ON THE MILLS-SELIGMAN AXIOMS FOR LIE ALGEBRAS OF CLASSICAL TYPE

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1. **Introduction.** Mills and Seligman in [8] presented a unified technique for the classification of those semisimple Lie algebras to which the methods of Killing and Cartan are applicable. They classified the Lie algebras L (assumed to be finite-dimensional, as will be also all algebras in the present paper), of characteristic not 2, 3, 5 or 7, which satisfy the following conditions: L has a Cartan subalgebra H such that L and H satisfy five axioms, which we may state as

(i) $LL = L$.

(ii) *The center of L consists of* 0 *alone.*

(iii) *H acts diagonally on L.*

(iv) *If α is a nonzero root then $L_\alpha L_{-\alpha}$ is one-dimensional.*

(v) *If α is a nonzero root and β is a linear functional on H, then there is a positive integer m such that $\beta + m\alpha$ is not a root.*

Here of course the roots are roots with respect to H, and (iii) means that if $h \in H$ and $a \in L_\alpha$, the root space for a nonzero root α, then $ah = \alpha(h)a$. By (ii) and (iii) H is necessarily abelian.

Mills and Seligman showed that if L is over a field F of characteristic $p > 7$ and satisfies the above axioms with respect to some Cartan subalgebra, then L is a direct sum of simple algebras of classical type, that is, each of the simple direct summands is the analogue over F of one of the simple Lie algebras (including the five exceptional algebras) over the complex numbers. More explicitly, the simple Lie algebras of classical type over F are obtained as follows: start with a complex simple Lie algebra, take a basis with integral structure constants as in [4, p. 24], reduce the structure constants modulo p, then take the scalar extension to an algebra over F, and divide by the center (which is nonzero only if the initial algebra is of type A_r with $p \mid r + 1$). The result of Mills and Seligman was extended by Mills [7] to include the case of characteristics 5 and 7.

Axiom (v) above, although essential for the use of root-string techniques, is rather artificial in that it does not correspond (because of the finite-dimensionality) to any step in the proof of the well-known results on complex semisimple Lie algebras.

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Our purpose in the present paper is the determination of the Lie algebras satisfying a set of axioms similar to those of Mills and Seligman, but without an axiom like (v) above; we shall prove that such algebras are direct sums of simple algebras which are either of classical type or of a certain explicitly determined class of algebras of rank one.

The axioms on L and H which we shall assume are

(i) $L^2 = L$.

(ii) *The center of* L *consists of* 0 *alone.*

(A) *For every nonzero root* α, L_α *is one-dimensional and* $\alpha(L_\alpha L_{-\alpha}) \neq 0$, *that is,* $L_\alpha + L_{-\alpha} + L_\alpha L_{-\alpha}$ *is a (split) three-dimensional simple algebra.*

Axiom (A) is formally stronger than (iii) and (iv) of Mills and Seligman, but still corresponds to a natural stage in the classification proof at characteristic 0. Axiom (A) was also used by Jacobson [5], together with (i) and the assumption that H is abelian (which is implied by (ii) and (A)), as a hypothesis in a preliminary investigation of the representation theory of such algebras.

Our classification must take account of the Albert-Zassenhaus algebras. By an Albert-Zassenhaus algebra is meant an algebra over a field F with a basis $\{u_\alpha \mid \alpha \in G\}$, where G is a finite additive subgroup of F, and with multiplication

$$u_\alpha u_\beta = \{\alpha h(\beta) - \beta h(\alpha) + \alpha - \beta\} u_{\alpha+\beta}, \qquad \alpha, \beta \in G, \tag{1.1}$$

where h is any additive mapping of G into F (see [1, p. 138]). Each of these algebras is a simple Lie algebra, for which u_0 spans a one-dimensional Cartan subalgebra with one-dimensional root spaces spanned by the u_α. The algebras satisfy axioms (i), (ii) and (A) above (with respect to (u_0)) but not (v) — the roots form an additive elementary p-group. These algebras are the only simple ones known which satisfy (i)–(iv) but not (v), as well as the only known simple algebras of rank one other than the three-dimensional algebra. Moreover the author has proved in [2] that if L has a one-dimensional Cartan subalgebra such that $e_\alpha L_{-\alpha} \neq 0$ for every nonzero root α and nonzero e_α in L_α, then L is either three-dimensional or an Albert-Zassenhaus algebra, provided that the base field F is algebraically closed of characteristic $p > 3$.

Our principal result, to whose proof almost all of this paper will be devoted, is the following.

MAIN THEOREM. *Let* L *be a finite-dimensional Lie algebra of characteristic* $p > 5$. *Suppose that* L *has a Cartan subalgebra* H *such that* (i), (ii) *and* (A) *are satisfied. Then* L *is a direct sum of simple algebras satisfying the same hypotheses, and each such simple algebra is either of classical type or has rank one. If the base field is perfect then the simple summands which are not of classical type are Albert-Zassenhaus algebras.*

If in addition L is restricted (that is, is a p-algebra) and if a simple direct summand is of rank one but not of classical type, then (as can be seen without using

[2]) the summand is the p-dimensional Witt algebra, that is, has a basis $\{u_i | i \in F_p\}$ (F_p the prime field) with $u_i u_j = (i-j)u_{i+j}$.

Information on representations of the Witt algebra constitutes the chief tool in the proof of the main theorem. We begin our proof in the next section by showing that if (v) fails then there is a root α such that $W_\alpha = \sum_{i \neq 0} L_{i\alpha}$ generates a Witt algebra. In the succeeding section we obtain explicit formulas for all irreducible representations of the Witt algebra of degree not greater than p. The remainder of the proof consists principally of the application of these formulas to the representations of W_α on $\sum_i L_{\beta+i\alpha}$ and $\sum_i L_{-\beta+i\alpha}$ when, for some i, $L_{\beta+i\alpha}L_{-\beta-i\alpha} \neq L_\alpha L_{-\alpha}$; this is done to show that certain algebras of rank greater than one cannot exist and that the L_β with $L_\beta L_{-\beta} = L_\alpha L_{-\alpha}$ generate a direct summand of L of rank one.

The assumption that $p \neq 5$ is used only in the proofs of Lemmas 2.2 and 4.1. The proof of Lemma 2.2 cannot be extended since when $p = 5$ there actually does exist a representation of the type considered in that proof, in which v is represented by a nonzero scalar. However it seems likely that our theorem remains true when $p = 5$.

2. **Proof that L contains the Witt algebra if not of classical type.** Throughout the proof of the main theorem we shall assume that L is a finite-dimensional Lie algebra over a field F of characteristic $p > 5$, with a Cartan subalgebra H, and that L satisfies Axioms (i), (ii) and (A), with respect to H. Until close to the end of the proof, in §5, we shall also assume that F is algebraically closed. All roots considered will be roots with respect to H. The letters i and j will always denote integers.

LEMMA 2.1. *If L is not a direct sum of simple algebras of classical type, then there is a nonzero root γ such that 2γ is also a root.*

Proof. Under the hypotheses, Axiom (v) of [8] must fail, that is, there exist roots α and β ($\alpha \neq 0$) such that $\beta + i\alpha$ is a root for all i. If $\beta + i\alpha = 0$ for some i then every $i\alpha$ is a root. Thus we may suppose that $\beta + i\alpha \neq 0$ ($i = 0, \cdots, p-1$). Suppose that none of 2α, $2\beta + 2\alpha$, $2\beta - 2\alpha$ is a root. For each root δ choose a nonzero element u_ρ in L_ρ. By the Jacobi identity,

$$(u_\beta u_{\beta+i\alpha})u_{-\beta} + (u_{\beta+i\alpha}u_{-\beta})u_\beta + (u_{-\beta}u_\beta)u_{\beta+i\alpha} = 0.$$

But for $i = \pm 2$, the first two terms vanish, so that $(\beta + i\alpha)(u_\beta u_{-\beta}) = 0$. This implies that $\alpha(L_\beta L_{-\beta}) = \beta(L_\beta L_{-\beta}) = 0$, which contradicts (A). Hence we may take one of $\alpha, \beta + \alpha$ or $\beta - \alpha$ as γ, and the lemma is proved.

Now suppose that α is nonzero root such that $i\alpha$ is a root for some $i = 2, \cdots, p-2$. Consider the subspace K of L spanned by all $L_{i\alpha}$ (including $L_0 = H$). Then K is a subalgebra, and its center I clearly is $\{h \in H | \alpha(h) = 0\}$. The quotient algebra K/I has rank one, with Cartan subalgebra H/I, and the roots of K/I consist of those mappings $i\bar{\alpha}$ of H/I which are induced by roots of L of the form $i\alpha$. It then follows easily from [9, pp. 42–47] (see [2, Lemma 3.1]) that K/I is a Witt algebra, with $i\bar{\alpha}$ a root for all i, and that there are nonzero elements $u_{i\alpha}$ in $L_{i\alpha}$ such that $u_{i\alpha}u_{j\alpha} \equiv (i-j)u_{(i+j)\alpha}$ (mod I). Since $I \subseteq H$ and $L_{i\alpha}L_{j\alpha} \subseteq L_{(i+j)\alpha}$, this implies that

$$u_{i\alpha}u_{j\alpha} = (i-j)u_{(i+j)\alpha} \text{ if } i+j \not\equiv 0 \pmod{p}. \tag{2.1}$$

We may assume that u_0 was chosen so that $u_\alpha u_{-\alpha} = 2u_0$. We may write $u_{2\alpha}u_{-2\alpha} = 4u_0 + 3\cdot 2\cdot 1 v$, where v is some element, possibly 0, in I. We shall show by induction that

$$u_{i\alpha}u_{-i\alpha} = 2iu_0 + (i+1)\,i(i-1)v, \quad i = 0,\cdots,p-1. \tag{2.2}$$

Indeed, for $i = 3,\cdots, p-1$, we have $(i-2)u_{i\alpha}u_{-i\alpha} = (u_{(i-1)\alpha}u_\alpha)u_{-i\alpha} = (u_{(i-1)\alpha}u_{-i\alpha})u_\alpha + u_{(i-1)\alpha}(u_\alpha u_{-i\alpha}) = -2(2i-1)u_0 + (1+i)u_{(i-1)\alpha}u_{-(i-1)\alpha} = (i-2)2iu_0 + (i-2)(i+1)\,i(i-1)v$, which proves (2.2).

It would be convenient if we could conclude directly that $v = 0$. However this is not possible without referring to the imbedding in L and using the assumption that L is centerless, since (2.1) and (2.2) actually define a $(p+1)$-dimensional Lie algebra with center (v) which is a nontrivial central extension of the Witt algebra. In order to show that v is in the center of L, we shall use a device motivated by the existence in the Witt algebra of two nonconjugate Cartan decompositions, each giving rise to a convenient multiplication table.

The Witt algebra W over F may be regarded as the Lie algebra of derivations of the p-dimensional commutative associative algebra $F[x]$ with $x^p = 1$. Thus W has a basis $f_0, f_1, \cdots, f_{p-1}$, where f_i is the derivation of $F[x]$ determined by $xf_i = x^{i+1}$, and these basis elements have the multiplication table

$$f_i f_j = (i-j)f_{i+j}, \tag{2.3}$$

where the subscripts are added modulo p. Taking $y = x-1$ as a generator of $F[x]$, so that this algebra becomes $F[y]$ with $y^p = 0$, we see that W also has a basis $e_{-1}, e_0, \cdots, e_{p-2}$, where $ye_i = y^{i+1}$, and for $-1 \leqq i, j \leqq p-2$,

$$e_i e_j = (i-j)e_{i+j}, \qquad e_k = 0 \text{ if } k > p-2. \tag{2.4}$$

Since $xe_i = (x-1)e_i = (x-1)^{i+1}$, we have

$$e_i = \sum_{j=0}^{i+1} \binom{i+1}{j} (-1)^j f_{i-j}, \qquad i = -1, \cdots, p-2 \tag{2.5}$$

and similarly, since $yf_i = (y+1)^{i+1}$,

$$f_i = \sum_{j=0}^{i+1} \binom{i+1}{j} e_{i-j}, \qquad i = 0, \cdots, p-1.$$

We are now ready to complete the proof that L contains the Witt algebra.

LEMMA 2.2. *Suppose that α is a nonzero root such that $k\alpha$ is a root for some $k = 2,\cdots,p-2$. Then all $i\alpha$ are roots, and there are nonzero elements $u_{i\alpha}$ in $L_{i\alpha}$ for $i = 0,\cdots,p-1$ such that $u_{i\alpha}u_{j\alpha} = (i-j)u_{(i+j)\alpha}$ for all i, j, that is, L_α and $L_{k\alpha}$ generate a Witt algebra.*

Proof. Choosing the elements $u_{i\alpha}$ in K so that (2.1) and (2.2) hold, what we must show is that the element v in I is zero. Let W_α be the subalgebra of L spanned by v and the $u_{i\alpha}$, $i = 0,1,\cdots,p-1$. Suppose that $v \neq 0$, and set

$$e'_i = \sum_{j=0}^{i+1} \binom{i+1}{j} (-1)^j u_{(i-j)\alpha}, \quad i = -1,\cdots,p-2 \quad (e'_k = 0 \text{ if } k > p-2).$$

Since $W_\alpha/(v)$ is the Witt algebra, we have

$$e'_i e'_j \equiv (i-j)e'_{i+j} \bmod (v).$$

Since $e'_0 = u_0 - u_{-\alpha}$, there is no term in v in $e'_i e'_0$, and it follows that e'_0 and v span a Cartan subalgebra of W_α with one-dimensional root spaces spanned by the e'_i, $i = -1,1,2,\cdots,p-2$. Therefore $e'_i e'_j$ has no term in v unless $i + j \equiv 0$ mod p. But e'_i is $u_{i\alpha}$ plus terms of lower index, so the only term in v in $e'_i e'_{p-i}$ is that of $u_{i\alpha}u_{(p-i)\alpha}$, that is, $e'_i e'_{p-i} = i(i+1)(i-1)v$. Now suppose that $\beta(v) \neq 0$ for some root β. Consider the representation space $M = \sum_i L_{\beta+i\alpha}$ for W_α under right multiplication. Since $\alpha(v) = 0$, $(\beta + i\alpha)(v) = \beta(v)$ for all i, and v is represented by a nonzero scalar. The elements $e'_{(p-1)/2}$, $e'_{(p+1)/2}$, e'_{p-2} and v span a subalgebra B of W_α containing v in its square.

Let N be an irreducible B-submodule of M. Since v is represented by a nonzero scalar of trace 0, N has dimension p, $M = N$, and M is irreducible under B. But since $p > 5$, e'_{p-2} is in the center of B, and so is represented by a scalar on M. Therefore $e'_{p-2}e'_2$, which is a nonzero multiple of v, is represented by 0, a contradiction. Hence $\beta(v) = 0$ for all roots β, and v is in the center of L and so vanishes. This completes the proof of the lemma.

3. **Representations of the Witt algebra.** Let W be the Witt algebra over a field F, with basis $e_{-1}, e_0, \cdots, e_{p-2}$ and multiplication given by (2.4). Let V be a vector space over F with basis $v_0, v_1, \cdots, v_{p-1}$, and let a, b be given scalars. For each e_j define a linear transformation on V by setting (for $i = 0,\cdots,p-1$)

$$(3.1) \quad \begin{aligned} v_i e_j &= [i + (j+1)a]v_{i+j} \quad \text{if} \quad (i,j) \neq (0,-1); \quad v_k = 0 \quad \text{if} \quad k > p-1; \\ v_0 e_{-1} &= b v_{p-1}. \end{aligned}$$

Note that if $j > p-2$ then either $v_{i+j} = 0$ or $i = 0$, $j = p-1$ and the coefficient of v_{i+j} vanishes. We claim that the linear mapping $\Delta = \Delta_{ab}$, determined by (3.1), of W to linear transformations on V, is a representation of W. Indeed if $j \neq k$ and either $i \neq 0$ or $j,\ k \neq -1$ then

$$\begin{aligned}(v_ie_j)e_k - (v_ie_k)e_j &= \{[i+(j+1)a][i+j+(k+1)a] - [i+(k+1)a][i+k(j+1)a]\}v_{i+j+k} \\ &= (j-k)[i+(1+j+k)a]v_{i+j+k} = (j-k)v_ie_{j+k} = v_i(e_je_k).\end{aligned}$$

Also, if $k>0$ then $(v_0e_{-1})e_k - (v_0e_k)e_{-1} = 0 - k(k+1)av_{k-1} = (-1-k)v_0e_{k-1} = v_0(e_{-1}e_k)$, and $(v_0e_{-1})e_0 - (v_0e_0)e_{-1} = [b(-1+a) - ba]v_{p-1} = -v_0e_{-1} = v_0(e_{-1}e_0)$. Hence Δ is a representation.

If $a=1$ and $b=0$ then $v_0, v_1, \cdots, v_{p-2}$ span a subspace of V which is invariant under all e_j. We shall denote by Δ'_{10} the restriction of Δ_{10} to the representation of W on this $(p-1)$-dimensional subspace.

Lemma 3.1. *Let $\Delta = \Delta_{ab}$ be the representation of W defined above, and suppose that $a \neq 0,1$. Then there exists a characteristic vector v of e_0^Δ which is annihilated by e_1^Δ and e_2^Δ, and for any such v, the characteristic value for e_0^Δ is $a-1$.*

Proof. Clearly v_{p-1} satisfies the conditions for v. Now suppose that $v \neq 0$, $ve_0 = cv$ and $ve_1 = ve_2 = 0$. Since V is spanned by characteristic vectors of e_0^Δ with distinct characteristic roots, we may suppose that $v = v_i$ for some i $(0 \leqq i \leqq p-1)$, and thus $c = i + a$. But $v_ie_1 = 0$ only if $i = p-1$ or $i + 2a = 0$, while $v_ie_2 = 0$ only if $i = p-1$ or $i = p-2$ or $i + 3a = 0$. Therefore the conditions that $v_ie_1 = v_ie_2 = 0$ and $a \neq 0,1$ imply that $i = p-1$ and $c = a-1$, and the lemma is proved.

Theorem 3.1. *Let W be the Witt algebra over an algebraically closed field F of characteristic $p>3$. The (nontrivial) irreducible representations of W of degree not greater than p are, up to equivalence, the mappings Δ_{ab} $(a,b \in F;\ (a,b) \neq (0,0),\ (1,0))$ and Δ'_{10} defined above. Two of the irreducible representations Δ_{ab}, Δ_{cd}, with $(a,b) \neq (c,d)$, are equivalent if and only if $b=d$ and either $a=1,\ c=0$ or $a=0,\ c=1$.*

Proof. The Witt algebra is a restricted Lie algebra, and for the basis elements e_i (satisfying (2.4)) we have $e_i^p = 0$ for $i = -1,1,2,\cdots,p-2$, and $e_0^p = e_0$. If Δ is a (not necessarily restricted) irreducible representation of a restricted Lie algebra K over F, then for any k in K, by Schur's Lemma $(k^\Delta)^p - (k^p)^\Delta$ is a scalar multiple of the identity transformation. In particaulr if Δ is an irreducible representation of W, there are scalars $\varepsilon_i = \varepsilon_i(\Delta)$ such that

$$(3.2)\qquad (e_0^\Delta)^p - e_0^\Delta = \varepsilon_0(\Delta)I,\ (e_i^\Delta)^p = \varepsilon_i(\Delta)I, \qquad i = -1,1,2,\cdots,p-2.$$

The irreducible representations of W over F have been considered by Chang [3]. For any given values of ε_i $(i = -1,\cdots,p-2)$ he determines the number of inequivalent irreducible representations Δ of W of each degree whose invariants $\varepsilon_i(\Delta)$ have the given values. His results on the irreducible representations Δ of W of degree not greater than p [3, pp. 172, 176] are as follows: $\varepsilon_i(\Delta) = 0$ for $i = 1,\cdots,p-2$; moreover for given values $\varepsilon_{-1},\varepsilon_0,0,\cdots,0$, of the invariants, the number of inequivalent irreducible representations of W is p, all of degree p, in case $\varepsilon_0 \neq 0$; $p-1$, all of degree p, in case $\varepsilon_0 = 0$ and $\varepsilon_{-1} \neq 0$; and, p, of which $p-2$ are of degree p, one of degree $p-1$ and one (the trivial representation) of degree 1, in case $\varepsilon_0 = \varepsilon_{-1} = 0$.

Let V' be an irreducible invariant subspace of V for the representation $\Delta = \Delta_{ab}$. Since V is spanned by characteristic vectors of e_0^Δ with distinct characteristic roots, and since V' contains a characteristic vector of e_0^Δ, some v_i $(0 \leqq i \leqq p-1)$ is in V'. Then V' is spanned by all $((v_i e_{j_1})\cdots)e_{j_r}$, so that V' is spanned by some of the v_j. Repeatedly operating with e_{-1}^Δ, we see that $v_0,v_1,\cdots,v_i$ are all in V'. But by the result of Chang, the dimension of V' is 1, $p-1$ or p. Hence either $V' = (v_0)$ or $V' = (v_0,\cdots,v_{p-2})$ or $V' = V$. By (3.1), (v_0) is invariant only if $(a,b) = (0,0)$, and $(v_0,\cdots,v_{p-2})$ is invariant only if $(a,b) = (1,0)$. Therefore Δ_{ab} is irreducible provided $(a,b) \neq (0,0)$, $(1,0)$, and Δ'_{10} is irreducible.

By (3.1), the representations $\Delta = \Delta_{ab}$ satisfy (3.2) with $\varepsilon_{-1}(\Delta) = 1\cdot 2\cdots(p-1)b = -b$, $\varepsilon_0(\Delta) = a^p - a$ and $\varepsilon_1(\Delta) = \cdots = \varepsilon_{p-2}(\Delta) = 0$. Hence for any values $\varepsilon_{-1},\varepsilon_0,0,\cdots,0$, there is a Δ_{ab} having invariants with these values, and the representations $\Delta_{a+k,b}$, for all k in F_p, also have invariants with the given values, but no other Δ_{cd} has this property. Moreover $\varepsilon_0(\Delta_{ab}) = 0$ if and only if $a \in F_p$.

By the result of Chang, it will complete the proof to show that Δ_{ab} and $\Delta_{a+k,b}$ $(k \in F_p, k \neq 0)$ are equivalent only if $a = 0$ and $k = 1$ or $a = 1$ and $k = -1$. But we see that this condition follows immediately from Lemma 3.1, taking into account the fact that if $a = 0$ or 1 then $v_{p-1}e_1 = v_{p-1}e_2 = 0$ and $v_{p-1}e_0 = -v_{p-1}$ or 0. Thus the theorem is proved.

Since Δ is restricted if and only if $\varepsilon_i(\Delta) = 0$ for $i = -1,0,\cdots,p-2$, we immediately get the following consequence.

COROLLARY 3.1. *Any (nontrivial) irreducible restricted representation of the Witt algebra, over an algebraically closed field of characteristic $p > 3$, is equivalent to one of the $p-1$ representations $\Delta'_{10},\Delta_{20},\cdots,\Delta_{-1,0}$, which are irreducible and inequivalent.*

4. **Application of the representation formulas.** If α is a root such that the root spaces $L_{i\alpha}$ generate a Witt algebra, we shall denote this Witt algebra by W_α. If also β is a root, then $\sum_i L_{\beta+i\alpha}$ is a representation space for W_α; we shall write

$$\sum_i L_{\beta+i\alpha} = M_{\beta\alpha},$$

and we shall write $\Gamma_{\beta\alpha}$ for the representation of W_α on $M_{\beta\alpha}$. Note that $M_{\beta\alpha}M_{-\beta\alpha} \subseteq H + W_\alpha$.

Lemma 4.1. *Suppose that α and β are roots and that the root spaces $L_{i\alpha}$ generate a Witt algebra W_α. If $M_{\beta\alpha}M_{-\beta\alpha} \nsubseteq W_\alpha$ then $M_{\beta\alpha}M_{-\beta\alpha} \subseteq \{h \in H \mid \alpha(h) = 0\}$.*

Proof. We shall take nonzero elements $u_{i\alpha}$ in $L_{i\alpha}$ $(i = 0, \cdots, p-1)$ such that $u_{i\alpha}u_{j\alpha} = (i-j)u_{(i+j)\alpha}$, and also a basis $e_{-1}, e_0, \cdots, e_{p-2}$ of W_α such that (2.4) holds.

If $M_{\beta\alpha}$ or $M_{-\beta\alpha}$ is one-dimensional then $M_{\beta\alpha} = L_\beta$, $M_{-\beta\alpha} = L_{-\beta}$, and $L_\beta L_{-\beta} \subseteq H$. Moreover $L_\alpha L_\beta = L_\alpha L_{-\beta} = 0$, so that $L_\alpha(L_\beta L_{-\beta}) = 0$, that is $\alpha(L_\beta L_{-\beta}) = 0$, and the conclusion holds in this case.

Now suppose $M_{\beta\alpha}$ is not one-dimensional. Then there is a root $\beta + i\alpha$ distinct from β, and so $(\operatorname{ad} u_0)^p$ does not vanish on $M_{\beta\alpha}$. Hence not all composition factors of $\Gamma_{\beta\alpha}$ are trivial. Therefore $\Gamma_{\beta\alpha}$ either is irreducible of degree $p-1$ or p, or contains a composition factor of degree $p-1$.

We shall now consider the possibilities for the representations of W_α on $M_{\beta\alpha}$ and $M_{-\beta\alpha}$ in a number of cases, showing that the first three cases cannot occur, and that in the remaining cases the conclusion of the lemma holds. We take a basis $h_1, \cdots, h_{r-1}, u_0$ of H such that $\alpha(h_i) = 0$, $i = 1, \cdots, r-1$. Under the hypothesis, there is an h_l $(1 \leqq l \leqq r-1)$ such that some element in $M_{\beta\alpha}M_{-\beta\alpha}$ has a nonzero component in h_l. In what follows, 0 and 1 will sometimes denote integers and sometimes elements of F; which is meant should be clear from the context. However i, j, k and n will denote integers.

Case 1. *$\Gamma_{\beta\alpha}$ and $\Gamma_{-\beta,\alpha}$ both irreducible of degree p, with $\Gamma_{\beta\alpha} \cong \Delta_{ab}$ and $\Gamma_{-\beta\alpha} \cong \Delta_{a',b'}$ where $a + a' \neq 1$ and either $a \neq 0, 1$ or $a' \neq 0, 1$:*

By symmetry we may suppose that $a \neq 0, 1$. There is a basis $v_0, \cdots, v_{p-1}$ of $M_{\beta\alpha}$ such that (3.1) holds, and similarly a basis $v'_0, \cdots, v'_{p-1}$ of $M_{-\beta\alpha}$ such that (3.1) holds with v_i, a, b replaced by v'_i, a', b'. We denote by c_{ij} the coefficient of h_l in the expression for $v_iv'_j$ as a linear combination of the given basis elements of H plus elements in other root spaces. Then $c_{ij} = 0$ if $i > p-1$ or $j > p-1$. We shall only consider the c_{ij} with nonnegative subscripts. By the Jacobi identity, $(v_iv'_j)e_k = (v_ie_k)v'_j + v_i(v'_je_k)$; taking the terms in h_l, we get

$$(4.1) \qquad 0 = [i + (k+1)a]c_{i+k,j} + [j + (k+1)a']c_{i,j+k}, \quad k \neq -1 \text{ or } i,j \neq 0,$$

and, when $k = -1$ and $i = 0$,

$$(4.2) \qquad 0 = bc_{p-1,j} + jc_{0,j-1}, \qquad j \neq 0.$$

With $k = 0$, (4.1) implies that $c_{ij} = 0$ unless $(i+j)1 + a + a' = 0$. Since $c_{ij} \neq 0$ for some i,j, $a + a' \in F_p$; we write

$$a + a' = n1, \qquad 1 \in F, \qquad 0 \leqq n \leqq p.$$

Hence if $c_{ij} \neq 0$ then $i + j + n = 0$, p or $2p$.

By (4.1), if $c_{i,j+1} = 0$ then $c_{i+1,j} = 0$ unless $i1 + 2a = 0$. In particular, since

$a \neq 0$, if $c_{0,j+1} = 0$ then $c_{1j} = 0$. Also, with $k = 2$ in (4.1) we see that if $c_{i-1,j+2} = 0$ $(i > 0)$ then $c_{i+1,j} = 0$ unless $(i-1)1 + 3a = 0$. Since $a \neq 1$, either $i1 + 2a \neq 0$ or $(i-1)1 + 3a \neq 0$ and hence

$$\text{if } c_{i,j+1} = c_{i-1,j+2} = 0 \text{ then } c_{i+1,j} = 0 \ (i > 0), \tag{4.3}$$
$$\text{if } c_{0,j+1} = 0 \text{ then } c_{1j} = 0.$$

Now suppose that $i + j + n = 2p$. Then c_{ij} is one of

$$c_{p-n+p,0}, c_{p-n+p-1,1}, \cdots, c_{p-n,p}, c_{p-n-1,p+1}$$

(all subscripts are nonnegative). Since the last two terms vanish, repeated application of (4.3) shows that all terms vanish, and $c_{ij} = 0$.

Next suppose that $i + j + n = p$. Then c_{ij} is one of $c_{p-n,0}, c_{p-n-1,1}, \cdots, c_{0,p-n}$. Since we have shown that $c_{p-1,p-n+1} = 0$, and since $n \neq 1$, (4.2) implies that $c_{0,p-n} = 0$. Then repeated application of (4.3) again shows that $c_{ij} = 0$.

Finally, if $i + j + n = 0$, then $i = j = n = 0$, and the only relevant c_{ij} is c_{00}. But (4.2) with $j = 1$ implies that $c_{00} = 0$. Hence Case 1 cannot occur.

Case 2. *Degree* p, *with one of* $\Gamma_{\beta\alpha}, \Gamma_{-\beta\alpha}$ *irreducible, and the other having a* $(p-1)$*-dimensional invariant subspace*:

By symmetry we may suppose that $\Gamma_{\beta\alpha}$ is irreducible, and hence $\Gamma_{\beta\alpha} \cong \Delta_{ab}$ where $(a,b) \neq (0,0), (1,0)$. Take a basis $v_0, \cdots, v_{p-1}$ of $M_{\beta\alpha}$ such that (3.1) holds.

Then $M_{-\beta\alpha}$ has a basis $v'_0, \cdots, v'_{p-1}$ such that

$$v'_i e_j = (i + j + 1)v'_{i+j}, \qquad i = 0, \cdots, p-2,$$

where $v'_i = 0$ if $i > p-1$, and $v'_{p-1}e_0$ is a linear combination of $v'_0, \cdots, v'_{p-2}$. Changing v'_{p-1} by subtracting a suitable linear combination of $v_0, \cdots, v_{p-2}$, we may suppose that $v'_{p-1}e_0 = 0$. By a weight argument, there exist scalars $b'_{-1}, b'_1, b'_2, \cdots, b'_{p-2}$ such that

$$v'_{p-1}e_{-1} = b'_{-1}v'_{p-2}, \ v'_{p-1}e_i = b'_i v'_{i-1}, \quad i = 1, 2, \cdots, p-2.$$

Write $(\mathrm{ad}\, e_{-1})^p = D$. Then D is a derivation which annihilates $M_{-\beta\alpha}$ and $M_{\beta\alpha}M_{-\beta\alpha}$. Therefore $(M_{\beta\alpha}D)M_{-\beta\alpha} = 0$. Since D acts as the scalar $-bI$ on $M_{\beta\alpha}$, this implies that $b = 0$, and hence $a \neq 0, 1$.

We continue using the notation c_{ij} of Case 1, and write $a' = 1$. In particular with n defined as before, it follows that $n \neq 1, 2$. Now (4.1) holds provided $j \neq p-1$ or $k = 0$. Moreover if $j \neq p-1$ then (4.2) and (4.3) hold. If $j = p-1$ and $k = 1$ or 2 then instead of (4.1) one has

$$0 = [i + (k+1)a]c_{i+k,p-1} + b'_k c_{i,k-1},$$

and hence, as in the proof of (4.3), if $c_{i,0} = c_{i-1,1} = 0$ $(i > 0)$ then $c_{i+1,p-1} = 0$, and if $c_{00} = 0$ then $c_{1,p-1} = 0$. We need only show that $c_{p-n,0} = c_{p-n-1,1} = 0$

in order to carry through in this case the previous proof that $c_{ij} = 0$ if $i + i + n = 2p$. If $i \neq 0$, an analogue of (4.2) obtained by taking $j = 0$ and $k = -$ says that $0 = ic_{i-1,0} + 0$. Since $n \neq 1$, it follows that $c_{p-n,0} = 0$. But then (4.1) with $i = p - n - 1$, $j = 0$ and $k = 1$ also gives $c_{p-n-1,1} = 0$.

No further use of (4.3) when $j = p - 1$ is made in Case 1 unless $n = 0$, in which case we must show that $c_{1,p-1} = 0 = c_{00}$. But since $b = 0$, (4.2) with $j = 1$ implies that $c_{00} = 0$, and we have already noted that this implies that $c_{1,p-1} = 0$. Hence Case 2 cannot occur.

Case 3. *Degree* p, *with one* $\Gamma_{\beta\alpha}, \Gamma_{-\beta\alpha}$ *irreducible and the other having a one-dimensional invariant subspace*:

We may suppose that $M_{\beta\alpha}$ is as in the proof of Case 2. Let v'_{p-1} be a basis of the one-dimensional invariant subspace of $M_{-\beta\alpha}$. Then $M_{-\beta\alpha}/(v'_{p-1})$ has a basis $\bar{v}_0, \cdots, \bar{v}_{p-2}$ such that $\bar{v}_i e_j = (i + j + 1)\bar{v}_{i+j}$, where $\bar{v}_k = 0$ if $k > p - 1$. Take an element v'_i in $\bar{v}_i$, $i = 0, \cdots, p-2$. Then $v'_i e_0 = (i + 1)v'_i + d_i v'_{p-1}$ for some scalar d_i. Changing v'_i by adding $d_i/(i + 1)v'_{p-1}$ to it, we may assume that $v'_i e_0 = (i + 1)v'_i$. By a weight argument,

$$v'_i e_j = (i + j + 1)v'_{i+j} \text{ if } i + j \neq p-1 \text{ and } (i,j) \neq (0, -1),\ (p - 1, -1);$$

$$v_k = 0 \text{ if } k > p - 1,$$

and there are scalars b'_i, $i = -1, 1, 2, \cdots, p-2$, such that

$$v'_0 e_{-1} = b'_{-1} v'_{p-1}, \quad v'_i e_{p-1-i} = b'_i v'_{p-1}, \quad i = 1, \cdots, p-2.$$

Write $a' = 1$, and define the c_{ij} and n as in Case 1. Then as in Case 2, $b = 0$ and hence $a \neq 0, 1$ and $n \neq 1, 2$. Now (4.1) holds with only the coefficient of $c_{i,j+k}$ changed for certain subscripts, and (4.2) holds except when $j \neq p-1$. It follows that the proof in Case 1 goes through here without change — the exceptional case of (4.2) is not used since $p - n + 1 \neq p - 1$ because $n \neq 2$. Hence Case 3 cannot occur.

Case 4. $\Gamma_{\beta\alpha}$ *and* $\Gamma_{-\beta\alpha}$ *both irreducible of degree* p, *with* $\Gamma_{\beta\alpha} \cong \Delta_{ab}$ *and* $\Gamma_{-\beta\alpha} \cong \Delta_{a'b'}$, *where* $a + a' = 1$ *or* $a = a' = 0$ *or* $a = a' = 1$:

Since the irreducible representations Δ_{0b} and Δ_{1b} are equivalent, we may assume that $a + a' = 1$. Let v_i and v'_i be as in Case 1. For any integer k, let k^* denote the integer such that $-1 \leqq k^* < p - 1$ and $k \equiv k^* \pmod p$. For all i, j, $v_i v_j$ is either 0 or a characteristic vector for e_0 with characteristic root $(i + a)1 + (j + a')1 = (i + j + 1)1$. Let d_{ij} denote the coefficient of $e_{(i+j+1)^*}$ in the expression of $v_i v_j$ as a linear combination of $e_{-1}, \cdots, e_{p-2}, h_1, \cdots, h_{r-1}$ (thus $v_i v_j = d_{ij} e_{(i+j+1)^*}$ unless $(i + j + 1)^* = 0$). In particular, $d_{ij} = 0$ if $i > p - 1$ or $j > p - 1$. As with the c_{ij}, we shall only consider d_{ij} with nonnegative subscripts. To prove the conclusion of the lemma, it suffices to show that all d_{ij} vanish.

We have $(v_i v'_j)e_k = (v_i e_k)v'_j + v_i(v'_j e_k)$; this gives

$$\begin{aligned} &[(i+j+1)^* - k][(i+j+1)^* + k]^{\perp} d_{ij} \\ (4.4)\qquad &= [i+(k+1)a]d_{i+k,j} + [j+(k+1)a']d_{i,j+k}, \qquad k \neq -1 \text{ or } i,j \neq 0, \end{aligned}$$

where, for any integer i, we write $i^{\perp} = 1$ if $i < p-1$ and $i^{\perp} = 0$ if $i \geqq p-1$. If $(i+j+1)^* \neq -1$ and $k = -1$ then the coefficient of d_{ij} in (4.4) is nonzero, so that if $d_{i-1,j} = d_{i,j-1} = 0$ then $d_{ij} = 0$ (if $(i,k) = (0,-1)$ then $d_{i-1,j}$ is replaced by $d_{p-1,j}$, since the term in $d_{i-1,j}$ in (4.4) must be replaced by $bd_{p-1,j}$; similarly if $(j,k) = (0,-1)$ then $d_{i,j-1}$ is replaced by $d_{i,p-1}$). It follows by induction that to prove that all d_{ij} vanish it suffices to prove that $d_{ij} = 0$ whenever $(i+j+1)^* = -1$.

Now suppose that $(i+j+1)^* = -1$. With $k = p-2$, (4.4) shows that $d_{ij} = 0$ unless one of i,j is 0 or 1. Hence by symmetry the only remaining d_{ij} that need to be considered are $d_{1,p-3}$ and $d_{0,p-2}$. With $i = 1$, $j = p-4$ and $k = 1$, (4.4) gives $0 = (1+2a)d_{2,p-4} + (-4+2a')d_{1,p-3}$, and hence $d_{1,p-3} = 0$ unless $a' = 2\cdot 1$. If $a' = 2\cdot 1$ then, with $i = 1, j = p-5$ (> 1 since $p \geqq 7$) and $k = 2$, (4.4) gives $0 = 0 + (-5+3a')d_{1,p-3} = d_{1,p-3}$. Hence always $d_{1,p-3} = 0$. Now with $i = 0$, $j = p-3$ and $k = 1$, (4.4) gives $0 = 0 + (-3+2a')d_{0,p-2}$. Also with $i = 0$, $j = p-4$ and $k = 2$, (4.4) gives $0 = 0 + (-4+3a')d_{0,p-2}$. Since either $-3\cdot 1 + 2a' \neq 0$ or $-4\cdot 1 + 3a' \neq 0$, $d_{0,p-2} = 0$. Hence always $d_{ij} = 0$ and the conclusion of the lemma holds in Case 4.

In the next three cases $\Gamma_{\beta\alpha}$ and $\Gamma_{-\beta\alpha}$ will be reducible of degree p. We will take a basis $v'_0, \cdots, v'_{p-1}$ of $M_{-\beta\alpha}$ with multiplication as in the appropriate one of Cases 2 and 3, and similarly for a basis of $M_{\beta\alpha}$, without the primes. We may write $a = a' = 1$. We shall use the same notation for d_{ij}, i^* and $i^{\perp}$ as in Case 4, except that now since $a + a' = 2$ we must replace $(i+j+1)^*$ in the definition of d_{ij} by $(i+j+2)^*$.

Case 5. *Degree p, $\Gamma_{\beta\alpha}$ and $\Gamma_{-\beta\alpha}$ both having an invariant subspace of dimension $p-1$*:

The analogue of (4.4) holds, with $(i+j+1)^*$ replaced by $(i+j+2)^*$, except that the term in $d_{i+k,j}$ or $d_{i,j+k}$ is also changed if $i = p-1$ or $j = p-1$. As in Case 4, it suffices to prove that $d_{ij} = 0$ whenever $(i+j+2)^* = -1$ and either $i = 1$ or $i = 0$ or, now, $i = p-1$. Therefore we must consider $d_{p-1,p-2}$, $d_{1,p-4}$, and $d_{0,p-3}$. With $i = p-1$, $j = p-5$ and $k = 3$, $((i+j+2)^* + k)^{\perp} = 0$, and the analogue of (4.4) gives $0 = b_3 d_{2,p-5} + (-5+4)d_{p-1,p-2}$, so that $d_{p-1,p-2} = 0$. With $i = 1$, $j = p-5$ and $k = 1$, the analogue of (4.4) gives $0 = 0 + (-5+2)d_{1,p-4}$ and $d_{1,p-4} = 0$. Then with $i = 0$, $j = p-4$, $k = 1$ we have $0 = 0 + (-4+2)d_{0,p-3}$. It follows that always $d_{ij} = 0$, and the conclusion of the lemma holds in Case 5.

Case 6. *Degree p, $\Gamma_{\beta\alpha}$ and $\Gamma_{-\beta\alpha}$ both having an invariant subspace of dimension one*:

The analogue of (4.4) holds, with $(i+j+1)^*$ replaced by $(i+j+2)^*$, except that the term in $d_{i+k,j}$ or $d_{i,j+k}$ is also changed if $i+k=p-1$ or $j+k=p-1$. As before, it suffices to prove that $d_{ij}=0$ whenever $(i+j+2)^*=1$ and either $i=1$ or $i=0$ (since, with $k=p-2$, $i+k=p-1$ only if $i=1$) But this is shown exactly as in Case 5, and the conclusion of the lemma holds in Case 6.

Case 7. *Degree* p, *one of* $\Gamma_{\beta\alpha}$ *having an invariant subspace of dimension* $p-1$, *the other having an invariant subspace of dimension one*:

This case is handled as in Cases 5 and 6. Suppose $\Gamma_{\beta\alpha}$ has the $(p-1)$-dimensional invariant subspace. Then we need only show that $d_{p-1,p-2}=d_{1,p-4}=d_{0,p-3}=0$, and this is shown exactly as in Case 5. Hence the conclusion of the lemma holds in Case 7.

Case 8. *Both* $\Gamma_{\beta\alpha}$ *and* $\Gamma_{-\beta\alpha}$ *have degree* $p-1$:

We simply set $v_{p-1}=v'_{p-1}=0$ and use the proof in Case 5. Hence the lemma's conclusion holds in Case 8.

We have considered all possibilities for $\Gamma_{\beta\alpha}$ and $\Gamma_{-\beta\alpha}$. Thus the lemma is proved.

5. Existence of a direct summand of rank one.

Lemma 5.1. *Suppose that* α *and* β *are roots such that the root spaces* $L_{i\alpha}$ *generate a Witt algebra* W_α. *If* $\beta+i\alpha$ *is also a root for some* $i=1,\cdots,p-1$, *then all* $\beta+i\alpha$ *are roots and (more important)* $L_\beta L_{-\beta}\subseteq W_\alpha$.

Proof. We shall use the notation $M_{\beta\alpha}$, $M_{-\beta\alpha}$ as in Lemma 4.1. Suppose that the hypotheses hold but that $L_\beta L_{-\beta}\nsubseteq W_\alpha$. Take nonzero elements u_β in L_β, $u_{\beta+i\alpha}$ in $L_{\beta+i\alpha}$ and $u_{-\beta}$ in $L_{-\beta}$, and write $u_\beta u_{-\beta}=h_\beta$. Then $\beta(h_\beta)\neq 0$ by (A) but $\alpha(h_\beta)=0$ by Lemma 4.1, so that $(\beta+i\alpha)(h_\beta)\neq 0$. By the Jacobi identity,

$$(u_\beta u_{\beta+i\alpha})u_{-\beta} = (u_\beta u_{-\beta})u_{\beta+i\alpha} + u_\beta(u_{\beta+i\alpha}u_{-\beta}).$$

But $u_{\beta+i\alpha}u_{-\beta}\in L_{i\alpha}\cap M_{\beta\alpha}M_{-\beta\alpha}$, which vanishes by Lemma 4.1. Therefore, $(u_\beta u_{\beta+i\alpha})u_{-\beta}=h_\beta u_{\beta+i\alpha}=-(\beta+i\alpha)(h_\beta)u_{\beta+i\alpha}\neq 0$ and hence $2\beta+i\alpha$ is a root. Since $\dim M_{\beta\alpha}>1$, $\dim M_{\beta\alpha}\geqq p-1$. It follows that $\dim M_{2\beta,\alpha}\geqq p-2$, so that there are distinct roots $\beta+j\alpha$ and $\beta+k\alpha$ such that $2\beta+2j\alpha$ and $2\beta+2k\alpha$ are roots. By Lemma 2.2, $L_{\beta+j\alpha}$ and $L_{2\beta+2j\alpha}$ generate a Witt algebra $W_{\beta+j\alpha}$, and similarly we get $W_{\beta+k\alpha}$. By Lemma 4.1, $\alpha(L_{\beta+j\alpha}L_{-\beta-j\alpha})=0$ and so $L_\alpha L_{-\alpha}\nsubseteq W_{\beta+j\alpha}$. We may now apply Lemma 4.1 to $W_{\beta+j\alpha}$ and $M_{\alpha,\beta+j\alpha}M_{-\alpha,\beta+j\alpha}$ and conclude that $(\beta+j\alpha)(L_\alpha L_{-\alpha})=0$. Similarly we have $(\beta+k\alpha)(L_\alpha L_{-\alpha})=0$. This implies that $\alpha(L_\alpha L_{-\alpha})=0$, contradicting (A). This proves the last statement of the lemma.

If not all $\beta+i\alpha$ are roots then $M_{\beta\alpha}$ and $M_{-\beta\alpha}$ both have dimension $p-1$ and so are as in Case 8 of the proof of Lemma 4.1. But the treatment of Case 8 did not use the hypothesis of Lemma 4.1 that $M_{\beta\alpha}M_{-\beta\alpha}\nsubseteq W_\alpha$, and showed that $\alpha(L_\beta L_{-\beta})=0$. But $L_\beta L_{-\beta}\subseteq W_\alpha$, and so, by (A), $\alpha(L_\beta L_{-\beta})\neq 0$, a contradiction. Thus the lemma is proved.

We are now ready to complete the proof that, under the hypotheses of the main theorem and the assumption that the base field is algebraically closed, L is a direct sum of algebras which are either simple of classical type or of rank one. Suppose that L is not a direct sum of simple algebras of classical type. Then by Lemmas 2.1 and 2.2, there is a root α such that the root spaces $L_\alpha, L_{2\alpha}, \cdots L_{(p-1)\alpha}$ generate a Witt algebra W_α. For each nonzero root β let h_β be a nonzero element of $L_\beta L_{-\beta}$, and write

$$S = \{\beta \mid \beta \text{ a root}, \ \beta \neq 0, \ h_\beta \in (h_\alpha)\}, \quad K = (h_\alpha) + \sum_{\beta \in S} L_\beta.$$

If $\beta, \gamma \in S$ and $L_\beta L_\gamma \neq 0$ then $\beta + \gamma$ is a root and $(L_\beta L_\gamma)L_{-\beta-\gamma} \subseteq (L_\beta L_{-\beta-\gamma})L_\gamma + L_\beta(L_\gamma L_{-\beta-\gamma}) \subseteq L_{-\gamma}L_\gamma + L_\beta L_{-\beta} \subseteq (h_\alpha)$. Hence K is a subalgebra of L. Since $\beta(h_\alpha) \neq 0$ if $\beta \in S$, it follows that (h_α) is a Cartan subalgebra of K. If $\beta \in S$ and β is not of the form $i\alpha$, write

$$J_\beta = \left(\sum_{i,j} L_{i\alpha + j\beta}\right) \cap K.$$

Then J_β is a subalgebra of K and (h_α) is also a Cartan subalgebra of J_β. By a result of Kaplansky [6, Theorem 4, p. 164] on Lie algebras of rank one, the roots form a group, so that J_β has dimension p^2, and all $i\beta$ are roots (this may also be proved more directly with the use of the Weyl-Jacobson lemma). If $\beta \in S$, $\gamma \notin S$ and $\gamma \neq 0$, then $L_\gamma L_{-\gamma} \nsubseteq W_\beta$, and so, by Lemma 5.1, $L_\gamma W_\beta = 0$.

Now let K^* be the subalgebra of L generated by all L_γ such that $\gamma \notin S$ and $\gamma \neq 0$. Then $K^*K = 0$. Every L_δ $(\delta \neq 0)$ is contained in K or K^*, and since H is spanned by the $L_\delta L_{-\delta}$ because $L = L^2$, we have $K + K^* = L$. Since $KK^* = 0$ and L is centerless, $K \cap K^* = 0$, and hence L is the direct sum of the ideals K and K^*. It is easy to see that the rank one algebra K is simple, or this may be concluded by using [2]. Therefore by induction it follows that L is a direct sum of simple algebras which are either of classical type or have rank one.

Now suppose that the base field F is not algebraically closed, and let E be its algebraic closure. It is obvious that L_E satisfies axioms (i) and (A), with respect to H_E. Since by (ii) the roots span the dual space of H, no nonzero element of H_E is annihilated by all roots, and it follows that L_E also satisfies axiom (ii). For each of the simple direct summands $L_i(E)$ of L_E, let L_i be the subalgebra of L generated by all L_α for nonzero roots such that $(L_\alpha)_E$ is in $L_i(E)$. Then $(L_i)_E = L_i(E)$, L_i is simple, and L is a direct sum of the L_i. Moreover, L_i is of classical type or of rank one according as $L_i(E)$ is, and L_i satisfies our axioms.

The only thing we need to do to complete the proof of the main theorem is to show that if F is perfect and L has a one-dimensional Cartan subalgebra $H = (u_0)$ such that Axiom (A) is satisfied, then L is an Albert-Zassenhaus algebra. Let E be the algebraic closure of L. Then by [2] L_E is an Albert-Zassenhaus algebra, the roots of L with respect to H form a group, and L_E has a basis $\{u_\alpha\}$ containing u_0 and satisfying (1.1), where each root α is identified with the scalar

$\alpha(u_0)$ and $u_\alpha \in (L_\alpha)_E$. The facts in [9, pp. 42–47] used in the proof of Lemma 3.1 of [2] are valid for F perfect. Hence elements $u'_{i\alpha}$ in $L_{i\alpha}$ $(i = 0, \cdots, p-1)$, with $u'_0 = u_0$, may be chosen so that $u'_{i\alpha}u'_{j\alpha} = (i-j)\alpha u'_{i\alpha+j\alpha}$, and this choice is unique even among elements in $(L_{i\alpha})_E$. It follows that $u_{i\alpha} = u'_{i\alpha} \in L_{i\alpha}$. Hence the basis elements u_α are in L, so that L is an Albert-Zassenhaus algebra. This completes the proof of the main theorem.

6. **Remarks on representations of the Witt algebra and on algebras of rank one.** We shall now indicate the form that Theorem 3.1, on the representations of the Witt algebra W, takes when the representations are expressed in terms of the basis elements $f_0, \cdots, f_{p-1}$ of W having multiplication table (2.3).

Let Z be a vector space over the base field F of W, with basis $z_0, z_1, \cdots, z_{p-1}$, and let a and c be given scalars. For each f_j define a linear transformation on Z by setting (for $i = 0, \cdots, p-1$)

$$z_i f_j = (c + i + ja) z_{i+j}, \tag{6.1}$$

where all subscripts are taken modulo p. Then we claim that the linear mapping $\Omega = \Omega_{ac}$, determined by (6.1), of W to linear transformations on Z, is a representation of W. This may be shown directly, or if $c \notin F_p$, by the use of part of the multiplication table of a suitable Albert-Zassenhaus algebra, as in [2, p. 22]. If $c \in F_p$, one of the more general Lie algebras of [1, Theorem 9, p. 133] may be used in the same way to show that Ω is a representation.

If $c = 0$ and $a = 1$ then $z_1, \cdots, z_{p-1}$ span an invariant $(p-1)$-dimensional subspace of Z. We shall denote by Ω'_{10} the restriction of Ω_{10} to this subspace. Since the subspace contains no one-dimensional invariant subspace Ω'_{10} is irreducible and equivalent to Δ'_{10}.

THEOREM 6.1. *Let W be the Witt algebra over an algebraically closed field F of characteristic $p > 3$. The (nontrivial) irreducible representations of W of degree not greater than p are, up to equivalence, the mappings Ω_{ac} $(a, c \in F$; $c \notin F_p$ or $a \neq 0, 1)$ and Ω'_{10}. The irreducible representation Ω_{ac} is equivalent to the representation Δ_{ab} of Theorem 3.1 where $b = c^p - a^p - (c - a)$. Two irreducible representations $\Omega_{a_1c_1}$ and $\Omega_{a_2c_2}$ are equivalent if and only if $c_2 = c_1 + i$, for some i in F_p, and either $a_1 = a_2$ or $(a_1, a_2) = (0,1)$ or $(1,0)$.*

Proof. Let $\Omega = \Omega_{ac}$ be one of the representations defined above. Since $e_{-1} = f_{-1}$, $[\Omega_{ac}(e_{-1})]^p = \varepsilon_{-1}I$ where $\varepsilon_{-1} = \prod_i (c + i - a) = c^p - a^p - (c - a)$. Write $z = z_0 + \cdots + z_{p-1}$. By (2.5), $e_0 = f_0 - f_{-1}$, $e_1 = f_1 - 2f_0 + f_{-1}$ and $e_2 = f_2 - 3f_1 + 3f_0 - f_{-1}$. Thus $ze_0 = \sum_i [(c+i)z_i - (c+i-a)z_{i-1}] = (a-1)z$, and similarly it may be seen that $ze_1 = ze_2 = 0$. If Z contains a one-dimensional subspace invariant under all $\Omega_{ac}(f_j)$ then the subspace is spanned by some z_i with $c + i = c + i + a = 0$. Now suppose that $c \notin F_p$ or $a \neq 0$, and that Z contains a proper invariant subspace Z'. Then Z' has dimension $p-1$ and is irreducible, the restriction of Ω to Z' is equivalent to Δ'_{10}, and $ze_0 \in Z'$. If $z \notin Z'$ then $a = 1$, while if $z \in Z'$ then z must correspond to a scalar multiple of v_{p-2} under the equivalence with Δ'_{10}, so that $a = 0$. But $c^p - a^p - (c-a) = 0$, so that $c \in F_p$.

It follows that if $c \notin F_p$ or $a \neq 0, 1$ then Ω_{ac} is irreducible. Comparing z with v of Lemma 3.1, we see that Ω_{ac} is equivalent to Δ_{ab}, where $b = \varepsilon_{-1} = c^p - a^p - (c-a)$. The rest of the theorem follows immediately from Theorem 3.1 and the fact that if c is a root of $(x-a)^p - (x-a) = b$ then the other roots are $c + i$ for i in F_p. Thus the theorem is proved.

As with Corollary 3.1, there is an immediate consequence for restricted representations.

COROLLARY 6.1. *Any (nontrivial) irreducible restricted representation of the Witt algebra, over an algebraically closed field of characteristic* $p > 3$, *is equivalent to one of* $\Omega'_{10}, \Omega_{20}, \cdots, \Omega_{-1,0}$. *These* $p-1$ *representations are inequivalent and irreducible, and are equivalent to* $\Delta'_{10}, \Delta_{20}, \cdots, \Delta_{-1,0}$, *respectively.*

In [2] an important tool in the proof is the use of the formulas (6.1) above for the representations Ω_{ac} with $c \notin F_p$. The proof in [2], at least for $p > 5$, could be brought into closer relationship with the proof in the present paper by using Theorem 6.1 above. This remark refers principally to the proof in [2, pp. 25–29] that f is skew-symmetric. That proof could be somewhat simplified by using arguments just like that in Case 4 of Lemma 4.1 above to eliminate all except a couple of possible values for $f(\alpha, \beta) + f(-\alpha, \beta)$. For the remaining values the system of d_{ij}'s, and hence the $u_{\alpha+i\beta}u_{-\alpha+j\beta}$, are determined uniquely up to scalar multiple. A computation in two ways of $(u_{i\beta}u_{\alpha})u_{-\alpha}$ then leads to the conclusion that f is skew-symmetric.

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