \in \Omega^{1,0}(\mathbb{X})$. Prove that the map σ is surjective.

—– END —–

Complex Analysis Qualifying Exam, 30 September 2020

Parts A and B

Answer all questions. Provide as many details as you can.

(1) Let G be a region (an open connected subset of $\mathbb{C}$) and suppose that f_n is analytic in G for each $n \geq 1$. Suppose that the sequence (f_n) converges uniformly to a function f on G. Show that f is analytic.

(2) Give Laurent series expansions of

$$f(z) = \frac{1}{z^2(1-z)}$$

in powers of z on two nonempty nonintersecting annuli, and specify the maximal region on which each expansion is valid.

(3) Let C be the circle $|z| = 3$, oriented counterclockwise. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\, ds \quad (|z| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

(4) Let a function f be analytic everywhere in a region (open connected subset of $\mathbb{C}$) D. Prove, using the Cauchy-Riemann equations, that if $f(z)$ is real-valued for all z in D then $f(z)$ must be constant throughout D.

(5) Prove that there exists no harmonic function u on $\Delta^* = \{0 < |z| < 1\}$ which is continuous on $\overline{\Delta^*} = \{|z| \leq 1\}$ such that $u \equiv 0$ on $\{|z| = 1\}$ and $u = 1$ at $z = 0$.

(6) If $f(z)$ is an entire function such that $z^{-1}\operatorname{Re}(f(z)) \to 0$ when $z \to \infty$ show that $f(z)$ is a constant function.

Part C

Answer all questions. Provide as many details as you can.

(7) Show that all meromorphic functions on the Riemann sphere have the form $\dfrac{p(z)}{q(z)}$, where $p(z)$ and $q(z)$ are coprime polynomials.

(8) Let $\{p_1, p_2, \ldots, p_{g+1}\}$ be $g+1$ distinct points on a compact Riemann surface Σ_g of genus g. Show that there is a non-constant meromporphic function on Σ_g with simple poles at some subset of $\{p_1, p_2, \ldots, p_{g+1}\}$.

(9) Let T^2 be a complex torus. Prove that there exists a two-sheeted cover $f\colon T^2 \to S^2$ with four branced points.

Complex Analysis – Qualifying Examination
September 2019

Part A and B

Answer all questions. Provide as much details as you could.

Problem 1 *Determine explicitly the largest disk about the origin where image under the mapping* $f(z) = z^2 - 2z$ *is one-to-one.Justify your answer.*

Problem 2 *Does there exist an analytic function* $f : D \to D$ *with* $f(\frac{1}{2}) = \frac{3}{4}$ *and* $f'(\frac{1}{2}) = \frac{2}{3}$*? Here* D *is the open unit disk.*

Problem 3 *Let* G *be a region and* $a \in G$*. Suppose that* f *is continuous on* G *and analytic on* $G \backslash \{a\}$*. Prove that* f *is analytic at* a*.*

Problem 4 *Given a Möbius transformation* $T(z) = \frac{az+b}{cz+d}$ *, determine necessary and sufficient conditions on* a, b, c, d *so that* T *map the domain* $D = \{z : \operatorname{Re} z > 0\}$ *onto* $G = \{z : \operatorname{Re} z < 0\}$*.*

Problem 5 *Prove Vitali's Theorem:* $H(G)$ *is the set of holomorphic functions on* G*. If* G *is a region and* $f_n \in H(G)$ *is locally bounded and* $f \in H(G)$ *has the property that* $A = \{z \in G : \lim_n f_n(z) = f(z)\}$ *has a limit point in* G*, then* f_n *converges in* f *in* $H(G)$*.*

Problem 6 *Let* G *be a region and* u *a non-constant harmonic functions on* G*. Show that* u *is an open map.*

Part C

Answer all questions. Provide as much details as you could.

Problem 7 *(a) Show that any compact Riemann surface admits a branched cover over the Riemann sphere* S^2*. (b) Let* M *be a compact Riemann surface of genus 5. Suppose* F *is a branched cover from* M *to* S^2*. Prove that* $R = 2(4+r)$*, where* R*=the total ramification index of* F*,* r*=number of sheets of the cover.*

Problem 8 *Prove that the Dolbeault cohomolgy group* $H^{1,1}(X) = \frac{\Omega^{1,1}}{\bar{\partial}\Omega^{1,0}}$ *is isomorphic to* $\mathbb{C}$ *for any compact Riemann surface* X*, and* $\mathbb{C}$*= complex number field.*

Problem 9 *Prove that the sum of residues of all the poles of a meromorphic one-form on a compact Riemann surface is equal to zero.*

END

Qualifying Examination - Complex Analysis (MATH 210ABC)

10:00am - 01:00pm, September 24, 2018

Instructions: Answer six (and only six) problems with a choice of two from each of part (A), part (B) and part (C). Circle the problems to be graded. Show your work to get credits.

Name: ______________________________

Part (A)

Problem 1. **(a)** Define a branch $f(z)$ of $\log\left(\frac{z-i}{z-1}\right)$ on a region containing 0. (You should specify the region and define the function explicitly.)

(b) Find the radius of convergence of the power series expansion of $f(z)$ centered at $z = 0$.

Problem 2. Let f be an analytic function on a region containing $\{z \in \mathbb{C} : |z| \leqslant 4\}$. Suppose that $|f(z)| \leqslant 1$ whenever $|z| \leqslant 4$. Prove that $|f(z_1) - f(z_2)| \leqslant \frac{8}{9}$ whenever $|z_1|, |z_2| \leqslant 1$.

Problem 3. Use the Residue Theorem to find

$$\int_0^{2\pi} \frac{1}{(2 + \sin\theta)^2}\, d\theta.$$

Part (B)

Problem 1. Let G be a region in $\mathbb{R}^2$. Suppose $u : G \longrightarrow \mathbb{R}$ is a continuous function which satisfies the Mean Value property. Prove that u is harmonic.

Problem 2. Show that there exists no one-to-one analytic map from $G = \{0 < |z| < 1\}$ onto the annulus $\Omega = \{r < |z| < R\}$, where $r > 0$.

Problem 3. State and prove the Hurwitz Theorem.

Part (C)

Problem 1. **(a)** State the uniformization theorem for Riemann surfaces.

(b) Prove that there exists no non-constant holomorphic map from the complex plane $\mathbb{C}$ to Σ_g, where $g \geqslant 2$ and Σ_g is a compact Riemann surface with genus g.

Problem 2. Prove that a topological 2-sphere S^2 admits only one complex structure.

Problem 3. **(a)** State the Riemann-Roch formula for Riemann surfaces.

(b) Prove that any non-trivial holomorphic one-form on Σ_g must vanish somewhere, where Σ_g is a compact Riemann surface of genus $g \geqslant 2$.

Qualifying Examination – Complex Analysis

10:00am-01:00pm, June 3, 2017

Instruction: Answer any seven questions from the following nine problems. Circle the problem numbers to be graded. Show your work to get credits.

1. Let $a > 0$. Find
$$\int_0^\infty \frac{\cos(ax)}{1+x^2}\,dx.$$

2. Is there a sequence of polynomials which converge uniformly to the function $1/z$ on $\{z : 1 < |z| < 2\}$? Justify your answer.

3. Let f be an analytic function on $B = \{z \in \mathbb{C} : |z| < 1\}$. Suppose that $\mathrm{Re}(f(z)) \neq 0$ whenever $0 < |z| < 1$. Prove that $\mathrm{Re}(f(0)) \neq 0$.

4. Find an explicit conformal transformation of an open set
$$U = \{z : |z| > 1\} \backslash (-\infty, -1]$$
to the unit disc.

5. Let $p(z)$ be a polynomial of degree $d \geq 2$, with distinct roots $a_1, \ldots, a_d$. Show that
$$\sum_{j=1}^{d} \frac{1}{p'(a_j)} = 0.$$

6. Let u be a positive harmonic function over the punctured complex plane $\mathbb{C}\backslash\{0\}$. Show that u must be a constant function.

7. Let $\mathcal{F} \subset H(G)$ be a locally bounded family, $H(G) = \{g : G \to \mathbb{C} \text{ analytic}\}$, G is a region in the complex plane $\mathbb{C}$. Prove that $\mathcal{F}$ is equicontinuous at each point of G. (Montel's Theorem is not allowed to apply).

8. Let $B = \{z \in \mathbb{C} : |z| < 1\}$ and $B^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Suppose that $u : B \to \mathbb{R}$ be a continuous function such that it is harmonic on B^*. Prove that there exists a hoarmonic function $\tilde{u} : B \to \mathbb{R}$ such that its restriction on B^* is equal to u.

9. Evaluate the integral
$$\int_{|z|=R} \frac{|dz|}{|z-a|^2},$$
where $a \in \mathbb{C}$, R is a positive real number, and $|a| < R$.

— END —

Qualifying Examination: Complex Analysis
10:00am-01:00pm, June 4, 2016

Instruction: Answer any eight questions from the following ten problems. Circle the problem numbers to be graded. Show your work.

Question 1. Denote by $B(a;R)$ the open disc with center a and radius R. Let $D = B(1;1)\backslash\overline{B(i;1)}$. Find a one-to-one conformal mapping from D onto $B(0;1)$.

Question 2. State and prove the Morera Theorem.

Question 3. Suppose that the radius of convergence of

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is R. (a) Prove that the radius of convergence of

$$g(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} a_n z^{n+1}$$

is equal to R. (b) On $B(0;R)$, $g'(z) = f(z)$.

Question 4. Evaluate the integral $\int_\gamma f$ where $\gamma(t) = e^{it}$, for $0 \leq t \leq 2\pi$ and $f(z) = \ln \frac{z-2i}{z+2i}$. Choose your definition of logarithm, and show your work.

Question 5. Let $\{f_n\}$ be a sequence of functions analytic on a region G. Suppose that this sequence converges to a function f uniformly. (a) Prove that f is analytic in G. (b) Let $f_n^{(k)}$ and $f^{(k)}$ be the k-th derivatives of f_n and f respectively. On every closed disk $\overline{B(a;R)}$ contained in G, for any $k \geq 1$, show that the sequence $\left\{f_n^{(k)}\right\}$ converges to $f^{(k)}$ uniformly on $\overline{B(a;R)}$.

Question 6. Let $f : D \longrightarrow \mathbb{C}$ be a non-constant analytic map, $D = \{z \in \mathbb{C} : |z| < 1\}$, such that $f(0) = 1, \operatorname{Re} f(z) \geq 0$. Prove that

$$|f(z)| \leq \frac{1+|z|}{1-|z|}$$

for all $|z| < 1$.

Question 7. Let G be a simply connected region that is not the entire complex plane. Without applying the Riemann Mapping Theorem, prove that there exists a non-constant bounded analytic function $f : G \longrightarrow \mathbb{C}$.

Question 8. Let $u : G \longrightarrow \mathbb{R}$ be a real-value continuous function satisfying the mean value property. Prove that u is harmonic, meaning it satisfies the equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Question 9. Find the Laurent series of the function

$$f(z) = \frac{(z-1)}{(z-2)(z-3)}$$

in the power series of z within the region $\{z \in \mathbb{C} : 2 < |z| < 3\}$ centered at $z = 0$.

Question 10. Let f be an entire function such that $|f(z)| = |f(1/z)|$ whenever z is nonzero. Prove that f is constant.

— END —

Qualifying Examination: Complex Analysis
10:00am - 01:00pm, June 6, 2015

Instruction: Answer any eight questions from the following ten problems. Circle the problem numbers to be graded. Show your work.

Problem 1 *State and prove the Harnack's Inequality for harmonic functions*

Problem 2 *State and prove the Rouche's Theorem*

Problem 3 *Let $\{f_n\}$ be a sequence of analytic functions on a region G such that $\{f_n\} \to f$ uniformly on compact subsets, and $a \in G$. Suppose that each f_n is one-to-one and $f^{'}(a) > 0$. Show that f is one-to-one and $f_n'(a)$ is non-zero for sufficiently large n.*

Problem 4 *Evaluate the integral*

$$\int_0^\infty \frac{1}{x^4+1}dx.$$

Problem 5 *Without using the Montel's Theorem prove directly that a family of locally bounded analytic functions are equicontinuous on any compact subset of the domain of definition.*

Problem 6 *Evaluate $\int_\gamma \frac{\log z}{z^2-25}dz$, where γ parametrizes $\partial B(4;2)$ once counter-clockwisely, where $\log z$ is the principal branch of logarithm.*

Problem 7 *Let $\gamma \subset \mathbf{C}$ be a piecewise differentiable curve, and let $\bar{\gamma}$ be the image of γ under the map $z \mapsto \bar{z}$. Show:*

a. If f is continuous on γ, then $z \mapsto \overline{f(\bar{z})}$ is continuous, and

$$\overline{\int_\gamma f(z)\,dz} = \int_{\bar{\gamma}} \overline{f(\bar{z})}dz$$

b. As an application of (a), show that if γ is the positively oriented unit circle, then

$$\overline{\int_{|z|=1} f(z)\,dz} = -\int_{|z|=1} \overline{f(z)}\frac{dz}{z^2}$$

Problem 8 *Let $D \subseteq \mathbf{C}$ be an open, connected subset of $\mathbf{C}$. Further, let f, g be holomorphic functions, defined on D so that $f(z) \neq 0 \neq g(z)$ for all $z \in D$, and let $(a_n)_{n\geq 1}$ be a convergent sequence of numbers $a_n \in D$ so that $a = \lim_{n\to\infty} a_n \in D$ and $a_n \neq a$ for all n. Show: If*

$$\frac{f'(a_n)}{f(a_n)} = \frac{g'(a_n)}{g(a_n)}$$

for all n, then there is a constant $c \in \mathbf{C}$ so that $f(z) = cg(z)$ for all $z \in D$.

Problem 9 *Let $f(z)$ be a holomorphic function, defined on the disk $|z| < R$. For $0 \le r < R$ we define*

$$M(r) = \sup_{|z|=r} |f(z)|$$

Show:

a. $M(r)$ is a continuous, non-decreasing function of r.

b. If $f(z)$ is not a constant function, then $M(r)$ is strictly increasing.

Problem 10 *Let $f(z)$ be an entire function. Furthermore, let $n \ge 0$ be a positive integer, and let $0 \le M, R$ be two positive constants. Show: If*

$$|f(z)| \le M|z|^n \textit{ for all } |z| \ge R,$$

then $f(z)$ is a polynomial of degree at most n.

END

Qualifying Examination: Complex Analysis

10:00am - 1:00pm, June 7, 2014

Instruction: Answer any eight questions from the following ten problems. Show your work to get credits. Put your name on all answer sheet.

Name: ______________________________________

Student ID Number: _____________________________

Circle the numbers of the eight problems you choose to be graded:

Problem	1	2	3	4	5	6	7	8	9	10
Score	__	__	__	__	__	__	__	__	__	__

Problem 1 *Using contour integration, find the value of*

$$\int_0^{2\pi} \frac{dt}{\cos(t)-2}.$$

Problem 2 *Let f be an entire function such that $|f(z)| < M\,|z|^\alpha$ for all $z \in \mathbb{C}$ with $|z| > R$, where M, R and α are constants with $0 < \alpha < 1$. Prove that f is a constant function.*

Problem 3 *Show that all the roots of the equation $e^z = 3z^2$ in the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ are real.*

Problem 4 *Let $f(z)$ be an entire function. Suppose in addition that $f(z) = f(\frac{1}{z})$ for all $z \neq 0$, prove that f is a constant function.*

Problem 5 *(a) Determine a Möbius transformation that maps the upper half of the unit disk onto the first quadrant. (b) Find a conformal map that maps the unit disk in the first quadrant one-to-one and onto the upper half plane. Provide the algebraic expression of the map.*

Problem 6 *Suppose that $\mathcal{F} \subset H(G) = \{\text{analytic functions on a region } G\}$ is a normal family. Denote the derivative of a function f by f'. (a) Show that the set $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is also a normal family. (b) Is the converse true? Give a proof if your answer is affirmative. Otherwise, give a counterexample.*

Problem 7 *Let* $\Omega \subset\subset G$ *be a relatively compact region, and* $f : \partial\Omega \longrightarrow \mathbb{C}$ *is a continuous function. (a) Describe, without proof, the Perron's solution for the Dirichlet problem on* Ω *with the boundary value* f *on* $\partial\Omega$*. (b) Let* Ω *be the punctured disk:* $\Omega = \{z \in \mathbb{C} : 0 < |z| < 1|\}$*. Find an example of a continuous function* $f(z)$ *on* $\partial\Omega$ *such that the Dirichlet problem on* Ω *is not solvable. Provide a proof.*

Problem 8 *Let* G *be a simply connected region which is not the entire complex plane. Suppose that* $\overline{z} \in G$ *whenever* $z \in G$*. Let* $a \in G \cap \mathbb{R}$ *and suppose that*

$$f : G \longrightarrow D = \{z \in \mathbb{C} : |z| < 1|\}$$

is a one-to-one analytic function such that $f(a) = 0$, $f'(a) > 0$ *and* $f(G) = D$*. Let* $G_+ = \{z \in G : \operatorname{Im} z > 0\}$*. Show that* $f(G_+)$ *must lie entirely above or entirely below the real axis.*

Problem 9 *A function of two real variables* $u(x, y)$ *is harmonic if it is twice continuously differentiable (i.e.* C^2*), and*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Prove that haromonic functions are infinitely differentiable (i.e. C^∞*).*

Problem 10 *Let* G *be a simply connected region which is not the entire complex plane. Without applying the Riemann Mapping Theorem, prove directly that there exists a non-constant analytic mapping* $f : G \longrightarrow \mathbb{C}$ *such that* $\mathbb{C} \backslash f(G)$ *contains a non-empty open set.*

END

Qualifying Examination: Complex Analysis
10:00am - 01:00pm, June 8, 2013

Instruction: Answer any eight questions from the following ten problems. Circle the problem numbers to be graded. Show your work.

Problem 1 *Determine a branch cut for the function* $f(z) = \sqrt{z^2-1}$. *Determine the radius of convergence for the power series representation of* $f(z)$ *centered at* $z = i$.

Problem 2 *State and prove the Morera Theorem*

Problem 3 *Suppose that* $\varphi(z)$ *is analytic in a domain* D. *Let* $a \in D$. *Prove that there exists a unique function* $f(z)$ *analytic on* D *such that on* $D\setminus\{a\}$,

$$f(z) = \frac{\varphi(z) - \varphi(a)}{z - a}.$$

Problem 4 *Let* $p(z)$ *be a polynomial. Show that*

$$\int_C p(z)d\bar{z} = -2\pi i R^2 p'(0),$$

where C *deontes the circle* $|z| = R$ *winding counter-clockwisely once.*

Problem 5 *Evaluate the integral*

$$\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx$$

where a *is a real number.*

Problem 6 *State and prove the Argument Principle.*

Problem 7 *Prove that a non-constant harmonic function on a region in* $\mathbb{R}^2$ *is an open map.*

Problem 8 ***Part A.*** *State the Maximum Modulus Principle for an analytic fucntion.* ***Part B.*** *State the classical Schwarz's Lemma for an analytic function* $f : \Delta \to \Delta$, *where* Δ *is the unit disk,* $f(0) = 0$. *Give a proof using Part A.* ***Part C.*** *Prove the uniqueness part of the Riemann Mapping Theorem.*

Problem 9 *Let G be a region and M be a fixed real number. Define*

$$H(G) = \{g : G \to \mathbb{C} : g \text{ is analytic}\}$$

and

$$\mathcal{F} = \left\{ g \in H(G) : \iint\limits_{G} |g(z)|^{100}\, dxdy \leq M \right\}.$$

Show that $\mathcal{F}$ is a normal family.

Problem 10 *State and prove the Harnack's Inequality for harmonic functions.*

END

Qualifying Examination: Complex Analysis
10:00am - 1:00pm, June 2, 2012

Answer any eight from the following problems. Show your work.

Problem 1 *(a) Determine a branch for the function* $f(z) = \ln \frac{z+i}{z-1}$ *so that* $f(z)$ *is analytic in a neighborhood* $z = 0$. *(b) Determine the radius of convergence of the power series expansion for* $f(z)$ *with center at* $z = 0$.

Problem 2 *(a) State the Cauchy Theorem on a simply connected region. (b) Suppose that* G *is simply connected and* f *is analytic on* G. *Show that there is an analytic funcitn* F *on* G *such that* $\frac{dF}{dz} = f$.

Problem 3 *Show that* $z^5 + 6z^3 - 10$ *has exactly two zeros, counting multiplicities, in the annulus* $2 < |z| < 3$.

Problem 4 *Assume that* $f(z)$ *is holomorphic for* $|z| < R$, *where* $R > 1$. *Prove*

$$\int_0^{2\pi} f(e^{it}) \cos^2 \frac{t}{2} dt = \pi f(0) + \frac{\pi}{2} f'(0).$$

$$\int_0^{2\pi} f(e^{it}) \sin^2 \frac{t}{2} dt = \pi f(0) - \frac{\pi}{2} f'(0).$$

Problem 5 *Let* $f(z)$ *be an entire function. Assume that there is a natural number* n *and real numbers* $R, M > 0$ *such that*

$$|f(z)| \leq M |z|^n$$

for all $|z| \geq R$. *Prove that* $f(z)$ *is a polynomial of degree* n *or less.*

Problem 6 *Let* $f = u + iv$ *be an analytic function on* $0 < |z| < 210$ *such that*

$$2011 < u < 2012.$$

Prove that f *can be analytically extended to* $\{z \in \mathbb{C} : |z| < 210\}$.

Problem 7 *Let* $u : G \longrightarrow \mathbb{R}$ *be a continuous function in a region* $G \subseteq \mathbb{R}^2$ *satisfying the mean value property. Prove that* u *is harmonic (i.e.* $\Delta u = 0$*).*

Problem 8 *Let* G *be a simply connected region which is not the whole plane. Without applying the Riemann mapping theorem, prove directly that there exists a non-constant analytic map* $f : G \longrightarrow D$, *the unit disk.*

Problem 9 *If* $f(z)$ *is an entire function such that* $z^{-1}\,\mathrm{Re}(f(z)) \to 0$ *when* $z \to \infty$. *Show that* $f(z)$ *is a constant function.*

Problem 10 *State and prove the Jensen's formula.*

Hint: you can assume the fact

$$\int_0^{2\pi} \log\left|1 - e^{i\theta}\right| d\theta = 0.$$

Qualifying Examination: Complex Analysis
10:00am - 1:00pm, June 4, 2011

Answer any eight from the following ten problems. Show your work.

Problem 1 *Find the poles and residues of* $\frac{\cos z}{\sin z}$.

Problem 2 *For the function*

$$\sqrt{\frac{z-1}{z+1}},$$

(a) determine the domain of its principal branch. (b) For the principal branch, determine the radius of covergence of its power series representation at $z = i$.

Problem 3 *Let*

$$n(\gamma; z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$$

be the winding number of a closed contour γ *about a point* z *not on* γ. *Prove that this is a constant function in the variable* z *on each connected component of* $\mathbb{C}\setminus\{\gamma\}$.

Problem 4 *Let* $G \subseteq \mathbb{C}$ *be an open region, and let* $\{f_n\}_{n\geq 1}$ *be a sequence of analytic functions defined on* G. *Show that the sequence* $\{f_n\}_{n\geq 1}$ *converges uniformly on compact subsets of* G *if and only if for each piecewise differentiable curve* γ, *the sequence of complex numbers*

$$\left\{\int_\gamma f_n(\zeta)\,d\zeta\right\}_{n\geq 1}$$

is a Cauchy sequence in $\mathbb{C}$.

Problem 5 *Prove that*

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}.$$

Problem 6 *Let* $a > 0$ *be a fixed real number. Compute the line integral*

$$\int_\gamma \frac{ds}{x^2+y^2-2ax+a^2},$$

where γ *is the circle with radius* $\rho > a$ *around the origin, oriented counter-clockwisely, and* s =*arc-length.*

Problem 7 *Suppose that a function* f *is analytic inside and on a positively oriented simple closed contour* γ, *and that* f *has no zero on* γ. *Show that if* f *has* n *zeroes* $\{z_k\}_{k=1}^n$ *inside* γ, *and the multiplicity of* z_k *is* m_k, *then*

$$\int_\gamma \frac{zf'(z)}{f(z)}dz = 2\pi i\left(\sum_{k=1}^n m_k z_k\right).$$

Problem 8 *Find the Laurent series of*

$$f(z) = \frac{1}{z(z-1)(z-3)}$$

about the origin in $\{z : 1 < |z| < 3\}$.

Problem 9 *(a) State the Rouchè's Theorem. (b) Give a proof of the Fundamental Theorem of Algebra using the Rouchè's Theorem.*

Problem 10 *Let* u *and* v *be two harmonic functions defined on the whole* $\mathbb{R}^2$ *such that*

$$u(x,y) \geq v(x,y)$$

for all $(x,y) \in \mathbb{R}^2$. *Prove that there exists a constant* c *such that for all* $(x,y) \in \mathbb{R}^2$,

$$u(x,y) = v(x,y) + c.$$

University of California at Riverside
Department of Mathematics
Spring 2010

Complex Analysis
Qualifying Examination

Choose 8 of the following 10 problems:

1. Show that $\int_0^\infty \frac{\cos x}{1+x^2} dx = \frac{1}{2}\frac{\pi}{e}$

2. Let $n \geq 1$ be an integer, and let $f(z)$ be an entire function so that $|f(z)| \leq |z|^n$ for all $z \in \mathbb{C}$.Prove that there is a constant c so that $f(z) = cz^n$.

3. Let D be an open, connected region so that $z \in D$ implies that also $\bar{z} \in D$, and let $f : D \to \mathbb{C}$ be a holomorphic function.

   (a) Show that the function $g : D \to \mathbb{C}$ defined by
   $$g(z) = \overline{f(\bar{z})}$$
   is also holomorphic.

   (b) Prove: If $f(r) \in \mathbb{R}$ for each $r \in D \cap \mathbb{R}$, then $f(z) = \overline{f(\bar{z})}$ for all $z \in D$.

4. Find explicitly a holomorphic bijection between $G = \{z : |z| < 1 \text{ and } \operatorname{Re}(z) > 0\}$ and the open unit disk $D = \{z : |z| < 1\}$.

5. Let G be a region in $\mathbb{C}$ and let $(a_k)_{k\geq 1}$ be a sequence of distinct points in G without limit points in G. Show that for an arbitrary sequence of complex numbers $(w_k)_{k\geq 1}$ there exists an analytic function f on G such that
   $$f(a_k) = w_k$$
   for all points a_k in the sequence.

6. Let G be an open subset of the complex plane and $f_n : G \to \mathbb{C}$ be analytic functions. Show that if
   $$\sum_{n=1}^{\infty} f_n(z)$$
   converges uniformly on compact subsets of G to f then f is analytic on G and
   $$f'(z) = \sum_{n=1}^{\infty} f_n'(z).$$

7. Let $B(a, R)$ be the open ball with center a and radius R, and let $\overline{B}(a, R)$ be the closure of $B(a, R)$. Let $u : \overline{B}(a, R) \to \mathbb{R}$ be a continuous function satisfying $u(z) \geq 0$ for all z. Show that if u is harmonic in $B(a, R)$ then for $0 \leq r < R$ and all θ we have:

$$\frac{R-r}{R+r}u(a) \leq u(a + re^{i\theta}) \leq \frac{R+r}{R-r}u(a).$$

8. Let a be a zero of the Riemann zeta-function ζ in the critical strip

$$0 \leq \mathrm{Re}(z) \leq 1.$$

Prove that $\bar{a}$, $1-a$, and $1-\bar{a}$ are also zeros of ζ.

9. State the open mapping theorem and sketch a proof of this theorem.

10. Let G be an open domain, let $f : G \to \mathbb{C}$ be a holomorphic functions and let F be a family of holomorphic functions, defined on G. Let A a closed, bounded disk of radius R and center z_0, and assume that $A \subseteq G$.

(a) Prove that $|z - z_0| < \frac{1}{2}R$ implies that

$$|f(z)| \leq \frac{1}{\pi R}\int_0^{2\pi} \left|f\left(z_0 + Re^{i\alpha}\right)\right| R d\alpha$$

(b) Prove that $|z - z_0| < \frac{1}{2}R$ implies that

$$|f(z)| \leq \frac{2}{\pi R^2}\int\int_A |f(\zeta)|\, dA$$

(Hint: Express $\int\int_A |f(\zeta)|\, dA$ in polar coordinates).

(c) Let $M > 0$ be a fixed constant, and assume that $\int\int_A |f(\zeta)|\, dA \leq M$ for each $f \in F$ and each compact subset $A \subseteq G$. Use Montel's theorem to prove that F is pre-compact in $H(G)$, where $H(G)$ is the space of all holomorphic functions on G, equipped with the topology of uniform convergence on compact sets.

Spring 2009 Complex Analysis Qualifying Exam

Directions: Complete 4 problems from Part A and 4 problems from Part B. Show all work. Indicate clearly on your paper which problems you have chosen. All problems have the same value.

Student ID Number: ______________________________

Part A. Complete four problems from this part. Circle the problem numbers of your choice.

Problem 1. (a) Let γ be a closed rectifiable path in $\mathbb{C}$. Show that the winding number $n(\gamma; a) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz$ is a continuous function on each connected component of $\mathbb{C}\setminus\{\gamma\}$. (b) Show that $n(\gamma; a) = 0$ if a in the unbounded component of $\mathbb{C}\setminus\{\gamma\}$.

Problem 2. (a) Show that if $f : \mathbb{C} \longrightarrow \mathbb{C}$ is analytic, then f is a conformal map at z_0 if $f'(z_0) \neq 0$. (b) Construct explicitly a conformal map from the (open) first quadrant to the interior of the disk centered at the origin with radius 2.

Problem 3. Let f_k be a sequence of functions analytic on a region G. Suppose that this sequence converges uniformly to a function f on G. Show that (a)

$$\lim_{k\to\infty}\int_\gamma f_k = \int_\gamma f,$$

for any closed rectifiable path γ in G, and (b) f is analytic.

Problem 4. Evaluate the integral

$$\int_\gamma \frac{z^{1/m}}{(z-1)^m}dz$$

where $\gamma(t) = 1 + \frac{1}{2}e^{it}$ and m is a positive integer.

Problem 5. f is an analytic function in a region G, and $z_0 \in G$. Show that there exists a unique analytic function g on G such that for all $z \in G\setminus\{z_0\}$

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}.$$

Part B. Complete four problems from this part. Circle the problem numbers of your choice.

Problem 6. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant analytic function. Prove that $M(r)$ is an increasing function in r and $\lim_{r\to\infty} M(r) = \infty$, where $M(r) = \max\{|f(z)| : |z| \le r\}$, $r \in \mathbb{R}$.

Problem 7. Determine the number of solutions of the equation $e^z - 5z^4 + 2 = 0$ inside the unit circle. Give reasons for your answer and show your method.

Problem 8. Let $f : \Delta \to \mathbb{C}$ be a non-constant analytic function defined on the unit disc Δ such that $\operatorname{Re} f(z) > 0$ and $f(0) = 1$. Prove that $|f(z)| \le \frac{1+|z|}{1-|z|}$, for all $z \in \Delta$. What can you say about f if there exists one point $z_0 \in \Delta$, $z_0 \neq 0$, which satisfies the equality $|f(z_0)| = \frac{1+|z_0|}{1-|z_0|}$? Give a proof of your answer.

Problem 9. Determine whether there exists a one-to-one, onto analytic mapping between the pair of domains A and B of $\mathbb{C}$ in (a), (b), (c), (d) respectively. Give reasons for your answers.

(a) $A = \mathbb{C}$, the whole complex plane. $B = \{z \in \mathbb{C} : |z| < 100\}$.

(b) $A = \{z \in \mathbb{C} : |z| < 1\}$, $B = \{x + iy : x, y \in \mathbb{R}, |x| < 1, |y| < 2\}$.

(c) $A = \mathbb{C}\setminus\{0\}$, $B = \mathbb{C}\setminus\{1, 2\}$.

(d) $A = \{z \in \mathbb{C} : |z| < 1\}$, $B = \mathbb{C}\setminus\{x + iy : y \in \mathbb{R}, x = 0, y \ge 0\}$.

Problem 10. (a) Let f be a continuous function on $\partial\Delta = \{z \in \mathbb{C} : |z| = 1\}$. Write explicitly a harmonic function u on $\Delta = \{z \in \mathbb{C} : |z| \le 1\}$ with boundary values f. No proof is needed.

(b) Evaluate the integral $\int_{|z|=1} \frac{d\theta}{\left|z-\frac{1}{2}\right|^2}$, $z = e^{i\theta}$, $\theta \in [0, 2\pi]$.

(c) Prove that a continuous function satisfying the mean-value equality must be harmonic.

SPRING 2008 COMPLEX ANALYSIS QUALIFYING EXAM

Directions: Complete 12 of the following 13 problems. Show all work. Indicate clearly on your paper which problems you have chosen. All problems have the same value.

Problem 1 *(a) Define radius of convergence for a power series* $\sum_{n=0}^{\infty} a_n(z-a)^n$. *(b) Let* R *be the radius of convergence for the given power series and* $0 < r < R$, *then* show *the series converges uniformly on* $\{z : |z-a| \leq r\}$.

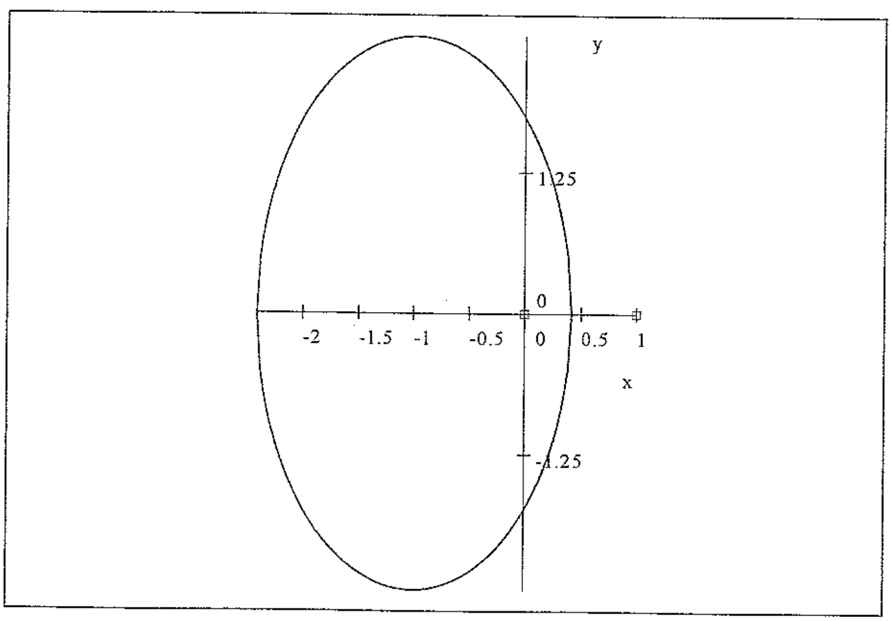
Problem 2 *(a) Determine a branch for the function* $f(z) = \ln \frac{z-i}{z-1}$ *so that* $f(z)$ *is analytic in a neighborhood* $z = 0$. *(b) Determine the radius of convergence of the power series expansion for* $f(z)$ *with center at* $z = 0$.

Problem 3 *Prove that if a complex-valued function is differentiable then it is analytic.*

Problem 4 *Prove that if a function is continuous on a region* G *and analytic on* $G\backslash\{a\}$, *then it is analytic throughout* G.

Problem 5 *(a) State the Cauchy Theorem on a simply connected region. (b) Suppose that* G *is simply connected and* f *is analytic on* G. *Show that there is an analytic* function F *on* G *such that* $\frac{dF}{dz} = f$.

Problem 6 *Consider the function* $f(z) = z^3$. *Plot the images of the following four objects on the same complex plane: (a) the point* $z = 0$, *(b) the point* $z = 1$, *(c) the point* $z = -1$, *and (d) the curve given below. Justify your answer.*



Problem 7 *f(z) is entire and* $|f(z)| \leq e^x$ *throughout the plane. What can be said about f(z)?*

Problem 8 *Consider the polynomials of degree n,* $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$, *and let* $M_P = max_{|z|\leq 1} |P(z)|$ *. For which P is* M_P *a minimum? Why?*

Problem 9 *Show that the infinite product*

$$\prod_{n=1}^{\infty}(1 - nz^n)$$

converges in $[|z| < 1]$ *and represents there an analytic function f(z). Show that the zeros of f(z) have all points on* $[|z| = 1]$ *as accumulation points.*

Problem 10 *A function w(z) is analytic in a convex domain D and Re w′(z) > 0 in D.*

a) *Prove that w(z) is 1-1 in D.*

b) *Using a), prove that , if* $w(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *in* $[|z|<1]$ *and* $\sum_{n=2}^{\infty} n|a_n| < 1$, *then* $w(z)$ *is* $1 - 1$ *in* $[|z|<1]$.

Problem 11 *If w(z) is an entire function and p(z) is a polynomial such that* $|w(z)|\leq|p(z)|$ *for all* $z \in \mathbb{C}$, *show that w(z) is a constant multiple of p(z).*

Problem 12 a) *Show that every conformal (1-1) map w(z) of the unit disk* $[|z|<1]$ *onto itself is a linear fractional transformation. Assume the fact that w(z) must extend continuously to the boundary* $|z|=1$, *on which* $|w| =1$.

b) *Show that there is no conformal map of* $\mathbb{C}$ *onto* $[|z|<1]$.

Problem 13 *Find the most general harmonic polynomial of the form* $ax^3+bx^2y+cxy^2+dy^3$. *Determine the conjugate harmonic function, and the corresponding analytic function of* $z=x+iy$.

Complex Analysis
Syllabus for the qualifying examination

1. Undergraduate material.

a) Complex numbers and their geometry
b) The Riemann-Stieltjes integral (chapter 6 in ref. 5)
c) Green's formula in two dimensions
d) Uniform convergence and equicontinuity of sequences of functions (chapter 7 in ref. 5); integrals of functions depending on parameters.

2. Elementary analytic functions and their mapping properties.

a) Linear fractional transformations and the Riemann sphere
b) Cross-ratio
c) The exponential and the logarithm
d) Trigonometric functions

3. The Cauchy-Riemann equations.

a) The operators ∂ and $\bar{\partial}$ in Cartesian and polar coordinates
b) The homogeneous equation $\bar{\partial}u = 0$; properties of $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$.
c) The inhomogeneous equation $\bar{\partial}u = f$.

4. Cauchy theorem and its consequences.

a) Proofs and Cauchy's formula
b) Cauchy's inequalities
c) The uniqueness principle
d) The maximum modulus principle and Schwarz lemma
e) The open mapping principle
f) Liouville's theorem and the fundamental theorem of algebra
g) Winding numbers, the argument principle and Rouche's theorem.

5. Singularities of analytic functions.

a) Classification of singularities,
b) Casorati-Weierstrass theorem,
c) Residue theorem,
d) Computation of definite integrals.

6. Taylor and Laurent series.

a) Cauchy-Hadamard formula for the radius of convergence,

b) Abel's theorem,
c) Laurent series,
d) Infinite products,
e) The expansions of elementary functions in infinite series and in infinite products.

7. Conformal transformations.

a) Riemann's mapping theorem,
b) The reflection principle,
c) Elementary conformal transformations.

8. Harmonic functions.

a) Maximum principle,
b) Mean value theorem,
c) Poisson and Jensen formulae,
d) Dirichlet problem,
e) Subharmonic functions.

References.

1) L. Ahlfors, Complex Analysis, McGraw-Hill.

2) J. Conway, Functions of One Complex Variable, 2nd edition, Springer.

3) K. Knopp, Theory of Functions, Parts I and II; Problem Book, Vol. I and II, Dover.

4) R. Narasimhan, Complex Analysis in One Variable, Birkhäuser.

5) W. Rudin, Principles of Mathematical Analysis, 3d. edition, McGraw-Hill (undergraduate material, only chapters 6 and 7)

6) S. Saks and A. Zygmund, Analytic Functions, Warsaw.

Qualifying Examination: Complex Analysis
June 2, 2007.

This is a closed book test. Please choose any eight questions to answer. If a problem has two parts, the weight of Part (a) and Part (b) are 20% and 80% respectively.

Question 1 *(a) State the Schwarz Lemma. (b) Let* $f(z)$ *be analytic in* $|z| < 1$ *and suppose* $|f(z)| \le 1$ *there. Suppose further that* $f(0) = f'(0) = 0$. *Prove* $|f(z)| < |z|^2$ *in* $[|z| < 1]$ *unless* $f(z) = e^{i\alpha} z^2$.

Question 2 *Compute the following integral by using residue theorems:*

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)(x^2+3)}.$$

Question 3 *Consider the polynomials of degree* n, $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$, *and let* $M_P = \max_{|z|\le 1} |P(z)|$. *For which* P *is* M_P *a minimum? Why?*

Question 4 *(a) State the Liouville's Theorem. (b)* $f(z)$ *is entire and* $|f(z)| \le e^x$ *throughout the plane. What can be said about* $f(z)$*?*

Question 5 *Find the central three terms* (c_{-1}, c_0, c_1) *of the Laurent expansion of* $f(z) = \frac{1}{z(z-1)(3z-2)}$ *valid in the annulus* $\frac{1}{6} < |z - \frac{1}{2}| < \frac{1}{2}$. *Is the coefficient of* c_{-1} *the residue of* $f(z)$ *at one of its poles? Why?*

Question 6 *Find the number of zeros for the function*

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $1 \le |z| \le 2$.

Question 7 *Prove that a continuous function on a domain is harmonic if and only if it satisfies the mean value property.*

Question 8 *(a) State the Hurwitz's Theorem. (b) Let* $\{f_n\}$ *be a sequence of analytic functions on a region* G *such that* $f_n \to f$ *in* $H(G)$, *and* $a \in G$. *Suppose that each* f_n *is one-to-one and*

$$f_n(a) = 0, \quad f_n'(a) > 0.$$

Show that f *is one-to-one and* $f'(a) > 0$.

Question 9 *(a) State the Riemann Mapping Theorem. (b) Explain why the conclusion in Riemann Mapping Theorem is not applicable to the complex plane* $\mathbb{C}$ *and the punctured disk* $B(0;1)\backslash\{0\}$.

Question 10 *Show that if* $\Pi_{n=1}^{\infty}(1 + w_n)$ *converges, then* $\lim_{n\to\infty} w_n = 0$.

Qualifying Examination: Complex Analysis
June 4, 2005. 2:30pm to 5:30pm

This is a closed book test. You choose three questions from Part A and three questions from Part B to answer. On this cover sheet below, please mark clearly your selection.

Name:

Student ID Number:

Selection from Part A

Section from Part B

1 Part A

Instruction: Choose three questions from Part A to answer.

Question 1 *(a) Define winding number of a closed rectifiable curve about a point on the complex plane.*

(b) Suppose that γ is a closed rectifiable curve on the complex plane. Show that if a and b are in the same path-connected component of $C\backslash\{\gamma\}$, then the winding numbers $n(\gamma; a)$ and $n(\gamma; b)$ are equal.

Question 2 *(a) For a given power series $\sum_{n=0}^{\infty} a_n z^n$, define the number R, $0 \le R \le \infty$, by*

$$\frac{1}{R} = \limsup |a_n|^{1/n}.$$

Show that if $|z| < R$, the series converges absolutely.

(b) Find the largest disk centered at "0" on which the power series $\sum_{n=0}^{\infty} z^{n!}$ is an analytic function.

Question 3 *Calculate*

$$\int_\gamma z^{-1/2} dz$$

where γ is the straight line from $+1$ to $-i$, then from $-i$ to -1.

Question 4 *Determine the largest disk about the origin whose image under the function $f(z) = z^3 + iz - 1$ is one-to-one.*

2 Part B

Instruction: Choose three questions from Part B to answer.

Question 5 *(a) State the Montel's Theorem. (b) Let Ω be a region in $\mathbf{C}$ and $\mathcal{F} = \{f \in H(\Omega) : \int\int_\Omega |f(z)|^2 dxdy < 10\}$, $H(\Omega)$ = family of all analytic functions on Ω. Prove that $\mathcal{F}$ is a normal family.*

Question 6 *(a) State the Riemann Mapping Theorem. (b) Prove the uniqueness part of the Riemann Mapping Theorem. (c) Describe the Riemann mapping $f : H \to \triangle$, $H = \{x + iy : x > 0\}$, $\triangle = \{z \in \mathbf{C} : |z| < 1\}$, such that $f(1) = 0$.*

Question 7 *Evaluate the integral*

$$\int_0^\infty \frac{dx}{(x^2+1)^2}.$$

Question 8 *Suppose that Ω is a bounded region in $\mathbf{C}$. Let $f : \Omega \to \mathbf{R}$ be a continuous function on $\overline{\Omega}$ satisfying the mean-valued equality*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta,$$

for all $z_0 \in \Omega$ and $\rho > 0$ with $\overline{B(z_0, \rho)} = \overline{\{z : |z - z_0| < \rho\}} \subset \Omega$. Give a <u>direct</u> proof of the statement that if f assumes the value $M = \max_{z \in \Omega} f(z)$ at an interior point of Ω, then it must be a constant function.

End

Complex Analysis Qualifying Examination 2004

• *In each problem, please show all your work (i.e., every step of your calculation). Each problem is worth 20 points for a total of 200 points.*

1. Find the most general harmonic polynomial of the form

$$ax^3 + bx^2y + cxy^2 + dy^3.$$

Determine the conjugate harmonic function, and the corresponding analytic function of $z = x + iy$.

2. Compute the Laurent expansion of $\frac{1}{(z-1)(z-2)}$ valid in the annulus $[1 < |z| < 2]$.

3. Use contour integrals to prove

$$\int_0^\infty \frac{x^{\alpha-1}}{x^2+1}\,dx = \frac{\pi}{2\sin\frac{\alpha\pi}{2}}, \quad 0 < \alpha < 2.$$

4. Show that all zeroes of

$$p(z) = z^4 + 6z + 3$$

lie inside the circle $|z| = 2$.

5. Find a conformal mapping from the first quarter of the unit disk to the unit disk.

6. (1) State and prove the Schwarz Lemma.
(2) Use the lemma to show that every one-to-one conformal mapping of a disk onto itself is given by a linear transformation.

7. State and prove the maximum-modulus Theorem for analytic functions.

8. Consider the function $w(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{n(n+2)}\right)$.
(a) Show that $w(z)$ is an entire function.
(b) Determine all the zeroes of $w(z)$ and justify your answer.
(c) Determine explicitly $w(1)$.

9. Let $f(z) = \sum_{n=-N}^{M} c_n z^n$, where M and N are nonnegative integers.
(a) Show that $c_k = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{k+1}}dw$, where $\gamma := re^{i\theta}$, $0 \le \theta < 2\pi$ and $r > 0$.
(b) Show that $c_k = \frac{1}{(k+N)!}[z^N f(z)]^{(k+N)}|_{z=0}$.

10. Consider the function $f(z) = \sum_{n=1}^{\infty} nz^n$.
(a) Show that $f(z)$ is one-to-one in $D := [|z| < 1]$.
(b) Determine the image $f(D)$. (Hint: Show that $f(e^{i\theta}) = -\frac{1}{4}\csc^2\frac{\theta}{2}$.)
(c) For any $z_0 \in D$ find the angle at z_0 between the two curves given by $Re\,(f(z) - f(z_0)) = Im\,(f(z) - f(z_0))$ and $f(z) - f(z_0) \ge 0$.

Complex Analysis Qualifying Examination 2003

• *In each problem, please show all your work (i.e., every step of your calculation). Each problem is worth 25 points for a total of 200 points.*

1. For which values of the real numbers α, β, γ, δ, ϵ, is the function $\delta y^3 + \alpha x^2 + y^2 + i\beta xy + i\gamma y^2 + i\epsilon x^3$ an analytic function of $z = x + iy$?

2. Determine the first three non-vanishing terms of the Laurent expansion of $w(z) = \frac{1}{z(z^2+2iz-1)}$ about $z = 0$ and its domain of convergence.

3. Evaluate the integral $\int_\gamma \frac{dz}{z^5 - z^4}$ for the following curves described counterclockwise:
(a) γ is the circle $[\,|z| = \frac{1}{2}]$
(b) γ is the circle $[\,|z| = 2]$

4. For all pairs of real numbers r and s, find the angle between the two curves $[Re\, e^z = r]$ and $[Im\, e^z = s]$ lying in $[0 \leq Im\, z < 2\pi]$. Why?

5. Let $f = u + iv$ be a holomorphic function on $0 < |z| < 2$ such that $-3 < u < 3$. Prove that we can extend u to $D = \{|z| < 2\}$.

6. Let $f = \frac{1}{z} + 1 + z$. Find $f_1(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$ for z inside a simple closed curve γ which goes around 0 counterclockwise and prove that f_1 does not depend on γ.

7. Find the Taylor series of the function $\arctan(z)$.

8. We say a family of functions on Ω has locally bounded integrals if for any $p \in \Omega$ there is a disk $D = \{|z-p| \leq r\}$ and a number C such that $\int\int_D |f(z)|dA < C$. Prove that a family of holomorphic functions with locally bounded integrals is normal.

Complex Analysis Qualifier Examination
June 24, 2000

Do any 7 of the following 8 problems.

1. Use contour integral methods to evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}$$

2. Suppose

$$\tan z = \sum_{n=0}^{\infty} a_n z^n$$

for all $z \in \mathbb{C}$ with $|z|$ sufficiently small. Prove that

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{2}{\pi}.$$

3. (a) State the general Cauchy Theorem.
(b) Prove that if $f(z)$ is analytic and $\neq 0$ in a simply connected open set $\Omega \subseteq \mathbb{C}$, then there exists an analytic function $F\colon \Omega \to \mathbb{C}$ such that

$$F'(z) = \frac{f'(z)}{f(z)}$$

in Ω.

4. Let $\{f_n\}$ be a sequence of analytic functions defined on an open set $\Omega \subseteq \mathbb{C}$. Suppose that the functions f_n converge to the function $f\colon \Omega \to \mathbb{C}$, uniformly on every compact subset of Ω. Prove that f is an analytic function on Ω.

5. Suppose that f is an analytic function on an open set $\Omega \subseteq \mathbb{C}$ satisfying the inequality $|f(z) - 1| < 1$. Prove that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0$$

for any closed curve γ in Ω.

6. (a) State the Riemann Mapping Theorem.
(b) Find a domain for which Riemann Mapping Theorem does not hold. Justify your choice.

The following problems are examples of how you can learn a lot about an entire function from seemingly very little information.

7. Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is an entire function and

$$f\left(\frac{1}{n}\right) = \sin\left(\frac{1}{n}\right)$$

for all $n = 1, 2, 3, \ldots$ What can you conclude about the function f, and why do you know this?

8. Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is an entire function and

$$|f(0)| \geq \sup_{|z|=1} |f(z)|.$$

What can you conclude about the function f, and why do you know this?

Qualifying exam for Complex Analysis
(3 hrs. exam, answer any seven questions)
June 28, 1999

1. State the general Cauchy's theorem and sketch a proof.

2. Evaluate the integral $\int_{-\infty}^{+\infty} \frac{x^2}{1+x^4} dx$.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $f'(a) \neq 0$ at a point $a \in \mathbb{C}$.
   (A) Consider f as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. Prove that the Jacobian matrix $Jf(a)$ of f at a is equal to λA, where λ is a positive number and A is a two by two orthogonal matrix.
   (B) If $f(z) = z^2 + 1$ and $a = (1, 0)$. Compute λ and A

4. Find a conformal map from the domain $A = \left\{z \in \mathbb{C} | 0 < \arg z < \frac{\pi}{3}\right\}$ onto the domain $B = \{z \in \mathbb{C} | |z| > 1\}$.

5. Suppose $f(z)$ is an entire function. Prove that if there exist positive numbers R, K and a positive integer n such that $|f(z)| < K|z|^n$ for all $|z| > R$, then $f(z)$ is a polynomial.

6. Find the Laurent series of the function $f(z) = \frac{(z-1)}{(z-4)(z-5)}$ in the power of z within the region $\{z \in \mathbb{C} | 4 < |z| < 5\}$.

7. Prove that there exists no non-constant positive harmonic function on $\mathbb{R}^2$.

8. (a) State the Riemann mapping theorem.
   (b) Let Ω be a bounded simply connected domain in $\mathbb{C}$ and $a \in \Omega$ be any fixed point. Support $f : \Omega \to \Omega$ is an analytic function such that $f(a) = a$ and $|f'(a)| = 1$. Prove that f is one-to-one and on-to.

Complex analysis

qualifying exam (August 3, 1998)

$\frac{5}{70}$ ① ⓐ State the Laurent's series theorem

$\frac{5}{70}$ ⓑ Find the Laurent's series expansion of $f(z) = \frac{1}{(z-1)(z-2)}$ about the origin in

$A = \{ z \in \mathbb{C} \mid 1 < |z| < 2 \}$

$\frac{5}{70}$ ② ⓐ State the Residue theorem

$\frac{5}{70}$ ⓑ Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

③ $\frac{5}{70}$ ⓐ State the Liouville's theorem

$\frac{5}{70}$ ⓑ Give a proof of the fundamental theorem of algebra

④ $h(x,y) = x^2 - y^2 + x + 1$

$\frac{5}{70}$ ⓐ Is $h(x,y)$ a harmonic function? Give a proof of your answer

$\frac{5}{70}$ ⓑ Find the harmonic conjugate of h, if exists

(5) $\frac{10}{70}$ Assume that $f(z)$ is analytic and satisfies the inequality $|f(z)-1|<1$ in a region Ω. Show that $\int_\gamma \frac{f'(z)}{f(z)}\,dz = 0$ for any closed curve γ in Ω.

(6) $\frac{5}{70}$ (a) State the Schwarz's lemma

$\frac{5}{70}$ (b) Let f and g be two analytic function from a region Ω into the unit disk Δ such that

(i) f and g are one-to-one and on-to

(ii) $f(p)=0$ and $g(p)=0$ for a given point $p \in \Omega$

(iii) $f'(p)=g'(p)$.

Prove that $f=g$ (The statement of Riemann mapping theorem will not be allowed

(7) $\frac{10}{70}$ Show that the successive derivatives of an analytic function at a point z can never satisfy $|f^{(n)}(z)| > n!\, n^n$

(8) (a) Write down the Cauchy-Riemann equations for an analytic function $f(z) = u(x,y) + i\,v(x,y)$, where $z = x+iy$

$\frac{5}{70}$

$\frac{5}{70}$ (b) Prove that $f(z) = x^2 + y^2$ is not an analytic function.

(9) $\frac{5}{70}$ (a) State the Rouché's theorem

$\frac{5}{70}$ (b) Determine how many zeros of the polynomial $g(z) = z^5 + 3z + 1$ lie in the disk $|z| < 2$

Choose any <u>seven</u> problems

Show all your work

Three hour exam.

Qualifying Examination in
Complex Analysis
(Spring, 1997)

10 1. Prove using complex numbers that the medians lying inside a triangle are concurrent.

10 2. Let $f(z)$ be an analytic function in the unit disk $|z| < 1$. Prove or disprove by examples each of the following statements:

a) f is continuous in $|z| < 1$;
b) f is uniformly continuous in $|z| < 1$;
c) Re f is indefinitely differentiable in $|z| < 1$;
d) $\lim_{z \to 1} f(z)$ exists;
e) $\lim_{r \nearrow 1} \int_{|z|=r} f(z)dz$ exists.

10 3. Let $f_n(z)$ be a uniformly bounded sequence of analytic function in the disk $|z| < 1$. Suppose that $\lim_{n \to \infty} f_n(z) = 0$ for every $|z| \leq \frac{1}{4}$. Prove that, for any $0 < r < 1$, the sequence $f_n(z)$ converges uniformly to 0 for $|z| \leq r$.

10 4. Let $f(z)$ be an entire function with the property that $f(z) \in \mathbb{R}$ for $|z| = 1$. Prove that $f(z)$ is constant.

10 5. Let f and g be two analytic functions from a region Ω into the unit disk such that

(i) f and g are one-to-one and on-to.
(ii) $f(p) = 0$ and $g(p) = 0$ for a given $p \in \Omega$
(iii) $f'(p) = g'(p)$ Prove that $f = g$.

10 6. a. State Rouchè's theorem.
b. Use Rouchè's theorem to give a proof of the fundamental theorem of algebra.

10 7. Evaluate the integral $\int_{-\infty}^{+\infty} \frac{x^2}{1+x^4} dx$

10 8. Prove that a continuous function satisfying the mean-valued inequality must satisfy the maximum principle.

Total : 80 points

Complex Analysis Qualifying Examination-Fall 1996

Answer any 8 problems:

1. Let $f_n, n \geq 1$, be a sequence of analytic functions defined in a domain Ω of the complex plane. Suppose that the sequence f_n converges uniformly on compact subsets of Ω to a function f. Prove that:
   a. The function f is analytic in Ω;
   b. (Hurwitz Theorem) Let p be a fixed positive integer. If each $f_n, n \geq 1$, has at most p zeroes in Ω, then either f is identically equal to zero, or f has at most p zeroes in Ω.
2. Find the general form of a fractionar linear transformation with two distinct fixed points. Identify the group formed by these transformations, corresponding to two prescribed fixed points.
3. Compute the first five coefficients of the power series at 0 of the function $(\cos z)^{1/2}$.
4. Let $\alpha \neq 0$ and β be real numbers. Prove that there exists an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with *rationnal* coefficients $a_n, n \geq 0$, such that $f(\alpha) = \beta$.
5. Find a conformal map between the sector $\{|z| < 1; Re(z) > 0, Im(z) > 0\}$ and the unit disk.
6. State and prove Schwarz lemma.
7. Let $u(x, y)$ be a harmonic function in the unit disk: $|z| < 1, z = x + iy$. Which of the following sets can coincide with the set of zeroes of u:
   1. $\{0\}$;
   2. $\{x = 0\} \cup \{y = 0\}$;
   3. $\{x = 0\} \cup \{x = y^2\}$;
   4. $\{x = 0\} \cap \{y \geq 0\}$, and why?
8. Compute $\int_{|z|=n} \tan(\pi z) dz$ for every positive integer n.
9. State and prove the fundamental theorem of Algebra.
10. Find the harmonic conjugate of $u(x, y) = x^3 - 3xy^2 + 2x + 1$
11. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function and it satisfies $|f(z)| < |z|^n$, for some n and all large $|z|$. Prove that f must be a polynomial.

Complex Analysis Qualifying Examination-Spring 1996

1. a). Find the harmonic conjugate of the function $u(x, y) = x^3 - 3xy^2$ in the complex plane.

b). Prove that the function $u(x, y) = ln(x^2 + y^2)$ is harmonic in the domain $G = \mathbf{C} \setminus \{0\}$; can one find its harmonic conjugate in G? Give reasons.

2. Can the successive derivatives of an analytic function $f(z)$ at a point a satisfy the inequalities:

$$|f^{(n)}(a)| \geq n!n^n?$$

Prove your answer.

3. Find the Laurent series of the function $f(z) = \frac{1}{(z-1)(z-3)}$ around the origin, in the annulus $A = \{z; 1 < |z| < 3\}$.

4. Let Δ denote the unit disk in the complex plane.

a). Does there exist an analytic function $f : \Delta \longrightarrow \Delta$ with $f(1/2) = 3/4$ and $f'(1/2) = 2/3$? Give reasons.

b). Is there an analytic function $f : \Delta \longrightarrow \Delta$ so that $f(0) = 2/3$ and $f'(0) = 3/4$? Is f unique, if it exists?

5. Let Ω be a bounded domain of the complex plane and let n be a positive integer. Define:

$$\mathcal{F} = \{z^n + a_1 z^{n-1} + \ldots + a_n; z \in \Omega, |a_j| \leq 5, 1 \leq j \leq n\}.$$

Prove that $\mathcal{F}$ is a normal family and find all its limit points in the topology of uniform convergence on compact subsets of Ω.

6. State and prove Rouché's Theorem. Give a significant application of this theorem.

Complex Analysis Qualifying Exam.

Oct. 29, 1995

1. Prove the existence and uniqueness of the Laurent series expansion.

2. State and prove the residue theorem.

3. Prove that the Cauchy-Riemann equations are satisfied by the function

$$f(x+iy) = \sqrt{|xy|}$$

at the point $x=0$, $y=0$.
Is the function f analytic at $(0,0)$?

4. Compute $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$.

5. Let $p(z)$ be a polynomial of degree d, and denote

$$M(r) = \sup_{|z|=r} |p(z)| \quad , \quad r>0.$$

Prove that the function $\dfrac{M(r)}{r^d}$ is monotonically decreasing for $r \to \infty$.

6. Let $u(z)$ be a continuous functio on the unit circle $|z|=1$.
Prove that the function

$$f(z) = \frac{1}{\pi i}\int\limits_{|\zeta|=1} \frac{u(\zeta)\,d\zeta}{\zeta - z} - \overline{f(0)}$$

is analytic in the unit disk $|z|<1$, and that $\operatorname{Re} f(z)$ extends continuously to $u(z)$ on $|z|=1$.

7. Find the power expansions of the function $1/(z^2+1)^2$

in neighborhoods of the points i and ∞, respectively.

8. Prove that in the right half-plan the equation

$$z = 3 - e^{-z}$$

has a unique root.

June 4, 1994

Qualifying Exam in Complex Analysis

1. Calculate $\int_{-\infty}^{\infty} \frac{x^2\,dx}{1+x^4}$.

2. Calculate the residues of the meromorphic function

$$f(z) = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

at its poles.

3. Show that $\sum_{n=1}^{\infty} \frac{z^{n^2}}{n^4}$ is an analytic function in the set $\{z \in \mathbb{C} : \quad |z| < 1\}$.

4. a) State the Riemann mapping theorem and prove its uniqueness part.
b) Let $U = \{z \in \mathbb{C}; Im(z) > 0\}$. Find a function f which is continuous on the closure of U, analytic on U, maps U one to one onto $\{z \in \mathbb{C}; |z| < 1\}$ and satisfies $f(1+i) = 0, f(-1) = 1$.

5. Show that when $f(z)$ is an analytic function in a simply connected domain Ω and $f(z) \neq 0$ for all $z \in \Omega$, then there is an analytic function $g(z)$ on Ω such that $f(z) = e^{g(z)}$.

6. Use the Cauchy integral formula to show that if a function is analytic on the complex plane and bounded, then the function is constant.

7. Use Green's Theorem to prove the following special case of the Cauchy integral theorem. If $f(z)$ is analytic on $|z| \leq 1$ and γ is the curve $|z| = 1$, then $\int_\gamma f(z)dz = 0$.

8. a) Evaluate $\int_\gamma z\,dz$ where γ is the line segment beginning at 1 and ending at $2+i$.
b) Find an upper bound for $\left|\int_\gamma e^z/z\,dz\right|$ where γ is the curve described in a) and justify your answer.

9. Suppose that D is a domain, $a \in D, f$ is an analytic function on $D\setminus\{a\}$ (D less

the set $\{a\}$) and $\lim_{z \to a}(z-a)f(z) = 0$. Show that f has an analytic extension defined on all of $\bar{D}$. Hint. Apply the Cauchy integral formula to an annulus centered at a.

10. Show that there does not exist an analytic function $f(z)$ on the annulus $1 < |z| < 3$ such that $e^{f(z)} = z$. Hint. Show that $f'(z) = 1/z$ and consider the integral $\int_\gamma 1/z \, dz$ where γ is the circle $|z| = 2$.

COMPLEX ANALYSIS QUALIFYING EXAM
June 5, 1993

1. (a) State *two* definitions of analytic functions.

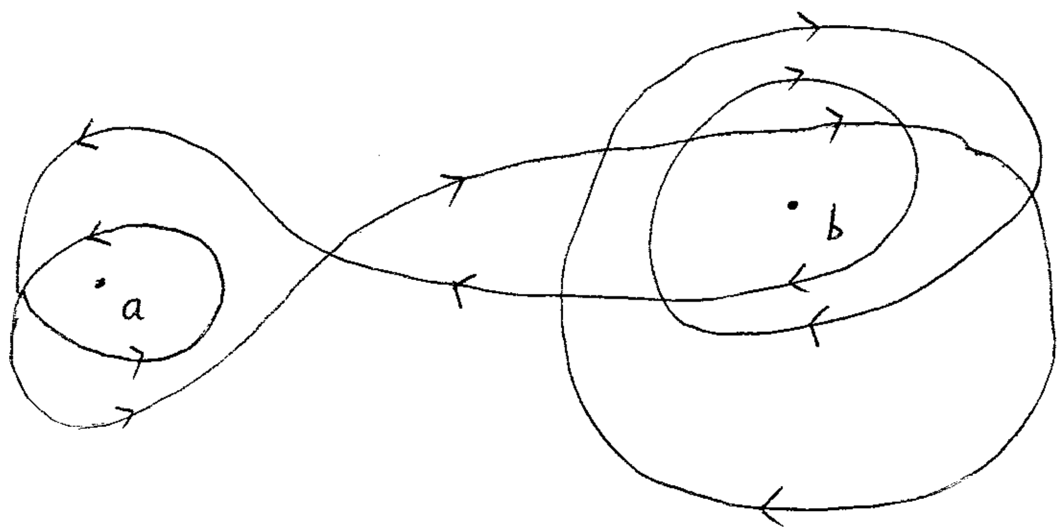
   (b) Suppose f and g are both non-constant analytic functions on a region G. Is $\bar{f} \cdot g$ analytic in G? Prove your answer.

2. (a) Write down the power series expansion of $\sqrt{z}$ at $z = 1$ and find its radius of convergence.

   (b) $f(z) = \frac{1}{z(z-1)(z-2)}$, find the Laurent expansion of $f(z)$ in the annulus $A = \{z | 1 < |z| < 2\}$ with center at the origin.

3. (a) State the Cauchy's integral formula.

   (b) Evaluate the line integral $\int_\gamma \frac{e^{-z}}{(z-a)(z-b)} dz$, where γ is the closed path shown in the diagram.



   (c) Evaluate $\int_\gamma \frac{\ln z}{z^4} dz$, where $\gamma(t) = 1 + \frac{1}{10} e^{it}, \quad 0 \leq t \leq 2\pi$

4. (a) State the Schwarz's lemma.

(b) Write down the automorphism φ_a of the unit disc $\Delta = \{z \big| |z| < 1\}$, $a \in \Delta$, which brings a to the origin. Write down the inverse of φ_a also.

(c) Does there exist an analytic function $f : \Delta \to \Delta$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{2}{3}$? Prove your answer.

5. Let f be an analytic function in the right half-plane $\mathbb{C}_+ = \{z \in \mathbb{C};\ \mathrm{Re}z > 0\}$, and let $a_1, a_2, \ldots$ devote the zeroes of f. Suppose that $|f(z)| \leq M$ for $z \in \mathbb{C}_+$.

a) Prove that for any $n \in \mathbb{N}$:

$$|f(z)| \leq |\frac{a_1 - z}{\bar{a}_1 + z} \cdots \frac{a_n - z}{\bar{a}_n + z}|M\ , \quad z \in \mathbb{C}_+$$

b) Prove that, if f is not identically zero, then

$$\sum_{n=1}^{\infty} \mathrm{Re}\ \left(\frac{1}{a_n}\right)$$

is a convergent series.

6. Prove the fundamental theorem of algebra.

7. Let $R \subset \mathbb{C}$ be the domain defined by:

$$R = \{z \in \mathbb{C} :\ \mathrm{Re}\ z < 0,\ \ \mathrm{Im}\ z \in (0, 2\pi)\}$$

Describe all conformal maps from the unit disk onto R.

8. State the symmetry principle and give an application of it to a concrete problem.

COMPLEX ANALYSIS QUALIFYING EXAM
June 5, 1992

1) Prove that, if $z_1, z_3, z_3 \in \mathbb{C}$, are the vertices of an equilateral triangle, then:

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1 .$$

2) Let f be an analytic function defined in a neighbourhood of the closed unit disk $\mathbb{D}$.

a) Compute $\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta$ in terms of the coefficients of the Taylor series of f at 0.

b) If $P(z)$ denotes a polinomial of degree n, find the minimum value of the integral:

$$\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta}) - P(e^{i\theta})|^2 \, d\theta .$$

For which polynomial P is this minimal value attained?

3) Assume that the entire function $F(z)$ satisfies the inequality $|F(x+iy)| < C e^{M|y|}$ for some positive constants C and M, and $x, y \in \mathbb{R}$. Then:

$$\frac{d}{dz}\left(\frac{F(z)}{\sin Mz}\right) = -\sum_{-\infty}^{\infty} \frac{M(-1)^n F\left(\frac{n\pi}{M}\right)}{(Mz - n\pi)^2} .$$

4) Let $\lambda > 1$ be fixed. Prove that the equation

$$z e^{\lambda - z} = 1$$

has exactly one solution in the unit disk, and that this solution is real and positive.

5) Let f be an analytic function in the unit disk. Prove that, if $f(0) = 0$, f is analytic at 1 and $f(1) = 1$, then $|f'(1)| > 1$.

Show that $f'(1)$ is in that case a real positive number.

Hint. Use Schwarz's Lemma.

(6) (a) Write down the Poisson's formula on disk radius r and state the conditions.

4 (b) Evaluate the integral $\displaystyle\int_{|z|=2} \frac{1}{|z-1|^2}\, dz$

4 (c) Prove that there exists no non-constant positive bounded harmonic function on $\mathbb{R}^2$.

(7) 5 (a) Evaluate the integral $\displaystyle\int_0^{\pi} \log \sin x \, dx$

5 (b) Let $f(z)$ be an analytic function in an open set U containg $\{z \mid |z| \leq r\}$. Suppose the only zero, which is of multiplicity one, of $f(z)$ lies on $\{z \mid |z| = r\}$, prove the following expression without assuming the Jensen's formula,

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta .$$

March 1986

Part I (Complex analysis Qualifying Exam

(Answer any six questions) B. Wong

① ⓐ Give three equivalent definitions of complex analytic functions

ⓑ Let $f(z) = Re(z^2)$ (i.e. Real part of z^2).

Is $f(z)$ an analytic function? prove your answer.

② ⓐ State Cauchy integral formula.

ⓑ Let $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$, find

$\int_\gamma \frac{\sin z}{z^n} dz$, where n is a positive even integer.

③ ⓐ State two versions of Cauchy's theorem.

ⓑ Let $f: G \longrightarrow \mathbb{C}$ be an analytic function on a simply-connected region G, prove that f has a primitive.

④ ⓐ State Goursat's theorem and Morera's theorem

ⓑ Let $f: \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous function such that $f: \mathbb{C} - \mathbb{R} \longrightarrow \mathbb{C}$ is analytic, where $\mathbb{R} = \{x+iy \in \mathbb{C} \mid y = 0\}$. Is f an analytic function over $\mathbb{C}$? prove your answer.

(5) (a) State maximum modulus principle for complex analytic functions

(b) Let G be a region and let f and g be two analytic functions on G such that $f(z) \cdot g(z) = 0$ for all $z \in G$. prove that either $f \equiv 0$ or $g \equiv 0$.

(6) Use residue theorem to evaluate $\displaystyle\int_0^\infty \frac{\cos x - 1}{x^2}\, dx$

(7) (a) Give the power series expansion of $\log z$ about $z = i$ and find the radius of convergence.

(b) Write down the Laurent series for the function $f(z) = \dfrac{1}{z(z-1)(z-3)}$ about $z = 0$ in the annulus $(0; 3, \infty) = \{ z \mid 3 < |z| < \infty \}$.

(3) 2 (a) State the Riemann mapping theorem

3 (b) prove the uniqueness of the Riemann mapping

3 (c) Construct a barrier for $\{z \mid |z|<1\}$ at each point of the boundary

2 (d) Let D be a simply-connected bounded region with smooth boundary in $\mathbb{C}$. Is the Dirichlet problem solvable for D. Give a proof or a Counterexample to your answer.

(Here you have the option to use the fact without proof that a conformal mapping between two simply connected bounded domains with smooth boundary extends continuously up to the boundary)

1984 Complex Analysis Qualifying Examination

Directions: Do 12 of the following 14 problems. Show all work. Indicate clearly on your paper which problems you have chosen. All problems have the same value.

#1. Describe a conformal mapping (or finite sequence of conformal mappings) which maps the semicircular domain ($[|z|<1, \operatorname{Im} z>0]$) onto the slit unit disk ($[|w|<1]$ less the non-negative real axis).

#2. State and prove Schwarz's Lemma.

#3. Let $f(z)$ be analytic in $[|z|<1]$ and suppose $|f(z)|\le 1$ there. Suppose further that $f(0)=f'(0)=0$. Prove $|f(z)|<|z|^2$ in $[|z|<1]$ unless $f(z)=e^{i\alpha}z^2$.
(Hint: model your proof after that of #2.)

#4. $f(z)$ is analytic in the unit semicircle, real and continuous on the diameter with $|f(z)|=1$ on the arc. Prove that $f(z)$ is a rational function.

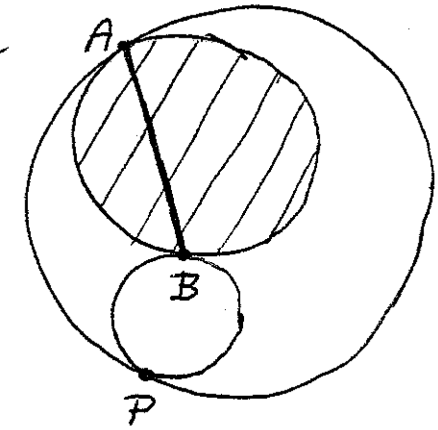
#5. Let $f(z)$ be an analytic function in $[|z|<1]$ whose real part has continuous boundary values $u(e^{i\theta})$. Given Poisson's formula, prove the Schwarz formula

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta \;+ iC \text{, where } C \text{ is a}$$

real constant.

#6. Compute the Laurent expansion of $\dfrac{1}{(z-1)(z-2)}$ valid in the annulus $[1<|z|<2]$.

#7. Three circles are mutually tangent as shown; the circumference of the shaded circle is mapped by a linear transformation onto a straight line L. Sketch the image of the configuration for the two cases



a) A, B map into finite points.

b) A maps into the point at ∞.

#8. Find the most general harmonic polynomial of the form $ax^3+bx^2y+cxy^2+dy^3$. Determine the conjugate harmonic function, and the corresponding analytic function of $z=x+iy$.

#9. State and prove a minimum principle (analogous to the maximum principle) for non-vanishing analytic functions.

#10. Let $f(z)$ be analytic in $[|z|<1]$ and satisfy the conditions

(a) $|f(z)| \leq 1$

(b) $\lim_{|z| \to 1} |f(z)| = 1$ (Hence defining $|f(z)| = 1$ on $[|z|=1]$ makes $|f(z)|$ continuous in $[|z| \leq 1]$.)

(c) $f(z) \neq 0$

Show that $f(z)$ is constant.

#11. Let $f(z)$ satisfy all the conditions of #10 except for (c). Show that $f(z)$ has only a finite number of zeros.

#12. Let $f(z)$ be a function as considered in #11 and $a_1, a_2, \ldots, a_n$ its zeros, each written down as many times as its multiplicity indicates. Show that $f(z)$ can be represented in the form $f(z) = \epsilon \prod_{i=1}^{n} \frac{z - a_i}{1 - \bar{a}_i z}$, ϵ being a constant of modulus 1. (Hint: use result of #10.)

#13. Compute by use of residue theorems:

(a) $\int_0^\infty \frac{dx}{(x^2+4)(x^2+3)}$ (b) $\int_0^\pi \frac{dx}{1+\beta \cos x}$, $|\beta|<1$

#14. Show that $f(z) = z + \frac{1}{\varepsilon} \sum_{n=2}^{\infty} \frac{z^n}{n^3}$, where $\varepsilon = \sum_{n=1}^{\infty} \frac{1}{n^2}$, is 1-1 in $[|z|<1]$.

Qualifying Exam. in Complex Analysis

and Final for Math 210 B. (June 8, 1982; 1:00 – 4:00 P.M.)

[Answer any five questions: open book.]

1.(a) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $\sup_{|z|\geq 1} \left|\frac{f(z)}{z^n}\right| < \infty$. Show that f must be a polynomial of degree m, $0 \leq m \leq n$.

(b) Using the expansion for $\mathrm{Log}(1+z)$, $\{z : |z| \leq 1\} - \{1\}$, show that

$$\mathrm{Log}(2 \sin\theta) = -\sum_{n\geq 1} \frac{\cos 2n\theta}{n},$$

$$\frac{\pi}{2} - \theta = \sum_{n\geq \phi} \frac{\sin 2n\theta}{n},$$

the series converging uniformly for $0 < \theta < \pi$. Deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \quad \text{and} \quad \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \cdots.$$

2. Let u, v be (real) conjugate harmonic functions in the plane such that $\frac{v(x,y)}{u(x,y)} = f(x)$, $x, y \in \mathbb{R}^2$. Using the facts that $g = u + iv$ is holomorphic and $\arg g = \mathrm{Im}(\mathrm{Log}\, g)$, show that $f(x) = \tan(\alpha x + \beta)$, for some real α, β. Deduce that $u(x,y) = a e^{-\alpha y} \cos x$, $v(x,y) = a e^{-\alpha y} \sin \alpha x$, $a = e^{\delta + i\beta}$ for some constant δ. [Here you may note that g is also harmonic.]

3. Consider $f(z) = \prod_{n\geq 0} (1 + z^{2^n})$, $|z| < 1$. Show that this product converges normally on compact subsets of the unit disc, and that $(1-z) f(z) = 1$. Similarly, verify that

$$\prod_{n\geq 2} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}.$$

4. (a) Using the infinite product representation of $\sin \pi z$, show

$$\cos \pi z = \prod_{n \geq 1} \left(1 - \frac{4z^2}{(2n-1)^2}\right).$$

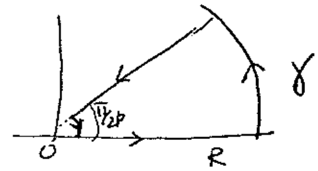
(b) Following the series representation for $\left(\frac{\pi}{\sin \pi z}\right)^2$ of your text, show

$$\frac{\pi}{\cos \pi z} = 2 \sum_{n \geq 1} (-1)^{n+1} \left[\frac{1}{(2n-1)-z} + \frac{1}{(2n-1)+z}\right] = \sum_{n \geq 1} \frac{(-1)^{n+1} 4(2n-1)}{(2n-1)^2 - z^2}.$$

Deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

5. For each $p > 1$, show that $\int_0^\infty \sin x^p \, dx$ is a convergent integral, and, with the method of residues, verify that (You may use contour γ below)

$$\int_0^\infty \sin x^p \, dx = \frac{1}{p} \Gamma\left(\frac{1}{p}\right) \sin \frac{\pi}{2p}.$$



[If $p = 2$, we get ~~the~~ Fresnel's integral with $p = 1/2$.]

6. Show that the following integral converges, and that

$$\int_0^\infty \frac{\sqrt{x} \operatorname{Log} x}{1 + x^2} \, dx = \frac{\pi^2}{2\sqrt{2}}.$$

[You may again use the contour shown above!]