Algebra Qualifying Exam- Part A

September 27, 2023

Choose four out of the following five problems.

1. A group G is indecomposable if $G \neq < e >$ and G is not the (internal) direct product of two of its proper subgroups. Show that the additive group $(\mathbb{Q}, +)$ is indecomposable.
2. Let p be an odd prime, then classify all groups of order $2p$ up to isomorphism.
3. Let G be a group. The subgroup of G generated by the set
$$\{aba^{-1}b^{-1} \mid a, b \in G\}$$
is called the commutator subgroup of G and denoted G'. Show the following:
   (a) if G is a group, then G' is a normal subgroup of G and G/G' is abelian,
   (b) let N be a normal subgroup of G. If G/N is abelian, then G' is a subgroup of N.
4. A group G in which every element has order a non-negative power of some fixed prime p is called a p-group. Show that a finite group G is a p-group if and only if $|G|$ is a power of p.
5. Show that if H is a subgroup of a finite group G of index $[G : H]$ equal to p, where p is the smallest prime dividing the order of G, then H is normal in G. Hint: Consider the set of all left cosets of H in G.

Algerba Qualifying Exam, Part B 2023

Attempt any four questions.

1. (10 points) Consider the following matrix with entries in the complex numbers.

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Let $T : \mathbf{C}^4 \to \mathbf{C}^4$ (where $\mathbf{C}$ is the field of complex numbers) be the linear transformation defined by this matrix. Find subspaces V_1 and V_2 of $\mathbf{C}^4$ such that the following hold:

$$\mathbf{C}^4 = V_1 \oplus V_2, \text{ and } \quad T(V_1) \subset V_1, \quad T(V_2) \subset V_2.$$

Determine its Jordan canonical form.

2. [10 points] Suppose that $R = \mathbf{C}[x]$ where $\mathbf{C}$ is the field of complex numbers. Consider the ideal I generated by $x^2 - 1$ and $x^3 - 1$. Prove that I is principal and find its generator.

3.[10 points] Consider the polynomial ring $k[x, y]$ in two variables with coefficients in a field k. Let I be the ideal generated by x and J be the ideal generated by y. Prove that $k[x, y]/(I + J)$ is a module over k and determine its dimension.

4. [10 points] Keep the assumptions of problem 3. Prove that $k[x, y]/I$ is a module for $k[x, y]$ and show that

$$k[x, y]/I \otimes_{k[x,y]} k[x, y]/J \cong k.$$

You must prove that maps are well defined and give all details of proof.

5. [10 points] Keep the assumptions of problem 3. Prove that I is a free $k[x, y]$-module. Determine whether there exists a submodule M of $k[x, y]$ such that $k[x, y] \cong I \oplus M$.

Algebra Qualifying Exam
Part C - 2023

Choose 4 out of the following 5 problems.

1. Let $g(x) = x^n - 1$.
   (a) Prove that the Galois group of $g(x)$ over $\mathbb{Q}$ is an abelian group.
   (b) Find the smallest n such that the Galois group is not cyclic.

2. Let F_1, F_2 be two intermediate subfields of a Galois extension K/k. Let $H_1 = \mathrm{Gal}(K/F_1)$ and $H_2 = \mathrm{Gal}(K/F_2)$. Show that H_1 and H_2 are conjugate (as subgroups) in $\mathrm{Gal}(K/k)$ if and only if there exists an automorphism $\tau \in \mathrm{Gal}(K/k)$ such that $\tau(F_1) = F_2$.

3. (a) Compute the minimal polynomial of $\sqrt{2} + \sqrt{-2}$.
   (b) Determine the Galois group of the Galois closure of $\mathbb{Q}(\sqrt{2} + \sqrt{-2})/\mathbb{Q}$.

4. Let L/K be a finite Galois extension. For any prime divisor p of $[L : K]$, show that there exists a subfield F of L such that $[L : F] = p$ and $L = F(\alpha)$ for some $\alpha \in L$.

5. Denote by $\mathbb{F}_2$ the finite field with two elements.
   (a) Show that $f(x) = x^4 + x^3 + x^2 + x + 1$ is irreducible and separable over $\mathbb{F}_2$.
   (b) Denote by K a splitting field of f over $\mathbb{F}_2$, and let $\beta \in K$ be a root of f. Show that $K = \mathbb{F}_2(\beta)$.
   (c) Determine $\mathrm{Gal}(K/\mathbb{F}_2)$.

2022 ALGEBRA QUAL - PART A

Choose 4 out of the following 5 problems.

1. Let G be a group of order 132. Prove that G is not a simple group.

2. Let n be an integer $\geqslant 5$. Let H be a proper subgroup of the alternating group A_n. Prove that $[A_n : H] \geqslant n$.

3. Let G_1 and G_2 be finite groups and let P be a Sylow p-subgroup of $G_1 \times G_2$. Prove that $P = P_1 \times P_2$ for some Sylow p-subgroups P_1 of G_1 and P_2 of G_2.

4. Denote by $\mathbb{Z}[i]$ the subring $\{a + bi : a, b \in \mathbb{Z}\}$ of $\mathbb{C}$. Prove that $\mathbb{Z}[i]$ is a Euclidean domain.

5. Let R be a principal ideal domain and I a proper ideal of R. Assume that R is a local ring with unique maximal ideal M. Prove that $I = M^n$ for some integer $n \geqslant 1$.

ALGEBRA QUALIFYING EXAMINATION, PART B. 2022

Solve 4 questions out of 5. Every question is worth 10 points. The total possible score is 40 points. All answers must be justified.

1. Which, if any, of the following statements are true? **Justify your answers.**

a) $\mathbb{Q}$ is a projective $\mathbb{Z}$-module;

b) $\mathbb{Q}\otimes_{\mathbb{Z}} -$ is exact.

2. Let R be a ring, let M, N be left R-modules and let $f \in \mathrm{Hom}_R(M,N)$.

a) Prove that if f is left invertible then f is injective.

b) Is it true that if f is injective then f is left invertible, ***as a morphism of left*** R***-modules***? Prove or provide a counterexample.

3. Let R be a unital ring and let V, W be left unitary R-modules. The map $\psi_{V,W} : V^*\otimes_R W \to \mathrm{Hom}_R(V,W)$, $v^*\otimes w \mapsto \langle \cdot, v^*\rangle w$, $v^*\in V^*$, $w\in W$ is a (well-defined) homomorphism of abelian groups.

a) Let $f \in \mathrm{Hom}_R(W,W')$ and let $f_* : \mathrm{Hom}_R(V,W) \to \mathrm{Hom}_R(V,W')$ be the corresponding natural homomorphism of abelian groups. Prove that the following diagram commutes:

$$\begin{tikzcd}
V^*\otimes_R W \arrow[r, "\psi_{V,W}"] \arrow[d, "\mathrm{id}_{V^*}\otimes f"'] & \mathrm{Hom}_R(V,W) \arrow[d, "f_*"] \\
V^*\otimes_R W' \arrow[r, "\psi_{V,W'}"'] & \mathrm{Hom}_R(V,W')
\end{tikzcd}$$

b) Prove that if W is free then $\psi_{V,W}$ is injective for any V.

4. Let D be a division ring, U, V, W be left finite dimensional D-vector spaces and $f\in \mathrm{Hom}_D(U,V)$, $g \in \mathrm{Hom}_D(V,W)$. Prove that

(a) $\operatorname{rank}(g\circ f) \le \min(\operatorname{rank} f, \operatorname{rank} g)$;

(b) $\dim\ker(g\circ f) \le \dim\ker f + \dim\ker g$;

Which, if any, of these statements remain true if we drop the assumption that our vector spaces are finite-dimensional?

5. Let V be a finite dimensional vector space over a field K of characteristic 0. Prove that $S = \{v_1,\dots,v_k\}\subset V$ is linearly dependent if and only if $v_1\wedge\cdots\wedge v_k = 0$.

Algebra Qualifying Exam, Part C. 2022

Choose 4 out of the following 5 problems.

1. Let K be a field and $f(x) \in K[x]$ be a separable, irreducible polynomial of degree 5. If a, b are distinct roots of $f(x)$ with $K(a) = K(b)$, show that $K(a)/K$ is Galois.

2. Let K be a subfield of $\mathbb{C}$ such that K is a Galois extension of $\mathbb{Q}$ with $[K : \mathbb{Q}]$ odd. Show that $K \subset \mathbb{R}$.

3. Let n be a positive integer. Show that $\mathbb{C}(t)/\mathbb{R}(t^n)$ is a Galois extension, and determine its Galois group. Here t is an indeterminate and $\mathbb{C}(t)$ is the rational function field.

4. Let L/K be an algebraic extension of fields of characteristic 0. Assume that for every $\alpha \in L$, the extension $K(\alpha)/K$ has degree ≤ 2. Show that $[L : K] \leq 2$.

5. Let $F \subset E$ be a Galois extension with the Galois group A_4 (the alternating group with 12 elements).
   (a) Find the number of distinct fields K such that $F \subset K \subset E$ and K is an extension of F of index 6. Justify your answer.
   (b) How many such extensions are Galois over F? Why?

2021 Algebra Qual, Part A

Solve 4 problems. Please clearly indicate the 4 problems to grade.

1. Find the number of elements of order 11 in a simple group of order $748 = 2^2 \cdot 11 \cdot 17$.

2. Let G be a finite group and p be a prime number. Let P be a Sylow p-subgroup of G and let H be a p-subgroup of G such that $H \subseteq N_G(P)$, where $N_G(P)$ denotes the normalizer of P in G. Show that $H \subseteq P$.

3. Let G be a finite simple group of even order. Prove that G is generated by elements of order 2.

4. Let G be a group and let Z denote its center. Suppose that G/Z is cyclic. Prove that G is abelian.

5. Let $D = \mathbb{Z}[\,i\sqrt{5}\,] = \{a + b\,i\sqrt{5} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$, where $i^2 = -1$. Show that the ring D is not a principal ideal domain.

2021 Algebra Qual, Part B

Do any 4 problems.

1. Recall that if k is a field, then the ring $M_{n\times n}(k)$ of $n \times n$ matrices with entries in k has no nontrivial 2-sided ideals.
   Prove that there exists a nontrivial 2-sided ideal of $M_{n\times n}(\mathbb{Z})$.

2. Let $R = \mathbb{C}[x]$, and let M be an R-module which is finite dimensional as a vector space over $\mathbb{C}$. Suppose there exists a nonzero vector $m \in M$ which is not an eigenvector for x. Prove the restriction map
$$\mathrm{Hom}_R(M, M) \to \mathrm{Hom}_{\mathbb{C}}(M, M)$$
   is *not* surjective.

3. Let R be a unital ring. Show that the following two conditions are equivalent:
   a) Every unital R-module is projective.
   b) Every unital R-module is injective.

4. Let R be a unital commutative ring, and let M, N, P be R-modules. Prove the following R-modules are isomorphic:
$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) \quad \text{and} \quad \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, P))$$

5. Let $R = \mathbb{C}[x]$ and $M = R/(f_c)$, where $c \in \mathbb{C}$ and $f_c(x) := x^2 + c$. Prove that the following two statements are equivalent:
   a) There exists an isomorphism $M \cong M' \oplus M''$, where M' and M'' are nontrivial R-modules.
   b) $c \neq 0$.

2021 Algebra Qual, Part C

Do any 4 problems. Please clearly indicate the 4 problems to grade.

1. Let K be a field. Let $F = K(x_1, \ldots, x_n)$, the field of rational functions in n indeterminates $x_1, \ldots, x_n$ over K. Let $\sigma : F \to F$ be a K-homomorphism. Let $E = \sigma(F)$. Prove that F/E is a finite extension.

2. Is $\mathbf{Q}(i\sqrt{3},\, 5^{1/3})/\mathbf{Q}$ a Galois extension? Justify your answer.

3. Let F/K be a normal algebraic extension. Let $f(x) \in K[x]$ be an irreducible polynomial. Suppose that $u, v \in F$ are two roots of $f(x)$. Prove that there exists a K-automorphism $\sigma : F \to F$ such that $\sigma(u) = v$.

4. Let F be a subfield of $\mathbf{C}$ such that $F/\mathbf{Q}$ is a finite Galois extension whose Galois group is isomorphic to A_5. Prove that $F \cap \mathbf{Q}(e^{2\pi i/n}) = \mathbf{Q}$ for every integer $n \geqslant 1$. (You may use the fact that A_5 is a simple group.)

5. Let K be a finite field. Let $f(x) \in K[x]$ be a monic irreducible polynomial. Prove that $f(x)$ divides $x^{q^n} - x$, where $q = |K|$ and $n = \deg f(x)$.

ALGEBRA QUALIFYING EXAMINATION, PART A

SEPTEMBER 29, 2020

Choose four problems out of five; indicate which ones should be graded. Each problem is worth 10 points. ***Show your work and justify all answers.***

1. Let M be an abelian monoid (and so $M \times M$ is an abelian monoid in a natural way). Define a relation $\equiv$ on $M \times M$ by $(\alpha, \beta) \equiv (\alpha', \beta')$, $\alpha, \alpha', \beta, \beta' \in M$ if and only if $\alpha + \beta' + \gamma = \alpha' + \beta + \gamma$ for some $\gamma \in M$. This is a congruence relation and so the set of equivalence classes with respect to $\equiv$ is a monoid. In fact, it is an abelian group called the *Grothendieck group* of M.

(a) Define a natural homomorphism of monoids i from M to its Grothendieck group. Prove that i is an isomorphism if and only if M is an abelian group.
(b) Find the Grothendieck group of the multiplicative monoid of all non-zero integers.
(c) Let $M = \mathbb{Z}_{\geq 0} \cup \{\infty\}$ where $\mathbb{Z}_{\geq 0}$ is the additive monoid of non-negative integers while $\infty + x = \infty = x + \infty$ for all $x \in M$. Prove that the Grothendieck group of M is trivial.

2. Let G be a group and suppose that G has a subgroup H of finite index $n > 1$.

(a) Construct a natural group homomorphism $G \to S_n$ and find its kernel.
(b) Suppose that G is simple. Use (a) to prove that G is isomorphic to a subgroup of A_n.

3.

(a) Find the maximal possible order of an element of S_5 and the number of elements of S_5 of that order without listing all of them.
(b) Identify the Sylow 2-subgroups of S_5 as abstract groups and find their number without listing all of them.

4. Let F be a field, let k be a positive integer and let R_k be the subset of $F[x]$ consisting of all polynomials with coefficients of x^i with $1 \leq i \leq k$ equal zero. Show that R_k is a subring of $F[x]$. Is R_3 a unique factorization domain? Prove or provide a counterexample.

5. Let R be a commutative ring. Given an ideal J of R, denote

$$\sqrt{J} = \{r \in R \,:\, r^n \in J \text{ for some positive integer } n\}.$$

(a) Prove that $\sqrt{I}$ is an ideal of R and that $\sqrt{\sqrt{I}} = \sqrt{I}$ for any ideal I of R.
(b) Let $R = \mathbb{Z}$. Find all ideals I in R such that $\sqrt{I} = I$.
(c) Let $S \subset R$ be multiplicative. Prove that $S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$ for any ideal I of R.

Algebra Qualifying Exam - Part B
UC Riverside - September 29 - 2020

Solve 4 of the 5 problems below. Please clearly indicate which 4 problems you want to be graded. Each question is worth 10 points.

(1) Let R be a ring and suppose we have a commutative diagram of R-modules and R-module homomorphisms, where the rows are exact sequences, as follows:

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & A_3 \arrow[r] \arrow[d, "h_3"] & 0 \\ 0 \arrow[r] & B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r] & 0. \end{tikzcd}$$

Show that if h_1 and h_3 are injective, then h_2 is injective.

(2) Let R be a principal ideal domain and let M be a finitely generated R-module. Show that the following are equivalent.

(a) M is free.

(b) M is projective.

(c) M is torsion free.

Note: You can use the structure theorem for finitely generated modules over a principal ideal domain.

(3) The inclusion of polynomial rings $\mathbb{Z}[x,y] \subset \mathbb{Z}[x,y,z,w]$ makes $\mathbb{Z}[x,y,z,w]$ into a $\mathbb{Z}[x,y]$-module. Show that

$$\frac{\mathbb{Z}[x,y]}{(x^3+y^2)} \otimes_{\mathbb{Z}[x,y]} \mathbb{Z}[x,y,z,w] \cong \frac{\mathbb{Z}[x,y,z,w]}{(x^3+y^2)\mathbb{Z}[x,y,z,w]}.$$

(4) Let V and W be two vector spaces over a field k, and let $f : V \to W$ be a linear map. Prove that f is surjective if and only if its dual map f^* is injective.

(5) Let k be a field and let $f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 \in k[x]$. Let us denote the class of x in $k[x]/(f(x))$ by $\overline{x}$ and let

$$\Psi : \frac{k[x]}{(f(x))} \to \frac{k[x]}{(f(x))}$$

be the k-linear function given by multiplication by $\overline{x}$. Find an ordered basis $\mathcal{B}$ of $k[x]/(f(x))$ such that the matrix that represents Ψ with respect to $\mathcal{B}$ is in rational canonical form. Justify your answer.

Qualifying Exam, 2020, Part C

Answer any four of the following questions. Each is worth 10 points. All work must be shown, all answers must be justified at each step. An answer should not just point to a theorem in the book.

1. Let $\bar{\mathbb{Q}}$ be the algebraic closure of the field $\mathbb{Q}$ of rational numbers. Suppose that D is an integral domain and $\bar{\mathbb{Q}} \supset D \supset \mathbb{Q}$. Prove that D is a field.

2. Let F be a field with eight elements. Find all subfields of F and explain why you have found them all.

3. Let $\zeta \in \mathbb{C}$ be a primitive fifth root of unity and let F be the splitting field of $x^5 - 1$ over $\mathbb{Q}$. Prove that $F \supset \mathbb{Q}$ is a Galois extension and determine its Galois group.

4. Determine the Galois group of $x^3 - x - 1$ over $\mathbb{Q}(\sqrt{2})$.

5. Consider the polynomial $f(x) = 3x^{11} + 4x + 1 \in \mathbb{K}[x]$, where $\mathbb{K}$ is a field. Let $F \supset \mathbb{K}$ be the splitting field of this polynomial. Give (with complete justification) a necessary and sufficient condition on $\mathbb{K}$ for F to be an algebraic and Galois extension of $\mathbb{K}$.

ALGEBRA QUALIFYING EXAMINATION, PART A

SEPTEMBER 23, 2019

Solve any four problems out of five; indicate which ones should be graded. Each problem is worth 10 points. The maximal possible score is 40 points. ***All answers must be justified!*** In particular, for questions "Is it true that...?" you should provide either a proof or a counterexample.

1. Let M be an abelian monoid written multiplicatively (and so $M \times M$ is an abelian monoid in a natural way). Define a relation $\equiv$ on $M \times M$ by $(\alpha, \beta) \equiv (\alpha', \beta')$, $\alpha, \alpha', \beta, \beta' \in M$ if and only if $\alpha\beta'\gamma = \alpha'\beta\gamma$ for some $\gamma \in M$.

(a) Show that $\equiv$ is a congruence relation and prove that the set $G(M)$ of equivalence classes for this relation is a group. Why is it important that M is abelian?

(b) Identify $G(M)$ when $M = \mathbb{Z}_{\geq 0}$ (the *additive* monoid of non-negative integers) and when $M = \mathbb{Z}_{>0}$ (the *multiplicative* monoid of positive integers);

(c) Define a natural homomorphism of monoids $i : M \to G(M)$. What property should M have to guarantee that i is injective?

2. Find, without listing all of them, the number of elements of order 2, the number of conjugacy classes and the number of Sylow 5-subgroups in S_5.

3. Let G be a finite group and let $p \mid |G|$ be a prime. Let H be the intersection of all Sylow p-subgroups of G. Prove or find a counterexample to the following statements.

(a) H is normal in G;

(b) Every subgroup of H is normal in G;

(c) Every normal p-subgroup of G is contained in H.

4. Let $R = \mathbb{Z}[\sqrt{-1}] = \{x + y\sqrt{-1} : x, y \in \mathbb{Z}\} \subset \mathbb{C}$ (the ring of Gaussian integers).

(a) Find a prime ideal in R;

(b) Find an ideal I in R such that $I \cap \mathbb{Z}$ is prime but I is not prime. Here we identify $\mathbb{Z}$ with a subring of R in a natural way.

You may assume to be known that R is a Euclidean domain with $\varphi : R \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ defined by $\varphi(x + y\sqrt{-1}) = x^2 + y^2$, $x, y \in \mathbb{Z}$.

5. Let $p \neq q$ be primes and $k > 0$. Let R be the ring $\mathbb{Z}_{p^k q}$ and let S be the image of $\{p^i : i > 0\} \subset \mathbb{Z}$ in R under the canonical projection $\mathbb{Z} \to R$.

(a) Is the natural homomorphism $R \to S^{-1}R$ injective?

(b) Find $S^{-1}R$. What are ideals in $S^{-1}R$?

Qualifying Exam, 2019, Algebra Part B.

Answer any two of the following questions. Each is worth 20 points.

1. Suppose that V is a complex vector space.

(i) Define the dual vector space V^* and prove that $\dim V = \dim V^*$ if $\dim V < \infty$. Next define the canonical map $V \to (V^*)^*$ and show that it is injective. Prove that if $\dim V = \infty$ then this map is not surjective. (In fact show that the identity map on $(V^*)^*$ will not be in the image).

(ii) Suppose that $T : V \to V$ is a linear transformation. Define the corresponding transformation $T^* : V^* \to V^*$.

(iii) Let V be a three dimensional complex vector space and $T : \mathbf{C}^3 \to \mathbf{C}^3$ be the linear transformation given by the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Determine the matrix of T^*.

2.(a) Consider the polynomial ring $k[x, y]$ in two variables with coefficients in a field k. Let I be the ideal generated by x and J be the ideal generated by y. Prove that $k[x, y]/I$ is a module for $k[x, y]$ and show that

$$k[x, y]/I \otimes_{k[x,y]} k[x, y]/J \cong k.$$

You must prove that maps are well defined and give all details of proof.

(b) Let I be as in part (a). Prove that I is a free $k[x, y]$-module. Determine whether there exists a submodule M of $k[x, y]$ such that $k[x, y] \cong I \oplus M$.

3. (a) Suppose that $R = \mathbf{Z}_2[x]$. Prove that R is a principal ideal domain. You must show that if I is an ideal in R then it must be principal. You may not quote a theorem for your answer.

(b) Consider the module $A = R/(1 + x^2)(1 + x^3)$ where R is as in part (a). Find the invariant factors and elementary divisors of A.

2019 Algebra Qual - Part C

Solve any 4 out of the following 5 problems. Indicate which ones should be graded.

(1) Suppose F/K is a field extension of degree n and $f \in K[x]$ is a polynomial of degree d. Prove that if f is irreducible over K and $\gcd(n,d) = 1$, then f is irreducible over F.

(2) Suppose F/K is a normal algebraic extension, M/K is any algebraic extension, and $\sigma : F \to M$ is a K-homomorphism. Prove that if $\sigma' : F \to M$ is any K-homomorphism, then the image of σ' is equal to the image of σ.

(3) Suppose F/K is a separable finite extension and M/K is an algebraic closure of K containing F. Suppose $u \in F$ is an element such that $\sigma(u) = u$ for every K-homomorphism $\sigma : F \to M$. Prove that $u \in K$.

(4) List all the intermediate fields K of $\mathbf{F}_{2^{100}}/\mathbf{F}_{2^{10}}$ and indicate in a diagram their inclusion relations. List the corresponding Galois groups $\mathrm{Aut}_K(\mathbf{F}_{2^{100}})$.

(5) Let $F = K(x_1, x_2)$ where K is a field and x_1, x_2 are indeterminates. Explain whether $\{x_1^2, x_2^2\}$ is a transcendence basis of F/K.

Algebra Qualifier, Part A

September 24, 2015

Do four out of the five problems. Cross out the number of the problem that you don't want me to grade.

1. Let G be a finite abelian group of order m and let p be a prime integer dividing m.

(a) Prove that if G has exponent n, that is $x^n = 1_G$ for each $x \in G$, then $(G : 1_G)$ divides n^k for some positive integer k.

(b) Using induction on $(G : 1_G)$, show that G has a subgroup of order p.

2. How many elements of order 7 are there in a simple group of order 168? Prove your answer.

3. (a) Define the characteristic of a ring.

(b) Give an example of a commutative ring R, of characteristic zero having a unique maximal ideal and a non-maximal prime ideal P such that the characteristic of R/P is not zero.

4. Let P be a p-Sylow subgroup of a finite group G and let H be a p-subgroup of G with $H \subseteq N_P$, the normalizer of P. Show that $H \subseteq P$.

5. Let G be a finite group of order $p^n q$, p and q primes with $p > q$. Show that G is not simple.

ALGEBRA QUALIFYING EXAMINATION, PART B

Solve any four problems out of five. Each problem is worth 10 points. The maximal possible score is 40 points. ***All answers must be justified!*** In particular, for questions "Is it true that...?" you should provide either a proof or a counterexample.

All rings are assumed to be unital and all modules are assumed to be unitary and left unless specified otherwise.

1. Let R be a ring and let M be an R-module. Then $\operatorname{End}_R M := \operatorname{Hom}_R(M, M)$ is a unital ring (usually non-commutative) and M is naturally a unitary $\operatorname{End}_R M$-module via $f.m = f(m)$, $m \in M$, $f \in \operatorname{End}_R M$.

(a) Suppose that M is such that $\mathrm{id}_M - f$ is invertible for any non-invertible $f \in \operatorname{End}_R M$. Prove that M is indecomposable as an R-module.

(b) If R is a division ring, prove that M is simple as an $\operatorname{End}_R M$-module. Is this statement still true if M is free but R is not a division ring?

2. Let R be a commutative ring and let E, F be free R-modules of the same finite rank. Prove that if $\psi \in \operatorname{Hom}_R(E, F)$ is surjective then it is an isomorphism. Is it true, under the same assumptions on E and F, that if ψ is injective then it is an isomorphism? Why do we need to assume that R is commutative?

3. Let M, M' be right R-modules and N, N' be left R-modules and let $f \in \operatorname{Hom}_R(M, M')$, $g \in \operatorname{Hom}_R(N, N')$.

(a) Is it always true that $\ker(f \otimes g) = \ker f \otimes_R N + M \otimes_R \ker g$?

(b) Give an example of an injective f such that $f \otimes 1 : M \otimes_R N \to M' \otimes_R N$ is not injective.

4. Let K be a field.

(a) Determine whether $K(x)$ is projective as a $K[x]$-module.

(b) Describe the $K[x]$-dual module of $K(x)$.

(c) Describe all isomorphism classes of $K[x]$-modules on which x acts nilpotently and which are 4-dimensional as K-vector spaces.

5. Let A and B be invertible $n \times n$-matrices over a field K. Prove that $A + \lambda B$ is invertible for all but finitely many $\lambda \in K$. Is this statement true if K is a commutative ring?

2018 ALGEBRA C QUAL

Name:

1. Let $K \to F \to L$ be field extensions, **not** assumed algebraic (much less finite-dimensional). For each of the following assertions, provide a proof or a counterexample.
   (i) If $K \to F$ and $F \to L$ are Galois and finite dimensional, then $K \to L$ is Galois.
   (ii) If $K \to L$ and $K \to F$ are Galois, then so is $F \to L$.
   (iii) If $K \to L$ is Galois, and $[F : K]$ is finite, then $F \to L$ is Galois.

2. Let $K \to F$ be a splitting field for a polynomial $f \in K[x]$ of degree $n \geq 1$. Prove that $[F : K]$ divides $n!$

3. Let L be an algebraic closure of the field $\mathbb{Z}_p$ where p is a prime number.
   (i) Prove that $\phi(x) = x^p$ defines a field automorphism of L, fixing $\mathbb{Z}_p$.
   (ii) Let n be a natural number. Prove that the fixed field F of ϕ^n, i.e.
   $F = \{x \in L : \phi^n(x) = x\}$ is a field with p^n elements.
   (iii) For F as above, prove that $\mathbb{Z}_p \to F$ is a Galois extension of degree n.

4. Let $K \to F$ be a Galois extension of degree 28. Prove that there exists a subfield E of F such that $[E : K] = 7$.

Algebra Qualifier, Part A

September 22, 2017

Do four out of the five problems. Cross out the number of the problem that you don't want me to grade.

1. Let G be a finite abelian group of order m and let p be a prime integer dividing m.

(a) Prove that if G has exponent n, that is $x^n = 1_G$ for each $x \in G$, then $(G : 1_G)$ divides n^k for some positive integer k.

(b) Using induction on $(G : 1_G)$, show that G has a subgroup of order p.

2. How many elements of order 7 are there in a simple group of order 168? Prove your answer.

3. (a) Define the characteristic of a ring.

(b) Give an example of a commutative ring R, of characteristic zero having a unique maximal ideal and a non-maximal prime ideal P such that the characteristic of R/P is **not** zero.

4. Let P be a p-Sylow subgroup of a finite group G and let H be a p-subgroup of G with $H \subseteq N_P$, the normalizer of P. Show that $H \subseteq P$.

5. Let G be a finite group of order $p^n q$, p and q primes with $p > q$. Show that G is not simple.

ALGEBRA QUALIFYING EXAMINATION, PART B

Solve 4 questions out of five. Every question is worth 10 points. The total possible score is 40 points. All answers must be justified.

All rings are assumed to be unital and all modules are assumed to be unitary and left unless specified otherwise. Given an R-module M and $m \in M$, denote $\operatorname{Ann}_R m = \{r \in R : rm = 0\}$.

1. Let R be a commutative ring and M be an R-module. Let $M^* = \operatorname{Hom}_R(M, R)$.

(a) Show that the assignments $\xi \otimes m \mapsto (m' \mapsto \xi(m')m)$, $m, m' \in M$, $\xi \in M^*$ define a homomorphism of R-modules $\psi : M^* \otimes_R M \to \operatorname{End}_R M$. Why do we need R to be commutative?

(b) Suppose that M is a free. Show that ψ is injective.

(c) Suppose that M is free of finite rank. Show that ψ is an isomorphism.

2. Let M be a cyclic R-module generated by some $m \in M$ and let N be an R-module. Let $I = \operatorname{Ann}_R m$.

(a) Prove that $\operatorname{Hom}_R(M, N) \cong \{n \in N : I \subset \operatorname{Ann}_R n\}$ as an abelian group.

(b) Assuming that R is commutative, prove that $\operatorname{End}_R M \cong R/I$ as a ring.

3. Let R be an integral domain and M be an R-module. Let $\tau(M) = \{m \in M : \operatorname{Ann}_R m \neq 0\}$. Prove that if $0 \to M \xrightarrow{f} M' \xrightarrow{g} M'' \to 0$ is an exact sequence of R-modules then $0 \to \tau(M) \xrightarrow{f|_{\tau(M)}} \tau(M') \xrightarrow{g|_{\tau(M')}} \tau(M'')$ is a left exact sequence of R-modules. Explain why we need to assume that R is an integral domain.

4. Let M, M' be right R-modules and N, N' be left R-modules and let $f \in \operatorname{Hom}_R(M, M')$, $g \in \operatorname{Hom}_R(N, N')$.

(a) Is it always true that $\ker(f \otimes g) = \ker f \otimes_R N + M \otimes_R \ker g$? Prove or provide a counterexample.

(b) Give an example of an injective f such that $f \otimes 1 : M \otimes_R N \to M' \otimes_R N$ is not injective.

5. Let R be a commutative ring.

(a) Let M be a finitely generated R-module and let $\mathfrak{m} \subset R$ be a maximal ideal such that $\mathfrak{m}M = 0$. Show that M admits a natural structure of a finite dimensional vector space.

(b) Let K be a field. Describe isomorphism classes of $K[x]$-modules M such that $\dim_K M = 5$ and $x^r M = 0$ for some $r > 0$.

Algebra Qualifying Examination, Part C

Questions 1 through 4 are worth 10 points each. Question 5 is worth 20 points. You may do either all of questions one through four or Question 5 and two of questions 1 through four. The total possible score is 40 points. All answers must be fully justified.

1. (a) Let L be a field extension of K. Let X be a subset of L consisting of algebraic elements such that $L = K(X)$. Prove that L is algberaic over K.
(b) Let x be transcendental over K and let $u \in K(x)$ satisfy $u = (x^2 + 1)^{-1}$. Prove that x is algebraic over $K(u)$ and that u is transcendental over K.

2. Suppose that F is a field extension of K.
(a) Find the unique maximal subfield E with $F \supset E \supset K$ with $\mathrm{Aut}_E F = \mathrm{Aut}_K F$.
(b) Suppose that F is Galois over K and assume that L is an intermediate field of the extension $F \supset K$. Give an example of F, L, K such that $F \supset K$ is Galois but the extension $L \supset K$ is not Galois.

3. Let $\mathbf{Q}$ be the field of rational numbers and let ζ be a primitive cube root of unity.
(a) Prove that $\mathbf{Q}(\sqrt[3]{2} + \zeta) = \mathbf{Q}(\sqrt[3]{2},\ \zeta)$.
(b) Determine the Galois group of the extensions $\mathbf{Q}(\sqrt[3]{2}) \supset \mathbf{Q}$, $\mathbf{Q}(\zeta) \supset \mathbf{Q}$ and $\mathbf{Q}(\sqrt[3]{2}, \zeta) \supset \mathbf{Q}$.

4. Let F be the splitting field over $\mathbf{Q}$ of $x^4 - 5$. Explain why F is a Galois extension of $\mathbf{Q}$. Detemine all the intermediate fields L of this extension.

5. Consider the polynomial $f(x) = x^6 + x^3 + 1 \in \mathbb{Z}_2[x]$ and let $F \supset \mathbb{Z}_2$ be the splitting field of this polynomial.
(a) Explain why F is a finite field.
(b) Prove that if r is a root of this polynomial then $r^9 = 1$ and $r^m \neq 1$ for $m = 3, 6, 7$.
(c) Recall the theorem that every element of a finite field must satisfy the polynomial $x^{p^n} - x$ for some positive integer n and some prime p. Use this theorem and parts (a) and (b) of the problem to determine the cardinality of F.
(d) What is the Galois group of F over $\mathbb{Z}_2$ and determine all the intermediate fields of the extension $F \supset \mathbb{Z}_2$.

2016 Algebra Qual - Part A

Choose 4 out of the following 5 problems.

(1) Let G be a group. Let $N \lhd G$ and $H < G$. Suppose that $[G : N]$ and $|H|$ are finite. Prove that if $[G : N]$ and $|H|$ are relatively prime, then $H < N$.

(2) Let G be a finite group. Suppose that $H < G$ and $H \neq G$.
(i) Prove that H has at most $[G : H]$ conjugates in G.
(ii) Prove that there exists a conjugacy class S in G such that $H \cap S = \emptyset$.

(3) How many elements of order 7 are there in a *simple* group of order 168?

(4) Let R be a commutative ring with identity. Let M be a *maximal* ideal of R.
(i) Prove that if R is a local ring, then $1 + x$ is a unit for every $x \in M$.
(ii) Prove that if $1 + x$ is a unit for every $x \in M$, then R is a local ring.

(5) Let R be a unique factorization domain and F its field of fractions. Let $f \in R[x]$ be a *monic* polynomial. Prove that if $c \in F$ is a root of f, then $c \in R$.

Algebra Qualifying Examination, Fall 2016, Part b

Answer any four of the following questions. All questions are worth 10 points

1. Let R be a commutative ring with identity and let a be a non–zero element in R. Suppose that P is a prime ideal properly contained in the principal ideal generated by a. Prove that $P = aP$. Suppose now that P is also principal. Prove that there exists $b \in R$ with $(1-ab)P = 0$ .What can you conclude about P if R is an integral domain and a is not a unit.

2. (a) Let R be a commutative ring with identity and regard R as a module for itself via left multiplication. Prove that this module is simple iff R is a field.
(b) Define a free module for a ring R. Suppose that R is a commutative ring with identity and satisfies the following condition: any submodule of a free module is free. Prove that R is a principal ideal domain.

3. Give examples to show that the following can happen for a ring R and modules M, N,
(i) $M \otimes_R N \ncong M \otimes_{\mathbb{Z}} N$, where $\mathbb{Z}$ is the ring of integers.
(ii) $u \in M \otimes_R N$ but $u \neq m \otimes n$ for any $m \in M$ and $n \in N$.
(iii) $u \otimes v = 0$ but $u, v \neq 0$.

4. Suppose that E is a three dimensional vector space over a field F and $f : E \to E$ is a non–zero linear transformation. Prove that there exists bases B_1 and B_2 of E such that the matrix of f is exactly one of the following.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5. Suppose that $D = (d_1, \cdots, d_n)$ is a diagonal matrix where the d_i, $1 \leq i \leq n$ are not necessarily distinct. What are the elementary and invariant factors of D? Suppose that A is similar to D. What can you say about its elementary divisors and invariant factors?

2016 Algebra Qual - Part C

(1) True/False: If E and F are extensions of $\mathbb{Q}$ such that $E \neq F$ and $[E:\mathbb{Q}] = [F:\mathbb{Q}] = 3$, then $[EF:\mathbb{Q}] = 9$. Prove or give a counterexample.

(2) Let p be a prime number and $a \in \mathbb{F}_p^*$.
(i) Prove that $x^p - x + a$ is irreducible and separable over $\mathbb{F}_p$.
(ii) What is the cardinality of the splitting field of $x^p - x + a$ over $\mathbb{F}_p$?

(3) Let n be a positive integer. Show that $\mathbb{C}(t^n) \subset \mathbb{C}(t)$ is a Galois extension and calculate its Galois group.

(4) Let ζ be a primitive 9th root of unity. Describe all the intermediate fields of the cyclotomic extension $\mathbb{Q} \subset \mathbb{Q}(\zeta)$. For each such intermediate field, give an explicit primitive element of the field as an extension of $\mathbb{Q}$.

QUALIFYING EXAMINATION, ALGEBRA, PART A, 2015

September 26, 2015

Solve any four questions; indicate which ones are supposed to be graded. Each question is worth 15 points. You must show all work and justify all statements either by referring to an appropriate theorem or by providing a full solution.

1.

(a) List all isomorphism classes of abelian groups of order 120. Is there a simple group of order 120?
(b) What is the maximal number of elements of order 5 in a group of order 120?
(c) How many conjugacy classes are there in S_5?

2. Let $p > q$ be primes.

(a) Describe *all* groups of order p^2 up to an isomorphism.
(b) Show that a group of order $p^n q$, $n > 0$, is solvable.

3. The action of a group G on a set X is called *transitive* if for every $x, x' \in G$ there exists $g \in G$ such that $gx = x'$.

(a) Show that the natural action of the symmetric group S_n on the set $\{1, \dots, n\}$ is transitive and find the stabilizer of an arbitrary element of that set.
(b) Suppose that a group G acts transitively on a set X. Prove that all subgroups $\operatorname{Stab}_G x$, $x \in X$ are conjugate and find $[G : \operatorname{Stab}_G x]$.

4. Recall that an element a of a ring is called nilpotent if $a^n = 0$ for some positive integer n. Prove the following statements for a *commutative unital ring* R:

(a) The set of all nilpotent elements in R is an ideal.
(b) R is local if and only if for all $x, y \in R$, $x + y = 1_R$ implies that x or y is a unit.
(c) If every non-unit in R is nilpotent then R is local.

5. Let $R = \mathbb{Z} \times \mathbb{Z}$ as an additive abelian group while the multiplication is defined by $(x, y) \cdot (x', y') = (xy' + yx', yy' - xx')$; then R is a commutative ring with unity $1_R = (0, 1)$. Answer the following questions (all answers must be justified).

(a) Is the ideal of R generated by $(0, 5)$ prime?
(b) Is R a domain? If so, describe its field of fractions.
(c) Choose a maximal ideal M in R and describe the localization of R at M.

Algebra Qualifier, Part B

September 26, 2015

Do 4 problems.

1. Show that if $\mathrm{Hom}_R(D, \ )$ preserves exactness of $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ for each D, then the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits.

2. Let B be an abelian group and let A be a subgroup of B. Show that any homomorphism $f : A \to \mathbb{Q}/\mathbb{Z}$ extends to a homomorphism $\overline{f} : B \to \mathbb{Q}/\mathbb{Z}$.

(3) (a) State the Hilbert basis theorem.

(b) Give an example of a Noetherian ring R and a Noetherian R-algebra B which is not a finitely generated R-algebra.

4. Let R be a principal ideal domain, let M be an R-module which can be generated by n elements, and let L be a submodule of M. Show that L can be generated by $\leq n$ elements.

5. Let R be an integral domain. Show that R is a PID if each finitely generated torsion-free R-module is free.

Algebra Qualifier 2015 - Part C

Do 4 out of the 5 problems.

(1) Let K be a field and $f \in K[x]$. Let n be the degree of f. Prove the theorem which states that there exists a splitting field F of f over K with $[F : K] \leqslant n!$.

(2) Let K be a subfield of $\mathbb{R}$. Let L be an intermediate field of $\mathbb{C}/K$. Prove that if L/K is a finite Galois extension of odd degree, then $L \subseteq \mathbb{R}$.

(3) Let K be a finite field of characteristic p. Prove that every element of K has a unique p-th root in K.

(4) Let $f(x) = x^5 - 4x + 2 \in \mathbb{Q}[x]$. Prove that $f(x) = 0$ is not solvable by radicals over $\mathbb{Q}$.

(5) Let F/K be a field extension whose transcendence degree is finite. Prove that if F is algebraically closed, then every K-monomorphism $F \to F$ is in fact an automorphism.

Algebra Qualifying Examination, Part A

You must show all work. You must justify all statements either by referring to or stating the appropriate theorem correctly or by providing a full solution. Referring to some problem in Hungerford is not an acceptable answer or reason. The hints are meant to suggest one approach to the problem. Attempt any four questions. Each question is worth 10 points.

1 (a). What are all the groups of order 9. You must explain your reasoning for the answer.

(b) Prove that a non–abelian group of order six is isomorphic to the dihredral group.

2. Let H be a subgroup of G and let $\varphi : G \to H$ be a homomorphism such that $\varphi(h) = h$. Assume that G is abelian. Prove that $G \cong H \times \ker \varphi$.

3. Prove that the symmetric group S_4 is generated by the elements $(1, 2, 3, 4)$ and $(1, 2)$. Write down all the Sylow sub groups of S_4. (Hint: As a start, write down for what primes p there will be a non–trivial Sylow p subgroup and what the cardinality of the subgroup is.)

4. (a) Suppose that H is a normal subgroup of G and $|H| = 2$. Prove that H is contained in the center of G.

(b) Suppose that H is a subgroup of G such that if $\varphi : G \to G$ is any automorphism of G then $\varphi(H) \subset H$. Prove that H is normal.

Hint: Think about the various ways in which G acts on itself.

5(a)How many elements of order 5 are there in a group of order 20.

5(b) Prove that a group of order 15 is not simple.

QUALIFYING EXAMINATION, ALGEBRA, PART B, FALL 2014

September 27

Solve four questions out of five; indicate which one is not supposed to be graded. Each item is worth 10 points. You must show all work and justify all statements either by referring to an appropriate theorem or by providing a full solution.

All rings are assumed to be unital and all modules are assumed to be unitary and left unless specified otherwise.

1. Let R be a commutative ring. Let $M \neq 0$ be a torsion R-module and $N \neq 0$ be a divisible R-module (that is, for each $n \in N$ and $r \in R \setminus \{0\}$, there exists $n' \in N$ such that $n = rn'$).
(a) Describe $M \otimes_R N$.
(b) Describe $N^* = \operatorname{Hom}_R(N, R)$ assuming that R is a domain and but not a field.

2. Let R be a ring and let M, N be left R-modules. Construct a natural homomorphism of abelian groups $M^* \otimes_R N \to \operatorname{Hom}_R(M, N)$. If $R = k$ is a field and M, N are finite dimensional k-vector spaces prove that this natural homomorphism is in fact an isomorphism of k-vector spaces.

3. Let R be a ring. Prove that if a projective R-module P is a homomorphic image of an R-module M, then $M = N \oplus Q$ for some submodules N, Q with $Q \cong P$.

4. Let k be a field. Describe all, up to an isomorphism, $k[x]$-module structures on k^5 such that x acts nilpotently. Describe $\operatorname{End}_{k[x]}(k[x]/(x^5))$.

5. Let R be a domain and let A be an $n \times n$ matrix over R. Prove that if a system of linear equations $Ax = 0$ has a non-zero solution then $\det A = 0$. Is the converse true?

PART C: Each problem is worth 10 points. Please do any 4 problems of your choice, and indicate which 4 you would like graded.

1. Let $f = x^3 - 7$, and let F be the splitting field of f over $\mathbf{Q}$. Determine the Galois group of f over $\mathbf{Q}$. Determine all intermediate fields lying between $\mathbf{Q}$ and F, and determine which of these fields are themselves Galois extensions of $\mathbf{Q}$.

2. Let f be a polynomial of degree n over a field K, and let F be the splitting field of f over K. Show that $[F : K]$ divides $n!$.

3.: Let K be a field, and $f \in K[x]$ an irreducible polynomial.

Part a: Show that if $Char(K) = 0$, then f is separable.
Part b: Show that if $K = \mathbf{Z}_p$ for some prime p, then f is separable. (Note: This is a special case of a more general result. You cannot just cite that result without proof).

4. Let p be any prime, and let n, m be positive integers Show that the field of order p^n has a subfield of order p^m if and only if m divides n, and that if the subfield exists then it is unique.

5. Let F be an extension of a field K. Define what it means for a subset $S \subseteq F$ to be a transcendence basis for F over K, and show that any extension possesses a transcendence basis.

Qualifying Examination, Algebra, Part A, Fall 2013

Answer four out of the five questions. Each question is worth 15 points.

1(a). Suppose that H is a normal subgroup of G and that $G \cong H \times K$. Prove that K is isomorphic to a normal subgroup of G.

1(b) Suppose that G, H, K are as in part (a). Prove that any homomorphism of groups $\varphi : H \to H$ gives rise to a canonical homomorphism from $\tilde{\varphi} : G \to G$. Give an example to show that φ is an isomorphism need not imply that $\tilde{\varphi}$ is an isomorphism.

2(a) Consider the symmetric group S_n acting on the set $S = \{1, 2, \cdots, n\}$ by permutations. What is the orbit of the element i? What is the stabilizer of the element i. Prove that the stabilizer subgroup of any two elements of S are conjugates.

2(b) Prove that the dihedral group D_3 is not isomorphic to the direct product of two subgroups.

3. Recall that a local ring is a non–zero commutative ring with a unique maximal ideal. Prove that if R is a local ring and $\varphi : R \to S$ is a surjective ring homomorphism onto a non–zero ring S, then S is local.

4. Consider the ring $\mathbf{C}[x]$ of polynomials in an indeterminate x with coefficients in the complex numbers. Find a maximal ideal containing the polynomial $f(x) = x^3 - 3x^2 + 2$. How many such maximal ideals are there? If we were working with the ring $\mathbb{R}[x]$ how many maximal ideals would there be containing $f(x)$. Find a polynomial $g(x) \in \mathbf{C}[x]$ such that the ideal generated by $f(x)$ and $g(x)$ is $\mathbf{C}[x]$.

5. Suppose that G is a group of order 105. For any prime p determine the cardinality of the Sylow p–subgroups of G. Prove that the intersection of any two distinct Sylow subgroups is the identity. Prove that G is not simple.

QUALIFYING EXAMINATION, ALGEBRA, PART B, FALL 2013

September 28

Solve five questions out of six; indicate which one is not supposed to be graded. Each item is worth 12 points.

All rings are assumed to be unital and all modules are assumed to be unitary and left unless specified otherwise.

1. Let R be a ring and let $I, J \subset R$ be ideals. Let

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

be a short exact sequence of R-modules.

(a) Prove that $IM_2 = 0$ implies that $IM_1 = 0 = IM_3$.

(b) Show that $IM_1 = 0 = JM_3$ implies that $IJM_2 = 0$. Give an example of a ring R, a short exact sequence of R-modules as above and an ideal $I \subset R$ such that $IM_1 = 0 = IM_3$ but $IM_2 \neq 0$.

2. Let R be a ring and let $M \neq 0$ be an R-module. Let $Z(R)$ be the center of R.

(a) Construct the natural homomorphism of rings $\psi : Z(R) \to \operatorname{End}_R M$ and describe its kernel.

The homomorphism ψ from part (a) induces a natural $Z(R)$-module structure on $\operatorname{End}_R M$; in particular, for any field $K \subset Z(R)$, $\operatorname{End}_R M$ is a K-vector space.

(b) Suppose that M and $K \subset Z(R)$ are such that $\operatorname{End}_R M$ is finite-dimensional as a K-vector space. Prove that M is a direct sum of finitely many indecomposable R-modules with the same property.

3. Let R be a commutative ring.

(a) Let $M \neq 0$ be a torsion R-module and $N \neq 0$ be a divisible R-module (that is, for each $n \in N$ and $r \in R \setminus \{0\}$, there exists $n' \in N$ such that $n = rn'$). Describe $M \otimes_R N$. Describe $N^* = \operatorname{Hom}_R(N, R)$ assuming that R is a domain and but not a field. Give an example of a divisible module.

(b) Find a necessary condition (on R) for a divisible R-module to be free. Is it also sufficient?

4. Let R be a commutative ring, E and F be free R-modules of the same finite rank. Prove that $f \in \operatorname{Hom}_R(E, F)$ is an epimorphism if and only if f is an isomorphism. Is there an analogous statement for f monic?

5. Let R be a ring, M, M' be right R-modules and N, N' be left R-modules. Let $f \in \operatorname{Hom}_R(M, M')$, $g \in \operatorname{Hom}_R(N, N')$. Is it always true that $\ker(f \otimes g) = M \otimes_R \ker g + \ker f \otimes_R N$? Prove or provide a counterexample.

6. Let K be a field. Describe all, up to an isomorphism, nilpotent $K[x]$-module structures on K^5 (that is, all elements of $xK[x]$ act nilpotently) and $\operatorname{End}_{K[x]}(K[x]/(x^5))$.

Algebra Qualifier, Part C

September 28, 2013

Do 4 out of the 5 problems. Indicate which of the 5 that I am not to grade.

(1.) Give an example of a finite extension of fields E/k having infinitely many intermediate fields F, $k \subseteq F \subseteq E$. Verify your assertions.

(2.) Let $q = p^n$, $p \in \mathbb{N}$ prime, and let F_q be a field with q elements and algebraic closure $\overline{F_q}$. Show that for each $m \in \mathbb{N}$, there exists a unique extension field $E \subseteq \overline{F_q}$ with $[E : F_q] = m$ and that the extension E/F_q is separable and normal.

(3.) Let K/k be a normal extension of fields and let K_0 be the maximal separable subextension of K. Show that K_0/k is normal.

(4.) Show that if k is a field of characteristic $p > 0$ and every algebraic extension E/k of the field k is separable, then $k^p = k$.

(5.) Prove or disprove the following. If K is a field and G is an arbitrary group of automorphisms of K, then $K^G = \{a \in K \mid \sigma(a) = a \text{ for all } \sigma \in G\}$ is a field with K/K^G algebraic.

Algebra Qualifier, Part A

September 29, 2012

Do four out of the five problems.

1. Let G be a finite group and let p be a prime integer.

(a) Show that if p divides $(G:1)$, then G contains an element of order p. (You may assume this holds if G is abelian.)

(b) Show that if each $x \in G$ has order p^k for some $k \geq 0$, then $(G:1)$ is a power of p.

2. Prove or disprove the following.

(a) The ring $\mathbb{Z}[\sqrt{-19}]$ is a PID.

(b) Each nonzero nonunit of $\mathbb{Z}[\sqrt{-19}]$ is a product of irreducible elements.

3. Let R be a commutative unitary ring. A multiplcative subset S of R is said to be *saturated* if $xy \in S$ for $x, y \in R$ implies $x, y \in S$.

(a) Show that if S is a saturated multiplicative subset of R, then $R \setminus S$ is a union of prime ideals of R.

(b) An element $a \in R$ is a *zero-divisor* if $ab = 0$ for some $b \neq 0$ in R. Show that the set of zero-divisors of R is a union of prime ideals of R.

4. How many elements of order 7 are there in a simple group of order 168? Prove your answer.

5. (a) Define the characteristic of a ring.

(b) Assume R is a commutative unitary ring having only one maximal ideal M. Show that the characteristic of R is either zero or a power of a prime.

(c) Show that if R/M has characteristic zero, where R is as in (b), then R contains a field.

(d) Give an example of a ring R, as in (b), of characteristic zero having a non-maximal prime ideal P such that the characteristic of R/P is **not** zero.

2012 RINGS QUALIFIER

Prove all your claims; proofs must be as self-contained as is feasible.

1.[20] Let R be a unitary commutative ring. Prove *from scratch* that R and $R \oplus R$ are not isomorphic as R-modules.

2. [40] In this problem only, advanced theorems may be used.

(i) Determine with proof the number of similarity classes of matrices over the field $\mathbb{C}$ of complex numbers, with characteristic polynomial $(x-1)^2(x-2)^3$

(ii) Determine with proof the number of similarity classes of 8×8 matrices over the field $\mathbb{C}$ of complex numbers, with minimal polynomial $(x-1)^3(x-2)$

(iii) Let k be an algebraically closed field. Let A be an $n \times n$ matrix over k. For $a \in k$, set

$$W_a(A) := \ker((A - aI_n)^n) \subset k^n.$$

A *quasi-eigenvector* for A is a nonzero vector $v \in W_a(A)$ for some $a \in k$. Prove that there exist $a_1, ..., a_m \in k$ such that

$$k^n = \bigoplus_{i=1}^{m} W_{a_i}(A).$$

(Hint: You may use the Jordan form theorem).

(iv) Let k be an algebraically closed field. Let A, B be $n \times n$ matrices over k such that $AB = BA$. Prove that there is a basis of k^n whose elements are quasi-eigenvectors for both A and B.

3. [20] Let $R = \mathbb{Z}_{20}, S = \{1, 4, -4\}$.

(i) Prove that the natural map $R \to S^{-1}R$ is surjective and identify its kernel.

(ii) What is the cardinality of $S^{-1}R$?

4. [20] (i) Prove *from scratch* that if R is a PID then R has the property that any submodule of a finitely generated R-module is finitely generated.

(ii) Give an example with proof of a commutative unitary ring R where this property fails.

Algebra Qualifying Examination, 2012, Part ~~A~~ C

Each question is worth 10 points. Answer any 4. Please remember, all answers need justification.

1. Let K be a field and let $K[x]$ be a polynomial ring in one variable and $K(x)$ the corresponding quotient field. Give an example of a monomorphism $\sigma : K(x) \to K(x)$ which is the identity on K but is not onto. (You must explain why your example is a homomorphism of fields and why it is a monomomorphism). Use this to deduce the following statement. Suppose that F is an extension field of K such that for every intermediate field E every monomorphism $\sigma : E \to E$ which is the identity on K is an automorphism. Then F is an algebraic extension of K.

2. Suppose that F is a field extension of K. Define the Galois group of F over K and the notion of F being Galois over K. Suppose that $E \supset L$ are intermediate fields of the extension such that $[E : L] = 2$. Prove that the Galois group of F over E is a subgroup of the Galois group of F over L of index at most 2. (You should prove the statement explicitly in this simple example and not just refer to the statement of the appropriate theorem).

3. Suppose that F is a finite–dimensional Galois extension of K and assume that the Galois group is the direct product $S_3 \times \mathbb{Z}_2$.

(i) If E is an intermediate extension, what are the possible values of $[E : K]$.
(ii) Prove that there exists two distinct intermediate fields L_1 and L_2 such that $[L_j : K] = 6$, and L_j is Galois over K, $j = 1, 2$. Are there any others with these properties?

4. Let F_7 be the cyclotomic extension of $\mathbb{Q}$ of order seven. If ζ is a primitive seventh root of unity, what is the irreducible polynomial over $\mathbb{Q}$ of $\zeta + \zeta^{-1}$. You must justify your answer.

5. Let F be a finite field of characteristic p. Prove that F is a separable extension of $\mathbf{Z}_p$. Suppose now that E is a finite–dimensional extension of F. Prove that E is separable over F. Now prove that any algebraic closure of F is algebraic and Galois over F.

6. Let $f \in \mathbb{K}[x]$ be a cubic whose discrminant is a square in K. If $charK \neq 2$ prove that f is either irreducible or factors completely in K. What happens if $charK = 2$?

Algebra Qualifying Examination, 2011, Part A

Each question is worth 10 points. A perfect score is 60 points

1(a). Suppose that G is a group and that H_1 and H_2 are two distinct subgroups of G. State and prove a sufficient condition for $H_1 \cup H_2$ to be a group.

1(b). Give an example of a group G and two subgroups K_1 and K_2 such that $K_1 \cup K_2$ is strictly contained in the subgroup generated by K_1 and K_2.

2. Suppose that G is a group and H, K are subgroups of G such that G is the internal direct product of H and K. Prove that H and K are normal subgroups of G and that $G/H \cong K$. Deduce that there does not exists a subgroup H of S_5 such that S_5 is isomorphic to the direct product of H and A_5.

3. Prove that a free abelian group is a free group if and only if it is cyclic. Give an example to show that a cyclic group may not be free.

4. How many subgroups of order 9 does the group $\mathbf{Z}_9 \oplus \mathbf{Z}_{27}$ have? How many non–isomorphic subgroups of order 9 does it have?

5. Suppose that H acts on a set S. Given $s \in S$ define the H–orbit of s. Prove that if $s' \in S$ is another element then either s' is in the orbit of s or that the orbits of s and s' are disjoint.

6. Prove that a group of order 200 is not simple.

7. Let t be an indeterminate and consider the ring of polynomials $\mathbf{R}[t]$ where $\mathbf{R}$ is the set of real numbers. Using the fact that $\mathbf{R}[t]$ is a principal ideal domain prove that the ideal generated by $t^2 + 2$ is maximal. Suppose now that we work with the ring $\mathbf{C}[t]$ where $\mathbf{C}$ is the set of complex numbers. Find a polynomial f such that the ideal generated by f is proper and strictly contains the ideal generated by $t^2 + 2$.

8. Suppose that R is the ring $\mathbf{Z}_6$ and $S = \{2, 4\}$ is a a subset of R. Prove that $S^{-1}R$ is a finite field and identify the field.

ALGEBRA QUALIFYING EXAMINATION, 2011, PART B

Each question is worth 10 points. A perfect score is 50 points.
All rings are assumed to be unital and all modules are assumed to be left and unitary unless specified otherwise.

1. Let I be a two-sided ideal in a ring R and let IM be the abelian subgroup of an R-module M generated by all elements of the form xm, $x \in I$, $m \in M$. Show that IM is an R-submodule of M, describe the natural left R-module structure on $R/I \otimes_R M$ and show that $R/I \otimes_R M \cong M/IM$ as left R-modules.

2. Prove Schur's lemma: if M is a simple module over a ring R then $\operatorname{End}_R M$ is a division ring. Is the converse true (prove or provide a counterexample)?

3. Let R be a ring and let M, N be left R-modules. Construct a natural homomorphism of abelian groups $M^* \otimes_R N \to \operatorname{Hom}_R(M, N)$. If $R = k$ is a field and M, N are finite dimensional k-vector spaces prove that this natural homomorphism is in fact an isomorphism of k-vector spaces.

4. Let K be a field. We say that a $K[x]$-module M is nilpotent if for every non-unit $p \in K[x]$, $p^n M = 0$ for n sufficiently large. Prove that a finitely generated nilpotent indecomposable $K[x]$-module is isomorphic to $K[x]/(x^k)$ for some $k > 0$.

5. Let R be a ring.

(a) (7 points) Prove that if a projective R-module P is a homomorphic image of an R-module M, then P is (isomorphic to) a direct summand of M (that is, $M = N \oplus P'$ for some submodules N, P' with $P \cong P'$).

(b) (3 points) Formulate and prove an analogous statement for injective modules.

6. Let R be a ring and M_i, N_i, $i = 1, 2, 3$ be R-modules. Consider a diagram

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, "f_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "\psi_2"'] & M_3 \arrow[r] & 0 \\ 0 \arrow[r] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \arrow[r] & 0 \end{tikzcd}$$

with exact rows (all maps are homomorphisms of R-modules). Suppose that there exists $\psi_1 \in \operatorname{Hom}_R(M_1, N_1)$ such that $g_1\psi_1 = \psi_2 f_1$. Prove that there exists $\psi_3 \in \operatorname{Hom}_R(M_3, N_3)$ that makes the diagram commute (that is, satisfies $g_2\psi_2 = \psi_3 f_2$). Which conditions should satisfy ψ_1 and/or ψ_2 to ensure that ψ_3 is surjective?

7. Let K be a field, $p \in K[x]$ be a monic polynomial and A be its companion matrix. Prove that p is the minimal polynomial of A. Write down the companion matrix of $q(x) = x^3 - x^2 + 2x - 1 \in \mathbb{Q}[x]$.

8. Let R be a domain and let A be an $n \times n$ matrix over R. Prove that if a system of linear equations $Ax = 0$ has a non-zero solution then $\det A = 0$. Is the converse true?

Algebra Part C, September 2011

1. Find a polynomial $f(x) \in \mathbb{Q}[x]$ with $\deg(f) = 101$ and f is not solvable by radicals. Justify your answer.

2. Give an example of an infinite field K with char $K = 5$ and a polynomial $f(x) \in K[x]$ such that the splitting field of f over K has degree 20 and is a cyclic extension of K. Justify your answer.

3. Let F be a cyclotomic extension of $\mathbb{Q}$ of order 24. Determine all intermediate fields and give the diagram of subfields of F. Among all these extensions which one(s) are normal? separable? radical? Justify your answer.

4. Let $F > E > K$ be fields. Prove or disprove each of the following statements.

(a). If F is Galois over E and E is Galois over K, then F is Galois over K.

(b). If F is purely inseparable over E and E is purely inseparable over K, then F is purely inseparable over K.

(c). If F is normal over E and E is normal over K, then F is normal over K.

ALGEBRA QUALIFIER. PART A

September 25, 2010

Do four out of the five problems. **All answers must be justified.**

1. Let G be a group and let A be an abelian group. Let $\theta : G \to \mathrm{Aut}(A)$ be a group homomorphism. Let $A \times_\theta G$ be the set $A \times G$ with the binary operation

$$(a, g)(a', g') = (a + \theta(g)(a'), gg').$$

(i) Prove that $A \times_\theta G$ is a group.
(ii) Find $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_m)$ such that the dihedral group D_m is isomorphic to $\mathbb{Z}_m \times_\theta \mathbb{Z}_2$. *Do not forget to prove the isomorphism!*

2.

(i) Let G be a group of order np where p is a prime. Find a sufficient condition for a subgroup of order p to be normal.
(ii) List all isomorphism classes of abelian groups of order 120.
(iii) Is there a simple group of order 120?
(iv) What is the maximal possible number of elements of order 5 in a group of order 120?

3. Let G be a group acting on a set X. We say that the action of G on X is transitive if for all $x, x' \in X$ there exists $g \in G$ such that $gx = x'$. Prove the following statements.

(i) For all $x \in X$, $Gx = X$. In particular, $[G : \mathrm{Stab}_G\, x] = |X|$ and if $|G|$ is finite then $|X|$ divides $|G|$.
(ii) The subgroups $\mathrm{Stab}_G\, x$ are conjugate for all $x \in X$.
(iii) Suppose that $|X| = k$ and let $\pi : G \to S_k = \mathrm{Bij}(X)$ be the group homomorphism given by the action. Then k divides $n = [G : \ker \pi]$ and n divides $k!$.

4. An element e in a ring R is said to be an idempotent if $e^2 = e$. The center $Z(R)$ of a ring R is the set of all elements $x \in R$ such that $xr = rx$ for all $r \in R$. An element of $Z(R)$ is called central. Two central idempotents $f, g \in R$ are called orthogonal if $fg = 0$. Suppose that R is a unital ring.

(i) If e is a central idempotent, then so is $1_R - e$, and e, $1_R - e$ are orthogonal.
(ii) eR and $(1_R - e)R$ are ideals and $R = eR \times (1_R - e)R$
(iii) If $R_1, \dots, R_n$ are rings with identity then the following statements are equivalent
  (a) $R \cong R_1 \times \cdots \times R_n$
  (b) R contains a set of orthogonal central idempotents $e_1, \dots, e_n$ such that $e_1 + \cdots + e_n = 1_R$ and $e_i R \cong R$, $1 \le i \le n$.
  (c) $R = I_1 \times \cdots \times I_n$ where I_k is an ideal of R and $R_k \cong I_k$.

5. An element a of a ring R is called nilpotent if $a^n = 0$ for some positive integer n. A ring is said to be local if it contains a unique maximal ideal. Prove the following statements.

(i) The set of all nilpotent elements in a commutative ring R is an ideal.

(ii) A commutative unital ring R is local if and only if for all $x, y \in R$, $x + y = 1_R$ implies that x is a unit.

(iii) Suppose that R is a commutative unital ring with the following property: if $x \in R$ is not a unit then x is nilpotent. Then R is local.

(iv) Let $\mathbb{Z}[i] = \{x + yi \,:\, x, y \in \mathbb{Z}\}$, $i^2 = -1$. Cleary, $\mathbb{Z}$ identifies with a subring of $\mathbb{Z}[i]$. Find a prime ideal in $\mathbb{Z}[i]$ and an ideal I such that $I \cap \mathbb{Z}$ is prime but I is not prime.

Qualifying Examination, Part B, 2010 Attempt as many questions as you like. A perfect score is 50.

Assume that all rings have identity.

1. (5 points)(a) Let V be a vector space over a field K of dimension r. Let $f \in \mathrm{Hom}_K(V, K)$. Prove that if f is non–zero, then it is surjective and determine the dimension of the kernel of f.

2. (7 points) (a) Suppose that R and S are commutative rings and that M is a (R, S)–bimodule. This means that M is a left R–module and a right S–module and the actions are compatible, i.e $r(ms) = (rm)s$, for all $r \in R$, $s \in S$ and $m \in M$. Let N be a left S–module. How does one define a left R–module structure on $M \otimes_S N$. What must you check to see that the action is well–defined? If we assume now in addition, that N is a (S, R)–bimodule what can you say about $M \otimes_S N$?

(b) (3 points) Suppose now that K is a field and let V, W be a vector space over K. Use (a) to show that $V \otimes_K W$ is also a vector space over K. What is the most natural way to find a basis for $V \otimes_K W$.

3. (5points) (a) Let V, W be a vector spaces over a field K. How does one define a vector space structure on $\mathrm{Hom}_K(V, W)$? Suppose now that $W = K$. Given a basis for V, how would you produce a natural basis for $V^* = \mathrm{Hom}_K(V, K)$? More generally, if $\dim V = r$ and $\dim W = s$ and you are given bases for V and W, find a natural basis for $\mathrm{Hom}_K(V, W)$.

(b) (10points) Let W be another vector space over K. Define the natural map of vector spaces $V^* \otimes W \to \mathrm{Hom}_K(V, W)$ and prove that it is an isomorphism of vector spaces.

4. (10 points) Let R be the polynomial ring $\mathbf{C}[t]$ in one variable with coefficients in the complex numbers and let I be the ideal generated by t^2 and let $M = R/I$. Prove that M has a proper non–zero submodule and that M cannot be written as a direct sum of proper non–zero submodules. Suppose now we take J to be the ideal generated by $t(t-1)$. Prove that the module $N = R/J$ is isomorphic to a direct sum of two proper non–zero submodules.

5. (5 points) Prove that an $n \times n$–matrix with entries in a field K is invertible iff 0 is not an eigenvalue of the matrix.

6. (10points) What is the companion matrix A of the polynomial $q = x^2 - x + 2$. Prove that q is the minimal polynomial of A.

7.(10 points) Suppose that P_1 and P_2 R–modules. Prove that $P_1 \oplus P_2$ is projective iff P_1 and P_2 are projective.

8. (10 points) Let $0 \to L \to M \to N \to 0$ be short exact sequence of R–modules such that we have have a short exact sequence

$$0 \to \mathrm{Hom}_R(N, L) \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, N) \to 0.$$

Prove that the original short exact sequence is split.

ALGEBRA QUALIFIER PART C

Problem 1. Prove that every finite field K is perfect.

Problem 2. Let E be a splitting field of the polynomial $f(x) = x^5 - 4x + 2$ over $\mathbb{Q}$. Show that this polynomial is irreducible in $\mathbb{Q}[x]$. Prove that the Galois group $\mathrm{Aut}_{\mathbb{Q}}(E)$ is isomorphic to S_5. Is it possible to find a radical extension F of $\mathbb{Q}$ such that $F \supset E$?

Problem 3. Let F be a field with p^n elements (here p is prime). Prove that F contains a primitive d-th root of unity for every divisor d of the number $p^n - 1$.

Problem 4. Let $K(x_1, x_2, \ldots, x_n)$ be the field of rational functions in n indeterminates. Show that $\{x_1, x_2, \ldots, x_n\}$ is a transcendence basis of $K(x_1, x_2, \ldots, x_n)$ over K.

Algebra Qualifying Exam

September 26, 2009

Part A

Do exactly four problems (at least one must be from 5-8).

1. Let $D_8 =< a, b >$ be the dihedral group of degree 8. What are the equivalent classes of D_8?

2. Let G be a group of order n. If $n > m!$ and G has m Sylow p-subgroups, then G is not simple.

3. Find all the Sylow 3-subgroups of S_6.

4. Let $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ be coproducts of the family $\{A_i |\ i \in I\}$ of objects of a category G. Prove that P and Q are equivalent.

5. Let $R = \mathbb{Z}_{96}$, and $S = \{2^i \in R |\, i = 1, 2, \ldots\}$.
   (a). Find $S^{-1}R$.
   (b). What are the ideals of $S^{-1}R$?

6. Find the center of the ring of all $n \times n$ matrices over $\mathbb{Z}_6$.

7. The ring $R = \{a + b\sqrt{2}\, i \mid a, b \in \mathbb{Z}\}$ is an Euclidean domain.

8. A UFD is integrally closed.

Algebra Qualifier '09 - Part B

Do any four problems.
Assume that all rings contain 1, and all modules are unitary, unless stated otherwise.

(1) Let U and W be subspaces of a finite dimensional vector space V. Prove that
$$\dim(U+W) + \dim(U \cap W) = \dim(U) + \dim(W).$$

(2) Prove that if m and n are coprime integers, then $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong 0$.

(3) Let R be a ring. Let M be a finitely generated R-module, and let N be a projective R-module. Prove that if $f : M \to N$ is a surjective homomorphism, then the kernel of f is finitely generated.

(4) Let $R = \mathbb{R}[x]$ and $A = Rz_1 \oplus Rz_2 \oplus Rz_3$, where
$\operatorname{ann}(z_1) = \big((x+1)^2(x^2+1)\big)$, $\operatorname{ann}(z_2) = \big((x^2+1)^2\big)$, $\operatorname{ann}(z_3) = \big(x^4-1\big)$.
Find the elementary divisors and the invariant factors of A.

(5) Let $A \in \mathrm{Mat}_n(\mathbb{C})$. Prove that A is conjugate to its transpose A^t.

Algebra Qualifier, Part C

September 26, 2009

Do four out of the five problems. Indicate which of the 4 that I am to grade.

1. Let K be a field of characteristic zero. Let $f \in K[X]$ be a cubic whose discriminant is a square in K. Show that either f is irreducible over K or factors completely over K.

2. Let G be a finite group of automorphisms of an integral domain A. Let $R = \{a \in A \mid \sigma(a) = a \text{ for each } \sigma \in G\}$. Show that each $a \in A$ satisfies a monic $f \in R[X]$. Further, if E and F are the quotient fields of A and R respectively, then E/F is separable.

3. Let $g_n(X)$ be the n-th cyclotomic polynomial over $\mathbb{Q}$, let p be a prime integer not dividing n, let $F = \mathbb{Z}/p\mathbb{Z}$. Suppose the canonical image $\overline{g}_n(X)$ of $g_n(X)$ in $F[X]$ remains irreducible in $F[X]$ and let $E = F[X]/(\overline{g}_n(X))$. Show that the Galois group $G(E/F)$ is isomorphic to the group of units $(\mathbb{Z}/n\mathbb{Z})^*$ of $\mathbb{Z}/n\mathbb{Z}$.

4. Show that if $f \in \mathbb{Z}/p\mathbb{Z}[X]$ is irreducible of degree n, then f divides $X^{p^n} - X$ in $\mathbb{Z}/p\mathbb{Z}[X]$.

5. (a) Define what it means for $f \in \mathbb{Q}[X]$ to be solvable by radicals.

(b) Show that $X^5 - 6X + 3$ is not solvable by radicals over $\mathbb{Q}$.

ALGEBRA QUALIFIER. PART A

September 27, 2008

Do four out of the five problems. **All answers must be justified.**

1. Let G be a group and let A be an abelian group. Let $\theta : G \to \mathrm{Aut}(A)$ be a group homomorphism. Let $A \times_\theta G$ be the set $A \times G$ with the binary operation

$$(a, g)(a', g') = (a + \theta(g)(a'), gg').$$

(i) Prove that $A \times_\theta G$ is a group.
(ii) Prove that the dihedral group D_m is isomorphic to $\mathbb{Z}_m \times_\theta \mathbb{Z}_2$ for some $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_m)$.

2. Prove that a finite group of order $p^n q$, $n > 0$, where $p > q$ are primes is not simple.

3. Let G be a group of order 120.

(i) List all isomorphism classes of abelian groups of order 120.
(ii) Can G be simple?
(iii) What is the maximal possible number of elements of order 5 in G?
(iv) How many conjugacy classes are there in S_6?

4. Let $\mathbb{Z}[i] = \{x + yi : x, y \in \mathbb{Z}\}$, $i^2 = -1$. This is a unital ring and $\mathbb{Z}$ identifies with a subring of $\mathbb{Z}[i]$.

(i) Is the ideal of $\mathbb{Z}[i]$ generated by 5 prime?
(ii) Is $\mathbb{Z}[i]$ a domain? If so, describe its field of fractions.
(iii) Choose a maximal ideal P in $\mathbb{Z}[i]$ and describe the localization of $\mathbb{Z}[i]$ at P.

5.

(i) Give an example of a category in which a morphism between two objects is epic if and only if it is surjective.
(ii) Give an example of a category $\mathcal{C}$ and of an epic morphism between two objects in $\mathcal{C}$ which is not surjective.

ALGEBRA QUALIFIER. PART B

September 27, 2008

Solve 4 out of the following 5 problems. All rings are assumed to be unital, and all modules are assumed to be unitary left modules unless otherwise stated.

(1) Prove that
$$\mathbb{Q}[x]/(x^5 - 4x + 2)$$
is a field. Show, on the other hand, that
$$\mathbb{Z}[x]/(x^5 - 4x + 2)$$
is not a field.

(2) Let R be a ring, and let A, B, C be three R-modules such that B is a submodule of A, and $C \cong A/B$. Prove that if C is a projective R-module, then $A \cong B \oplus C$.

(3) Let R be a commutative ring and I an ideal of R. Let A be a R-module and denote by IA the submodule of A generated by all elements of the form ra with $r \in I$ and $a \in A$. Prove that there is an isomorphism of R-modules
$$(R/I) \otimes_R A \cong A/IA.$$

(4) Let V and W be two vector spaces over a field k, and $f : V \longrightarrow W$ be a linear map. Prove that f is surjective if and only if its dual map f^* is injective.

(5) Find the Jordan normal form of the following matrix over the field of complex numbers:
$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Fields qualifier '08

ALGEBRA QUALIFIER. PART C

Name in block letters:

Do any 3 problems.

1. Let $f(x) = x^5 - x + 1 \in \mathbb{F}_5[x]$.
(a) Prove that f has no root in $\mathbb{F}_{25}$ (Hint: what polynomial identity holds for any element of $\mathbb{F}_{25}$?).
(b) Determine the splitting field and the full Galois correspondence for the polynomial $x^5 - x + 1$
(i) over $\mathbb{F}_5$;
(ii) over $\mathbb{F}_{25}$;
(iii) over $\mathbb{F}_{125}$.

2. Let F be a splitting field of $f \in K[x]$ over K. Prove that if an irreducible polynomial $g \in K[x]$ has a root in F, then g splits in linear factors over F. (This result is part of the theorem characterizing normal extensions and you may not, of course, quote this theorem or its corollaries).

3. Disprove (by example) or prove the following:
if $K \to F$ is an extension (not necessarily Galois) with $[F : K] = 6$ and $\mathrm{Aut}_K(F)$ isomorphic to the Symmetric group S_3, then F is the splitting field of an irreducible cubic in $K[x]$.

4. If $\mathbb{Z}_p \to F$ is a field extension of degree n then $x \mapsto x^p$ is a $\mathbb{Z}_p$-automorphism of F of order exactly n whose fixed field is $\mathbb{Z}_p$.

Algebra Qualifier, Part A

September 29, 2007

Do four out of the five problems.

1. If G is a group, a *left G-module* is an abelian group $(A, +)$ with a group action $\sigma : G \times A \to A$ such that if we denote $\sigma(g, a)$ by ga, we have $g(a + b) = ga + gb$ for all $g \in G$, a, $b \in A$. If A is a G-module, the *(external) semidirect product* $A \times_\sigma G$ is the set $A \times G$ with product defined by $(a, g)(b, h) = (a + gb, gh)$.

(a) Show that $A \times_\sigma G$ is a group.

(b) Show that $S_3 \cong \mathbb{Z}_3 \times_\sigma \{1, -1\}$ for some G-module action $\sigma : \{1, -1\} \times \mathbb{Z}_3 \to \mathbb{Z}_3$.

2. Let R be a PID and define $\varphi : R \setminus \{0\} \to \mathbb{N}_0$ by $\varphi(a) = n$ if $a = up_1p_2 \cdots p_n$ for a unit u and prime elements p_i of R. Consider the condition:

(*) for every a_1, $a_2 \in R$ there exists $d \in R$ such that $a_1R + a_2R = (a_1 + da_2)R$.

(a) Show that if R satisfies the condition (*), then R is a Euclidean domain with Euclidean function φ.

(b) Show that (*) implies that $\mathcal{U}(R) \to \mathcal{U}(R/A)$ is onto for each ideal A of R, where $\mathcal{U}(T)$ denotes the group of units of the commutative ring T.

(c) Show that $\mathbb{Z}$ does not satisfy (*).

3. Let S_5 operate on itself by conjugation. How many orbits does S_5 have?

4. Let G be a finite group of order $p^n q$, p and q primes with $p > q$. Show that G is not simple.

5. Give examples of the following and explain how you know they have the properties claimed.

(a) A Noetherian integral domain that is not a PID.

(b) An integral domain R of characteristic zero having a non-maximal prime ideal P such that the characteristic of R/P is not zero.

Algebra Qualifier, Part A

September 29, 2007

Do four out of the five problems.

1. If G is a group, a *left G-module* is an abelian group $(A,+)$ with a group action $\sigma : G \times A \to A$ such that if we denote $\sigma(g,a)$ by ga, we have $g(a+b) = ga + gb$ for all $g \in G$, a, $b \in A$. If A is a G-module, the *(external) semidirect product* $A \times_\sigma G$ is the set $A \times G$ with product defined by $(a,g)(b,h) = (a+gb, gh)$.

(a) Show that $A \times_\sigma G$ is a group.

(b) Show that $S_3 \cong \mathbb{Z}_3 \times_\sigma \{1,-1\}$ for some G-module action $\sigma : \{1,-1\} \times \mathbb{Z}_3 \to \mathbb{Z}_3$.

2. Let R be a PID and define $\varphi : R \setminus \{0\} \to \mathbb{N}_0$ by $\varphi(a) = n$ if $a = up_1p_2\cdots p_n$ for a unit u and prime elements p_i of R. Consider the condition:

(*) for every a_1, $a_2 \in R$ there exists $d \in R$ such that $a_1R + a_2R = (a_1 + da_2)R$.

(a) Show that if R satisfies the condition (*), then R is a Euclidean domain with Euclidean function φ.

(b) Show that (*) implies that $\mathcal{U}(R) \to \mathcal{U}(R/A)$ is onto for each ideal A of R, where $\mathcal{U}(T)$ denotes the group of units of the commutative ring T.

(c) Show that $\mathbb{Z}$ does not satisfy (*).

3. Let S_5 operate on itself by conjugation. How many orbits does S_5 have?

4. Let G be a finite group of order p^nq, p and q primes with $p > q$. Show that G is not simple.

5. Give examples of the following and explain how you know they have the properties claimed.

(a) A Noetherian integral domain that is not a PID.

(b) An integral domain R of characteristic zero having a non-maximal prime ideal P such that the characteristic of R/P is not zero.

ALGEBRA QUALIFIER. PART B

September 29, 2007

Solve four problems out of five. All rings are assumed to be unital and all modules are assumed to be unitary and left unless specified overwise.

1. Give an example of a ring R and an R-module M such that

(i) $- \otimes_R M$ is not exact

(ii) $\mathrm{Hom}_R(M, -)$ is not exact.

2. Let R be a ring and M_i, N_i, $i = 1, 2, 3$ be R-modules. Consider a diagram

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, "f_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "\psi_2"'] & M_3 \arrow[r] & 0 \\ 0 \arrow[r] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \arrow[r] & 0 \end{tikzcd}$$

with exact rows (all maps are homomorphisms of R-modules).

(i) Suppose that there exists $\psi_3 \in \mathrm{Hom}_R(M_3, N_3)$ such that $g_2\psi_2 = \psi_3 f_2$. Prove that there exists $\psi_1 \in \mathrm{Hom}_R(M_1, N_1)$ making the diagram commute. Which conditions should satisfy ψ_2 and/or ψ_3 to ensure that ψ_1 is injective?

(ii) Suppose that there exists $\psi_1 \in \mathrm{Hom}_R(M_1, N_1)$ such that $g_1\psi_1 = \psi_2 f_1$. Prove that there exists $\psi_3 \in \mathrm{Hom}_R(M_3, N_3)$ making the diagram commute. Which conditions should satisfy ψ_1 and/or ψ_2 to ensure that ψ_3 is surjective?

3. Let K be a field.

(i) Determine, with a proof, whether the field of rational functions $K(x)$ is a projective $K[x]$-module.

(ii) Describe the $K[x]$-module dual of $K(x)$.

(iii) Will the same results remain true if K is replaced by an integral domain R?

4. Let R be a domain, A be a $n \times n$ matrix over R.

(i) Prove that if the system of linear equations $Ax = 0$ has a non-trivial solution then $\det A = 0$.

(ii) Prove, or provide a counterexample to, the converse.

(iii) Which, if any, of these statements remain true if we drop the assumption that R is a domain? Prove or provide a counterexample.

5. Let R, S be commutative rings, $\varphi : R \to S$ be a ring homomorphism.

(i) Extend φ to a ring homomorphism $\bar{\varphi} : \mathrm{Mat}_n(R) \to \mathrm{Mat}_n(S)$ and show that $\det(\bar{\varphi}(A)) = \varphi(\det(A))$ for all $A \in \mathrm{Mat}_n(R)$.

(ii) Use part (i) to prove that the constant term of the characteristic polynomial of a matrix $A \in \mathrm{Mat}_n(R)$ equals $(-1)^n \det(A)$.

Algebra Qualifier Part C

Attempt any five, all are worth 10 points.

1. Consider the polynomial $x^5 - ax - 1 \in \mathbf{Z}[\mathrm{x}]$. Find all possible values of $a \in \mathbf{Z}$ for which the polynomial is irreducible in $\mathbf{Z}[\mathrm{x}]$.

2. Prove that $\mathbf{Q}(\sqrt{2}+\sqrt{3}) = \mathbf{Q}(\sqrt{2},\sqrt{3})$.

3. Let F be a Galois extension of a field K of degree 27. Prove that there exist Galois extensions of K contained in F of degree 3 and 9.

4. Prove that if the Galois group of the splitting field of a cubic over the rational numbers is Z_3 then all roots of the cubic are real.

5. Prove that a finite dimensional extension of a finite field is Galois.

6. Prove that in a finite field of characteristic p every element has a unique pth root. Give an example to show that the condition that the field be finite is necessary.

ALGEBRA QUALIFIER. PART A

September 27, 2007

Do four out of the five problems. **All answers must be justified.**

1. Let G be a group and let A be an abelian group. Let $\theta : G \to \mathrm{Aut}(A)$ be a group homomorphism. Let $A \times_\theta G$ be the set $A \times G$ with the binary operation

$$(a, g)(a', g') = (a + \theta(g)(a'), gg').$$

(i) Prove that $A \times_\theta G$ is a group.
(ii) Prove that the dihedral group D_m is isomorphic to $\mathbb{Z}_m \times_\theta \mathbb{Z}_2$ for some $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_m)$.

2. Prove that a finite group of order $p^n q$, $n > 0$, where $p > q$ are primes is not simple.

3. Let G be a group of order 120.
(i) List all isomorphism classes of abelian groups of order 120.
(ii) Can G be simple?
(iii) What is the maximal possible number of elements of order 5 in G?
(iv) How many conjugacy classes are there in S_6?

4. Let $\mathbb{Z}[i] = \{x + yi : x, y \in \mathbb{Z}\}$, $i^2 = -1$. This is a unital ring and $\mathbb{Z}$ identifies with a subring of $\mathbb{Z}[i]$.
(i) Is the ideal of $\mathbb{Z}[i]$ generated by 5 prime?
(ii) Is $\mathbb{Z}[i]$ a domain? If so, describe its field of fractions.
(iii) Choose a maximal ideal P in $\mathbb{Z}[i]$ and describe the localization of $\mathbb{Z}[i]$ at P.

5.
(i) Give an example of a category in which a morphism between two objects is epic if and only if it is surjective.
(ii) Give an example of a category $\mathcal{C}$ and of an epic morphism between two objects in $\mathcal{C}$ which is not surjective.

ALGEBRA QUALIFIER. PART B

September 27, 2008

Solve 4 out of the following 5 problems. All rings are assumed to be unital, and all modules are assumed to be unitary left modules unless otherwise stated.

(1) Prove that
$$\mathbb{Q}[x]/(x^5 - 4x + 2)$$
is a field. Show, on the other hand, that
$$\mathbb{Z}[x]/(x^5 - 4x + 2)$$
is not a field.

(2) Let R be a ring, and let A, B, C be three R-modules such that B is a submodule of A, and $C \cong A/B$. Prove that if C is a projective R-module, then $A \cong B \oplus C$.

(3) Let R be a commutative ring and I an ideal of R. Let A be a R-module and denote by IA the submodule of A generated by all elements of the form ra with $r \in I$ and $a \in A$. Prove that there is an isomorphism of R-modules
$$(R/I) \otimes_R A \cong A/IA.$$

(4) Let V and W be two vector spaces over a field k, and $f : V \longrightarrow W$ be a linear map. Prove that f is surjective if and only if its dual map f^* is injective.

(5) Find the Jordan normal form of the following matrix over the field of complex numbers:
$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Fields qualifier '08

Name in block letters:

Do any 3 problems.

1. Let $f(x) = x^5 - x + 1 \in \mathbb{F}_5[x]$.
(a) Prove that f has no root in $\mathbb{F}_{25}$ (Hint: what polynomial identity holds for any element of $\mathbb{F}_{25}$?).
(b) Determine the splitting field and the full Galois correspondence for the polynomial $x^5 - x + 1$
(i) over $\mathbb{F}_5$;
(ii) over $\mathbb{F}_{25}$;
(iii) over $\mathbb{F}_{125}$.

2. Let F be a splitting field of $f \in K[x]$ over K. Prove that if an irreducible polynomial $g \in K[x]$ has a root in F, then g splits in linear factors over F. (This result is part of the theorem characterizing normal extensions and you may not, of course, quote this theorem or its corollaries).

3. Disprove (by example) or prove the following:
if $K \to F$ is an extension (not necessarily Galois) with $[F : K] = 6$ and $\mathrm{Aut}_K(F)$ isomorphic to the Symmetric group S_3, then F is the splitting field of an irreducible cubic in $K[x]$.

4. If $\mathbb{Z}_p \to F$ is a field extension of degree n then $x \mapsto x^p$ is a $\mathbb{Z}_p$-automorphism of F of order exactly n whose fixed field is $\mathbb{Z}_p$.

Algebra

Syllabus for the qualifying examination

The topics marked with an asterisk are considered undergraduate material and will be only briefly reviewed in the graduate sequence Math 201A-B-C.

Groups

1.* Basic properties of groups and homomorphisms
2.* Cosets and Lagrange's Theorem
3.* Normal subgroups, fundamental homomorphism theorems
4.* Symmetric groups, Cayley's theorem
5.* Alternating and dihedral groups
6. Groups operating on a set
7. Sylow theorems
8. Construction of products and coproducts in the categories of groups and abelian groups
9. Free groups, presentations of groups

References

1. T. W. Hungerford, *Algebra*, Springer-Verlag, New York, 1974, Chapter 1, sections 1-6 of Chapter 2.
2. N. Jacobson, *Basic Algebra I*, W. H. Freeman and Company, San Francisco, 1974, Chapter 1.
3. S. Lang, *Algebra*, Addison Wesley, Menlo Park, CA, 1984, Chapter 1.

Rings and fields

1.* Homomorphism theorems and ideals
2.* Construction of the quotient field of an integral domain
3.* Definition and elementary properties of polynomial rings over fields
4.* Consequences of the Euclidean algorithm for the ideal theory and unique factorization in the ring of integers and the ring of polynomials in one variable over a field
5.* Elementary properties of field extensions including the degree of a finite extension
6. Maximal and prime ideals
7. Euclidean domains, principal ideal domains and unique factorization domains
8. Elementary properties of localization
9. Definition and universal properties of polynomial rings and groups rings
10. Unique factorization in polynomial rings
11. Galois groups of quadratic and cubic extensions
12. Finite fields and their Galois theory

References

1. T. W. Hungerford, *Algebra*, Chapter 3 and sections 1, 5 of chapter 5.
2. N. Jacobson, *Basic Algebra I*, Chapter 2 and sections 1, 13 of chapter 4.
3. S. Lang, *Algebra*, Chapter 2 and sections 1, 5 of chapter 7.

Modules and linear algebra

1.* Basic properties of bases and dimension of vector spaces
2.* The relationship between matrices and linear transformations
3.* Inner products and orthogonal bases, Gram-Schmidt process
4.* Determinants, eigenvalues, Cayley-Hamilton Theorem
5. Definitions of Noetherian and Artinian modules and inheritance of these properties by submodules and factor modules
6. Free modules and invariance of basis number
7. Tensor product and Hom and their behavior with respect to exact sequences and products and coproducts
8. Structure of modules over principal ideal domains and applications to linear algebra including rational and Jordan canonical forms
9. Definitions of tensor, symmetric and exterior algebras and their behavior on direct sums and free modules

References

1. Serge Lang, *Linear Algebra*, Springer-Verlag, New York
2. T. W. Hungerford, *Algebra*, Sections 1-6 of chapter 4, and chapter 7.
3. N. Jacobson, *Basic Algebra I*, Chapter 3.
4. S. Lang, *Algebra*, Sections 1-6 of chapter 3, sections 1-6 of chapter 13, chapter 15.

Algebra Qualifier, Part A
Fall 2006

1. Find all normal subgroups of D_{24}.

2. Find the Sylow 2-subgroups of S_5.

3. Let $H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}_{12} \right\}$ be a set of 3×3 matrices. Prove or disprove that H is a nilpotent group under ordinary matrix multiplication.

4.
(i) Determine the units in $\mathbb{Z}[\sqrt{-5}\,] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$.
(ii) Is 7 irreducible? Prove your answer.

5. Prove or disprove that the ring of Gaussian integers $\mathbb{Z}[i]$ is a PID.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Algebra Qualifier, Part B.

Attempt any four, all questions are worth 10 points.

1.(a) Let R be a ring with identity and M a left module for R. Recall that M is indecomposable if M cannot be written as a direct sum of two non–zero submodules. Prove that if $f : M \to M$ is a homomorphism of modules then $f^2 = f$ implies that either $f = 0$ or $f = id$.

(b) Suppose now that M is decomposable. Prove that there exists $f : M \to M$ a homomorphism of modules such that $f^2 = f$ and f different from zero and the identity.

2. Suppose that R is a ring with identity and $e \in R$ is such that $e^2 = e$.
(a) Prove that $(1 - e)$ has the same property.
(b) Prove that $Re \cap R(1 - e) = \{0\}$ and hence that $R = Re \oplus R(1 - e)$.
(c) Prove that the R–module Re is projective.

3. Let R be a ring with identity. Regard R as a right R–module in the usual way and let M be a right R–module. Prove that $Hom_R(R, M) \cong M$ as abelian groups.

4. Consider the ring $R = C[x]$ of polynomials in an indeterminate x with coefficients on C.

(a) Let M be a torsion free module for R with two generators. Prove that M is free of rank at most two.

(b) Prove that if M is a cyclic R–module and $M \neq R$ then M is torsion. Under what condition on the torsion ideal will M be simple?

5(a). Prove that if A and B are invertible $n \times n$ matrices with entries in an integral domain R, then $A + rB$ is invertible in the quotient field K for all but finitely many r.

(b) Prove that the minimal polynomial of a linear transformation of an n–dimensional vector space has degree at most n.

6 Suppose that Φ and Ψ are commuting linear transformation of an n–dimensional vector space E. Prove that if E_1 is a Φ–invariant subspace of E then E_1 is also Ψ–invariant. Use this to prove that if Φ and Ψ both have linear elementary divisors then there exists a basis of E with respect to which the matrix Φ and the matrix Ψ are both diagonal.

ALGEBRA QUAL PART C
10/29/06

Answers should be self-contained as much as possible. Points for each problem as indicated.

1. [4×10] Let $K \subseteq F$ be a finite-dimensional extension.
   (i) Define what it means for F to be separable over K.
   (ii) Prove from scratch that if K is a finite field then F is separable over K.
   (iii) Prove that if K is of characteristic zero then F is separable over K.
   (iv) Give an example of a non-separable finite-dimensional extension.

2.[20] Let K be a field with 9 elements. Prove from scratch that K has an extension of degree 2 and that any two such are isomorphic over K.

3. [4×10] Let $u = \sqrt{3+\sqrt{2}}$.
   (i) Determine the minimal polynomial of u over $\mathbb{Q}$.
   (ii) Prove that $F = \mathbb{Q}(u)$ is a splitting field of f over $\mathbb{Q}$.
   (iii) Prove that $\sqrt{7} \in F$.
   (iv) Determine the Galois group of F over Q.

Algebra Qualifier Part 1, Fall 2005

Answer any four questions.

1. Consider the additive quotient group $\mathbf{Q}/\mathbf{Z}$, where $\mathbf{Q}$ is the set of rational numbers and $\mathbf{Z}$ the set of integers.

(a) Show that every coset contains exactly one element $q \in \mathbf{Q}$ with $0 \leq q \leq 1$.

(b) Show that every element in $\mathbf{Q}/\mathbf{Z}$ has finite order but that there are elements of infinitely large order.

(c) Consider now the group $\mathbf{R}/\mathbf{Z}$ where R is the set of real numbers. Prove that any element of finite order in $\mathbf{R}/\mathbf{Z}$ is in $\mathbf{Q}/\mathbf{Z}$.

2. (a) Exhibit two distinct Sylow 2 subgroups of S_5 and an element of S_5 that conjugates one into another.

(b) How many elements of order 7 are there in a simple group of order 168.

3(a) Prove that D_8 is not isomorphic to $D_4 \times \mathbf{Z}_2$.

3(b) Prove that if G is a group of order p^n, then it has a normal subgroup of order p^k for all $0 \leq k \leq n$.

4. Let F be a field and let R be the subset of $F[x]$ consisting of polynomials whose coefficient of x is 0. Prove that F is a subring of R. Prove also that R is not a UFD by showing that $x^6 \in R$ has two different factorizations in R into irreducibles.

5. Let R be a commutative ring. Assume that $R[x]$ is a principal ideal domain. Prove that (a) R is a domain, (b) the ideal (x) of $R[x]$ generated by x is prime, (c) explain why (x) is then maximal and (d) conclude finally that R must be a field.

Hint: Recall the equivalent conditions for an ideal in a commutative ring to be prime (resp. maximal), and recall also the evaluation homomorphism ev_a for $a \in R$.

October 2005 UCR Rings qualifier

Prove all your claims; proofs must be as self-contained as is feasible. 100 points suffice or a full score.

1.(15 pts) Let $\phi : R \to S$ be a homomorphism of commutative unitary rings, $I < R$,$J < S$ ideals. Prove or disprove each of the following assertions.
(i) $\phi(I)$ is an ideal in S
(ii) If J is a prime ideal, so is $\phi^{-1}(J)$
(iii) If J is a maximal ideal, so is $\phi^{-1}(J)$

2. (20 pts) Let R be a commutative integral domain, $S \subset R$ a multiplicative system and M an R-module. Prove directly from the definitions that

$$S^{-1}M \simeq S^{-1}R \otimes_R M$$

(isomorphic as R-modules).

3. (20 pts) (i) Determine with proof $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{R}, \mathbb{Z})$.
(ii) Determine with proof whether $\mathbb{R}$ is a projective $\mathbb{Z}$-module.

4.(20 pts) Determine with proof whether the following $\mathbb{Z}$-modules are injective: (i)$\mathbb{Q}$, (ii)$\mathbb{Z}/12$

5.(40 pts)(i) Let

$$M = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Determine with proof the invariant factor and elementary divisor decompositions of the $\mathbb{Q}[x]$-module corresponding to M.
(ii) Let ditto with $\mathbb{Q}$ replaced by $\mathbb{R}$.
(iii) Let

$$M' = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}.$$

Determine with proof whether M and M' are similar as matrices over $\mathbb{Q}$ or $\mathbb{R}$.
(iv) Ditto with M' replaced by

$$M" = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}.$$

Algebra Qualifier, Part C

October 30, 2005

Do four out of the five problems.

1. Let p be a prime integer and let F_{p^n} denote the field with p^n elements in a fixed algebraic closure of F_p.

(a) Show that $F_{p^n} \subseteq F_{p^m}$ if and only if $n|m$.

(b) Let q be another prime integer and let $E = \cup_{i=1}^{\infty} F_{p^{(q^i)}}$. Show that E is an infinite extension of F_p which is not algebraically closed.

2. (a) Factor the polynomial $X^9 - 1$ over $\mathbb{Q}$.

(b) Is $X^6 + X^3 + 1$ irreducible over $\mathbb{Q}$? Explain.

(c) Give the Galois group of $X^6 + X^3 + 1$ over $\mathbb{Q}$. Explain.

3. Define what it means for a polynomial $f \in \mathbb{Q}[X]$ to be solvable by radicals over $\mathbb{Q}$.

(b) Show that $X^5 - 4X + 2$ is not solvable by radicals over $\mathbb{Q}$.

4. Give an example of a finite extension E/F of fields with infinitely many intermediate fields and explain why your example works.

5. Define the symmetric algebra $S(M)$ of a k-module M and state the universal property that it satisfies.

Algebra Qualifier, Part A

April 10, 2004

Do four out of the six problems.

1. Let $G = \mathbb{Q}/\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Z}$ being considered as additive groups. Prove that for any positive integer n, G has a unique subgroup $G(n)$ of order n, and that $G(n)$ is cyclic.

2. Let G be a group. Prove that if the group of automorphisms $\mathrm{Aut}(G)$ of G is cyclic, then G is abelian.

3. Let G be a finite group and $H \triangleleft G$ a normal subgroup of prime order p. Prove that H is contained in each Sylow p-subgroup of G.

4. Let $H < G$ be a proper subgroup of finite index in a group G. Prove that G does not equal the union of all subgroups of G conjugate to H.

5. Prove that a group of order 40 has a normal Sylow subgroup.

6. Let G be a finite group. Describe all group homomorphisms $\varphi : G \to F_2$, where F_2 denotes the free group on two elements.

Algebra Qualifier, Part B

April 10, 2004

Do four out of the six problems.

1. Let I and J be ideals in a commutative unitary ring R such that $I + J = R$ and let A be a left R-module. Show that $A/(I \cap J)A \cong A/IA \times A/JA$.

2. Let p be a prime integer and let $Z(p^\infty)$ be the subgroup of the additive group $\mathbb{Q}/\mathbb{Z}$ generated by the cosets of the form $1/p^i + \mathbb{Z}$, $i \in \mathbb{Z}$, $i \geq 0$. Let A be a subgroup of a finitely generated group B. Show that any homomorphism $f : A \to Z(p^\infty)$ extends to a homomorphism $g : B \to Z(p^\infty)$.

3. Give an example of a monomorphism $f : A \to B$ of abelian groups and an abelian group C such that the induced homomorphism $\mathrm{Hom}_{\mathbb{Z}}(B, C) \to \mathrm{Hom}_{\mathbb{Z}}(A, C)$ is not onto.

4. (a) Sketch the construction of $A \otimes_R B$.
(b) State the universal property of $A \otimes_R B$.

5. Let R be a commutative ring with identity and let X be an indeterminate. Let M be a maximal ideal of $R[X]$ such that $M \cap R = P$ is a maximal ideal of R. Show that $M = fR[X] + PR[X]$ for some monic $f \in R[X]$ such that the image $\overline{f}$ of f in $(R/P)[X]$ is irreducible. ($\overline{f}$ is obtained from f by reducing coefficients mod P.)

6. Give an example of an integral domain R which is not a UFD although each element of R factors into irreducibles in R. Justify your assertions.

Algebra Qualifier, Part C

April 10, 2004

Do four out of the six problems.

1. Suppose that F is an extension field of K. Let u, $v \in F$ be algebraic over K and assume that the degree of u is prime to the degree of v over K. What is the degree of the extension $K(u, v)$ over K? You must provide a proof for your answer.

2(a). Let F be an extension field of K. Define the Galois group of F over K. Suppose that L is a subfield of F containing K. When do we say that L is closed? Let G be the Galois group of F over K. Let H be a subgroup of G. When do we say that H is closed?

2(b). Prove that if H is a subgroup of G then its fixed field is closed.

3. Prove that a field F generated by an infinite set of separable elements is separable.

4. Suppose that K is the field of rational numbers and let f be an irreducible polynomial of degree 3 and discriminant D. Suppose $D \neq 0$. Prove that $D > 0$ iff f has three real roots. What conclusion can you draw if $D < 0$?

5. Suppose that $f \in K[x]$ is a monic polynomial whose roots are distinct and form a field. Prove that char$K \neq 0$. What can you say about f in this case?

6. Define a cyclotomic extension of order n of a field K. Define the n-th cyclotomic polynomial over K. Suppose that F_8 is a cyclotomic extension of order 8 over the field Q of rational numbers. Determine the Galois group of F_8 over Q.

Algebra Qualifier, Part A

October 30, 2004

Do four out of the five problems.

1. Let R be a UFD in which each nonzero prime ideal is maximal. Show that

(a) if a, $b \in R$ and $(a,b) = 1$, then $ax + by = 1$ for some x, $y \in R$.

(b) Show that each ideal of R that is generated by two elements is principal.

2. (a) Let G be a finite group and H a normal subgroup of G. Show that if H and G/H are solvable, then G is sovable also.

(b) Show that if H and K are solvable normal subgroups of G with $HK = G$ and $H \cap K = \{e\}$, then G is solvable.

3. Let S be a multiplicative subset of the commutative ring R and let I be an ideal of R. Show that $S^{-1}\mathrm{Rad}(I) = \mathrm{Rad}(S^{-1}I)$, where $\mathrm{Rad}(J) = \{x \in R \mid x^n \in J$ for some positive integer $n\}$.

4. Let p be a prime integer and let G be a finite p-group. Show that G has a nonzero center.

5. Let G be a finite simple group having a subgroup H of index n. Show the G is isomorphic to a subgroup of S_n.

October 2004 UCR Rings qualifier key

1. Let R be a ring.

(i) Prove that if R is unitary, then any left (resp. right, 2-sided) ideal is contained in a maximal left (resp. right, 2-sided) ideal.

$\because$ use Zorn's Lemma.

(ii) Prove that if R is unitary and commutative, then an ideal I of R is maximal iff R/I is a field.

$\because$easy.

2.(i) Prove that any nontrivial subgroup of $\mathbb{Z}^2$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$ but not to both.

$\because$ For subgroup A, let $A_1 = A \cap \mathbb{Z}e_1, A_2 = A/A_1$, iso to subgroup of $\mathbb{Z}e_2$. Because $\mathbb{Z}$ is a PID, A_1, A_2 both iso to zero or $\mathbb{Z}$ and the surjection $A \to A_2$ splits, so $A \simeq A_1 \oplus A_2$. Finally, $\mathbb{Z} \not\simeq \mathbb{Z}^2$, indeed there is no injective map $\mathbb{Z}^2 \to \mathbb{Z}$ because the images of e_1 and e_2 would be proportional.

(ii) Let A be a subgroup of $\mathbb{Z}^2$ isomorphic to $\mathbb{Z}^2$. Prove that $\mathbb{Z}^2/A$ is finite.

$\because$ In the above, $A_1 = me_1, A_2 = ne_2, m, n \neq 0$. Then $\mathbb{Z}^2/A$ is generated by 2 elements and killed by mn, hence finite.

3.(i) Decompose the following as direct sum of cyclic groups:

(a) $$\mathbb{Z}_{12} \otimes \mathbb{Z}_9$$

$\because$ $\mathbb{Z}_{12} \simeq \mathbb{Z}_4 \oplus \mathbb{Z}_3, \mathbb{Z}_4 \otimes \mathbb{Z}_9 = 0$ because 4 is a unit mod 9, $\mathbb{Z}_3 \otimes \mathbb{Z}_9$ is cyclic, killed by 3 and surjects to $\mathbb{Z}_3$, hence $\simeq \mathbb{Z}_3$.

(b) $$\mathrm{Hom}(\mathbb{Z}_{12}, \mathbb{Z}_9)$$

$\because= \mathrm{Hom}(\mathbb{Z}_3, \mathbb{Z}_9) =$ group of elements of order 3 in $\mathbb{Z}_9 \simeq \mathbb{Z}_3$.

(ii) Let A, B be finitely generated abelian groups. Prove that $\mathrm{Hom}(A, B)$ is finitely generated.

$\because$ A surjection $\mathbb{Z}^n \to A$ yields an injection $\mathrm{Hom}(A, B) \to \mathrm{Hom}(\mathbb{Z}^n, B) = B^n$ and a subgroup of a finitely generated abelian group is finitely generated.

4.(i) Let

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Determine with reason the invariant factor and elementary divisor decompositions of the $\mathbb{R}[x]$-module corresponding to M.

$\because$ min poly$= x(x-1)$, char poly$= x^2(x-1)$ so inv factos are $x, x, (x-1)$, elem divisors are $x, x(x-1)$.

(ii) Let M be an $n \times n$ matrix such that $M^2 = M$. Prove that there exists $0 \leq r \leq n$ such that M is similar to the block matrix

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$\because$ If $M \neq 0, I_n$, min poly is $x(x-1)$ with distinct roots so M is diagonalizable with eigenvalues 0,1.

Algebra Qaulifier Part C

PLEASE CIRCLE THE Four PROBLEMS YOU WANT TO BE GRADED.

1. Let F be a splitting field over $\mathbb{Q}$ of the polynomial $x^4 - 4x^2 - 1$. Let $g(x) = x^3 + 6x^2 - 12x - 12$. Does $g(x)$ have a root in F? Prove your answer.

2. Let F be a splitting field over $\mathbb{Q}$ of the polynomial $x^4 - 5$. Find all the intermediate fields of F over $\mathbb{Q}$. Indicate the ones which are Galois over $\mathbb{Q}$. Prove your answer.

3. Let F_{12} be a cyclotomic extension of $\mathbb{Q}$ of order 12. Determine $Aut_{\mathbb{Q}}F_{12}$ and all intermediate fields.

4. Construct a field with 49 elements and give the rules for its addition and multiplication. If a is a generator, what is the multiplicative inverse of $1 + a$ in terms of your set of *minimal* generator(s)?

5. Let a, b be nonzero in some extension field of K. Assume that a is separable over K and b is purely inseparable over K. Prove that $K(a, b) = K(ab) = K(a + b)$.

Algebra Qualifier, Part A

September 27, 2003

Do four out of the six problems.

1. Prove that a group cannot be the union of two proper subgroups.

2. Let $\mathbb{Q}_{add}$ be the additive group of rational numbers. Prove that any subgroup of $\mathbb{Q}_{add}$ generated by two distinct elements is isomorphic to $\mathbb{Z}$. Use this to prove that $\mathbb{Q}_{add}$ is not isomorphic to $\mathbb{Q}_{add} \times \mathbb{Q}_{add}$.

3. Let G_1 and G_2 be two non-trivial non-isomorphic simple groups. Prove that any proper non-trivial normal subgroup of $G_1 \times G_2$ coincides with G_1 or G_2.

4. Prove that a free group that is abelian is either trivial or is isomorphic to $\mathbb{Z}$.

5. Describe up to isomorphism all groups of order 121. Prove your answer.

6. Prove that if S is a Sylow subgroup of a finite group G, then $N_G(N_G(S)) = N_G(S)$.

Algebra Qualifier, Part B

September 27, 2003

Do four out of the six problems.

1. Let I and J be ideals of a commutative unitary ring R. Show that $I + J = R$ implies that $I \cap J = IJ$, and if R is a PID, the converse holds.

2. Let A and B be commutative unitary rings. Given that $A \otimes_{\mathbb{Z}} B$ is a commutative unitary ring with multiplication satisfying $(a_1 \otimes_{\mathbb{Z}} b_1)(a_1 \otimes_{\mathbb{Z}} b_1) = (a_1 a_2) \otimes_{\mathbb{Z}} (b_1 b_2)$, for $a_i \in A$, $b_i \in B$, show that the coproduct of A and B exists in the category of commutative unitary rings and unitary ring homomorphisms.

3. Let R be a commutative ring with unique maximal ideal M. Let A be the smallest subring of R containing the multiplicative identity 1 of R. Show that A is ring isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}/p^n\mathbb{Z}$ for some p, $n \in \mathbb{N}$ with p prime. (Hint: Consider the idempotents of R.)

4. Let R is a commutative Noetherian ring. (So each ideal of R is finitely generated.) Show that each submodule N of a finitely generated R-module M is finitely generated.

5. Let $0 \to A \to B \to C \to 0$ be an exact sequence of left R-modules.
(a) What part of the sequence $0 \to \mathrm{Hom}_R(C, D) \to \mathrm{Hom}_R(B, D) \to \mathrm{Hom}_R(A, D) \to 0$ is exact for every left R-module D? (No proofs required.)
(b) Show the the remaining part is exact if and only if the sequence $0 \to A \to B \to C \to 0$ splits.

6. Let K be a field and let M and N be finitely generated modules over the polynomial ring $K[X]$.
Suppose M has invariant factors $(X-1)$, $(X-1)(X-2)^2$, $(X-1)(X-2)^2(X-3)$, and N has invariant factors $(X-2)(X-3)^2$, $(X-2)^2(X-3)^2(X-5)$.
(a) Give the elementary divisors of $M \oplus N$.
(b) Give the invariant factors of $M \oplus N$.

Algebra Qualifier, Part C

September 27, 2003

1. A complex number is said to be an algebraic number if it is algebraic over Q and an algebraic integer if it is a root of a monic polynomial in $Z[x]$.

(a) Prove that u is an algebraic number iff there exists an integer n such that nu is an algebraic integer.

(b) If $r \in Q$ is an algebraic integer prove that $r \in Z$.

(c) If u is an algebraic number prove that $u + n$ is algebraic for all $n \in Z$.

(d) Deduce **from the above**, that the sum and product of two algebraic integers is an algebraic integer.

2 Let F be a field extension of K. Define the Galois group G of F over K. Now define what one means by a stable subfield of this extension. Prove that if E is a stable intermediate field, then the Galois group of F over E is a normal subgroup of G. Conversely, prove that if H is a normal subgroup of G, then the fixed field of H is a stable subfield of the extension $F \supset K$.

3. Let F be an extension field of K. Define the maximal algebraic extension of K in F. Suppose that for every extension $F \supseteq K$, the maximal extension is K, prove that K is algebraically closed. Now suppose that K is algebraically closed and let F be any extension of K. Prove that the maximal algebraic extension of K in F is K.

4. Let F be a finite dimensional extension of Z_3. Deduce that F is Galois over Z_3. Prove that $\phi : F \to F$ given by $\phi(u) = u^3$ is a Z_3–automorphism of F. Show that ϕ generates the Galois group of F over Z_3.

5(a). Let $K = Q(i)$. Let $F \subset C$ be a field that contains a root of the polynomial $x^4 - 2 \in K[x]$. Prove that F is the splitting field of this polynomial. What is the Galois group of this polynomial? Determine the subfields of this extension.

5(b). Let E be an algebraic extension of a field K. Prove that there exists an extension F of E such that F is normal over K and no proper subfield of F containing E is normal over K.

Algebra Qualifier, Part A

September 28, 2002

Do 4 problems.

1. Show that if a positive integer d divides the order n of a finite cyclic group G, then G has a unique subgroup of order d.

2. Show that there are no simple groups of order 200.

3. Let H be a subgroup of a group G with $(G : H) = n$.

(a) Show H contains a normal subgroup K such that $(G : K)$ divides $n!$.

(b) Show that if G is finite and $n = p$ is the smallest prime dividing $|G|$, then H is normal.

4. Show that each finite p-group is solvable.

5. Show that in a principal ideal domain every non-zero prime ideal is maximal.

6. Let R be a commutative unitary ring. An ideal I of R is said to be *primary* if a, $b \in R$, and $ab \in I$ imply $a \in I$ or $b^n \in I$ for some $n \in \mathbb{N}$. Assume I is primary and S is a multiplicative subset of R such that $S \cap I = \emptyset$. Show that $S^{-1}I$ is a primary ideal of $S^{-1}R$.

Algebra qualifier 2002 Fall

choose five problems

1.) Let

$$A=\begin{bmatrix} 0 & 1 & & & & & & & \\ -1 & 0 & & & & & & 0 & \\ & & 0 & 1 & & & & & \\ & & -1 & 0 & & & & & \\ & & & & 0 & 1 & & & \\ & & & & & 0 & & & \\ & 0 & & & & & 0 & 1 & 0 \\ & & & & & & & 0 & 1 \\ & & & & & & & & 0 \end{bmatrix}$$

diagonal

(i) Find the minimal polynomial, the rational canonical form.

(ii) How many independent eigenvectors does A have? (Explain)

2) Let V be a n-dimensional vector space over k.

(i) Define the tensor algebra $T(V)$, the alternating algebra $\Lambda(V)$, and the symmetric algebra $S(V)$, and make them graded k-algebra.

(ii) What are the dimensions of $T^r(V)$, $\Lambda^r(V)$ and $S^r(V)$? (expl

(iii) Prove that $\Lambda^r V = 0$ for $r > n$.

Prove or disprove 3 – 9

3). Let R be any ring, and let A, B, C, D be R-modules

If $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, then

(i) $0 \to \mathrm{Hom}_R(D, A) \to \mathrm{Hom}_R(D, B) \to \mathrm{Hom}_R(D, C) \to 0$ is a short exact sequence of R-modules.

(ii) $0 \to \mathrm{Hom}_R(C, D) \to \mathrm{Hom}_R(B, D) \to \mathrm{Hom}_R(A, D) \to 0$ is a short exact sequence of R-modules.

4). Let R, A, B, C, D be as in (3), and $0 \to A \to B \to C \to 0$

i) $0 \to A \otimes_R D \to B \otimes_R D \to C \otimes_R D \to 0$

ii) $A \otimes_R B \cong A \otimes_R C \Rightarrow B \cong C$

5) (i) a torsion-free module is projective.

(ii) a projective module is torsion-free.

(iii) a torsion-free module is free.

6) (i) a free module is torsion-free.

(ii) a projective module is free.

(iii) a free module is projective.

7. (i) a submodule of a free module is free.

(ii) a submodule of a finitely generated module is finitely generated.

8). $X^{10} + \pi X^7 + \pi X + \pi$ is irreducible over $\mathbb{Q}$.

9).

$$\begin{array}{ccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 \end{array}$$

α_4 is one-to one and α_1, α_3 are onto

$\Rightarrow$ α_2 is onto.

Algebra Qualifying Examination, Part 3

Answer any four questions.

1. Prove that if $f \in K[x]$ is a polynomial of degree n, then there exists an extension of K in which f as a root. Consider the example $f(x) = x^3 - 5x - 2 \in Q[x]$, let u be a real root of this polynomial. What is the natural basis of $Q(u)$ and write the element $x^4 - 3x + 1$ as a linear combination of the basis elements.

2. Let F be a finite–dimensional Galois extension of K and E an intermediate field. Prove that there exists an unique smallest field L between E and F which is Galois over K and prove that

$$Aut_L F = \cap_\sigma \sigma (Aut_E F) \sigma^{-1}.$$

3. Prove that F is an algebraic closure of K iff F is algebraic over K and for every algebraic field extension E_1 of another field K_1 and isomorphism of fields $\sigma : K_1 \to K$, σ extends to a monomorphism $E \to F$.

4. (a) Compute the Galois group of $x^3 - 10$ over $Q(\sqrt{2})$.

(b) Prove that if F is Galois over E, E Galois over K and F is a splitting field of polynomials in $K[x]$, then F is Galois over K.

5.Let F be an algebraic closure of Z_p. Prove that

(a) F is algebraic Galois over Z_p.

(b) The map $\phi : F \to F$ given by $u \to u^p$ is a non–identity Z_p–automorphism of F.

(c) What is the fixed field of the subgroup of $Aut_{Z_p} F$ generated by ϕ.

Algebra Qualifier. Part A

November 17, 2001

Do 4 problems.

1. Show that the symmetric group S_3 is not isomorphic to a direct product of two of its proper subgroups.

2. Show that a group of order 77 has exactly one subgroup of order 11.

3. Prove that if G is a group and $G/C(G)$ is cyclic, then G is abelian, where $C(G)$ is the center of G.

4. Describe explicitly all free groups which are abelian.

5. Prove that if H is a cyclic normal subgroup of G, then every subgroup of H is normal in G.

6. How many elements of order 7 are there in a simple group of order 168? Prove your answer.

Algebra Qualifier, Part B

November 17, 2001

Do 4 problems.

1. Let R be an integral domain. Show that R is a PID if each finitely generated torsion-free R-module is free.

2. Show that if $\mathrm{Hom}_R(D,\ )$ preserves exactness of $0 \to A \to B \to C \to 0$ for each D, the sequence $0 \to A \to B \to C \to 0$ splits.

3. If X, Y are independent indeterminates over the commutative ring R, show $R[X] \otimes_R R[X] \cong R[X, Y]$ as R-algebras.

4. Let B be an abelian group and let A be a subgroup of B. Show that any homomorphism $f : A \to \mathbb{Q}/\mathbb{Z}$ extends to a homomorphism $\overline{f} : B \to \mathbb{Q}/\mathbb{Z}$.

5. Let R be a commutative unitary ring. Show that each injective R-module is divisible.

6. Prove that 3×3 matrices A and B over a field K are similar if they have the same minimal polynomials and the same characteristic polynomials.

Algebra Qualifying Exam, Part C

Answer any four of the following six questions.

1. (a) Suppose that $F = \mathbf{Q}(\alpha_1, \alpha_2, \cdots, \alpha_n)$ where $\alpha_i^2 \in \mathbf{Q}$ for $i = 1, 2, \cdots, n$. Prove that $\sqrt[3]{2} \notin \mathbf{Q}$.
(b) Determine the degree of the extension of $\mathbf{Q}(\sqrt{32\sqrt{2}})$ over $\mathbf{Q}$.

2. (a) Determine the splitting field and its degree over $\mathbf{Q}$ for $x^6 - 4$.
(b) Prove that the polynomial $x^{p^n} - x$ over $\mathbf{Z_p}$ is separable.

3. For any integer $r \geq 1$, let $\mathbf{F_{p^r}}$ be a finite field of cardinality p^r. Prove that $\mathbf{F_{p^r}} \subset \mathbf{F_{p^s}}$ if and only if r divides s. (Hint. First prove that r divides s if and only if $x^r - 1$ divides $x^s - 1$.

4. Determine the Galois group of $(x^3 - 2)(x^3 - 3)$ over $\mathbf{Q}$. Determine the subfields which contain $\mathbf{Q}(\rho)$ where ρ is a primitive cube root of unity.

5. Determine the splitting field of $x^p - x - 1$ over $\mathbf{Z_p}$ and prove explicitly that the Galois group is cyclic.

6. Determine the Galois group of $(x^2 - 2)(x^2 - 3)(x^2 - 5)$ and let F/Q be the splitting field of this polynomial. Determine all subfields $L \subset F$ which are galois over $\mathbf{Q}$.

Algebra Qualifying Examination, 2000

Answer any 6.

1. Give examples of abelian groups A, B satisfying (i) $A \otimes B = 0$, (ii) $A \otimes A \cong A$.

2. Let R be a ring with identity. Assume that M is a simple module for R. Prove that M is cyclic and that any non–zero R module endomorphism of M is an isomorphism.

3. Prove that a module P over a ring R is projective if and only if P is a summand of a free module.

4. If A is any abelian group, compute $Hom(Z_m, A)$. What can you say about Z_m^*?

5. Prove that a free module over a pid is torsion free and give an example to show that the converse is false.

6. What are the invariant factors of $Z_n \oplus Z_m$, regarded as modules over Z.

7. Prove that if $m > n$, any alternating multilinear form on $(R^n)^m = 0$, here R is any commutative ring.

8. Prove that if q is the minimal polynomial of a linear transformation of vector space E, then $deg\ q \leq dim\ E$.

Choose 5 problems. (20% each)

1). (a) Find all subgroups of $\mathbb{Z}_{48}$.

(b) Find all ideals of the ring $\mathbb{Z}_{48}$.

(c) Which ideals in (b) are maximal? Which are prime?

2) Find all normal subgroups of D_{11}.

Prove or disprove the following

3) A group of order 375 is simple.

4). There is a non abelian group G s.t. $|G|=125$ and $|C(G)|=25$.

5). $\mathbb{R}[X,Y]$ is a PID, but $\mathbb{Z}[X]$ is not a PID.

(X, Y are indeterminants)

6). $\mathbb{Z}[i]$ is a UFD.

7). Let p be a fixed prime number.

Let $R=\{\frac{a}{b} \mid a, b \in \mathbb{Z},\ p \nmid b\}$. Then R is a local ring

Algebra Qualifier Part C: Fall 2000

1. Let K be a field containing a primitive n th root of 1, ζ, where the characteristic of K is either 0 or does not divide n, and let F be a field extension of degree n over K.

(i) Prove that if $n = 2$ then F is cyclic Galois over K.

(ii) Assume F is cyclic Galois over K (any n), and let σ be a generator of $Gal(F/K)$, considered as an endomorphism of F as K-vector space. Show that ζ is an eigenvalue of σ.

(iii) Conclude from (ii) or prove otherwise that, with assumptions as in (ii), F is a radical extension of K.

2. Let G be a finite group. Prove that there exists a finite Galois field extension $K \subseteq F$ with Galois group G.

3. Let $K \subseteq F$ be a field extension. Prove or disprove the following

(i) If $Gal(F/K) \simeq S_3$ then F is a splitting field over K of some cubic polynomial.

(ii) Ditto, assuming in addition that F is Galois over K.

4. Let $f(x) = x^{11} + 5x^3 + 10 \in Q[x]$ and let α be a root of f in C. Let $g(x) = x^{19} + 6x^5 - 12$. Does g have a root in $Q(\alpha)$?. Prove your answer.

5. Find with proof a transcendence base over Q of

$$Q(x^5, x^2 + y^5, x^3 + y^4).$$

Algebra Qualifier, Parts A and B

December 10, 1999

Do 8 problems.

1. Let $K \subseteq H$ be subgroups of the finite group G, which are not necessarily normal. Show $(G : K) = (G : H)(H : K)$, where the notation $(A : B)$ denotes the number of left cosets of B in A.

2. Let the group G operate on the set S, and suppose that $s,\ t\ \in\ S$ are in the same orbit under the operation. Show that the isotropy groups G_s and G_t are conjugate. That is, there exists $g \in G$ such that $g^{-1}G_s g = G_t$. (Recall that $G_s = \{x \in G \,|\, gs = s\}$.)

3. Show that each group of order p^2, p prime, is abelian.

4. (a) Define *free group* $F(X)$ *on a set* X.

(b) Define *coproduct* $(G, \{\phi_i\}_{i \in I})$ *of a family* $\{G_i \mid i \in I\}$ of groups.

(c) Show that if (G, ϕ_1, ϕ_2) is a coproduct of G_1 and G_2 with G_i isomorphic to the additive group of integers Z, then $G \cong F(\{a,\ b\})$.

(d) Is $F(\{a,\ b\})$ abelian?.

5. Show that the direct sum of an arbitrary family of injective abelian groups is injective. (Hint: divisible.)

6. (a) Determine the units in $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a,\ b \in \mathbb{Z}\}$.

(b) Show that $1 + \sqrt{-5}$ and 2 are irreducible in $\mathbb{Z}[\sqrt{-5}]$.

7. (a) Give an example of an integral domain R and ideals I and J of R such that $IJ \neq I \cap J$.

(b) Show that if $I + J = R$ then $IJ = I \cap J$.

8. Give an example of a monomorphism $f : A \to B$ of abelian groups and an abelian group C such that the induced homomorphism $Hom_{\mathbb{Z}}(B, C) \to Hom_{\mathbb{Z}}(A, C)$ is not onto.

9. Show that if R is a PID and $M = R/a_1R \oplus R/a_2R \oplus \cdots \oplus R/a_mR$, with $R \neq a_1R \supseteq a_2R \supseteq \cdots \supseteq a_mR$, then M cannot be generated by fewer than m-elements.

10. Let

$$A = \begin{bmatrix} 2 & & & & & & & \\ & -1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & 1 & 1 & & & \\ & & & & & -1 & & \\ & & & & & & 2 & \\ & & & & & & & -1 \end{bmatrix}.$$

Find the minimal polynomial and the rational canonical form of A.

Algebra Qualifier, Part C

December 10, 1999

Do 4 problems.

1. Let F be a splitting field of $X^{18}-1$ over $\mathbb{Q}$. Determine all intermediate fields $F \supseteq \mathbb{Q}$. How many of them are Galois over $\mathbb{Q}$?

2. Find all roots of unity in $\mathbb{Q}(\sqrt{11})$.

Prove or disprove.

3. The polynomial $X^{625}-X-1$ is irreducible over $\mathbb{Z}_5$.

4. Let F be a degree 4 extension of $\mathbb{Q}$. If any proper subfield of F is Galois over $\mathbb{Q}$, then F is Galois over $\mathbb{Q}$.

5. For field extensions, being separable (respectively, purely inseparable) is transitive.

6. Let $F > k$ be fields.
$[F:k] = \infty \Rightarrow \text{tr.d.}_k F > 0$.
$[F:k] < \infty \Rightarrow \text{tr.d.}_k F = 0$.
$\text{tr.d.}_k F < \infty \Rightarrow \text{tr.d.}_k F = [F:k]$.

Algebra Qualifier, Parts A and B

December 10, 1999

Do 8 problems.

1. Let $K \subseteq H$ be subgroups of the finite group G, which are not necessarily normal. Show $(G : K) = (G : H)(H : K)$, where the notation $(A : B)$ denotes the number of left cosets of B in A.

2. Let the group G operate on the set S, and suppose that $s,\ t\ \in\ S$ are in the same orbit under the operation. Show that the isotropy groups G_s and G_t are conjugate. That is, there exists $g \in G$ such that $g^{-1}G_s g = G_t$. (Recall that $G_s = \{x \in G \mid gs = s\}$.)

3. Show that each group of order p^2, p prime, is abelian.

4. (a) Define *free group* $F(X)$ *on a set* X.

(b) Define *coproduct* $(G, \{\phi_i\}_{i\in I})$ *of a family* $\{G_i \mid i \in I\}$ of groups.

(c) Show that if (G, ϕ_1, ϕ_2) is a coproduct of G_1 and G_2 with G_i isomorphic to the additive group of integers Z, then $G \cong F(\{a,\, b\})$.

(d) Is $F(\{a,\, b\})$ abelian?.

5. Show that the direct sum of an arbitrary family of injective abelian groups is injective. (Hint: divisible.)

6. (a) Determine the units in $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a,\, b \in \mathbb{Z}\}$.

(b) Show that $1 + \sqrt{-5}$ and 2 are irreducible in $\mathbb{Z}[\sqrt{-5}]$.

7. (a) Give an example of an integral domain R and ideals I and J of R such that $IJ \neq I \cap J$.

(b) Show that if $I + J = R$ then $IJ = I \cap J$.

8. Give an example of a monomorphism $f : A \to B$ of abelian groups and an abelian group C such that the induced homomorphism $Hom_{\mathbb{Z}}(B, C) \to Hom_{\mathbb{Z}}(A, C)$ is not onto.

9. Show that if R is a PID and $M = R/a_1R \oplus R/a_2R \oplus \cdots \oplus R/a_mR$, with $R \neq a_1R \supseteq a_2R \supseteq \cdots \supseteq a_mR$, then M cannot be generated by fewer than m-elements.

10. Let

$$A = \begin{bmatrix} 2 & & & & & & & \\ & -1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & 1 & 1 & & & \\ & & & & & -1 & & \\ & & & & & & 2 & \\ & & & & & & & -1 \end{bmatrix}.$$

Find the minimal polynomial and the rational canonical form of A.

Algebra Qualifier, Part C

December 10, 1999

Do 4 problems.

1. Let F be a splitting field of $X^{18} - 1$ over $\mathbb{Q}$. Determine all intermediate fields $F \supseteq \mathbb{Q}$. How many of them are Galois over $\mathbb{Q}$?

2. Find all roots of unity in $\mathbb{Q}(\sqrt{11})$.

Prove or disprove.

3. The polynomial $X^{625} - X - 1$ is irreducible over $\mathbb{Z}_5$.

4. Let F be a degree 4 extension of $\mathbb{Q}$. If any proper subfield of F is Galois over $\mathbb{Q}$, then F is Galois over $\mathbb{Q}$.

5. For field extensions, being separable (respectively, purely inseparable) is transitive.

6. Let $F > k$ be fields.

$[F : k] = \infty \Rightarrow \text{tr.d.}_k F > 0.$

$[F : k] < \infty \Rightarrow \text{tr.d.}_k F = 0.$

$\text{tr.d.}_k F < \infty \Rightarrow \text{tr.d.}_k F = [F : k].$

ALGEBRA QUALIFIER - PART A - Nov. 14, 1998

Separate paper for each part.
Choose four problems.

1. (a) Find all the subgroups of $\mathbb{Z}_{24}$.

   (b) Find all the ideals in the ring $\mathbb{Z}_{24}$.

   (c) Which of these ideals are maximal and which are prime?

In problems 2-5a, prove or disprove.

2. If $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products of the family $\{A_i | i \in I\}$ of objects of a category G, then P and Q are equivalent.

3. A group of order 250 is simple.

4. Let R be an integral domain. Then
   R is Euclidean $\Leftrightarrow$ R is a PID $\Leftrightarrow$ R is a $U.F.D.$

5. (a) Let $R = \mathbb{Z}_6, S_1 = \{1, 3\}$, and $S_2 = \{1, 2, 4\}$. Then $S_i^{-1}R$ is a field.

   (b) How many elements does $S_i^{-1}R$ have?

ALGEBRA QUALIFIER PART B, 11/14/98

Throughout R denotes a commutative ring with unit element, M a unitary R-module and k a field.

1. (i) Prove that if R is an integral domain with fraction field K and M is a torsion-free R-module, then M is isomorphic to an R-submodule of the K-vector space $M \otimes_R K$.

(ii) Prove (from scratch) that if R is a principal ideal domain and M a finitely generated torsion-free R-module such that $M \otimes_R K$ is 1-dimensional as K-vector space, then $M \simeq R$ as R-modules (i.e. M is free of rank 1).

(iii) Give (with proof) an example of two nonzero $\mathbb{Z}$-modules whose tensor product is zero. (iv) Prove that if R is an integral domain but not a field then its field of fractions is torsion-free, but is not a submodule of a free R-module.

2. (i) State with proof a correspondence between {pairs (V, L) where V is a finite-dimensional k-vector space and L is a linear transformation on V} and {finitely generated torsion $k[X]$-modules M}.

(ii) Determine with proof the pair (V, L) corresponding to a cyclic $k[X]$-module $k[X]/(f)$, $f \in k[X]$ nonzero, and compute the minimal polynomial and the characteristic polynomial of L.

(iii) In the general case, interpret with proof the minimal polynomial and the characteristic polynomial of a linear transformation L in terms of the corresponding module M.

3. Let V be an n-dimensional k-vector space.

(i) State the definition of $\bigwedge^m(V)$;

(ii)Prove that if $m > n$ then $\bigwedge^m(V) = 3D(0)$.

(iii) Prove that the dimension of $\bigwedge^2(V)$ is $\binom{n}{2}$.

Algebra Qualifying Exam - Part C - Nov. 14, 1998

Separate paper for each part.

Answer any three questions. F will always denote an extension field of K.

1(a) Let $u \in F$ be an element of odd degree over K, prove that $K(u) = K(u^2)$.
(b) Compute the Galois group of $x^3 - x - 1$ over Q.
(c) If $f \in K[x]$ has degree n and F is its splitting field prove that [F:K] divides $n!$.

2. Prove that $Aut_K K(x)$ is finite. Deduce that if K is finite then $K(x)$ is not a Galois extension. Determine the closed subgroups of $Aut_K K(x)$ when K is infinite.

3. Prove that in a finite field of characteristic p, every element has a unique p^{th} root in it. If F is a finite extension of a finite field K of characteristic p, prove that $Aut_K F$ is cyclic.

4. Let F be an algebraically closed extension of K of finite transcendence degree, prove that every K–monomorphism of F to itself is actually an automorphism.

Give an example of a field extension which is transcendental of finite degree and give an example of a field extension of infinite transcendental degree.

Algebra Qualifying Exam. 1996. Part 1

Attempt any 5.

1. Let F be a field. Prove that F contains a unique smallest subfield F_0 and that F_0 is isomorphic to either $\mathbb{Q}$ or to $\mathbb{Z}_p$ for some prime p.

2. Let R be a commutative ring with identity. Prove that a polynomial ring in more than one variable is not a principal ideal domain.

3. Let P be a prime ideal in a commutative ring with 1 and let D be the set $R - P$. Show that the ring of fractions $D^{-1}R$ is defined and that it has a unique maximal ideal.

4. Let $\phi : R \to S$ be a homomorphism of rings. Prove that the inverse image in R of a prime ideal in S is either R or a prime ideal.

5. Let A and B be finite groups. Prove that the number of Sylow–p subgroups in $A \times B$ is the product of the number of Sylow-p subgroups in A and B.

6. Let G be a cyclic group of order n and assume that k is prime to n. Prove that the map $x \to x^k$ is surjective. Now prove that the same result holds for any finite group G of order n and for k prime to n.

7. Let C be a normal subgroup of A and D a normal subgroup of B. Prove that $C \times D$ is a normal subgroup of $A \times B$ and that the corresponding quotient group is isomorphic to $A/C \times B/D$.

Algebra Qualifier, Part II

1. Describe all maximal ideals in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Prove your answer.

2. Let R be a ring. A left R-module M is called *simple* iff $M \neq \{0\}$ and M has no proper submodule. Prove that for any simple left R-module the endomorphism ring $\mathrm{Hom}_R(M, M)$ is a division ring.

3. Consider the $\mathbb{C}[x]$-module structure on $\mathbb{C}^n$ defined by

$$p(x) \cdot v = (p(A))(v),$$

where $v \in \mathbb{C}^n, p(x) \in \mathbb{C}[x]$, and A is the $n \times n$-matrix

$$\begin{pmatrix} 0 & & & 0 \\ i & 0 & & \\ & i & \ddots & \\ 0 & & i & 0 \end{pmatrix}$$

(where $i^2 = -1$). Decompose $\mathbb{C}^n$ according to the structure theorem for modules over principle ideal domains.

4. Let V be an n-dimensional vector space over $\mathbb{R}$. Establish a canonical isomorphism between $\Lambda^k(V)$ and $\Lambda^{n-k}(V^*) \otimes_{\mathbf{R}} \Lambda^n(V)$, where k is any fixed positive integer less or equal n and V^* is the space dual to V. (A canonical isomorphism between two vector spaces V' and V'' is an isomorphism which does not depend on the choice of bases in V' and V'').

Algebra Qualifier, Part C. November, 1996

1. Suppose E is a Galois extension field of degree 7 over F. What can you say about a group of automorphisms of E whose elements fix each element of F? Prove your answer.

2. Let E be the field obtained from the rationals Q by adjoining a cube root of 3. Determine the Galois group of E (over Q).

3. Let f be a polynomial of degree n over F and let E be a splitting field of f over F. Prove that the degree of E over F is at most n! .

4. What can you say about the cardinality of an algebraic closure of the field of 3 elements? Sketch a proof of your answer.

5. Let F be a field of 4 elements. Prove that there exists an extension field of degree 2 over F.

Algebra Qualifying Exam, 1996, Part 1

Attempt any 5.

1. Let F be a field. Prove that F contains a unique smallest subfield F_0 and that F_0 is isomorphic to either $\mathbb{Q}$ or to $\mathbb{Z}_p$ for some prime p.

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Algebra Qualifier, Part II

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Algebra Qualifier, Part C.   November, 1996

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Algebra Qualifying Exam, 1996, Part 1

Attempt any 5.

1. Let F be a field. Prove that F contains a unique smallest subfield F_0 and that F_0 is isomorphic to either $\mathbb{Q}$ or to $\mathbb{Z}_p$ for some prime p.

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Algebra Qualifier, Part II

1. Describe all maximal ideals in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Prove your answer.

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Algebra Qualifier, Part C. November, 1996

1. Suppose E is a Galois extension field of degree 7 over F. What can you say about a group of automorphisms of E whose elements fix each element of F? Prove your answer.

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August 26, 1995

Algebra Qualifier. Part I

1. Define normal subgroup. Give an example of a subgroup that is normal and one that is not.

2. Let the group G operate on the set S, and suppose that $s,\ t\ \in\ S$ are in the same orbit under the operation. Show that the isotropy groups G_s and G_t are conjugate. That is, there exists $g \in G$ such that $g^{-1}G_s g = G_t$. (Recall that $G_s = \{g \in G \mid gs = s\}$.)

3. Let p be a prime integer and let G be a finite p-group. Show that G has a nonzero center.

4. Let G be a finite group of order $p^n q$, p and q primes with $p > q$. Show that G is not simple.

5. Show that every group of order p^2, p a prime, is abelian.

6. Let p be a prime integer and let $Z(p^\infty)$ be the subgroup of the additive group $\mathbb{Q}/\mathbb{Z}$ generated by the cosets of the form $1/p^i$, $i \in \mathbb{Z}$, $i \geq 0$. Let A be a subgroup of a finitely generated group B. Show that any homomorphism $f : A \to Z(p^\infty)$ extends to a homomorphism $g : B \to Z(p^\infty)$.

Algebra Qualifying Exam Part B   August 27, 1995

In problems 1, 2 and 4, R is a commutative ring with 1, and all R-modules considered are assumed to be unitary.

1. a) Define: an R-module is <u>projective</u>.
   b) Prove that if R is a field then any R-module is projective.

2. a) Given R-modules M and N, and their tensor product

$$A = M \otimes_R N$$

as an abelian group, state a way of making each element of R act on A so that A becomes an R-module.
   b) Prove that the action you gave in part a) is well-defined.

3. a) Suppose F is a given field, and that c and d are two distinct elements of F. Determine the number of similarity classes of matrices (over F) with characteristic polynomial

$$(x-c)^2(x-d)^2 .$$

   b) State the result for finite abelian groups which corresponds to your result for part a).

4. Let M be an R-module, and let H be the set of all m in M such that $rm = 0$ implies $r = 0$ or $m = 0$.
   a) Prove that if R is a principal ideal domain (PID) and M is cyclic then H is a submodule of M.
   b) Is the result in part a) true if the hypothesis that R be a PID is just deleted? -- Give a proof or counterexample.

Algebra Qualifying Exam. 1995. Part 3

1. Let $\alpha = \sqrt{2+\sqrt{2}}$.

(a) Find the minimum polynomial of α over $\mathbb{Q}$. What are its other roots?

(b) Prove that $\mathbb{Q}(\alpha)$ is the splitting field of α over $\mathbb{Q}$.

(c) Write down all the automorphisms of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

(d) Prove that the Galois group of $\mathbb{Q}(\alpha)/\mathbb{Q}$ is cyclic.

(e) Determine the intermediate fields of the extension $\mathbb{Q} \subset \mathbb{Q}(\alpha)$.

2. (a) Let p be a prime and let $F = \mathbb{Z}_p$. Determine the number of irreducible polynomials of type $x^2 + cx + d$ where $c, d \in F$.

(b) Let $f(x)$ be one of the polynomials described in (a). Prove that $K = F[x]/(f)$ is a field containing p^2 elements, and that the elements of K have the form $a + b\alpha$ where $a, b \in F$ and α is a root of f in K. Prove also that every such element $a + b\alpha$ with $b \neq 0$ is a root of an irreducible quadratic polynomial in $F[x]$.

(c) Show that every polynomial of degree 2 in $F[x]$ has a root in K. (Hint: Any two fields containing p^2 elements are isomorphic.)

3. Determine the splitting field of the polynomial $f(x) = (x^2 - 2x - 1)(x^2 - 2x - 7)$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

August 26, 1995

Algebra Qualifier, Part I

1. Define normal subgroup. Give an example of a subgroup that is normal and one that is not.

2. Let the group G operate on the set S, and suppose that $s,\ t\ \in\ S$ are in the same orbit under the operation. Show that the isotropy groups G_s and G_t are conjugate. That is, there exists $g \in G$ such that $g^{-1}G_s g = G_t$. (Recall that $G_s = \{g \in G \mid gs = s\}$.)

3. Let p be a prime integer and let G be a finite p-group. Show that G has a nonzero center.

4. Let G be a finite group of order $p^n q$, p and q primes with $p > q$. Show that G is not simple.

5. Show that every group of order p^2, p a prime, is abelian.

6. Let p be a prime integer and let $Z(p^\infty)$ be the subgroup of the additive group $\mathbb{Q}/\mathbb{Z}$ generated by the cosets of the form $1/p^i$, $i \in \mathbb{Z}$, $i \geq 0$. Let A be a subgroup of a finitely generated group B. Show that any homomorphism $f : A \to Z(p^\infty)$ extends to a homomorphism $g : B \to Z(p^\infty)$.

Algebra Qualifying Exam Part B   August 27, 1995

In problems 1, 2 and 4, R is a commutative ring with 1, and all R-modules considered are assumed to be unitary.

1. a) Define: an R-module is <u>projective</u>.
   b) Prove that if R is a field then any R-module is projective.

2. a) Given R-modules M and N, and their tensor product

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as an abelian group, state a way of making each element of R act on A so that A becomes an R-module.
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3. a) Suppose F is a given field, and that c and d are two distinct elements of F. Determine the number of similarity classes of matrices (over F) with characteristic polynomial

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   b) State the result for finite abelian groups which corresponds to your result for part a).

4. Let M be an R-module, and let H be the set of all m in M such that $rm = 0$ implies $r = 0$ or $m = 0$.
   a) Prove that if R is a principal ideal domain (PID) and M is cyclic then H is a submodule of M.
   b) Is the result in part a) true if the hypothesis that R be a PID is just deleted? -- Give a proof or counterexample.

Algebra Qualifying Exam, 1995, Part 3

1. Let $\alpha = \sqrt{2+\sqrt{2}}$.
   (a) Find the minimum polynomial of α over $\mathbb{Q}$. What are its other roots?

   (b) Prove that $\mathbb{Q}(\alpha)$ is the splitting field of α over $\mathbb{Q}$.

   (c) Write down all the automorphisms of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

   (d) Prove that the Galois group of $\mathbb{Q}(\alpha)/\mathbb{Q}$ is cyclic.

   (e) Determine the intermediate fields of the extension $\mathbb{Q} \subset \mathbb{Q}(\alpha)$.

2. (a) Let p be a prime and let $F = \mathbb{Z}_p$. Determine the number of irreducible polynomials of type $x^2 + cx + d$ where $c, d \in F$.

   (b) Let $f(x)$ be one of the polynomials described in (a). Prove that $K = F[x]/(f)$ is a field containing p^2 elements, and that the elements of K have the form $a + b\alpha$ where $a, b \in F$ and α is a root of f in K. Prove also that every such element $a + b\alpha$ with $b \neq 0$ is a root of an irreducible quadratic polynomial in $F[x]$.

   (c) Show that every polynomial of degree 2 in $F[x]$ has a root in K. (Hint: Any two fields containing p^2 elements are isomorphic.)

3. Determine the splitting field of the polynomial $f(x) = (x^2 - 2x - 1)(x^2 - 2x - 7)$.

Typeset by $\mathcal{AMS}$-TEX

August 26, 1995

Algebra Qualifier, Part I

1. Define normal subgroup. Give an example of a subgroup that is normal and one that is not.

2. Let the group G operate on the set S, and suppose that $s,\ t\ \in\ S$ are in the same orbit under the operation. Show that the isotropy groups G_s and G_t are conjugate. That is, there exists $g \in G$ such that $g^{-1}G_s g = G_t$. (Recall that $G_s = \{g \in G \,|\, gs = s\}$.)

3. Let p be a prime integer and let G be a finite p-group. Show that G has a nonzero center.

4. Let G be a finite group of order $p^n q$, p and q primes with $p\ >\ q$. Show that G is not simple.

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Algebra Qualifying Exam Part B August 27, 1995

In problems 1, 2 and 4, R is a commutative ring with 1, and all R-modules considered are assumed to be unitary.

1. a) Define: an R-module is <u>projective</u>.
   b) Prove that if R is a field then any R-module is projective.

2. a) Given R-modules M and N, and their tensor product

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as an abelian group, state a way of making each element of R act on A so that A becomes an R-module.
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4. Let M be an R-module, and let H be the set of all m in M such that $rm = 0$ implies $r = 0$ or $m = 0$.
   a) Prove that if R is a principal ideal domain (PID) and M is cyclic then H is a submodule of M.
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Algebra Qualifying Exam, 1995, Part 3

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   (a) Find the minimum polynomial of α over $\mathbb{Q}$. What are its other roots?

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August 26, 1995

Algebra Qualifier, Part I

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Algebra Qualifying Exam Part B August 27, 1995

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Algebra Qualifying Exam. 1995. Part 3

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

August 26, 1995

Algebra Qualifier, Part I

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Algebra Qualifying Exam Part B August 27, 1995

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   (c) Write down all the automorphisms of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

   (d) Prove that the Galois group of $\mathbb{Q}(\alpha)/\mathbb{Q}$ is cyclic.

   (e) Determine the intermediate fields of the extension $\mathbb{Q} \subset \mathbb{Q}(\alpha)$.

2. (a) Let p be a prime and let $F = \mathbb{Z}_p$. Determine the number of irreducible polynomials of type $x^2 + cx + d$ where $c, d \in F$.

   (b) Let $f(x)$ be one of the polynomials described in (a). Prove that $K = F[x]/(f)$ is a field containing p^2 elements, and that the elements of K have the form $a + b\alpha$ where $a, b \in F$ and α is a root of f in K. Prove also that every such element $a + b\alpha$ with $b \neq 0$ is a root of an irreducible quadratic polynomial in $F[x]$.

   (c) Show that every polynomial of degree 2 in $F[x]$ has a root in K. (Hint: Any two fields containing p^2 elements are isomorphic.)

3. Determine the splitting field of the polynomial $f(x) = (x^2 - 2x - 1)(x^2 - 2x - 7)$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

February 12, 1994

Algebra Qualifier. Part 1

1. Give an example of a polynomial $p(x) \in Z_{10}[x]$ of degree n for some n which has more than n zeros. Can you give an example of such a polynomial in $Z_{13}[x]$. Expalin your answer.

2. (a) Find all the subgroups of Z_{18}.
(b) Find all the ideals in the ring Z_{18}.
(c) Which of these ideals are maximal and which are prime?

3. Prove that every group of order 45 has a normal subgroup of order 9.

4. Let G be a group of order p^n. Assume that the center $Z(G)$ has order $\geq p^{n-1}$. Prove that G is abelian.

5. Let G be a finite group and let N be a normal subgroup of G. Let $\phi : G \to G/N$ be the canonical homomorphism. If H is a subgroup of order G/N prove that $\phi^{-1}(H)$ is a subgroup of order $|H||N|$.

February 12. 1994

Algebra Qualifier. Part II

1. Find all possible Jordan forms for 8×8 matrices with minimal polynomial $x^2(x-1)^3$.

2. Let

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C' \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C'' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram of R-modules and homomrphisms such that each row is exact. If two of the vertical maps are injective. what about the third? Consider all three cases.

3. Prove or disprove that every 2×2 matrix over $Z[X]$ is diagonalizable by row and column operations.

4. Which of the implications: free $\Leftrightarrow$ projective $\Leftrightarrow$ torsionfree hold? Give proofs or counterexamples.

5. Compute $Hom_Z(Z_2, R)$ where R is the additive group of real numbers.

February 12, 1994

Algebra Qualifier. Part III

1. Let $K \subseteq F$ be fields and $a, b \in F$ algebraic over K.

(a) State and prove an upper bound for the degree of the product ab over K.

(b) Give an example showing your bound is sharp.

2. Let F be a field and $\sigma \in Aut(F)$. Let $K = \{u \in F | \sigma(u) = u\}$.

(a) Prove that K is a field.

(b) If $\sigma^n =$ identity and $\sigma^k \neq$ identity for $k < n$, prove $[F : K] = n$.

3. Determine the irreducible factorization and Galois group of the polynomial

$X^{16} - X$

(a) over F_4.

(b) over F_8.

4. (a) Define Galois extension.

(b) Let K be a subfield of the complex numbers which is a Galois extension of Q, the rationals. Prove or disprove that complex conjugation takes K onto itself and defines an automorphism of K.

5. Assume π is a transcendental number.

(a) Show $X^3 + \pi X + 6$ is irreducible over $Q(\pi)$ ($Q =$ the rational numbers).

(b) Determine the Galois group of $X^3 + \pi X + 6$ over $Q(\pi)$.

ALGEBRA QUALIFYING EXAM September 10, 1994

MATH 201A

1. Describe all groups G which contain no proper subgroup.

2. Let G be a cyclic group of order n and H a cyclic group of order m. Describe all homomorphisms from G to H.

3. (a) Prove that every subgroup of index 2 is normal.
(b) Give an example of a subgroup of index 3 in a group G which is not normal. (Hint: Take G to be of order 6.)

4. Prove that a group G of order 224 cannot be simple. (Hint: Show by counting that if G has more than one Sylow 2-subgroup, then it has only one Sylow 7-subgroup.)

5. For which integers $n \geq 5$ is the cyclic subgroup of the symmetric group S_n generated by $(1\,2\,3\,4\,5)$ a normal subgroup of S_n ?

MATH 201B

1. Let

$$p(x) = x^2 + x + 1, \quad q(x) = x^4 + 3x^3 + x^2 + 6x + 10.$$

Find all integers $n \geq 10$ such that $p(x)$ divides $q(x)$ in the ring $\mathbb{Z}_n[x]$.

2. Let R be a commutative ring and I an ideal of R. Let M be an ideal of R containing I, and let $\overline{M} = M/I$ be the corresponding ideal of R/I. Prove that M is maximal if and only if $\overline{M}$ is maximal.

3. A module M for a ring R is called *simple* if $M \neq \{0\}$ and M has no proper submodule, i.e. if $N \subseteq M$ is a submodule, then $N = \{0\}$ or M. Let M and M' be two simple modules and let $\phi : M \to M'$ be a module homomorphism. Prove that ϕ is either zero or an isomorphism.

4. Let I be an ideal in a ring R. Prove that, if R/I is a free R-module, then $I = \{0\}$.

5. (a) Prove that a module P over a ring R is projective if and only if, for any surjective homomorphism of R-modules

$$\phi : N \to N',$$

the corresponding homomorphism of abelian groups

$$hom_R(P, N) \to hom_R(P, N')$$

is surjective.

(b) Prove that

$$\mathbb{Z}_2 \bigotimes_{\mathbb{Z}} \mathbb{Z}_5 = 0.$$

September 8, 1993

Algebra Qualifier. Part I

1. Prove that every group of order 15 is cyclic.

2. Recall that the commutator subgroup of a group G is the subgroup G' generated by the elements $aba^{-1}b^{-1}$ where a, $b \in G$.

(i) Prove that $G' \trianglelefteq G$ and that G/G' is abelian.

(ii) If $G = S_4$ prove that $G' = A_4$. Find the commutator subgroup of A_4.

3. Show that the group generated by the two elements a, b with relations $a^3 = 1$, $b^2 = 1$, $ba = a^2b$ is isomorphic to S_3.

4. Find the number of orbits in $\{1, 2, 3, 4, 5, 6, 7, 8\}$ under the action of the subgroup of S_8 generated by $(1, 3)$ and $(2, 4, 7)$.

5. Let R be the ring Z_6 and $S = \{1, 2, 4\}$. Prove that $S^{-1}R$ is a field and identify the field. If $T = \{1, 3\}$, is $T^{-1}R$ a field?

6. Let R be a local ring and $f : R \to R'$ a surjective ring homomorphism. Prove that R' is local.

7. Let R be an integral domain and X an indeterminate over R. Show that for a, $b \in R$, a a unit of R, $X \mapsto aX + b$ extends to an automorphism of $R[X]$ which is the identity on R. Calculate the inverse of this automorphism.

Algebra Qualifier. Part II

In problems $1-4$, give an example and show that your example has the desired properties.

1. A ring that does not have the invariant dimension property.

2. A module D and an exact sequence $0 \to A \to B \to C \to 0$, which shows that neither $Hom_R(\cdot, D)$ nor $Hom_R(D, \cdot)$ is an exact functor.

3. A projective module which is not free.

4. A submodule of a finitely generated module which is not finitely generated.

5. What is $C \otimes_Z Z_{10}$, where C is the complex numbers?

6. Let R be a commutative ring with identity and M be an R-module.

(a) Outline the construction of the tensor algebra $T(M)$ and the exterior algebra $\bigwedge(M)$.

(b) Give the universal property of $T(M)$.

(c) Show that if M is free of dimension n and $1 \leq k \leq n-1$, then $\bigwedge^k(M) \cong \bigwedge^{n-k}(M)$.

7. Let

$$A = \begin{bmatrix} -1 & & & & & & & & \\ & 2 & & & & & & & \\ & & -1 & & & & & & \\ & & & 1 & & & & & \\ & & & & 1 & 1 & & & \\ & & & & & 1 & & & \\ & & & & & & -1 & & \\ & & & & & & & 2 & \\ & & & & & & & & -1 \end{bmatrix}.$$

Find the minimal polynomial, the rational canonical form, and all eigenvectors of A.

Algebra Qualifier. Part III

1. (a) Define "simple algebraic extension F/K of fields".

(b) Show that if F/K is a simple algebraic extension of degree n then $|Aut_K(F)| \leq n$.

(c) Give an example of a simple algebraic extension F/K where $|Aut_K(F)| < n$.

2. Assuming that e is transcendental over the rationals Q, prove that so is $e + \sqrt{2}$.

3. Let F be a finite field and F_1, F_2 subfields of F. State and prove a formula for $|F_1 \cap F_2|$ in terms of $|F_1|$ and $|F_2|$.

4. Compute the Galois group of $X^3 + 3$ over the rationals Q, R, the reals R, and Z_7.

5. Prove or disprove that the degree of the splitting field of a polynomial of degree n divides $n!$.

Dec. 8, 97.

Algebra Qualify Exam
Part 1

1. Let $|G| = p^n$. Prove that for each $0 \leq k \leq n$ G has a normal subgroup of order p^k.

2. Let G be a group containing an element of order different from 1 or 2. Prove that G has a non-identity automorphism.

3. Prove that a commutative ring with identity is local iff for all $r, s \in R, r + s = 1_R$ implies r or s is a unit.

4. Prove that $\mathbb{Z}[x]$ is not a prinicipal ideal domain.

5. Prove that in the ring $\mathbb{Z}$ the following conditions are equivalent

   a. I is prime
   b. I is maximal
   c. $I = (p)$ with p prime.

Algebra qualifer, Part II

II.1 Describe all maximal ideals in $\mathbb{Z}[x]/((x-a)^n)$, where $n, a \in \mathbb{Z},\ n \geq 1$. Prove your answer .

II.2. Let A be a left module over a ring R. Construct a canonical R-module homomorphism

$$\theta : A \to A^{**},$$

where, for any left or right R-module B, $B^* := \mathrm{Hom}_R(B, R)$. Give a sufficient condition on R and A for θ to be an isomorphism.

II.3. Let R be a non-zero commutative ring with unity such that every submodule of any free R-module of finite rank is itself free. Prove that R is a principal ideal domain.

II.4. Prove that any maximal linearly independent subset X of a (left) vector space V over a division ring D is a basis of V.

Algebra Qualifying Exam
Part C

1. (i) Define separable extension, perfect field.
   (ii) Give an example of an imperfect field.
   (iii) Prove that every finite field is perfect.

2. Prove from scratch that every finite field admits algebraic extensions of arbitrarily large degree.

I.P.
Z.R.

~~3. State and prove the theorem on the primitive element.~~

4. Let $K \subset E \subset F$ be fields. Prove or disprove:
   (i) If F is Galois over K, then F is Galois over E.
   (ii) If F is Galois over K, then E is Galois over K.
   (iii) If F is Galois over E and E is Galois over K, then F is Galois over K.

5. Compute the Galois group of $X^4 - 5$ over Q.