Geometry of the Hopf Fibration

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UC Riverside

Math 232a    Winter 2014
Outline.

1 History

2 Problem Set: $S^3 \to S^2$
   - $h$ maps $S^3$ into $S^2$.
   - Fibers and orbits of $S^1$ acting on $S^3$.
   - Upstairs and Downstairs.
   - Characteristics of $h$.

3 Problem Set: $S^7 \to S^4$
Consider continuous, surjective maps of the form $S^n \to S^{n-1}$.

For $n = 1$, we have $S^0 = \{-1, 1\} \subset \mathbb{R}$, a disconnected space. Since $S^1$ is connected, there is no surjective map $S^1 \to S^0$.

For $n = 2$, $S^2$ is simply connected, while $S^1$ is not. No non-nullhomotopic continuous, surjective map exists.

However, for $n = 3$, such a map $h : S^3 \to S^2$ does exist. This is the Hopf map, one of the most basic (but nontrivial) examples of a fibration, or fiber bundle. This shows that $S^3$ is locally like the product $S^1 \times D^2$, though globally it is different.

The discovery presented here follows in the order of Dr. Wilhelm’s first homework assignment.
Nonnullhomotopic Continuous, Surjective Maps Between Spheres.

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Problem 1: \( h : S^3 \rightarrow S^2 \left( \frac{1}{2} \right) \).

- Consider \( S^3 \) as the unit sphere in \( \mathbb{C} \oplus \mathbb{C} \), and \( S^2 \left( \frac{1}{2} \right) \) as the sphere of radius \( \frac{1}{2} \) in \( \mathbb{C} \oplus \mathbb{R} \).
- Define the Hopf Fibration \( h : S^3 \rightarrow S^2 \left( \frac{1}{2} \right) \) via
  \[
h : (a, c) \mapsto \left( a\overline{c}, \frac{1}{2} (|a|^2 - |c|^2) \right).
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- We will first show that \( \text{Im}(h) \subseteq S^2 \left( \frac{1}{2} \right) \).
Problem 1:

$h : S^3 \rightarrow S^2 \left( \frac{1}{2} \right)$. 

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Utilizing the provided construction of \( h \), let \((a, c) \in \mathbb{C} \oplus \mathbb{C}\). Then

\[
|h(a, c)|^2 = |a|^2|c|^2 + \frac{1}{4} \left( |a|^2 - |c|^2 \right)^2
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= |a|^2|c|^2 + \frac{1}{4} \left( |a|^4 - 2|a|^2|c|^2 + |c|^4 \right)
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P. Rajan, F. Rebro, A. Yassine
Geometry of the Hopf Fibration
Problem 2:

\[ H : S^1 \times S^3 \to S^3. \]

Identify \( S^1 \) with the unit circle in \( \mathbb{C} \).

Consider the \( S^1 \)-action on \( S^3 \),

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via

\[ H : (\omega, (a, c)) \mapsto (\omega a, \omega c), \]

where the multiplication takes place in \( \mathbb{C} \).

Then the orbits of the action coincide with the fibers of \( h \).
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Let $O_{(a,c)} = \{ \omega(a, c) \mid \omega \in S^1 \}$ be the orbits generated by the action of the circle on a point $(a, c) \in S^3$.

We first show that $O_{(a,c)} \in h^{-1}(p)$ where $p = h(a, c)$.
Let $\omega \in S^1$. Then

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\begin{align*}
    h(\omega(a, c)) &= \left( \omega a \bar{\omega} c, \frac{1}{2} (|\omega a|^2 - |\omega c|^2) \right) \\
    &= \left( \omega a \cdot \frac{1}{\omega} \bar{c}, \frac{|\omega|^2}{2} \cdot (|a|^2 - |c|^2) \right) \\
    &= \left( a \bar{c}, \frac{1}{2} (|a|^2 - |c|^2) \right) \\
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\[\text{P. Rajan, F. Rebro, A. Yassine} \quad \text{Geometry of the Hopf Fibration}\]
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\[ H : S^1 \times S^3 \to S^3. \]

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Let \( \omega \in S^1 \). Then

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h(\omega(a, c)) = \left( \omega a \omega c, \frac{1}{2} (|\omega a|^2 - |\omega c|^2) \right)
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H : S^1 × S^3 → S^3.

Let O_{(a,c)} = \{ ω(a, c) | ω ∈ S^1 \} be the orbits generated by the action of the circle on a point (a, c) ∈ S^3.
We first show that O_{(a,c)} ∈ h^{-1}(p) where p = h(a, c).
Let ω ∈ S^1. Then

\[ h(ω(a, c)) = \left( ωa\overline{ωc}, \frac{1}{2} (|ωa|^2 - |ωc|^2) \right) \]
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On the other hand, consider \((\alpha, \delta) \in h^{-1}(p)\) for some \(h(a, c) = p \in S^2 (\frac{1}{2})\). First, consider nonzero values for \(\alpha\) and \(\delta\). By construction of \(h\),

\[ |\alpha|^2 - |\delta|^2 = |a|^2 - |c|^2. \]

Since \((\alpha, \delta), (a, c) \in S^3\), we also have

\[ |\alpha|^2 + |\delta|^2 = |a|^2 + |c|^2 = 1. \]

Summing (or subtracting) accordingly, we find

\[ |\alpha|^2 = |a|^2 \quad \text{and} \quad |\delta|^2 = |c|^2. \]
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|\alpha|^2 = |a|^2 \quad \text{and} \quad |\delta|^2 = |c|^2.
\]
Problem 2: \[ H : S^1 \times S^3 \rightarrow S^3. \]

On the other hand, consider \((\alpha, \delta) \in h^{-1}(p)\) for some \(h(a, c) = p \in S^2\left(\frac{1}{2}\right)\). First, consider nonzero values for \(\alpha\) and \(\delta\).

By construction of \(h\),

\[ |\alpha|^2 - |\delta|^2 = |a|^2 - |c|^2. \]

Since \((\alpha, \delta), (a, c) \in S^3\), we also have

\[ |\alpha|^2 + |\delta|^2 = |a|^2 + |c|^2 = 1. \]

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Problem 2: \[ H : S^1 \times S^3 \rightarrow S^3. \]

Thus, there exist \( \omega_0, \omega_1 \in S^1 \) such that

\[ \alpha = \omega_0 a \quad \text{and} \quad \delta = \omega_1 c. \]

Since \( h(\alpha, \delta) = h(a, c) \),

\[ a\bar{c} = \alpha \bar{\delta} \]

\[ = \omega_0 a \omega_1 c. \]

Dividing by \( a\bar{c} \),

\[ 1 = \frac{\omega_0 \bar{\omega}_1}{\omega_0} = \frac{\omega_0}{\omega_1}, \]

and there exists a \( \omega = \omega_0 = \omega_1 \in S^1 \) such that \( \omega(a, c) = (\alpha, \delta) \).
Problem 2: \( H : S^1 \times S^3 \rightarrow S^3 \).

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\[
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\[
a\bar{c} = \alpha \delta = \omega_0 a \bar{\omega_1 c}.
\]

Dividing by \( a\bar{c} \),

\[
1 = \frac{\omega_0}{\omega_1} \quad \Rightarrow \quad \frac{\omega_0}{\omega_1} = \omega = \omega_0 = \omega_1 \in S^1
\]

and there exists a \( \omega = \omega_0 = \omega_1 \in S^1 \) such that \( \omega(a, c) = (\alpha, \delta) \).
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Problem 2: \[ H : S^1 \times S^3 \to S^3. \]

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\]

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Problem 2:

\[ H : S^1 \times S^3 \rightarrow S^3. \]

Note that \((\alpha, \delta), (a, c) \neq (0, 0)\), as the origin is not in \(S^3\). Suppose one coordinate, say \(a\), is zero. Then we still find that

\[ |\alpha|^2 = |a|^2 \quad \text{and} \quad |\delta|^2 = |c|^2, \]

and that there exists a \(\omega_1 \in S^1\) such that \(\delta = \omega_1 c\). However,

\[ \omega_1 a = \omega_1 \cdot 0 \]
\[ = \alpha, \]

and we can simply consider \(\omega = \omega_1\). The group action of \(S^1\) on \(S^3\) is called the Hopf Action.
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Problem 3(a): \( U(2) \) Acting on \( \mathbb{C}^2 \).

- \( U(2) \) is the group of \( 2 \times 2 \) matrices over \( \mathbb{C} \) such that
  \[
  AA^* = A^*A = id,
  \]
  where \( A^* \) denotes the conjugate transpose of \( A \).

- \( U(2) \) acts naturally on \( S^3 \) by matrix multiplication:
  \[
  U(2) \times \mathbb{C}^2 \rightarrow \mathbb{C}^2
  \]
  via
  \[
  \begin{pmatrix}
  \alpha & \beta \\
  \delta & \gamma
  \end{pmatrix}
  \begin{pmatrix}
  a \\
  c
  \end{pmatrix}
  \mapsto
  \begin{pmatrix}
  \alpha & \beta \\
  \delta & \gamma
  \end{pmatrix}
  \begin{pmatrix}
  a \\
  c
  \end{pmatrix}.
  \]
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  \[
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  \]
  via
  \[
  \left( \begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix} \right) \rightarrow \begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix}.
  \]
Problem 3(a): \( U(2) \) Acting on \( \mathbb{C}^2 \).

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\[
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\]

via

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta \\
\end{pmatrix}
\begin{pmatrix}
a \\
c \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\alpha & \beta \\
\delta & \gamma \\
\end{pmatrix}
\begin{pmatrix}
a \\
c \\
\end{pmatrix}.
\]
Problem 3(a): **$U(2)$ Acting on $\mathbb{C}^2$.**

- $U(2)$ is the group of $2 \times 2$ matrices over $\mathbb{C}$ such that
  
  $$AA^* = A^*A = id,$$

  where $A^*$ denotes the conjugate transpose of $A$.

- $U(2)$ acts naturally on $S^3$ by matrix multiplication:

  $$U(2) \times \mathbb{C}^2 \rightarrow \mathbb{C}^2$$

  via

  $$
  \left( \begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix} \right) \mapsto \begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix}.
  $$
Problem 3(a): A Linear Map.

- $U(2)$ acts isometrically on $\mathbb{C}^2$, and preserves the unit 3-sphere $S^3$.
- Note that $A \in U(2)$ is a linear map, and thus

\[
A \begin{bmatrix} \omega a \\ \omega c \end{bmatrix} = A \cdot \omega \begin{bmatrix} a \\ c \end{bmatrix} = \omega A \begin{bmatrix} a \\ c \end{bmatrix}.
\]

- Thus any element of $U(2)$ commutes with the Hopf action.
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- Thus any element of $U(2)$ commutes with the Hopf action.
Problem 3(a): Induced Maps on $h(S^3)$.

For any $p \in S^2 \left(\frac{1}{2}\right)$, any two $\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} \hat{a} \\ \hat{c} \end{bmatrix} \in h^{-1}(p)$ are related by the Hopf action. Thus, any $A \in U(2)$ induces a smooth map on $\hat{A}$ on $h(S^3) \subseteq S^2 \left(\frac{1}{2}\right)$ such that the diagram:

\[\begin{array}{ccc}
S^3 & \xrightarrow{A} & S^3 \\
\downarrow h & & \downarrow h \\
h(S^3) & \xrightarrow{\hat{A}} & h(S^3)
\end{array}\]

commutes. We will return to this after part (b) and (c).
Consider the subgroup of $U(2)$ of matrices of the form

$$A_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad \text{where } \theta \in [0, 2\pi).$$

For $(a, c) \in S^3$, we find

$$A_\theta \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} a \cos \theta - c \sin \theta \\ a \sin \theta + c \cos \theta \end{bmatrix},$$

so

$$h\left( A_\theta \begin{bmatrix} a \\ c \end{bmatrix} \right) = h\left( \begin{bmatrix} a \cos \theta - c \sin \theta \\ a \sin \theta + c \cos \theta \end{bmatrix} \right) = \left( a \cos \theta - c \sin \theta \right)\left( a \sin \theta + c \cos \theta \right), \quad \frac{1}{2} |a \cos \theta - c \sin \theta|^2 - \frac{1}{2} |a \sin \theta + c \cos \theta|^2.$$

1st slot 2nd slot
Problem 3(b): Real Rotation - The First Slot.

For the first slot, we find

\[
(a \cos \theta - c \sin \theta)(a \sin \theta + c \cos \theta) = a\bar{a} \cos \theta \sin \theta - c\bar{c} \cos \theta \sin \theta + a\bar{c} \cos^2 \theta - c\bar{a} \sin^2 \theta \\
= \cos \theta \sin \theta (a\bar{a} - c\bar{c}) + \text{Re} a\bar{c} \cos^2 \theta - i \text{Im} a\bar{c} \cos^2 \theta \\
- (\text{Re} a\bar{c} - i \text{Im} a\bar{c}) \sin^2 \theta \\
= \sin 2\theta \cdot \frac{1}{2} (|a|^2 - |c|^2) + \text{Re} a\bar{c} (\cos^2 \theta - \sin^2 \theta) \\
+ i \text{Im} a\bar{c} (\cos^2 \theta + \sin^2 \theta) \\
= \sin 2\theta \cdot \frac{1}{2} (|a|^2 - |c|^2) + \text{Re} a\bar{c} (\cos 2\theta) + i \text{Im} a\bar{c}.
\]
Problem 3(b): Real Rotation - The Second Slot.

For the second slot, we have
\[ \frac{1}{2} |a \cos \theta - c \sin \theta|^2 - \frac{1}{2} |a \sin \theta + c \cos \theta|^2 \]

\[ = \frac{1}{2} (a \cos \theta - c \sin \theta)(\bar{a} \cos \theta - \bar{c} \sin \theta) - \frac{1}{2} (a \sin \theta + c \cos \theta)(\bar{a} \sin \theta + \bar{c} \cos \theta) \]

\[ = \frac{1}{2} \left[ a\bar{a} \cos^2 \theta + c\bar{c} \sin^2 \theta - (a\bar{c} + c\bar{a}) \cos \theta \sin \theta - a\bar{a} \sin^2 \theta - c\bar{c} \cos^2 \theta - (a\bar{c} + c\bar{a}) \cos \theta \sin \theta \right] \]

\[ = \frac{1}{2} \left[ \cos^2 \theta - \sin^2 \theta \right] \left( |a|^2 - |c|^2 \right) - 2 \cos \theta \sin \theta (a\bar{c} + c\bar{a}) \]

\[ = \cos 2\theta \cdot \frac{1}{2} (|a|^2 - |c|^2) - \sin 2\theta \cdot \text{Re} a\bar{c}. \]
Recalling that we consider $S^2\left(\frac{1}{2}\right) \subseteq \mathbb{C} \times \mathbb{R}$, we can let

$$(x + iy, t) = \left(a\overline{c}, \frac{1}{2}(|a|^2 - |c|^2)\right),$$

then we find

$$\hat{A}_\theta \left(a\overline{c}, \frac{1}{2}(|a|^2 - |c|^2)\right) = \hat{A}_\theta(x + iy, t)$$

$$= (\sin 2\theta \cdot t + \cos 2\theta \cdot x + iy, \cos 2\theta \cdot 2 - \sin 2\theta \cdot x).$$

Thus, $\hat{A}_\theta$ fixes the imaginary $y$, and produces a rotation of angle $2\theta$ around the imaginary $y$-axis.
Problem 3(b): Complex Rotation.

We can also consider the subgroup of $U(2)$ of the form

$$A_\omega = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix},$$

where $|\omega_1| = |\omega_2| = 1$ and $\omega_1 \overline{\omega}_2 = \omega$. Then

$$A_\omega \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \omega_1 a \\ \omega_2 c \end{bmatrix},$$

so

$$h(A_\omega \begin{bmatrix} a \\ c \end{bmatrix}) = \left( a\overline{c}\omega_1 \overline{\omega}_2, \frac{1}{2}|a|^2 - \frac{1}{2}|c|^2 \right).$$

Since $h$ commutes with any $A \subseteq U(2)$, utilizing coordinates on $S^2 \left( \frac{1}{2} \right) \subseteq \mathbb{C} \times \mathbb{R}$,

$$\hat{A}_\omega (x + iy, t) = (\omega_1 \overline{\omega}_2 (x + iy), t).$$

This shows that $\hat{A}_\omega$ produces a rotation of angle $\text{Arg}(\omega)$ in the complex plane (i.e., rotation around the $t$-axis).
Problem 3(a): Generating $U(2)$.

- The group $U(2)$ can be generated by the linear maps of the form $A_\theta$ and $A_\omega$.
- Hence, any induced map $\hat{A}$ downstairs can be written as some composition of maps of the form $\hat{A}_\theta$ and $\hat{A}_\omega$.
- As all $\hat{A}_\theta$ and $\hat{A}_\omega$ and linear (and thus smooth), any $A \in U(2)$ induces a smooth map $\hat{A}$ downstairs.
Problem 3(a): Generating $U(2)$. 

- The group $U(2)$ can be generated by the linear maps of the form $A_\theta$ and $A_\omega$.

- Hence, any induced map $\hat{A}$ downstairs can be written as some composition of maps of the form $\hat{A}_\theta$ and $\hat{A}_\omega$.

- As all $\hat{A}_\theta$ and $\hat{A}_\omega$ and linear (and thus smooth), any $A \in U(2)$ induces a smooth map $\hat{A}$ downstairs.
Problem 3(a): Generating $U(2)$.

- The group $U(2)$ can be generated by the linear maps of the form $A_\theta$ and $A_\omega$.

- Hence, any induced map $\hat{A}$ downstairs can be written as some composition of maps of the form $\hat{A}_\theta$ and $\hat{A}_\omega$.

- As all $\hat{A}_\theta$ and $\hat{A}_\omega$ and linear (and thus smooth), any $A \in U(2)$ induces a smooth map $\hat{A}$ downstairs.
Example: Rotation to North Pole.
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Problem 3(b): \( h \) is Onto.

- As shown graphically, any point \( p \in S^2 \left( \frac{1}{2} \right) \) can be transferred to the north pole \( n = (0, 1) \in \mathbb{C} \oplus \mathbb{R} \) by composing isometries
  
  \[
  \hat{A}_\theta \circ \hat{A}_\omega
  \]

  for some \( \theta \in [0, 2\pi) \) and some \( \omega = e^{i\phi} \).

- Since the maps commute, letting \( N \) be the North pole of \( S^3 \),

  every point in \( S^2 \left( \frac{1}{2} \right) \) can be realized as \( h(P) \) for some \( P \in S^3 \).
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for some $\theta \in [0, 2\pi)$ and some $\omega = e^{i\phi}$.

- Since the maps commute, letting $N$ be the North pole of $S^3$,

\[
\begin{array}{c}
N \quad \xrightarrow{(A_{\theta} \circ A_{\omega})^{-1}} \quad P \\
\downarrow h \quad \downarrow h \\
(\hat{A}_{\theta} \circ \hat{A}_{\omega})^{-1} \quad \downarrow (\hat{A}_{\theta} \circ \hat{A}_{\omega})^{-1} \\
n \quad \xrightarrow{p}
\end{array}
\]

every point in $S^2\left(\frac{1}{2}\right)$ can be realized as $h(P)$ for some $P \in S^3$. 

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Geometry of the Hopf Fibration
Problem 3(b): $h$ is Onto.

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for some $\theta \in [0, 2\pi)$ and some $\omega = e^{i\phi}$.

- Since the maps commute, letting $N$ be the North pole of $S^3$,

$$\begin{align*}
N &\xrightarrow{(A_\theta \circ A_\omega)^{-1}} P \\
\downarrow h &\hspace{1cm} \downarrow h \\
n &\xrightarrow{(\hat{A}_\theta \circ \hat{A}_\omega)^{-1}} P
\end{align*}$$

every point in $S^2\left(\frac{1}{2}\right)$ can be realized as $h(P)$ for some $P \in S^3$. 
Problem 3(c): Orientation-Preserving Isometries.

- Any orientation preserving isometry of $S^2 \left( \frac{1}{2} \right)$ is an element of $SO(3)$, which can be thought of as orientation around some axis.
- Let the vector $v \in S^2 \left( \frac{1}{2} \right)$ be the axis of rotation, and let $\phi$ be the angle of rotation.
- Let $U \in SO(3)$ is the matrix of the rotation.
  - It can be realised by first transferring $v$ to $n$ via some
    $$\hat{A}_\theta \circ \hat{A}_\omega;$$
  - then rotating around the $t$-axis via
    $$\hat{A}_{e^{i\phi}};$$
  - then returning $v$ through
    $$\left( \hat{A}_\theta \circ \hat{A}_\omega \right)^{-1}.$$
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- Any orientation preserving isometry of $S^2\left(\frac{1}{2}\right)$ is an element of $SO(3)$, which can be thought of as orientation around some axis.
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- Let $U \in SO(3)$ is the matrix of the rotation.
  - It can be realised by first transferring $\nu$ to $n$ via some
    \[ \hat{A}_\theta \circ \hat{A}_\omega; \]
  - then rotating around the $t$-axis via
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  - then returning $\nu$ through
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Problem 3(c): Orientation-Preserving Isometries.

Together, we find

\[ U = \left( \hat{A}_\theta \circ \hat{A}_\omega \right)^{-1} \circ \hat{A}_{e^{i\phi}} \circ \hat{A}_\theta \circ \hat{A}_\omega \]

\[ = \hat{A}_{-\omega} \circ \hat{A}_{-\theta} \circ \hat{A}_{e^{i\phi}} \circ \hat{A}_\theta \circ \hat{A}_\omega. \]

Each transformation on the right side has a nice lift upstairs to symmetries of the Hopf fibration, mapping \( S^3 \to S^3 \).
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Each transformation on the right side has a nice lift upstairs to symmetries of the Hopf fibration, mapping \( S^3 \to S^3 \).
Problem 3(d): \( h \) is a Quotient Map.

- Recall: “For a surjective map \( f : X \rightarrow Y \) to be a quotient map, it is a sufficient condition for \( f \) to be either closed or open.”

- We have already shown \( h \) is a surjective map.

- Let \( F \in S^3 \) be closed. Since our domain \( S^3 \) is compact, \( F \) is compact.

- Since \( h \) is continuous, \( h(F) \) is compact.

- Since \( S^2 \left( \frac{1}{2} \right) \) is Hausdorff, \( h(F) \) is closed, and \( h \) is a closed map.

- Thus, \( h \) is indeed a quotient map.
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Problem 3(e): The Map $\hat{A}$ is a Diffeomorphism.

Consider the commutative diagram:

\[
\begin{array}{ccc}
S^3 & \xrightarrow{A} & S^3 \\
| & \downarrow{h} & | \\
S^2 & \xrightarrow{\hat{A}} & S^2 \\
| & \downarrow{h} & | \\
S^2 & \xrightarrow{A^{-1}} & S^2
\end{array}
\]

$A^{-1} \circ A = I$

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Problem 3(e): The Map \( \hat{A} \) is a Diffeomorphism.

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| & & | \\
\downarrow{h} & \hat{A} & \downarrow{h} \\
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| & & | \\
\downarrow{A^{-1}} & \hat{A}^{-1} & \downarrow{A^{-1}} \\
S^3 & \xrightarrow{\hat{A}^{-1}} & S^3
\end{array}
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Problem 3(e): The Map $\hat{A}$ is a Diffeomorphism.

- We have shown that $h$ is onto and $A$ is an isometry, so $h \circ A$ is onto.

- Since the diagram commutes,

  $$\hat{A} \circ h = h \circ A,$$

  so $\hat{A}$ is necessarily onto.

- By construction, $\hat{A}$ is a smooth map (in fact, linear).

- Since $\hat{A} : S^2 \left( \frac{1}{2} \right) \to S^2 \left( \frac{1}{2} \right)$ is surjective, it is also injective.

- From our diagram, $\hat{A}$ has an inverse, $\hat{A}^{-1}$, which is also smooth.

- Hence, $\hat{A}$ is a diffeomorphism.
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- Hence, $\hat{A}$ is a diffeomorphism.
Problem 4: \( h : S^3 \rightarrow S^2 \left( \frac{1}{2} \right) \) is a Riemannian Submersion.

- As with the previous approaches, we consider the North pole

\[ N = (1, 0) \in S^3 \subseteq \mathbb{C} \oplus \mathbb{C}, \]

and the north pole,

\[ n = \left( 0, \frac{1}{2} \right) \in S^2 \left( \frac{1}{2} \right) \subseteq \mathbb{C} \oplus \mathbb{R}. \]

- Then

\[ h : N \rightarrow n. \]

- Moreover,

\[ h^{-1}(n) = (e^{i\theta}, 0), \]

a circle in the first complex plane.
Problem 4: \( h: S^3 \rightarrow S^2(\frac{1}{2}) \) is a Riemannian Submersion.
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\]
Problem 4: \( h : S^3 \to S^2 \left( \frac{1}{2} \right) \) is a Riemannian Submersion.

- The tangent space at the North pole is
  
  \[ T_N S^3 = \text{span} \{ [N, (i, 0)], [N, (0, 1)], [N, (0, i)] \}, \]
  
or ignoring footpoints,
  
  \[ T_N S^3 = \text{span} \{ (i, 0), (0, 1), (0, i) \}. \]

- Since \((i, 0)\) is tangent to the Hopf fiber it gets killed when we apply the differential map

  \[ h_* : T_N S^3 \to T_n S^2 \left( \frac{1}{2} \right). \]

- Thus, we can consider the action of \( h_* \) on \((0, 1)\) and \((0, i)\).

- The span \{\((0, 1), (0, i)\)\} is called the horizontal space.
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  \[
  T_N S^3 = \text{span} \{ (i, 0), (0, 1), (0, i) \}.
  \]
- Since \((i, 0)\) is tangent to the Hopf fiber it gets killed when we apply the differential map
  \[
  h_* : T_N S^3 \to T_n S^2 \left( \frac{1}{2} \right).
  \]
- Thus, we can consider the action of \( h_* \) on \((0, 1)\) and \((0, i)\).
- The span \{\((0, 1), (0, i)\)\} is called the horizontal space.
Problem 4: \( h: S^3 \to S^2(\frac{1}{2}) \) is a Riemannian Submersion.

- The basis vector \((0,1)\) can be realized as the derivative of a geodesic \(\gamma(t)\) passing through the footpoint, \(N\), where \(\gamma(0) = N\).

- This can be represented as

\[
\gamma(t) = (1,0) \cdot \cos t + (0,1) \cdot \sin t.
\]

- This is true, for if we take its derivative, we find

\[
\gamma'(t)\bigg|_{t=0} = (1,0) \cdot (-\sin(0)) + (0,1) \cdot \cos(0) = (0,1),
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and we have \(\gamma(0) = N\), while \(\gamma'(0) = (0,1)\), as needed.
Problem 4: \( h : S^3 \rightarrow S^2 \left( \frac{1}{2} \right) \) is a Riemannian Submersion.

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\[
\gamma(t) = (1, 0) \cdot \cos t + (0, 1) \cdot \sin t.
\]

3. This is true, for if we take its derivative, we find

\[
\gamma'(t)\bigg|_{t=0} = (1, 0) \cdot (-\sin(0)) + (0, 1) \cdot \cos(0)
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Problem 4: \( h: S^3 \to S^2 (\frac{1}{2}) \) is a Riemannian Submersion.

Applying the differential map \( h_* \),

\[
\begin{align*}
    h_*((0,1)) &= \left. \frac{d}{dt} h(\gamma(t)) \right|_{t=0} \\
    &= \left. \frac{d}{dt} h(\cos t, \sin t) \right|_{t=0} \\
    &= \left. \frac{d}{dt} \left( \cos t \sin t, \frac{1}{2} (\cos^2 t - \sin^2 t) \right) \right|_{t=0} \\
    &= \left. \frac{d}{dt} \left( \frac{\sin 2t}{2}, \frac{\cos 2t}{2} \right) \right|_{t=0} \\
    &= (\cos 2t, -\sin 2t) \bigg|_{t=0} \\
    &= (1,0).
\end{align*}
\]

Note that the length of the basis vector \((0,1) \in \mathbb{C} \oplus \mathbb{C}\) is preserved under the map \( h_* \).
Problem 4: \( h : S^3 \to S^2 \left( \frac{1}{2} \right) \) is a Riemannian Submersion.

In a similar manner, we can construct a geodesic

\[
\alpha(t) = (1, 0) \cdot \cos t + (0, i) \sin t \\
= (\cos t, i \sin t),
\]

such that \( \alpha(0) = N \) and \( \alpha'(0) = (0, i) \).
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Problem 4: $h : S^3 \to S^2 \left( \frac{1}{2} \right)$ is a Riemannian Submersion.

Applying the differential map $h_*$ to this second basis vector $(0, i)$,

$$h_*((0, i)) = \left. \frac{d}{dt} h(\alpha(t)) \right|_{t=0}$$

$$= \left. \frac{d}{dt} h(\cos t, i \sin t) \right|_{t=0}$$

$$= \left. \frac{d}{dt} \left( \cos t \cdot i \sin t, \frac{1}{2} \left( \cos^2 t - |i^2 \sin^2 t| \right) \right) \right|_{t=0}$$

$$= \left. \frac{d}{dt} \left( -i \cdot \frac{\sin 2t}{2}, \frac{\cos 2t}{2} \right) \right|_{t=0}$$

$$= (-i \cos 2t, -\sin 2t) \bigg|_{t=0}$$

$$= (-i, 0).$$

Length is again preserved.
Problem 4: $h : S^3 \rightarrow S^2(\frac{1}{2})$ is a Riemannian Submersion.

Moreover, $h_\ast$ preserves inner products.

Note that since we can carry any point in $S^3$ to $N$ through the isometries $A_\omega$ and $A_\theta$, it is enough to prove $h$ is a Riemannian submersion relative to $N$.

Since $h_\ast$ preserves both inner products and lengths on the basis vectors of the horizontal space, it is an isometry, and $h$ is a Riemannian submersion.
Problem 4: \( h : S^3 \rightarrow S^2 \left( \frac{1}{2} \right) \) is a Riemannian Submersion.

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Problem 5: $S^7 \rightarrow S^4$ is Analogous to $S^3 \rightarrow S^2$.

- We considered the map $h : S^3 \rightarrow S^2$ as one from $\mathbb{C} \oplus \mathbb{C}$ to $\mathbb{C} \oplus \mathbb{R}$.

- In a similar manner, we can construct $h : S^7 \rightarrow S^4 \left( \frac{1}{2} \right)$, and consider the ambient spaces in terms of quaternions. I.e.,

$$S^7 \subset \mathbb{H} \oplus \mathbb{H} \quad \text{and} \quad S^4 \left( \frac{1}{2} \right) \subset \mathbb{H} \oplus \mathbb{R}.$$ 

- By approaching the spaces from a quaternion point of view, we can use the map

$$h : (a, c) \mapsto \left( ac, \frac{1}{2} \left( |a|^2 - |c|^2 \right) \right).$$

- As the map is essentially an adaptation to the quaternions, the proof that $h$ is onto is similar to the previous complex case.
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Problem 6: $S^3$ Acting on $S^7$.

- Again similar to the complex case, we can define an $S^3$-action on $S^7$, defined by

\[ H : S^3 \times S^7 \to S^7 \]

via

\[ H : (\omega; (a, c)) \mapsto (a\omega, c\omega). \]

- In a manner akin to the complex case, we can show that the orbits of this cycle coincide with the fibers of $h$. However, we will again forgo the calculations for brevity.

(a concept overlooked prior to problem 5).
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Problem 7(a):  Upstairs and Downstairs, Part Deux.

- Recall that $\text{Sp}(2)$ is the group of $2 \times 2$ symplectic matrices with quaternion entries. That is, it is the group of $2 \times 2$ matrices with quaternion entries where

$$AA^* = A^*A = id,$$

where $A^*$ is the conjugate transpose of $A$.

- Here, we find there is a natural action of $\text{Sp}(2)$ on $\mathbb{H}^2$ by

$$\text{Sp}(2) \times \mathbb{H}^2 \to \mathbb{H}^2,$$

$$(\begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix}) \mapsto \begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix}.$$
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\[
\text{Sp}(2) \times \mathbb{H}^2 \rightarrow \mathbb{H}^2,
\]

\[
\left( \begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix} \right) \mapsto \begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix}.
\]
Problem 7(a): Upstairs and Downstairs, Part Deux.

For any $A \in \text{Sp}(2)$, we can also show that there is an induced map $\hat{A}$ such that the diagram:

\[
\begin{array}{ccc}
S^7 & \xrightarrow{A} & S^7 \\
\downarrow{h} & & \downarrow{h} \\
h(S^7) & \xrightarrow{\hat{A}} & h(S^7)
\end{array}
\]

commutes.
Problem 7(b): Upstairs and Downstairs, Part Deux.

Moreover, the analogous rotation

\[ A_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad \theta \in [0, 2\pi) \]

induces a map downstairs

\[ \hat{A}_\theta : \mathbb{H} \oplus \mathbb{R} \to \mathbb{H} \oplus \mathbb{R} \]

which causes rotation in the plane formed by the first real coordinate of \( \mathbb{H} \) and the last \( \mathbb{R} \) by an angle of exactly \( 2\theta \), fixing the complex entries in \( \mathbb{H} \).

Similarly a map

\[ A_\omega = \begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix}, \quad \omega \in \mathbb{H}, \ |\omega| = 1 \]

induces a map \( \hat{A}_\omega \) which produces rotation in \( \mathbb{H} \oplus \mathbb{R} \) that fixes the last \( \mathbb{R} \)-entry.
Problem 7(b):  \( h : S^7 \to S^4 \left( \frac{1}{2} \right) \) is Onto.

- Since the rotations \( \hat{A}_\omega \) and \( \hat{A}_\theta \) in \( S^4 \left( \frac{1}{2} \right) \) can isometrically transfer any point in \( S^4 \left( \frac{1}{2} \right) \) to the point \( (0, \frac{1}{2}) \), it suffices to show surjectivity at \( (0, \frac{1}{2}) \).

- However, the image of the North pole in \( S^7 \), \((1,0)\), is the point \( (0, \frac{1}{2}) \in S^4 \left( \frac{1}{2} \right) \subseteq \mathbb{H} \oplus \mathbb{R} \).

- Thus, \( h \) is surjective.
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- Thus, $h$ is surjective.
Consider another group of maps $G$ of the form

$$g_{\omega_1, \omega_2} : H \oplus H \to H \oplus H$$

via

$$(a, c) \mapsto (\omega_1 a, \omega_2 c).$$

This induces the map $\hat{g}_{\omega_1, \omega_2} : H \oplus \mathbb{R} \to H \oplus \mathbb{R}$ via

$$\hat{g}_{\omega_1, \omega_2}(a, c) = \left( \omega_1 \overline{ac\omega_2}, \frac{1}{2}|a|^2 - |c|^2 \right).$$

Note that this fixes the last coordinate, acting only on the quaternion coordinate of $S^4 \left( \frac{1}{2} \right)$. 
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P. Rajan, F. Rebro, A. Yassine
Geometry of the Hopf Fibration
Problem 7(c): Orientation-Preserving Isometries.

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Note that this fixes the last coordinate, acting only on the quaternion coordinate of $S^4 \left(\frac{1}{2}\right)$. 
Problem 7(c): Orientation-Preserving Isometries.

- Any orientation-preserving isometry $T \in \text{SO}(5)$ will have one fixed line, since it will have 1 as an eigenvalue.

- Hence, we can view such a transformation as a rotation being realised by an element of $\text{SO}(4)$ rotating about a vector $v$.

- We can send $v$ to $(0, \frac{1}{2}) \in S^4 \left( \frac{1}{2} \right)$ utilizing a composition $U_p$ of some $\hat{A}_\omega$ and $\hat{A}_\theta$ from the circle subgroups.

- This allows us to express such a $T$ as

$$T = U_v^{-1} \circ \hat{g}_{\omega_1, \omega_2} \circ U_v,$$

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Note: The Group of Symmetries $\Sigma$.

- Since $\text{Sp}(2) = \text{Gl}(2, \mathbb{H}) \cap \text{O}(8)$, every element of the subgroup $\text{Sp}(2) < \text{Gl}(2, \mathbb{H})$ is $\mathbb{H}$-linear.
- In the last proof, the group $G$ of symmetries of the form $g_{\omega_1, \omega_2}$ were not a subgroup of $\text{Sp}(2)$, as these maps are not $\mathbb{H}$-linear (the quaternionic multiplication is not commutative).
- For example, consider right multiplication defined as

$$R_v(u_1, u_2) = (u_1 v, u_2 v).$$

Then

$$R_v[(u_1, u_2)w] = (u_1 wv, u_2 wv)$$

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- Moreover, these two actions commute.

- These actions also overlap, as they both contain multiplication by $-1$.

- Hence they combine to give an action of $\text{Sp}(2) \times S^3/\sim$ on $S^7$, where $\sim$ is the two element subgroup consisting of the identity and the antipodal map.

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