

# UCR Math Dept Topology Qualifying Exam

September 2018

- There are three parts in this exam and each part has three problems. You should complete two (and only two) problems of your choice in each part. Each problem is worth 10 points.
- Support each answer with a complete argument. State completely any definitions and basic theorems that you use.
- This is a closed book test. You may only use the test, something to write with, and the paper provided. All other material is prohibited. Write each of your solutions on separate sheets. Write your ID number on every sheet you use.
- The time for the exam is 3 hours.

## Part I

1. Let

$$p : (E, e_0) \longrightarrow (B, b_0)$$

be a covering map of pointed spaces. Show that

$$p_* : \pi_1(E, e_0) \longrightarrow \pi_1(B, b_0)$$

is injective. Here  $p_*$  is the map that  $p$  induces  $\pi_1(E, e_0) \longrightarrow \pi_1(B, b_0)$ .

2. Let  $h : S^1 \longrightarrow S^1$  be nullhomotopic.

(a) Show that  $h$  has a fixed point.

(b) Show that  $h$  maps some point  $x$  to its antipode  $-x$ .

3. Let  $X$  be a compact metric space. Let

$$C_1 \supset C_2 \supset C_3 \supset \cdots$$

be an infinite sequence of closed, nonempty subsets. Prove that

$$\bigcap_{i=1}^{\infty} C_i \neq \emptyset.$$

## Part II

4. Suppose that  $X$  is a connected, locally arcwise connected, Hausdorff space and  $f : X \rightarrow S^1$  is a continuous map such that  $f(x_0) = 1$  for some  $x_0 \in X$ . Prove that  $f$  is homotopic to a constant map if and only if the induced map of fundamental groups from  $\pi_1(X, x_0)$  to  $\pi_1(S^1, 1)$  is trivial.

5. Let  $A$  be a simplex in  $\mathbb{R}^n$  for some  $n$ , and let  $B \subset A$  be a union of faces in  $A$ . Prove that the homology groups  $H_q(B)$  are isomorphic to the homology groups  $H_{q+1}(A, B)$  for all  $q > 0$ . Give a counterexample to the corresponding statement if  $q = 0$  and  $A$  is a 1-simplex.

6. Prove that if  $f : S^n \rightarrow S^n$  is continuous and 1-1, then  $f$  is a homeomorphism using Invariance of Domain. Also prove that if  $m > n$ , then there is no continuous 1-1 mapping from  $S^m$  to  $S^n$ .

## Part III

7. Consider the map

$$\phi_t : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} (\cos t)x - (\sin t)y \\ (\sin t)x + (\cos t)y \\ z \end{pmatrix}$$

a. Let  $S^2 \subset \mathbb{R}^3$  be the unit sphere: show that  $\phi_t$ , when restricted to  $S^2$ , defines a one-parameter family of diffeomorphisms. (Note: this entails showing that for a *fixed*  $t$  each  $\phi_t$  is 1. bijective *in*  $S^2$ ; 2. smooth with smooth inverse; 3.  $\phi_s \circ \phi_t = \phi_{s+t}$ .)

b. Determine the vector field generating this one-parameter family of diffeomorphisms.

8. Show that the subset of  $\mathbb{R}^3$  defined by the equation

$$(1 - z^2)(x^2 + y^2) = 1$$

is a smooth manifold.

9. Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined as

$$F(x, y) = (x + y^2, x^2).$$

Denoting by  $u, v$  the Cartesian coordinates of the target, determine  $F^*(vdu + dv)$ .

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September 2017

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## Part I

1. Let  $p : X \rightarrow Y$  be a surjective local homeomorphism from a compact Hausdorff space  $X$  to a Hausdorff space  $Y$ . Show that  $p$  is a covering map.
2. Let  $X$  be a compact metric space. Prove that  $X$  is separable. That is prove that  $X$  contains a countable dense subset.
3. Let  $D$  be the unit disk in the plane, and  $f : D \rightarrow D$  a homeomorphism. Show that  $f$  sends the unit circle  $S^1 \subset D$  into itself.

## Part II

4. Let  $S$  be a closed orientable surface of genus 2. Let  $C$  be a circle in  $S$  that bounds a 2-disk, and let  $X$  be the space obtained from  $S$  by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to  $C$ .

Compute the homology groups of  $X$ .

5. Let  $F$  be the free group on two generators  $F = \langle a, b \rangle$ . Consider the homomorphism  $\phi : F \rightarrow \mathbb{Z}/3$  that sends  $a$  to  $\bar{2}$  and  $b$  to  $\bar{1}$ . Give a set of free generators for the kernel of  $\phi$ .

**Hint:** Construct an appropriate covering space.

6. The suspension of  $\mathbb{R}P^2$  is defined as the quotient space obtained from  $\mathbb{R}P^2 \times [0, 1]$  by collapsing each ‘end’  $\mathbb{R}P^2 \times \{0\}$  and  $\mathbb{R}P^2 \times \{1\}$  to a point.

a. Compute the fundamental group of the suspension of  $\mathbb{R}P^2$ .

b. Compute the homology groups of the suspension of  $\mathbb{R}P^2$ .

### Part III

7. Let  $V, W$  be the following vector fields on  $\mathbb{R}^3$ :

$$V = \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}; \quad W = y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

a. Find the flowline  $c(t)$  of  $[V, W]$  such that  $c(0) = (1, 1, 1)$ .

b. Decide whether there exists a choice of coordinates  $(x', y', z')$  in a neighborhood of  $(1, 1, 1)$  such that

$$V = \frac{\partial}{\partial x'}; \quad W = \frac{\partial}{\partial y'}.$$

8. Consider the subset  $\Sigma$  of  $\mathbb{C}^3$  given by the intersection of the sphere

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$$

and the cone

$$z_1^2 + z_2^2 + z_3^2 = 1.$$

Prove that  $\Sigma$  is a smooth submanifold of  $\mathbb{C}^3$  by showing that it is the level set of a regular value of a suitable map of  $\mathbb{C}^3$ . (Remark: recall that if  $z = x + iy \in \mathbb{C}$ ,  $|z|^2 = x^2 + y^2$  and  $z^2 = x^2 - y^2 + 2ixy$ .)

9. Show with an example that there exist smooth maps  $F : M \rightarrow N$ ,  $G : N \rightarrow P$  between manifold that have constant rank, but whose composition  $G \circ F : M \rightarrow P$  doesn't have constant rank. (Remark: there exist examples with  $M = \mathbb{R}$ ,  $N = \mathbb{R}^2$ ,  $P = \mathbb{R}$ .)

# UCR Math Dept Topology Qualifying Exam

December 2015

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## Part I

1. (a) State whether the following is true or false, and give reasons for your answer: If  $U$  and  $V$  are disjoint open subsets of a topological space  $X$ , then their closures are also disjoint.  
  
(b) Let  $\mathbb{R}$  have the Euclidean topology, and let  $X$  be connected. Suppose that  $f: X \rightarrow \mathbb{R}$  is continuous and locally constant, that is, every point has an open neighborhood on which  $f$  is constant. Prove that  $f$  is constant. Show that the statement is false if  $X$  is not connected.
2. Suppose that  $X$  is a compact metric space,  $Y$  is a metric space, and  $f: X \rightarrow Y$  is continuous and onto. Prove that  $f$  is a quotient map.
3. (a) Prove that  $S^1 \times S^1$  is not a retract of  $D^2 \times D^2$ .  
  
(b) Let  $X$  and  $Y$  be Hausdorff, and let  $Y$  be connected. Suppose that  $p: X \rightarrow Y$  is a covering map, and the fiber  $p^{-1}[\{y_0\}]$  consists of 2 points for some  $y_0 \in X$ . Prove that for every  $y \in Y$ , the fiber  $p^{-1}[\{y\}]$  consists of 2 points.

## Part II

4. (a) (4 pts) Show that  $S^1 \times S^1$  and  $(S^1 \vee S^1) \vee S^2$  have isomorphic homologies in all dimensions, but their universal covers do not.

(b) (3 pts) Show that any continuous map  $S^2 \longrightarrow S^1 \times S^1$  is nullhomotopic.

(c) (3 pts) Find a continuous map  $S^2 \longrightarrow (S^1 \vee S^1) \vee S^2$  that is not nullhomotopic.

5. Recall that a topological manifold is a second countable, locally Euclidean, Hausdorff space.

Let  $X$  be a topological manifold. Calculate

$$H_*(X, X \setminus \{x\})$$

for all  $x$  in  $X$ .

6. Let  $X$  be a path connected, semilocally simply connected topological space whose fundamental group is finite of odd order.

(a) Prove that  $X$  cannot have any connected 2-sheeted covering spaces.

(b) Provide an example of a space  $X$  whose fundamental group has odd finite order.



### Part III

7a. Let  $M$  and  $N$  be smooth manifolds and  $\pi: M \rightarrow N$  a smooth submersion. Prove that  $\pi$  is an open map, and that it is a quotient map if  $\pi$  is surjective.

b. Let  $M$  be a compact smooth manifold. Prove that there exists no smooth submersion  $M \rightarrow \mathbb{R}^k$  for any  $k \geq 1$ .

8. Let  $M$  be an embedded  $m$ -dimensional smooth submanifold of  $\mathbb{R}^n$ , and let

$$UM = \{(x, v) \in T\mathbb{R}^n \mid x \in M, v \in T_x M, |v| = 1\},$$

the set of all unit tangent vectors to  $M$ , called the *unit tangent bundle of  $M$* . Prove that  $UM$  is an embedded  $(2m - 1)$ -dimensional submanifold of  $T\mathbb{R}^n$ .

9. Let  $G$  be a Lie group.

a. Prove that the identity component  $G_0$  is the only connected open subgroup of  $G$ , and that every connected component of  $G$  is diffeomorphic to  $G_0$ .

b. Let  $U$  be any neighborhood of the identity element  $e$ . Prove that there exists a neighborhood  $V \subseteq U$  of  $e$  such that  $gh^{-1} \in U$  whenever  $g, h \in V$ .

# UCR Math Dept Topology Qualifying Exam

December 2014

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- The time for the exam is 3 hours.

## Part I

- (1) (a) Prove that every closed subspace of a compact space is compact.  
(b) Prove that every compact subspace of a Hausdorff space is closed.
- (2) Let  $X$  and  $Y$  be topological spaces, and recall that  $[X, Y]$  denotes the set of homotopy classes of maps  $X \rightarrow Y$ .
  - (a) Prove that every contractible space is path-connected.
  - (b) Prove that if  $Y$  is contractible, then  $[X, Y]$  has a single element.
  - (c) Prove that if  $X$  is contractible and  $Y$  is path-connected, then  $[X, Y]$  has a single element.
- (3) Recall that  $\mathbf{R}P^2$  is defined to be the quotient space obtained by identifying antipodal points of  $S^2$ .
  - (a) Compute the fundamental group of  $\mathbf{R}P^2$ .
  - (b) Let  $X$  be the quotient space obtained from  $D^2$  by identifying antipodal points of the boundary  $S^1$ . Prove that  $X$  is homeomorphic to  $\mathbf{R}P^2$ .

## Part II

- (4) (a) Suppose that  $p: (E, e) \rightarrow (B, b)$  is a basepoint preserving covering space projection where  $E$  is path-connected, and that  $f: (X, x) \rightarrow (B, b)$  is basepoint preservingly homotopic to a constant map where  $X$  is also path-connected. Prove that there is a continuous lifting  $g: (X, x) \rightarrow (E, e)$  such that  $f = pg$ .
- (b) Suppose that  $(B, b)$  is a path-connected space with basepoint such that  $\pi_1(B, b)$  is cyclic of order 4. Prove that, up to equivalence of covering spaces, there is exactly one connected covering space  $p: (E, e) \rightarrow (B, b)$  such that  $p$  is not a homeomorphism and  $E$  is not simply connected.
- (5) Let  $X$  be a graph with vertices  $A, B, C, D, E, F$  and edges  $AB, AC, BC, BD, BE, CE, CF, DE$  and  $EF$ .
- (a) Draw a sketch of  $X$ , and find a maximal tree in  $X$ . [*Hint:* Put  $A$  at the top, with  $B$  and  $C$  on the next line and  $D, E, F$  on the third line.]
- (b) If  $Y \rightarrow X$  is a connected 3-sheeted covering space, find the nonnegative integer  $m$  such that the fundamental group of  $Y$  is free on  $m$  generators.
- (6) One can use Mayer-Vietoris sequences and homotopy invariance in (a) singular homology theory to prove the following formula, in which  $n$  is a positive integer and  $X$  is an arbitrary nonempty space:

$$H_q(S^n \times X) \cong H_q(X) \oplus H_{q-n}(X)$$

Use this formula to compute the homology groups of  $S^2 \times S^2$ , and explain why this shows that  $S^2 \times S^2$  and  $S^4$  are not homeomorphic (or even homotopy equivalent).

## Part III

- (7) Consider the function  $\Phi: \mathbf{R}^4 \rightarrow \mathbf{R}^2$  defined by

$$\Phi(x, y, z, t) = (x + y, x + y^2 + z^2 + t^2 + y).$$

Prove that  $(0, 1)$  is a regular value, and show that  $\Phi^{-1}(0, 1)$  is diffeomorphic to  $S^2$ .

- (8) Let  $c \in \mathbf{R}$  and let  $f: \mathbf{R}^n \setminus \{\mathbf{0}\} \rightarrow \mathbf{R}$  be a function that is *positively homogeneous of degree  $c$* , i.e., such that for all  $\lambda > 0$ ,

$$\mathbf{x} \in \mathbf{R}^n \setminus \{\mathbf{0}\},$$

$$f(\lambda \mathbf{x}) = \lambda^c f(\mathbf{x}).$$

Show that

$$Ef = cf,$$

where  $E$  is the vector field whose value at  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{R}^n \setminus \{\mathbf{0}\}$  is given by

$$E_{\mathbf{x}} = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}.$$

(9) Recall that

$$SL_n(\mathbf{R}) = \{A \in GL_n(\mathbf{R}) \mid \det A = 1\}.$$

- (a) Prove that  $d(\det)_I: T_I GL_n(\mathbf{R}) \rightarrow T_1 \mathbf{R}$  is onto.
- (b) Prove that  $d(\det)_I: T_I GL_n(\mathbf{R}) \rightarrow T_1 \mathbf{R}$  is onto implies that for all  $A \in SL_n(\mathbf{R})$ ,  $d(\det)_A: T_I GL_n(\mathbf{R}) \rightarrow T_1 \mathbf{R}$  is onto.

# UCR Math Dept Topology Qualifying Exam

November 2013

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## Part I

- (1) (a) Show that there exists a homeomorphism  $f: S^1 \setminus \{p\} \rightarrow \mathbf{R}$ , where  $p \in S^1$ .  
(b) Prove that no homeomorphism  $f: S^1 \setminus \{p\} \rightarrow \mathbf{R}$  can be extended to a continuous map  $\hat{f}: S^1 \rightarrow \mathbf{R}$   
(c) Prove that no continuous map  $g: S^1 \rightarrow \mathbf{R}$  can be injective.
- (2) Consider the interval  $[0, 1]$  endowed of the following topology:  $U \subset [0, 1]$  is open if either  $(0, 1) \subset U$  or  $\frac{1}{2} \notin U$ .  
(a) Is this topology comparable with the standard topology?  
(b) What is the closure of  $\{\frac{1}{2}\}$ ?  
(c) What is the closure of  $\{\frac{1}{4}\}$ ?
- (3) Let  $f: X \rightarrow Y$  be a continuous map between Hausdorff spaces, and let  $U \subset X$  be an open subset.  
(a) Give an example of the situation above where the closure of  $f(U)$  in  $Y$  fails to be compact.  
(b) Prove that if we assume furthermore that  $U$  is contained in a compact subset  $K \subset X$ , then the closure of  $f(U)$  in  $Y$  is compact.

## Part II

- (4) Prove that if  $p: X \rightarrow B$  is a regular covering map and  $G$  is its group of covering transformations, then there is a homeomorphism  $k: X/G \rightarrow B$  such that  $p = k \circ \pi$ , where  $\pi: X \rightarrow X/G$  is the projection.
- (5) Let  $f: (X, A) \rightarrow (Y, B)$  be a map of pairs of spaces such that both  $f: X \rightarrow Y$  and its restriction to  $A \rightarrow B$  are homotopy equivalences.
- Prove that  $f_*: H_n(X, A) \rightarrow H_n(Y, B)$  is an isomorphism for every  $n \geq 0$ .
  - Prove that  $f$  need not be a homotopy equivalence of pairs, in that there need not be a map  $g: (Y, B) \rightarrow (X, A)$  such that  $fg$  and  $gf$  are homotopic to the identity via maps of pairs. (Hint: Consider the inclusion  $(D^n, S^{n-1}) \rightarrow (D^n, D^n \setminus \{0\})$ .)
- (6) (a) Take the quotient map  $T^2 \rightarrow S^2$  by collapsing the subspace  $S^1 \vee S^1$ . Use homology to prove that this map is not nullhomotopic.
- (b) Use covering spaces to prove that, on the other hand, any map  $S^2 \rightarrow T^2$  is nullhomotopic.

## Part III

- (7) Regarding  $S^1$  as the equator of  $S^2$ , we obtain  $\mathbb{R}P^1$  as a submanifold of  $\mathbb{R}P^2$ . Show that  $\mathbb{R}P^1$  is not a regular level surface of any  $C^1$  map  $\mathbb{R}P^2 \rightarrow \mathbb{R}$ . (Hint: no connected neighborhood of  $\mathbb{R}P^1$  in  $\mathbb{R}P^2$  is separated by  $\mathbb{R}P^1$ .)
- (8) Let  $M$  be a compact, smooth embedded submanifold of  $\mathbb{R}^k$ . Show that  $M$  has a *tubular neighborhood*. That is, there is a neighborhood  $N$  of  $M$  and a smooth deformation retraction of  $N$  onto  $M$ . You may assume that the normal bundle of  $M$  is a smooth embedded submanifold of  $\mathbb{R}^k \times \mathbb{R}^k$ .
- (9) (a) If  $F: M \rightarrow N$  is a smooth map,  $c \in N$ , and  $F^{-1}(c)$  is an embedded submanifold whose codimension is equal to the dimension of  $N$ , must  $c$  be a regular value of  $F$ ? Give a proof or provide a counterexample.
- (b) Suppose that  $\pi: M \rightarrow N$  is a smooth map such that every point of  $M$  is the image of a smooth section of  $\pi$ . Show that  $\pi$  is a submersion.

# UCR Math Dept Topology Qualifying Exam

December 2012

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## Part I

1. Let  $(X, d)$  a metric space endowed with the metric topology. Prove or disprove the following statement: for all  $x \in X$ ,  $\epsilon > 0$ , the "closed ball"

$$\bar{B}_d(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$$

is the closure of the "open ball"

$$B_d(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}.$$

2. Let  $X$  be a compact topological space; let  $A_1 \supset A_2 \supset A_3 \supset \dots$  be an (infinite) nested sequence of closed, nonempty subsets. Prove that  $\bigcap_{i=1}^{\infty} A_i \neq \emptyset$ .

3. Endow  $\mathbf{R}^n$  with the standard (Euclidean) metric  $d$ . Let  $A$  a closed subset; show that for each  $x \in \mathbf{R}^n$ , there exist a point  $y \in A$  of minimum distance from  $x$  (i.e. the continuous function  $d(\cdot, x) : A \rightarrow \mathbf{R}$  attains minimum value for some at  $y \in A$ ). (Note:  $A$  is not assumed to be bounded.)

## Part II

4. Recall that a continuous mapping  $g : X \rightarrow Y$  is a retract if there is a continuous mapping  $r : Y \rightarrow X$  such that  $r \circ g$  is the identity.

(a) Prove that if  $g$  is a retract, then  $g$  induces an 1 – 1 homomorphism of fundamental groups.

(b) Let  $W \subset S^1 \times S^1$  be the union of the circles  $S^1 \times \{1\}$  and  $\{1\} \times S^1$ . Prove that the inclusion of  $W$  in  $S^1 \times S^1$  is not a retract.

5. Let  $Y$  be the complete graph on 5 vertices; *i.e.*,  $Y$  has 5 vertices and for each pair of vertices there is a unique edge joining them. Suppose that  $X$  is a 5-sheeted covering space of  $Y$ . Compute the fundamental groups of  $Y$  and  $X$ . [Hint: Both are free groups on some finite number of generators. What are these numbers?]

6. Let  $M$  be a topological space whose fundamental group is finite of odd order.

(a) Prove that  $M$  cannot have any connected 2-sheeted covering spaces.

(b) Give a counterexample to the preceding statement if we drop the connectedness assumption.

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## Part III

7. Let  $M$  be a manifold with boundary, and let

$$\begin{aligned} H^n &\equiv \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}, \\ \partial H^n &\equiv \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}, \text{ and} \\ \text{Int}H^n &\equiv \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}. \end{aligned}$$

Recall that

$\partial M \equiv \{p \in M \mid \text{there is a coordinate chart } \phi : U \rightarrow H^n \text{ with } p \in U \text{ and } \phi(p) \in \partial H^n\}$   
and

$\text{Int}M \equiv \{p \in M \mid \text{there is a coordinate chart } \phi : U \rightarrow H^n \text{ with } p \in U \text{ and } \phi(p) \in \text{Int}H^n\}$ .

Show that  $\partial M \cap \text{Int}M = \emptyset$ .

8. Show that  $S^n \times \mathbb{R}$  is parallelizable for all  $n \geq 1$ .

9. Let  $M$  be a smooth compact manifold. Show there is no submersion  $F : M \rightarrow \mathbb{R}^k$  for any  $k > 0$ .

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# UC Riverside Mathematics Department — Mathematics 205 Qualifying Exam

December 2011

- There are three parts in this exam and each part has three problems. You should complete two (and only two) problems of your choice in each part. Each problem is worth 10 points.
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- The default time for the examination is 3 hours.

## Part I

1. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  be continuous maps between the topological spaces  $X$  and  $Y$ . Suppose  $f(g(y)) = y$  for all  $y \in Y$ .

(a) Prove that if  $Y$  is connected, and  $f^{-1}(y)$  is connected for all  $y \in Y$ , then  $X$  is connected.

(b) Prove that if  $Y$  is arcwise connected, and  $f^{-1}(y)$  is arcwise connected for all  $y \in Y$ , then  $X$  is arcwise connected.

2. Let  $X$  and  $Y$  be topological spaces, and  $X \times Y$  their product. Assume that  $Y$  is compact.

(a) Prove that the projection map

$$p_X : X \times Y \rightarrow X : (x, y) \mapsto x$$

is closed.

(b) Let  $f : X \rightarrow Y$  be a function. Let

$$Z = \{(x, y) \in X \times Y \mid y = f(x)\}.$$

Suppose that  $Z$  is closed in  $X \times Y$ . Prove that  $f$  is continuous.

3. (a) Let  $A$  be a subspace of a Hausdorff space  $X$ . Suppose there is a continuous map  $f : X \rightarrow A$  such that  $f(a) = a$  for all  $a \in A$ . Prove that  $A$  is closed in  $X$ .

(b) Let  $X$  be a compact Hausdorff space. Let  $C$  be a nonempty closed subset of  $X$ , and  $U = \{x \in X \mid x \notin C\}$ . Prove that the quotient space  $X/C$  is homeomorphic to the one-point compactification of  $U$ .

## Part II

4. Let  $p : E \rightarrow B$  be a covering map over a connected space  $B$ . Assume that for some  $b_0 \in B$ ,  $p^{-1}(b_0)$  has exactly  $k$  elements. Prove that for all  $b \in B$ ,  $p^{-1}(b)$  has exactly  $k$  elements.

5. (a) Let  $M$  be a smooth manifold, and let

$$f : M \rightarrow \mathbb{R}$$

be a smooth map. Assume that  $t \in \mathbb{R}$  is a regular value. Show that for all  $x \in \Sigma := f^{-1}(t)$  we can identify  $T_x \Sigma$  with the kernel of  $df_x$ , where

$$df_x : T_x M \rightarrow \mathbb{R}$$

is the differential map.

(b) Using the above result, show that the submanifolds of  $\mathbb{R}^3$  defined by the equations  $x^2 + y^2 + z^2 = 1$  and  $z - x^2 - y^2 = 0$  intersect transversally.

6. Let  $p : E \rightarrow B$  be a covering map over a compact space  $B$ . Assume that for all  $b \in B$ ,  $p^{-1}(b)$  is finite. Show that  $E$  is compact.

## Part III

**Default hypothesis:** Unless explicitly stated otherwise, all spaces in the problems below are assumed to be Hausdorff and locally arcwise connected.

7. Suppose that  $(X, x_0)$  and  $(Y, y_0)$  are connected and locally simply connected pointed spaces and  $f : (X, x_0) \rightarrow (Y, y_0)$  is continuous and basepoint preserving. Let  $p : (\tilde{X}, \xi) \rightarrow (X, x_0)$  and  $q : (\tilde{Y}, \eta) \rightarrow (Y, y_0)$  be universal covering space projections. Prove that there is a unique lifting of  $f$  to a basepoint preserving map  $F : (\tilde{X}, \xi) \rightarrow (\tilde{Y}, \eta)$  in the sense that  $q \circ F = f \circ p$ . Also prove that if  $g$  is basepoint preservingly homotopic to  $f$  and  $G$  is the corresponding lifting of  $g$ , then  $F$  and  $G$  are basepoint preservingly homotopic.

8. (a) Let  $G$  be a finite abelian group. Give an example of an arcwise connected space with basepoint  $(X, x_0)$  such that  $\pi_1(X, x_0) \cong G$ .

(b) Suppose that  $X$  is the union of the open arcwise connected subsets  $U$  and  $V$ , and assume that  $U \cap V$  is simply connected. Prove that  $\pi_1(X)$  is abelian if and only if both  $\pi_1(U)$  and  $\pi_1(V)$  are **AND** one of them is trivial.

9. (a) Let  $p : X \rightarrow Y$  be a finite covering of (arcwise) connected spaces, where  $Y$  and  $X$  are locally simply connected. Prove that there is a finite covering space projection  $W \rightarrow X$  such that  $W$  is connected and the composite  $W \rightarrow X \rightarrow Y$  is a regular covering space projection.

(b) Suppose that  $p : X \rightarrow Y$  is a covering space projection such that  $X$  and  $Y$  are finite graph complexes, so that  $\pi_1(X)$  and  $\pi_1(Y)$  are free groups on  $g_X$  and  $g_Y$  generators respectively. Given that the covering is  $n$ -sheeted, derive an equation expressing  $g_X$  in terms of  $n$  and  $g_Y$ .

# UCR Math Dept Topology Qualifying Exam

December 2010

- There are three parts in this exam and each part has three problems. You should complete two (and only two) problems of your choice in each part. Each problem is worth 10 points.
- Support each answer with a complete argument. State completely any definitions and basic theorems that you use.
- This is a closed book test. You may only use the test, something to write with, and the paper provided. All other material is prohibited. Write each of your solutions on separate sheets. Write your ID number on every sheet you use. Please do not write your name on the exam.
- The time for the exam is 3 hours.

## Part I

- (1) Let  $A$ ,  $B$  and  $A_\alpha$  be subsets of a topological space  $X$ . Determine if the following equalities hold true and, if not, determine if any inclusion holds true.
  - (a)  $\overline{\bigcap A_\alpha} = \bigcap \overline{A_\alpha}$
  - (b)  $\overline{A - B} = \overline{A} - \overline{B}$ .
- (2) Let  $f : X \rightarrow Y$  be a closed surjective function between two topological spaces, and let  $g : Y \rightarrow Z$  be a function to a third topological space such that  $g \circ f : X \rightarrow Z$  is continuous. Show that  $g$  is continuous.
- (3) Let  $f : S^1 \rightarrow \mathbb{R}$  be a continuous function.
  - (a) Prove that  $f$  cannot be surjective.
  - (b) Prove that  $f(S^1)$  is a closed, finite interval.
  - (c) Prove that  $f$  cannot be injective.

## Part II

- (4) If  $M$  is a compact  $n$ -manifold without boundary, show it admits a smooth immersion into  $\mathbb{R}^{2n}$ .
- (5) Show that a connected smooth manifold is “smoothly homogeneous” in the sense that given any  $p, q \in M$  there is a diffeomorphism

$$\Phi : M \longrightarrow M$$

so that  $\Phi(p) = q$ .

You may assume that any two points  $p, q \in M$  can be connected by a smooth embedded curve,  $\gamma$ , and that a vector field with compact support is “complete” in the sense that it generates a global flow.

- (6) Recall that the mapping cone of a map  $f : X \rightarrow Y$  is the quotient space of  $(X \times I) \amalg Y$  given by identifying each point  $(x, 1) \in X \times I$  to  $f(x) \in Y$ , and, for all  $x \in X$ , identifying all  $(x, 0)$  to a single point. Recall also that  $\mathbb{R}P^n$  is  $S^n/\mathbb{Z}_2$  where the  $\mathbb{Z}_2$ -action is by the antipodal map. Let  $p : S^n \rightarrow \mathbb{R}P^n$  be the corresponding quotient map. Show that  $\mathbb{R}P^{n+1}$  is homeomorphic to the mapping cone of  $p$ .

## Part III

- (7) Consider the space  $\bigvee_{i=1}^3 S_i^1$ , the wedge of three circles at a common basepoint.
- Describe its universal cover.
  - Describe one other covering space of this space.
  - Compute its fundamental group.
- (8) Suppose that any map  $X \rightarrow X$  has a fixed point, and that  $A \subseteq X$  is a retract of  $X$ . Prove that any map  $A \rightarrow A$  has a fixed point.
- (9) Recall that the suspension of a space  $X$  is the quotient space of  $X \times I$  given by identifying  $X \times \{0\}$  to a point and  $X \times \{1\}$  to a point. Prove that if  $X$  is path-connected, then  $SX$  is simply connected.

UCR Math Dept Topology Qualifying Exam

December 2009

- There are three parts in this exam and each part has three problems. You should complete two (and only two) problems of your choice in each part. Each problem is worth 10 points.
- Support each answer with a complete argument. State completely any definitions and basic theorems that you use.
- This is a closed book test. You may only use the test, something to write with, and the paper provided. All other material is prohibited. Write each of your solutions on separate sheets. Write your ID number on every sheet you use.
- The time for the exam is 3 hours.

Part I

1. Let  $X$  be a compact metric space. Prove that  $X$  is separable.
  
2. Prove or give a counterexample to each of the following:
  - i: If  $f : X \rightarrow Y$  is a continuous map of topological spaces, then  $X$  is compact if and only if its image  $f(X)$  is compact.
  
  - ii: If  $\mathcal{U}_1$  and  $\mathcal{U}_2$  are compact Hausdorff topologies on the same set  $X$ , then neither one of them is properly contained in the other.

*Hint:* Think about the continuity of the identity map from  $(X, \mathcal{U}_1)$  to  $(X, \mathcal{U}_2)$ .
  
3. A topological space  $X$  is **irreducible** if whenever  $X$  is decomposed as
$$X = A \cup B$$
with  $A$  and  $B$  closed subsets, then either  $A = X$  or  $B = X$ .  
Prove that an irreducible Hausdorff space consists of at most one point.

## Part II

4. Let  $X$  be the space obtained by gluing a copy of  $S^1$  and a copy of  $S^2$  along a point. Compute  $\pi_1(X)$  and describe the (uniquely defined)  $n$ -fold cover  $\tilde{X}$  of  $X$ .

5. Let  $D$  be the unit disk in the plane, and  $f : D \rightarrow D$  a homeomorphism. Show that  $f$  sends the unit circle  $S^1 \subset D$  into itself.

*Hint:* Assume, by contradiction, that there is a point  $p \in S^1$  that is sent in the interior of  $D$ . Consider then the map  $f$  restricted to  $D^2 \setminus \{p\}$ .

6. Denote by  $T$  a torus with one disk removed. Let  $C$  be the boundary circle. Prove that  $T$  cannot retract to  $C$ .

*Hint:* In the free group  $G$  generated by  $a, b$ , let  $H$  be the subgroup generated by  $aba^{-1}b^{-1}$ . Can we extend the identity map

$$id : H \rightarrow H$$

to a group homomorphism  $G \rightarrow H$ ?

## Part III

7. Let  $M$  be a compact  $n$ -manifold and  $f : M \rightarrow \mathbb{R}^{n+1}$  differentiable with  $0 \notin f(M)$ .

Show that there's a line through the origin of  $\mathbb{R}^{n+1}$  that meets only finitely many points of  $f(M)$ .

8. Let  $M$  and  $N$  be smooth  $n$ -manifolds. If  $M$  is compact, show that any immersion  $f : M \rightarrow N$  is a cover.

9. Suppose that  $f$  is a smooth function defined on an open subset  $U$  of  $\mathbb{R}^n$ . Recall that the gradient of  $f$  is the vector field

$$\nabla f : p \mapsto \left( \frac{\partial}{\partial x^1} f, \frac{\partial}{\partial x^2} f, \dots, \frac{\partial}{\partial x^n} f \right) \Big|_p.$$

Assume that  $\nabla f$  is never 0.

(1): Prove that a smooth vector field  $Z$  on  $U$  is perpendicular to  $\nabla f$  if and only if the directional derivative  $Z(f)$  is zero.

(2): Suppose that  $X$  and  $Y$  are smooth vector fields on  $U$  which are perpendicular to  $\nabla f$ . Prove that the same is true for the Lie bracket  $[X, Y]$ .



# UCR Mathematics Department

## Mathematics 205 Qualifying Examination

December 6, 2008

- There are three parts to this examination, and each part has three problems. You should complete two (and only two) problems of your choice from each part. The point value for every problem is 10 points.
- Support each answer with a complete argument, and state completely any definitions and basic theorems that are used.
- This is a closed book test. You may only use the test, writing instruments, and the paper provided. All other materials are prohibited. Use a new sheet of paper to start each problem, indicate the sequence of pages, and write your student ID number on each page.
- The default time for this examination is 3 hours; students who have submitted appropriate documentation will have the time period adjusted in keeping with the instructions on these documents.

## Part I

1. Suppose that  $(X, d)$  is a metric space. Show that the topology on  $X$  determined by the metric  $d$  is the coarsest (or smallest) of all topologies  $\mathbf{T}$  on  $X$  such that the function  $d : X \times X \rightarrow \mathbb{R}$  is continuous, where  $X \times X$  has the product topology associated to  $\mathbf{T}$ .

2. Let  $X \equiv C^0(\mathbb{R}, \mathbb{R})$  be the set of continuous functions from  $\mathbb{R}$  to itself. Given a continuous function  $g$  in this set and a continuous function

$$\varepsilon : \mathbb{R} \longrightarrow (0, \infty)$$

define

$$U_\varepsilon(g) \equiv \{h \in C^0(\mathbb{R}, \mathbb{R}) : |h(x) - g(x)| < \varepsilon(x) \text{ for all } x \in \mathbb{R}\}.$$

(a) Show that the collection of all such sets  $U_\varepsilon(g)$  forms a basis for a topology on  $X$  (this topology is called the **fine  $C^0$ -topology**.)

(b) Show that the fine  $C^0$ -topology is not first countable, and explain why it is also not metrizable.

3. (a) [1 point] State the definition of an open map.

(b) [1 point] State the definition of a closed map.

(c) [2 points] State the definition of a quotient map.

(d) [3 points] Show that a surjective (onto) open map is a quotient map.

(e) [3 points] Is the converse true? As always you must justify your answer.

## Part II

**GENERAL CONDITIONS:** In working the problems below, assume that all spaces in the statements of the problems are Hausdorff, connected and locally arcwise connected.

1. (a) Suppose that  $p : E \rightarrow B$  is a covering space projection (covering map), where  $E$  is compact and simply connected, and let  $b \in B$ . Prove that  $\pi_1(B, b)$  is finite.

(b) Given an example of a covering space projection as in (a) such that  $p$  is not a homeomorphism.

2. (a) Given an example of two subsets  $K \subset U$  in  $\mathbb{R}^2$  such that  $K$  is compact,  $U$  is open, and  $K$  is a deformation retract of  $U$ .

(b) Suppose that  $X$  is a space such that  $x \in X$  and  $\pi_1(X, x)$  is generated by two elements. Let  $E$  be the Figure 8 space  $A \cup B$  where  $A$  and  $B$  are homeomorphic to the circle and  $A \cap B = \{e\}$  (one point). Prove that there is a continuous basepoint preserving function  $f : (E, e) \rightarrow (X, x)$  such that the associated homomorphism of fundamental groups is surjective.

3. (a) Prove that the Cartesian product of two covering space projections is a covering space projection.

**Definition:** Given two spaces with base points  $(X, x)$  and  $(Y, y)$ , a base point preserving continuous map  $f : (X, x) \rightarrow (Y, y)$  is said to be a *homotopy equivalence of pointed spaces* if there is a base point preserving continuous map  $f' : (Y, y) \rightarrow (X, x)$  such that the composites  $f' \circ f$  and  $f \circ f'$  are base point preservingly homotopic to the identity maps on  $(X, x)$  and  $(Y, y)$  respectively; the map  $f'$  is said to be a pointed homotopy inverse to  $f$ .

(b) Suppose that we are given basepoint preserving continuous maps  $f : (X, x) \rightarrow (Y, y)$  and  $g : (Y, y) \rightarrow (Z, z)$  such that  $g$  and  $g \circ f$  are homotopy equivalences of pointed spaces. Prove that  $f$  is also a homotopy equivalence of pointed spaces. [*Hint:* Let  $h$  and  $k$  be pointed homotopy inverses to  $g \circ f$  and  $g$ , show that both  $k$  and  $f \circ h$  are basepoint preservingly homotopic to  $k \circ g \circ f \circ h$ , and finally show that  $h \circ g$  is a pointed homotopy inverse to  $f$ .]

## Part III

1. Consider the space  $GL_n(\mathbb{R})$  of real  $n \times n$  invertible matrices. Give the definition of *left-invariant* vector field on  $GL_n(\mathbb{R})$ . Show that the space of left-invariant vector fields can be identified with  $\mathcal{M}(n)$ , the space of all  $n \times n$  matrices. [Hint: Prove that a left-invariant vector field is determined by its value at the identity matrix  $I \in GL_n(\mathbb{R})$ .]

2. (a) Decide whether the set of points in  $\mathbb{R}^3$  that satisfy the system of equations  $x^2 + y^2 + z^2 = 1$ ,  $x^2 + y^2 - z^2 = 0$  is a smooth submanifold of  $\mathbb{R}^3$ .

(b) Decide whether the set of points in  $\mathbb{R}^3$  that satisfy the equation  $x^2 + y^2 - z^2 = 0$  is a smooth submanifold of  $\mathbb{R}^3$ . [Hint: Every point of a submanifold must have an open neighborhood such that ... ]

3. Consider the following vector fields on  $\mathbb{R}^2$ :

$$X = y \frac{\partial}{\partial x} - 4x \frac{\partial}{\partial y}, \quad Y = xy \frac{\partial}{\partial x}$$

(a) Compute the Lie derivative  $\mathcal{L}_Y X = [Y, X]$ .

(b) Write, in parametric form, the integral curve  $(x(t), y(t))$  to  $X$  passing through  $(1, 0)$  at  $t = 0$ .

UC Riverside Mathematics Department — Mathematics 205 Qualifying Exam

January 2008

- There are three parts in this exam and each part has three problems. You should complete two (and only two) problems of your choice in each part. Each problem is worth 10 points.
- Support each answer with a complete argument. State completely any definitions and basic theorems that you use.
- This is a closed book test. You may only use the test, something to write with, and the blue books (or approved substitutes). All other material is prohibited. Write each of your solutions in a separate blue book (or alternative). Write your ID number on the cover of every blue book or alternative that you use.
- The time for the examination is 3 hours.

Part I

1. Let  $X$  be a compact metric space, and let  $\mathcal{U}$  be an open covering of  $X$ . Prove that there is a positive real number  $\eta$  such that if  $y, z \in X$  satisfy  $d(y, z) < \eta$ , then there is an open subset in  $\mathcal{U}$  which contains both  $y$  and  $z$ .
2. (i) Let  $X$  be a topological space. Prove that  $X$  is Hausdorff if and only if the diagonal in  $X \times X$  (all points of the form  $(x, x)$  for some  $x \in X$ ) is a closed subset in the product topology.  
(ii) Let  $f : X \rightarrow Y$  be a continuous 1–1 onto map of compact Hausdorff spaces. Prove that  $f$  is a homeomorphism, and give a counterexample to this result if the compactness assumption on  $X$  is dropped.
3. (i) Let  $X$  and  $Y$  be topological spaces, take the product topology on  $X \times Y$ , and let  $q : X \times Y \rightarrow X$  denote projection onto the first coordinate. Prove that  $q$  is an open mapping, and give an example to show that  $q$  is not necessarily a closed mapping.

Part II

4. Let  $T^2 = S^1 \times S^1$  be the torus with its usual topology. Let  $X \subseteq T^2$  be the subset  $S^1 \times \{*\}$ , where  $*$   $\in S^1$  is any point.  
(i) Is  $X$  a retract of  $T^2$ ? Prove your answer.  
(ii) Is  $X$  a deformation retract of  $T^2$ ? Prove your answer.
5. (i) Thinking of  $S^1$  as the unit circle in the complex plane, show that the map  $f: \mathbb{R} \rightarrow S^1$  given by  $f(x) = \exp(ix)$  is a covering map.  
(ii) Prove that the plane  $\mathbb{R}^2$  is not homeomorphic to the cylinder  $\mathbb{R} \times S^1$ .
6. Compute the fundamental group of  $S^2$  with  $n$  points removed.

## Part III

7. Let  $M_n(\mathbb{C})$  be the space of  $n \times n$  matrices with complex entries.

(i) Show that  $M_n(\mathbb{C})$  can be identified with a real vector space, and determine its (real) dimension.

(ii) Denote by  $GL_n(\mathbb{C})$  the subspace of invertible matrices. Prove that it is an open subset of  $M_n(\mathbb{C})$ .

(iii) On  $M_n(\mathbb{C})$  we can naturally define the determinant map  $\det : M_n(\mathbb{C}) \rightarrow \mathbb{C}$ . Let  $SL_n(\mathbb{C}) = \det^{-1}(1)$ ; show that it is a smooth submanifold of  $GL_n(\mathbb{C})$ , and compute its (real) dimension.

8. Consider the following vector fields on  $\mathbb{R}^2$ :

$$X := \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad Y := y \frac{\partial}{\partial x} + 4x \frac{\partial}{\partial y}.$$

(i) Compute the Lie derivative  $\mathcal{L}_X Y$ .

(ii) Write in parametric form the integral curve  $(x(t), y(t))$  to  $Y$  passing through the point  $(1, 1)$  at  $t = 0$ .

9. Show that the set of points of  $\mathbb{R}^3$  that satisfy the equations

$$x^2 + y^2 - z^2 = 1, \quad x + y = 1$$

is a smooth submanifold of  $\mathbb{R}^3$ .

UC Riverside Mathematics Department — 205 Qualifying Exam

February 2007

- There are three parts in this exam and each part has three problems. You should complete two (and only two) problems of your choice in each part. Each problem is worth 10 points.
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- This is a closed book test. You may only use the test, something to write with, and the blue books. All other material is prohibited. Write each of your solutions in a separate blue book. Write your ID number on the cover of every blue book that you use.
- The time for the exam is 3 hours.

Part I

- (i) Let  $X$  be a topological space that is expressible as a union  $A \cup B \cup C$ , where  $A$ ,  $B$  and  $C$  are connected subsets such that  $A \cap B$  and  $B \cap C$  are nonempty (but  $A \cap C$  might be empty). Prove that  $X$  is connected.

(ii) Let  $X$  be a metric space and let  $x \in X$ . Prove that  $\{x\}$  is nowhere dense in  $X$  if and only if  $\{x\}$  is not open in  $X$  (hence  $x$  is not *isolated* in  $X$ ).

(iii) Let  $X$  be a countable complete metric space. Prove that  $X$  has at least one isolated point.
- (i) Let  $X$  be the unit interval  $[0, 1]$  with the usual topology, and let  $\mathcal{E}$  be the equivalence relation given by  $s \equiv t$  if and only if  $s$  and  $t$  are nonzero multiples of each other. Show that the quotient space  $X/\mathcal{E}$  has only finitely many equivalence classes, describe the open subsets of  $X/\mathcal{E}$  in the quotient topology explicitly, and use the latter to explain why  $X/\mathcal{E}$  with the quotient topology is not Hausdorff.

(ii) Suppose that  $X$  is a separable metric space and  $A$  is a subspace of  $X$ . Prove that  $A$  is separable.
- (i) Let  $X$  be a topological space, let  $A \subset B \subset X$ , and suppose that  $B$  is dense in  $X$ . Prove that  $A$  is dense in  $X$  if  $A$  is dense in  $B$ .

(ii) Suppose that  $X$  is a compact metric space. Prove that  $X$  is complete with respect to the given metric.

(iii) Given a metric space  $(X, \mathbf{d})$ , the *diameter*  $d(X)$  of  $X$  is defined to be the least upper bound of the distances  $\mathbf{d}(y, z)$ , where  $y$  and  $z$  run through all the points of  $X$ , if this set of distances is bounded and  $\infty$  otherwise. If  $X$  is compact, prove that  $d(X) \neq \infty$  and that there exists a pair of points  $u, v \in X$  such that  $\mathbf{d}(u, v) = d(X)$ .

## Part II

4. Show that  $S^n$  and  $S^2$  are not homeomorphic if  $n > 2$ .
5. Let  $X$  be a topological space with finite fundamental group. Show that every map  $f : X \rightarrow S^1$  is homotopic to a constant map.
6. Let  $X_\phi$  be the topological space obtained by attaching the Möbius band  $M$  along its boundary  $\partial M$  to a torus  $S^1 \times S^1$  with attaching map  $\phi : \partial M \rightarrow S^1 \times S^1$ . Determine the fundamental group of  $X_\phi$  when:
- $\phi$  is nullhomotopic;
  - $\phi$  identifies  $\partial M$  with a circle  $S^1 \times \{y_0\}$ .

## Part III

7. The Hopf fibration  $h : S^3 \rightarrow S^2$  is defined by

$$h(a, c) = (a\bar{c}, |a|^2 - |c|^2),$$

where  $S^3$  is the unit sphere in  $\mathbb{C} \oplus \mathbb{C}$  and  $S^2$  is the unit sphere in  $\mathbb{C} \oplus \mathbb{R}$ . The Hopf fibration extends to a map  $\hat{h} : (\mathbb{C} \oplus \mathbb{C}) \setminus \{0\} \rightarrow S^2$  by setting  $\hat{h}(v) = h(|v|^{-1} \cdot v)$ . The *Canonical Line Bundle over  $\mathbb{C}P^1 \cong S^2$*  is defined to be

$$E = \left\{ (p, v) \in S^2 \times (\mathbb{C} \oplus \mathbb{C}) \mid \text{either } v = 0 \text{ or } \hat{h}(v) = p \right\}$$

and the complex projective plane  $\mathbb{C}P^2$  is the quotient of the circle action on  $S^5$ , sending  $(\omega, (a, b, c)) \in S^1 \times S^5$  to  $(\omega a, \omega b, \omega c)$ , or equivalently it is the quotient of the  $\mathbb{C}^*$  action on the space  $(\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}) \setminus \{0\}$  given by (coordinatewise) scalar multiplication.

Show that  $\mathbb{C}P^2$  is homeomorphic to the one point compactification of the canonical line bundle over  $\mathbb{C}P^1$  or equivalently that  $\mathbb{C}P^2 \setminus \{pt\}$  is homeomorphic to  $E$ .

8. Let  $\frac{\partial}{\partial x}$  and  $\frac{\partial}{\partial y}$  be the coordinate vector fields on  $\mathbb{R}^2$ . Let

$$\begin{aligned} X &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \\ Y &= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}. \end{aligned}$$

Compute the Lie bracket  $[X, Y]$ .

9. Suppose that  $m < p$  and  $f : S^m \rightarrow S^p$  is a continuous function. Show that  $f$  is homotopic to a constant map.



# MATHEMATICS 205ABC QUALIFYING EXAMINATION TOPICS

January, 2006

## Background material

1. "Undergraduate material" in the syllabi for real and complex analysis
2. Items 1–3 in the "undergraduate material" on groups in the syllabus for algebra
3. Items 1–2 in the "undergraduate material" on rings and fields in the syllabus for algebra
3. Items 1–4 in the "undergraduate material" on modules and linear algebra in the syllabus for algebra
4. Basics of set theory including the standard basic results on transfinite cardinal numbers (the Axiom of Choice will be used as needed, but Zorn's Lemma and its applications are not required).

## General Topology

1. Metric and topological spaces, open and closed sets
2. Continuous functions, homeomorphisms, related concepts
3. Constructions on spaces and mappings: Subspaces, products, quotients
4. Connectedness, local connectedness and path connectedness
5. Compactness (excluding Tychonoff's Theorem)
6. Countability and separation properties, specializations to metric spaces
7. Local compactness, one point compactification
8. Complete metric spaces, completions (no existence proofs), Baire's Theorem

## Introduction to Algebraic Topology

1. Homotopy of continuous mappings, basic properties
2. Construction of the fundamental group, important general properties
3. Fundamental groups of important examples, applications
4. Covering spaces
5. The Seifert-van Kampen Theorem
6. Computational applications
7. Lifting criterion, existence and classification of coverings

## Introduction to the theory of manifolds

1. Topological manifolds, partitions of unity
2. Local theory of smooth functions
3. Differential manifolds, global theory of smooth functions
4. Constructions on smooth manifolds
5. Tangent bundles, regular mappings, diffeomorphisms
6. Vector fields, integral curves, Lie brackets, completeness
7. Vector bundles, cross sections, constructions on vector bundles
8. Differential forms and exterior differential calculus (local theory)

UCR Mathematics Department 205 Qualifying Exam

January 2006

- There are three parts in this exam and each part has four problems. You should complete two (and only two) problems of your choice in each part. Each problem is worth 10 points.
- Support each answer with a complete argument. State completely any definitions and basic theorems that you use.
- This is a closed book test. You may only use the test, something to write with, and the blue books. All other material is prohibited. Write each of your solutions in a separate blue book. Write your ID number on the cover of every blue book that you use.
- The time for the exam is 3 hours.

Part I

1. Let  $X$  be a topological space, let  $p \in X$ , and let  $C$  be the union of all connected subsets of  $X$  containing  $p$ . Prove that  $C$  is connected.
2. Prove that a compact metric space must be complete and bounded, and give an example of a complete and bounded metric space that is not compact.
3. Suppose that  $X$  is a  $T_3$  (i.e.,  $T_1$  and regular) space and  $A$  is a closed subset of  $X$ . Define an equivalence relation on  $X$  whose equivalence classes are the one point subsets  $\{x\}$  for all  $x \in X \setminus A$  together with  $A$ , and let  $X/A$  be the set of all such equivalence classes with the quotient topology. Prove that the topological space  $X/A$  is Hausdorff.
4. Let  $X$  be a set and let  $U$  and  $V$  be topologies on  $X$  such that  $U$  strictly contains  $V$ . Prove the following statements:
  - (1) If  $(X, U)$  is compact Hausdorff then  $(X, V)$  is not Hausdorff.
  - (2) If  $(X, V)$  is compact Hausdorff then  $(X, U)$  is not compact.[Hint: Consider the identity map from  $(X, U)$  to  $(X, V)$ .]

Part II

5. Show that every finitely presented group is the fundamental group of a topological space
6. Show that if  $n \geq 2$ , then every continuous map  $f : S^n \rightarrow S^1$  is null-homotopic.
7. Let  $p : (E, e_0) \rightarrow (B, b_0)$  be a covering space projection. Show that  $p_*\pi_1(E, e_0)$  is a normal subgroup of  $\pi_1(B, b_0)$  if and only if for every pair of points  $e_1, e_2 \in p^{-1}(b_0)$ , there is a covering transformation  $h : E \rightarrow E$  so that  $h(e_1) = e_2$ .

8. Recall that  $X$  is called *locally simply connected* if and only if for all open subsets  $U$  of  $X$  and all  $u \in U$ , there is a simply connected, open set  $V$  so that  $u \in V \subset U$ .

Let  $X$  be the so-called "Hawaiian Earring". Specifically,

$$X = \bigcup_{n=1}^{\infty} C_n \subset \mathbb{R}^2$$

is a subspace of  $\mathbb{R}^2$ , where  $C_n$  is the circle of radius  $\frac{1}{n}$  with center  $(\frac{1}{n}, 0)$ . Show that the cone on  $X$  is simply connected but not locally simply connected. Conclude that the cone on the Hawaiian Earring is semi-locally simply connected.

### Part III

9. Let  $U$  be an open subset of  $\mathbb{R}^n$ , and let  $W$  be an open neighborhood of  $U \times \{0\}$  in  $U \times \mathbb{R}$ . Prove that there is a positive real valued smooth function  $h : U \rightarrow \mathbb{R}$  such that  $W$  contains the set

$$\{(x, t) \in U \times \mathbb{R} : |t| \leq h(x)\}.$$

[Hint: For each  $u \in U$  there is an open neighborhood  $V_u$  of  $u$  in  $U$  such that such a function exists.]

10. Let  $M$  and  $N$  be smooth manifolds, and let  $f : M \rightarrow N$  be a smooth mapping. Define the graph function  $G_f : M \rightarrow M \times N$  by the formula  $G_f(x) = (x, f(x))$  for all  $x \in M$ . Prove that  $G_f$  is a smooth immersion. [Hint: Look locally, or look at the projections of  $G_f$  onto  $M$  and  $N$ .]

11. Suppose that  $M$  is a smooth manifold and  $g$  and  $h$  are smooth Riemannian metrics on  $M$ . Let  $u$  and  $v$  be smooth nonnegative functions on  $M$  such that  $u(x) + v(x)$  is positive for all  $x \in M$  (hence for each  $x$  either  $u(x) > 0$  or  $v(x) > 0$ ). Prove that the linear combination  $u \cdot g + v \cdot h$  is also a Riemannian metric on  $M$ .

12. (a) Compute the Lie bracket  $[X, Y]$  of the vector fields  $X$  and  $Y$  defined on  $\mathbb{R}^2$ , where

$$X = x^2 \frac{\partial}{\partial y} \quad \text{and} \quad Y = y \frac{\partial}{\partial x}.$$

(b) Differential forms  $\omega$  and  $\theta$  on  $\mathbb{R}^4$  are defined by the equations

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$

and

$$\theta = x_1 dx_2 \wedge dx_3 \wedge dx_4 + x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4.$$

Compute the expression  $\omega \wedge \omega - d\theta$ .

UCR Math Dept Topology Qualifying Exam

January 2005

- There are three parts in this exam and each part has three problems. You should complete two (and only two) problems of your choice in each part. Each problem is worth 10 points.
- Support each answer with a complete argument. State completely any definitions and basic theorems that you use.
- This is a closed book test. You may only use the test, something to write with, and the blue books. All other material is prohibited. Write each of your solutions in a separate blue book. Write your ID number on the cover of every blue book that you use.
- The time for the exam is 3 hours.

Part I

1. (a) Let  $U \subset K$  be contained in a Hausdorff space  $X$ , with  $U$  open and  $K$  compact, and assume that  $f : X \rightarrow Y$  is a continuous map into another Hausdorff space  $Y$ . Prove that the closure of  $f(U)$  is a compact subset of  $f(K)$ .  
(b) Let  $f, X, Y$  be as in the preceding problem, and assume in addition that  $f$  is an onto, open mapping and  $X$  is locally compact. Prove that  $Y$  is also locally compact.
2. Let  $S$  be the set of all connected subsets of the cartesian plane. Show that the cardinality of  $S$  is strictly greater than the cardinality of the plane. (Hint: Consider all subsets of the form

$$(0,1)^2 \cup \{B \times \{1\}\}$$

where  $B$  is a subset of  $(0,1)$ . Under what conditions is such a subset connected? Why?)

3. The logical implications of the statements below have the form  $A \implies B \implies D$  and  $A \implies C \implies D$  if they are suitably labeled as  $A, B, C, D$ . Give a labeling of the statements for which this is true.
  - (1) The topological space  $X$  is compact and metrizable.
  - (2) The topological space  $X$  is first countable.
  - (3) The topological space  $X$  is metrizable.
  - (4) The topological space  $X$  is second countable.

Part II

4. Let  $F_n$  be the free group of rank  $n$ . Show that for all  $n \geq 2$ ,  $F_2$  contains a subgroup isomorphic to  $F_n$ .

5. Let

$$Y = \{(x, 0, 0) \in \mathbb{R}^3; x \geq 0\} \cup \{(0, y, 0) \in \mathbb{R}^3; y \geq 0\} \cup \{(0, 0, z) \in \mathbb{R}^3; z \geq 0\}$$

be a subset of  $\mathbb{R}^3$ . Calculate the fundamental group of  $X = \mathbb{R}^3 \setminus Y$ .

6. Suppose that the universal covering space of  $X$  is compact, show that the fundamental group of  $X$  is a finite group.

### Part III

7. Suppose that  $M$  is a smooth  $n$ -manifold and that

$$\pi : M' \longrightarrow M$$

is a covering map. Show that  $M'$  has a unique smooth structure relative to which  $\pi$  is locally a diffeomorphism.

8. Prove that every smooth manifold admits a Riemannian metric.

9. To define the Hopf fibration  $h : S^3 \longrightarrow S^2$  we think of  $S^3$  as the unit sphere in  $\mathbb{C} \oplus \mathbb{C}$  and  $S^2$  as the unit sphere in  $\mathbb{C} \oplus \mathbb{R}$ . With respect to these coordinates the formula for  $h$  is

$$h : (a, c) \mapsto (2a\bar{c}, |a|^2 - |c|^2).$$

- (a) Show that the image of  $h$  is indeed contained in  $S^2$ .
- (b) Show that  $h$  is a quotient map.
- (c) Identify  $S^1$  with the unit circle in  $\mathbb{C}$ . Consider the  $S^1$ -action on  $S^3$ ,

$$H : S^1 \times S^3 \longrightarrow S^3,$$

$$H : (\omega; (a, c)) \longrightarrow (\omega a, \omega c),$$

where the multiplication takes place in  $\mathbb{C}$ . Show that the orbits of this circle action coincide with the fibers of  $h$ . (The fibers of a map are the preimages of points.)

UCR Department of Mathematics  
Topology Qualifying Exam

June 17, 2004

- There are three parts in this exam. You should complete two problems of your choice in each part. Each problem is worth 10 points.
- Support each answer with a complete argument. State completely any definitions and basic theorems that you use.
- This is a closed book test. You may only use the test and papers to write on. All other material is prohibited. Write your answer to each problem separately and write the last four digits of your ID number on the top of the first page of each answer sheet.
- The time for the exam is 3 hours.

Part I

- X 1. (i) Suppose that  $X$  is a set and  $\mathcal{U}$  and  $\mathcal{V}$  are topologies for  $X$ . Prove that  $\mathcal{U} \cap \mathcal{V}$  is also a topology for  $X$ .
- (ii) Suppose that  $\sim$  is the binary relation on  $\mathbf{R}$  such that  $x \sim y$  if and only if  $y$  is a positive multiple of  $x$ . Determine whether the quotient space is Hausdorff and prove that your conclusion is correct.
2. (i) Let  $X$  and  $Y$  be topological spaces, let  $a \in X$  and  $b \in Y$ , and let  $A$  and  $B$  be the connected components of  $A^c$  and  $B^c$  in  $X$  and  $Y$  respectively. Prove that  $A \times B$  is the connected component of  $(a, b)$  in  $X \times Y$ .
- (ii) Let  $X$  be a topological space and suppose that  $x \in X$  is not a limit point of  $X$ . Prove that the one point subset  $\{x\}$  is open in  $X$ .
3. Let  $X$  be a topological space such that every point in  $X$  lies in a maximal compact subset. Prove that  $X$  is compact.
- X 4. (i) Suppose that  $X$  is a separable metric space and  $A$  is a subspace of  $X$ . Prove that  $A$  is also separable.
- (ii) Give an example of a topological space that is metrizable but does not have a complete metric, and give reasons for your answer. [Hint: Look at dense subsets of the real numbers.]

Part II

- X 5. Let  $B^3 = \{(x, y, z) \in \mathbf{R}^3; x^2 + y^2 + z^2 \leq 1\}$  be the 3-ball. Let  $H$  be a solid torus in the interior of  $B^3$  obtained by revolving the disk  $D = \{(x, 0, z) \in \mathbf{R}^3; (x - 1/2)^2 + z^2 \leq 1/9\}$  about the  $z$ -axis. Let  $X = \overline{B^3} \setminus H$ . Calculate the fundamental group  $\pi_1(X)$ .

UCR Department of Mathematics  
Topology Qualifying Exam

January 3, 2004

- There are three parts in this exam. You should complete two problems of your choice in each part. Each problem is worth 10 points.
- Support each answer with a complete argument. State completely any definitions and basic theorems that you use.
- This is a closed book test. You may only use the test and papers to write on. All other material is prohibited. Write your answer to each problem separately and write the last four digits of your ID number on the top of the first page of each answer sheet.
- The time for the exam is 3 hours.

Part I

1. Let  $A$  and  $B$  be subsets of a topological space  $X$ , and let  $C$  be a subset of  $A \cap B$  that is closed in each of  $A$  and  $B$  with respect to the subspace topologies. Prove that  $C$  is a closed subset of  $A \cup B$ .
2. (a) Let  $f$  and  $g$  be continuous functions from a topological space  $X$  to a Hausdorff space  $Y$ . Prove that the set of all points  $x \in X$  such that  $f(x) = g(x)$  is a closed subset of  $X$ .  
(b) Let  $A$  be a subset of the Hausdorff space  $X$ , and let  $r : X \rightarrow A$  be a continuous map such that the restriction of  $r$  to  $A$  is the identity. Prove that  $A$  is a closed subset of  $X$ .
3. Let  $(X, d)$  be a metric space, and  $A$  is a subset of  $X$ . Show that the function  $d_A : X \rightarrow \mathbb{R}$ ,  $d_A(x) = \inf \{d(x, a) ; a \in A\}$ , is continuous.
4. Recall that if  $X$  is a metric space, then a contraction mapping  $\varphi : X \rightarrow X$  is any map that satisfies

$$\text{dist}(\varphi(x), \varphi(y)) \leq k \text{dist}(x, y),$$

for all  $x, y \in X$  and some  $k \in [0, 1)$ .

Show that any contraction mapping from a compact metric space to itself has a unique fixed point.

*Hint:* Let  $x_0$  be any point in  $X$  and consider the recursively defined sequence

$$x_{n+1} \equiv \varphi(x_n).$$

Show that  $\{x_{n+1}\}_{n=1}^{\infty}$  is Cauchy, and that its limit is the desired fix point.

5. Let  $\omega$  be a  $C^\infty$  differential  $k$ -form on  $\mathbb{R}^n$  such that

$$\int_M \omega = 0$$

for every compact oriented smooth  $k$ -manifold  $M \subset \mathbb{R}^n$  without boundary. Use Stokes Theorem to show that  $\omega$  is closed, that is,  $d\omega = 0$ .

*Hint:* To show that  $d\omega$  is 0 at a point  $p$ , let  $M$  a very small sphere whose center is  $p$ .

6. Let  $p \in \mathbb{R}^n$ , and let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$ -mapping such that  $df_p$  is one-to-one. Show that there is an  $\epsilon > 0$  so that if  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a  $C^1$ -mapping with  $\|dg_p\| < \epsilon$ , then  $h \equiv f + g$  is one-to-one in a neighborhood of  $p$ .

## Part II

7. Let  $p : E \rightarrow B$  be a covering map and  $B$  be path connected. Show that the sets  $p^{-1}(b)$ ,  $b \in B$ , all have the same cardinality.

8. Let  $p_i = (i, 0) \in \mathbb{R}^2$ ,  $i = 1, 2, \dots, n$ , and  $X = \mathbb{R}^2 \setminus \{p_1, p_2, \dots, p_n\}$ . Use Seifert-van Kampen Theorem to prove that the fundamental group  $\pi_1(X)$  is a free group of rank  $n$ .

9. (1) Let  $X$  be a topological space. Let  $f, g : X \rightarrow S^1$  be two continuous maps. Show that if  $f(x)$  and  $g(x)$  are not antipodal to each other for every  $x \in X$ , then  $f$  and  $g$  are homotopic.

(2) Find two non-homotopic continuous maps  $f, g : S^1 \rightarrow S^1$  such that there is exactly one point  $x_0 \in S^1$  where  $f(x_0) = -g(x_0)$ .

## Part III

10. Identify  $\mathbb{R}^4$  with  $\mathbb{C}^2$  so that  $(x, y, u, v) \in \mathbb{R}^4$  corresponds to  $(z_1, z_2) \in \mathbb{C}^2$  for  $z_1 = x + iy$  and  $z_2 = u + iv$ . Let  $V$  be the zero locus of the polynomial  $P(z_1, z_2) = z_1^3 + z_2^2$ , i.e.

$$V = \{(z_1, z_2); z_1^3 + z_2^2 = 0\} \subset \mathbb{R}^4.$$

Show that  $V \setminus \{(0, 0)\}$  is a 2-dimensional smooth manifold.



**11.** Let  $M$  be a smooth manifold. For  $f, g$  smooth functions on  $M$  and  $X, Y$  smooth vector fields on  $M$ , we have

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

Suppose that a smooth function  $f$  on  $M$  satisfies  $[fX, Y] = f[X, Y]$  for all smooth vector fields  $X$  and  $Y$  on  $M$ . What can one say about  $f$ ?

**12.** (a) State what it means for a smooth vector field on a smooth manifold to be complete.

(b) Give an example of a smooth manifold  $M_1$  such that every vector field on  $M_1$  is complete.

(c) Give an example of a smooth manifold  $M_2$  which has vector fields that are not complete.

UCR Department of Mathematics  
Topology Qualifying Exam  
January 4, 2003

- There are three parts in this exam and each part has four or five problems. You should complete three (and only three) problems of your choice in each part. Each problem is worth 10 points.
- Support each answer with a complete argument. State completely any definitions and basic theorems that you use.
- This is a closed book test. You may only use the test and papers to write on. All other material is prohibited. Write your answer to each problem separately and write the last four digits of your ID number on the top of the first page of each answer sheet.
- The time for the exam is 3 hours.

Part I

1. Recall that a metric space  $(X, d)$  is called bounded if there is a number  $D > 0$  so that

$$d(a, b) \leq D, \quad \forall a, b \in X.$$

Show that the boundedness is a metric rather than a topological property of  $X$ . That is to exhibit a topological space that admits both a bounded metric and an unbounded metric.

2. Show that  $\mathbb{Q}$ , as a subspace of  $\mathbb{R}$ , is not locally compact.
3. Use the connectedness of the unit interval  $[0, 1]$  to show that any continuous map  $f : [0, 1] \rightarrow [0, 1]$  has a fixed point. That is to show that for each such  $f$ , there is a  $t_0$  in  $[0, 1]$  so that  $f(t_0) = t_0$ .
4. Let  $p : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the projection onto the first factor,  $p(x, y) = x$ . Let

$$Z = \{(x, y) \in \mathbb{R}^2 ; x \geq 0 \text{ or } y = 0\}$$

be a subspace of  $\mathbb{R}^2$ . Show that the restriction of  $p$  to  $Z$  is a quotient map that is neither open nor closed.

## Part II

5. Let  $S^1 \times S^1 = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| = 1, |z_2| = 1\}$  be a torus.

(1) Show that for any  $2 \times 2$  integer matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = \pm 1,$$

the map  $h_A(z_1, z_2) = (z_1^a z_2^b, z_1^c z_2^d)$  is a homeomorphism of  $S^1 \times S^1$  to itself.

(2) We have two 2-fold covering maps from  $S^1 \times S^1$  to itself:

$$p_1(z_1, z_2) = (z_1^2, z_2) \quad \text{and} \quad p_2(z_1, z_2) = (z_1, z_2^2).$$

Give a third example of a 2-fold covering map  $p : S^1 \times S^1 \rightarrow S^1 \times S^1$ .

6. Let  $p : X \rightarrow Y$  be a covering map and  $Y$  is Hausdorff. Show that if  $X$  is compact, then  $p$  is a finite fold covering map.

7. Let  $X$  be the union of the unit 2-sphere  $S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$  and the line segment  $\{(x, 0, 0) \in \mathbb{R}^3; -1 \leq x \leq 1\}$ . Use Seifert-van Kampen Theorem to calculate the fundamental group  $\pi_1(X)$ .

8. Let  $X$  be a topological space. Let  $f, g : X \rightarrow S^2$  be two continuous maps. Show that if  $f(x)$  and  $g(x)$  are not antipodal to each other for every  $x \in X$ , then  $f$  and  $g$  are homotopic.

### Part III

9. (1) Give an example of a compact (without boundary) 4-dimensional differentiable manifold whose fundamental group is infinite abelian.

(2) Let  $M$  be a compact (without boundary) manifold with a finite fundamental group. Prove that its universal covering space is homeomorphic to a closed and bounded subset in  $\mathbb{R}^N$ .

10. Let  $M$  be an  $n$ -dimensional differentiable manifold. Let  $f : M \rightarrow \mathbb{R}^n$  be an one-to-one immersion. Prove that there exists an imbedding  $F : M \rightarrow \mathbb{R}^{n+1}$  such that  $F(M)$  is a closed subset of  $\mathbb{R}^{n+1}$ .

11. Prove that for  $n > 1$ , there is no immersion  $f : S^n \rightarrow T^n$ , where  $S^n$  is the  $n$ -sphere, and  $T^n$  the  $n$ -torus.

12. (1) Let  $B_2 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$ . Prove that the 1<sup>st</sup> de Rham cohomology group of  $B_2 \setminus \{(0, 0)\}$  is non-trivial.

(2) Let  $M$  be a compact (without boundary) oriented  $n$  dimensional differentiable manifold. Prove that its  $n^{\text{th}}$  de Rham cohomology group is non-trivial.

13. Let  $m < p$  and  $M$  be an  $m$ -dimensional differentiable manifold. Show that every continuous map  $\varphi : M \rightarrow S^p$  is homotopic to a constant map.

6. Let  $F(a, b, c)$  be the free group of rank 3 generated by  $a, b, c$  and  $F(x, y)$  be the free group rank 2 generated by  $x, y$ . Find an explicit injective homomorphism  $\phi : F(a, b, c) \rightarrow F(x, y)$ . Also describe a complete list of cosets of the subgroup  $\phi(F(a, b, c))$  of  $F(x, y)$ .

7. Suppose that  $X$  is a compact topological space, and  $p : \tilde{X} \rightarrow X$  is the universal covering space of  $X$ . Show that  $\tilde{X}$  is compact if and only if the fundamental group of  $X$  is finite.

### Part III

8. Show that there is no one to one  $C^\infty$ -map  $g : S^2 \rightarrow S^1$ .

9. Let  $U$  be an open subset of euclidean space and let  $X$  and  $Y$  be smooth vector fields on  $U$ . Show that

$$[X, Y] \equiv 0$$

if and only if the local flows generated by  $X$  and  $Y$  commute.

10. To define the Hopf fibration  $h : S^3 \rightarrow S^2$  we think of  $S^3$  as the unit sphere in  $\mathbb{C} \oplus \mathbb{C}$  and  $S^2$  as the unit sphere in  $\mathbb{C} \oplus \mathbb{R}$ . With respect to these coordinates the formula for  $h$  is

$$h : (a, c) \mapsto (2a\bar{c}, |a|^2 - |c|^2).$$

(i) Show that the image of  $h$  is indeed contained in  $S^2$ .

(ii) Show that  $h$  is a quotient map.

UCR Department of Mathematics  
Topology Qualifying Exam  
September 29, 2001

- There are three parts in this exam and each part has three problems. You should complete two (and only two) problems of your choice in each part. Each problem is worth 10 points.
- Support each answer with a complete argument. State completely any definitions and basic theorems that you use.
- This is a closed book test. You may only use the test and papers to write on. All other material is prohibited. Write your answer to each problem separately and write the last four digits of your ID number on the top of the first page of each answer.
- The time for the exam is 3 hours.

Part I

1. Let  $(X, d)$  be a metric space, and  $A, B$  be two compact subsets of  $X$ . By definition,  $d(A, B) = \inf\{d(x, y); x \in A, y \in B\}$ . Show that there are points  $x_0 \in A$  and  $y_0 \in B$  such that  $d(x_0, y_0) = d(A, B)$ .
2. Let  $Y$  and  $Y_1$  be subsets of a topological space  $X$  such that  $Y \subset Y_1 \subset \bar{Y}$ . Show that if  $Y$  is connected, so is  $Y_1$ . If  $Y$  is path connected, can we conclude that  $Y_1$  is also path connected?
3. An equivalence relation on  $\mathbb{R}^2$  is defined as follows: Two point  $(x_1, y_1)$  and  $(x_2, y_2)$  are equivalent,  $(x_1, y_1) \sim (x_2, y_2)$ , if  $y_1 - x_1^2 = y_2 - x_2^2$ . Show that the quotient space  $\mathbb{R}^2 / \sim$  is homeomorphic to  $\mathbb{R}^1$ .

Part II

4. Find a connected graph which can be realized as a double covering space of  $S^1 \vee S^1$  (the figure 8 graph). Use your construction to show that the free group of rank 3 is isomorphic to a subgroup of the free group of rank 2.
5. Show that every map  $\mathbb{R}P^2 \rightarrow S^1$  is homotopic to a constant. (Hint: Can such a map be lifted to a map  $\mathbb{R}P^2 \rightarrow \mathbb{R}^1$ ?)
6. Take the 2-sphere  $S^2$  and identify its north and south poles, we get a quotient space  $X$  of  $S^2$ . Use Seifert-van Kampen Theorem to calculate the fundamental group  $\pi_1(X)$ .

### Part III

7. Is it possible to provide the surface of the cube

$$\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n ; |x_i| \leq 1, i = 1, 2, \dots, n\}$$

with a differentiable structure? Explain.

8. Show that  $S^n \times \mathbb{R}^1$  is parallelizable, i.e., its tangent bundle is trivial. (Hint: Identify  $S^n \times \mathbb{R}$  with a subset of  $\mathbb{R}^{n+1}$ .)

9. Show that there is no smooth one-to-one map  $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ .

UCR, Department of Mathematics  
Topology Qualifying Exam  
January 2001

- There are three parts in this exam and each part has three problems. You should complete two (and only two) problems of your choice in each part. Each problem is worth 10 points.
- Support each answer with a complete argument. State completely any definitions and basic theorems that you use.
- This is a closed book test. You may only use the test and papers to write on. All other material is prohibited. Write your answer to each problem separately and write the last four digits of your ID number on the top of the first page of each answer.
- The time for the exam is 3 hours.

Part I

1. Let  $f : X \rightarrow Y$  be a continuous map. The graph of  $f$  is

$$\Gamma_f = \{(x, y) \in X \times Y; y = f(x)\} \subset X \times Y.$$

Show that  $\Gamma_f$  with subspace topology is homeomorphic to  $X$ .

2. A subset  $A$  of a topological space  $X$  is called *everywhere dense* or simply *dense* if  $\bar{A} = X$ . A subset  $B$  of  $X$  is called *nowhere dense* if  $X \setminus \bar{B}$  is dense.

- (a) Prove that  $B$  is nowhere dense iff for every non-empty open subset  $U$ , there is a non-empty open subset  $V \subset U$  such that  $V \cap B = \emptyset$ .
- (b) Let  $f : \mathbb{R}^1 \rightarrow \mathbb{R}^2$  be an imbedding. Show that  $f(\mathbb{R}^1)$  is nowhere dense in  $\mathbb{R}^2$ .

3. Let  $(X, d)$  be a compact metric space and  $f : X \rightarrow X$  be an isometry, i.e.  $d(f(x_1), f(x_2)) = d(x_1, x_2)$  for all  $x_1, x_2 \in X$ , then  $f$  must be surjective.

Part II

4. Use van Kampen Theorem to calculate  $\pi_1(\mathbb{R}P^2)$ .



5. Suppose  $\tilde{X}$  is the universal covering space of a space  $X$ . Show that if  $X$  is compact and its fundamental group is finite, then  $\tilde{X}$  is also compact.
6. Let  $X$  be a path-connected space. Show that the 0-th singular homology group  $H_0(X; \mathbb{Z})$  is isomorphic to  $\mathbb{Z}$ .

### Part III

7. Identify  $\mathbb{R}^4$  with  $\mathbb{C}^2$  so that  $(x, y, u, v) \in \mathbb{R}^4$  corresponds to  $(z_1, z_2) \in \mathbb{C}^2$  for  $z_1 = x + \mathbf{i}y$  and  $z_2 = u + \mathbf{i}v$ . Let  $V$  be the zero locus of the polynomial  $P(z_1, z_2) = z_1^3 + z_2^2$ , i.e.

$$V = \{(z_1, z_2); z_1^3 + z_2^2 = 0\} \subset \mathbb{R}^4.$$

Show that  $V \setminus \{(0, 0)\}$  is a 2-dimensional smooth manifold.

8. Calculate the de Rham cohomology  $H_{DR}^1(S^1)$  directly from its definition.
9. Let  $(x, y)$  be a Cartesian coordinate system of  $\mathbb{R}^2$  and  $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$  be the vector fields on  $\mathbb{R}^2$  associated with this coordinate system. For a constant  $2 \times 2$  matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , we define a vector field

$$V_A = (ax + by) \frac{\partial}{\partial x} + (cx + dy) \frac{\partial}{\partial y}.$$

For two constant  $2 \times 2$  matrices  $A$  and  $B$ , verify

$$[V_A, V_B] = -V_{[A, B]}.$$

UCR, Department of Mathematics  
Topology Qualifying Exam  
January 2000

- There are three parts in this exam and each part has three problems. You should complete two (and only two) problems of your choice in each part. Each problem is worth 10 points.
- Support each answer with a complete argument. State completely any definitions and basic theorems that you use.
- This is a closed book test. You may only use the test, something to write with, and the blue books. All other material is prohibited. Write each of your solutions in a separate blue book. Write your ID number on the cover of every blue book that you use.
- The time for the exam is 3 hours.

Part I

1. Let  $f, g : X \rightarrow Y$  be two continuous maps and  $Y$  is a Hausdorff space. Show that the subset

$$E = \{ x \in X ; f(x) = g(x) \}$$

is closed.

2. Let  $(X, d)$  be a compact metric space and  $f : X \rightarrow X$  is a continuous map. If there is a constant  $\alpha$ ,  $0 \leq \alpha < 1$ , such that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for every  $x, y \in X$ , show that there is a unique point  $x \in X$  such that  $f(x) = x$ .

3. Let  $X$  be a compact Hausdorff space that is connected and contains more than one point. Prove that  $X$  is uncountable.

Part II

4. Let  $X = S^2 \cup D^2$ , where

$$S^2 = \{ (x, y, z) \in \mathbb{R}^3 ; x^2 + y^2 + z^2 = 1 \}$$

and

$$D^2 = \{ (x, y, 0) \in \mathbb{R}^3 ; x^2 + y^2 \leq 1 \}.$$

Calculate the homology groups of  $X$ .

5. Let  $U$ ,  $V$  and  $W$  be open convex subsets of  $\mathbb{R}^2$  such that  $U \cap V$  and  $V \cap W$  are nonempty but  $U \cap W$  is empty.

- (1) Prove that  $U \cup V \cup W$  is connected.
- (2) What is the fundamental group of  $U \cup V \cup W$ ?
- (3) Give an example to show that your answer of (2) is false if all three pairwise intersections are nonempty.

6. Let  $f : S^2 \rightarrow S^2$  be a continuous map, and assume that it has non-zero degree. Prove that  $f$  is onto.

### Part III

7. Prove that the special linear group

$$SL(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

is a smooth manifold. What is its dimension?

8. Think of  $S^1 = \{ e^{i\theta} \in \mathbb{C} ; \theta \in [0, 2\pi) \}$  as a smooth manifold.

- (1) Prove that  $d\theta$  is a well defined 1-form on  $S^1$ .
- (2) A general 1-form  $\omega$  on  $S^1$  can be written as  $\omega = f(\theta) d\theta$ , where  $f(\theta)$  is a smooth function and  $f(\theta + 2\pi) = f(\theta)$ . Show that such a 1-form  $\omega$  is closed if and only if

$$\int_0^{2\pi} f(\theta) d\theta = 0.$$

- (3) Use (2) to show that the de Rham cohomology  $H^1(S^1) \cong \mathbb{R}$ .

9. Let  $X = \{(0, y) \in \mathbb{R}^2 \mid -\infty < y < \infty\}$  and  $Y = \{(x, \sin 1/x) \in \mathbb{R}^2 \mid 0 < x < \infty\}$ . Show that  $X \cup Y$  is an immersed submanifold of  $\mathbb{R}^2$  but not an imbedded submanifold. Show that  $Y$  is an imbedded submanifold.

UCR, Math Dept

Topology Qualifying Exam, 1998

TIME: 3 hours

This is a closed book test. Write each of your solutions in a separate blue book. This test has three parts. Do two and only two problems from each part. Each problem is worth 10 points.

You may only use the test, something to write with, and the blue books. All other material is prohibited.

Support each answer with a complete argument. State completely any definitions and basic theorems that you use.

Part 1: Do two (and only two) of the following three problems:

1. Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a continuous map such that

$$\lim_{|x| \rightarrow \infty} |f(x)| = \infty$$

and let  $\bar{f}$  be the extension of  $f$  to one point compactifications such that  $\bar{f}(\infty_m) = \infty_n$  (where  $\infty_k$  denotes the point at infinity in the one point compactification of  $\mathbb{R}^k$ ). Prove that  $\bar{f}$  is continuous.

2. Let  $X$  and  $Y$  be topological spaces, let  $f : X \rightarrow Y$  be continuous, onto and closed, let  $R$  be the equivalence relation on  $X$  that is given by

$$(u, v) \in R \Leftrightarrow f(u) = f(v)$$

let  $X/R$  be the associated set of equivalence classes with the quotient topology, and let  $g : X/R \rightarrow Y$  be the well defined map

$$g(CLASS[x]) = f(x).$$

Prove that  $g$  is a homeomorphism.. [ It is not necessary to show that  $R$  is an equivalence relation or that  $g$  is well defined. Explain why it suffices to prove that  $g$  is continuous, 1-1, onto, and closed, and prove these things.]

3. Recall that if  $X$  is a set, then the diagonal of  $X$  is

$$\Delta(X) \equiv \{(x, x) \in X \times X \mid x \in X\}.$$

Let  $X$  be a topological space. Show that  $X$  is Hausdorff if and only if  $\Delta(X)$  is closed in  $X \times X$ .

**Part 2: Do two (and only two) of the following three problems:**

4. (a) Let  $X, Y$  be compact Hausdorff spaces and  $p : Y \rightarrow X$  is a covering map. Show that  $p^{-1}(x)$  contains only finitely many points for every  $x \in X$ .

(b) Suppose  $X$  is a compact Hausdorff space whose universal cover is  $S^n$ . Show that the fundamental group of  $X$  is finite. (Hint: Use part a.)

5. Let  $B$  be the set of all rays in the first octant of the 3-space (the first octant is  $\{(x, y, z) \in \mathbb{R}^3; x \geq 0, y \geq 0, z \geq 0\}$ ).

(a) Show that  $B$  is homeomorphic to a disk,  $D^2$ .

(b) Show that any  $3 \times 3$  matrix with positive entries has a positive eigenvalue.

*Hint for part (b):* Observe that the linear map  $v \mapsto Av$  induces a continuous map  $B \rightarrow B$ , where  $B$  is as in part (a).

6. Show that any finitely presented group is the fundamental group of a topological space.

**Part 3: Do two (and only two) of the following three problems:**

7. Prove that a  $C^\infty$  map  $\phi : S^2 \rightarrow S^1$  can not be one-to-one.

8. Is every vector field on the real line complete?

9. (a) Consider the 2-form

$$\omega = \frac{x dy \wedge dz - y dx \wedge dz + z dx \wedge dy}{r},$$

on  $R^3 \setminus 0$ , where  $r = \sqrt{x^2 + y^2 + z^2}$ . Show that  $\omega$  is not exact.

(b) Regarding  $r$  as a function on  $R^3 - 0$ , show  $dr \wedge \omega = dx \wedge dy \wedge dz$ .

## Topology Qualify Exam (1997)

### Instructions:

- (1) Solve each problem in a separate blue book. You may only turn in 6 proposed solutions with the full credit of 60 points.
- (2) Support each answer with a complete argument. State completely definitions and basic theorems used in your argument.

• *Work out two of the following three problems:*

1. Recall that a *subbasis*  $\mathcal{S}$  for a topology on  $X$  is a collection of subsets whose union equals  $X$ . The topology generated by a subbase  $\mathcal{S}$  is defined to be the collection  $\mathcal{T}$  of all unions of finite intersections of elements in  $\mathcal{S}$ . Now let  $\mathcal{S}$  the collection of all infinite subsets of  $\mathbb{R}$ .

(a) Show that  $\mathcal{S}$  is a sub-basis for a topology on  $\mathbb{R}$ . (2 points)

(b) What is the topology generated by  $\mathcal{S}$ ? (It is a familiar one.) (8 points)

2. (a) Provide an example of a quotient map which is not open. (5 points)

(b) Provide an example of a quotient map which is not closed. (5 points)

3. Show that the following two different metrics on  $\mathbb{R}^n$  (where  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$ ):

$$d_2(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$
$$d_\infty(x, y) = \max\{|x_i - y_i|; i = 1, \dots, n\}$$

define the same topology on  $\mathbb{R}^n$ . (10 points)

• *Work out two of the following three problems:*

4. What is the fundamental group of  $\mathbb{R}P^2 \setminus \{\text{pt}\}$ ? (10 points)

5. Is the map  $f : (0, 3) \rightarrow S^1$  defined by  $f(x) = e^{2\pi i x}$  a covering map? (10 points)

6. Show that any continuous map  $f : S^1 \rightarrow S^1$  which is not onto has a fixed point, i. e. a point  $z \in S^1$  such that  $f(z) = z$ . (10 points)

• *Work out two of the following three problems:*

7. Prove that the de Rham cohomology group  $H^1(\mathbb{R}^2 \setminus \{(0, 0)\}) \neq 0$ . (10 points)

8. Let  $M$  be a compact orientable differentiable manifold without boundary. Prove that  $M$  is not smoothly contractible. (10 points)

9. Recall that a differentiable vector field  $X$  on a differentiable manifold  $M$  may be described as a  $\mathbb{R}$ -linear mapping  $X : \mathcal{D} \rightarrow \mathcal{D}$ , where  $\mathcal{D}$  is the set of differentiable functions on  $M$ , satisfying the Leibniz rule  $X(fg) = fX(g) + X(f)g$ .

(a) Let  $X, Y$  be two differentiable vector fields on a differentiable manifold  $M$ . Is  $XY$ , defined to be  $(XY)(f) = X(Y(f))$ , a vector field in general? If yes, give a proof. If no, give a counter example. (5 points)

(b) Let

$$X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$$
$$Y = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}$$

in  $\mathbb{R}^3$  with coordinates  $(x, y, z)$ . Compute  $[X, Y]$ . (5 points)



You may answer any 2 of problems 1 - ~~4~~<sup>3</sup>

1. Show that in a Hausdorff space, a compact subset is closed.
2. Let  $X$  and  $Y$  be compact spaces. Prove that  $X \times Y$  is a compact space.
3.
  - a. Let  $A, B$  be subsets of a space  $X$ . If there is a homeomorphism  $h : X \rightarrow X$  such that  $h(A) = B$ , then the closures of  $A$  and  $B$ ,  $\bar{A}$  and  $\bar{B}$ , are homeomorphic.
  - b. Consider the graph  $A$  of the function  $y = \sin(1/x)$  for  $x > 0$  in  $\mathbb{R}^2$ . Let  $B$  be the positive half of the  $x$ -axis. Show that  $A$  and  $B$ , with subspace topology, are homeomorphic.
  - c. Use <sup>(a)</sup>~~(1)~~ to prove that for the subsets  $A$  and  $B$  of  $\mathbb{R}^2$  in <sup>(b)</sup>~~(2)~~, there is no homeomorphism  $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $h(A) = B$ .
  - d. We may think of <sup>(2)</sup>~~(2)~~ as exhibiting such a phenomenon that the real line  $\mathbb{R}^1$  can be embedded in the plane in at least two different ways. Do you think that a similar phenomenon might happen if the real line is replaced by a closed interval? (Just tell your intuition.)

(problem 4 doesn't exist)

You may answer any 2 of problems 5 - 8

5. Calculate the fundamental group of the space obtained from the torus  $S^1 \times S^1$  by identification of any two distinct points.
6. Let  $X$  be a topological space, which can be presented as a union of open connected sets  $U$  and  $V$ . Prove that:
  - a. if  $U \cap V$  is disconnected then  $X$  has a connected infinite-fold covering space, and
  - b. if  $U$  and  $V$  are simply-connected and  $U \cap V$  is a union of two simply-connected disjoint open sets, then  $\pi_1(X)$  is an infinite cyclic group.
7. Calculate the fundamental group of  $\mathbb{R}^4 \setminus T$ , where  $T$  is the torus  $S^1 \times S^1$  standardly embedded into  $\mathbb{R}^3 \subset \mathbb{R}^4$ .
8. Prove that any two continuous maps  $S^2 \rightarrow S^1 \times S^1$  are homotopic.

You may answer any 2 of problems 9 – 11.

10 9. Let  $M$  be a differentiable manifold. Prove that there exists a proper differentiable map  $f : M \rightarrow \mathbb{R}^N$ , for any positive integer  $N$ .

10 10. (a) Suppose  $w$  is a exact  $n$ -form on a closed orientable  $n$ -dimensional differentiable manifold  $M$ . Compute  $\int_M w$  (give reasons).  $\frac{3}{10}$

(b) Let  $M$  be a (connected) differentiable manifold. Compute  $H^0(M)$  (Give reasons).  $\frac{3}{10}$

(c) Compute  $H_c^1(\mathbb{R}^1)$  (Give reasons).  $\frac{4}{10}$

10 11. (a) Classify all two dimensional orientable closed manifolds. Describe their fundamental groups and universal covering spaces.  $\frac{5}{10}$

(b) Let  $M$  be an  $n$ -dimensional closed differentiable manifold. Prove that there exists no immersion from  $M$  into  $\mathbb{R}^n$ .  $\frac{5}{10}$

Remark: In problems 2 and 3, closed = compact without boundary.

10 11

# Topology Qualifying Examination

WEDNESDAY, JANUARY 17, 1996

Department of Mathematics, UCR

Name \_\_\_\_\_

1. Show that if  $X$  is a compact Hausdorff space and if  $x, y \in X$  are in different components of  $X$ , then there are disjoint open sets  $U, V \subset X$  with  $x \in U, y \in V$  and  $U \cup V = X$ .
2. Let  $(X_i)_{i \in I}$  be a family of topological spaces. Show that the Cartesian product  $\prod_{i \in I} X_i$  is a regular Hausdorff space if and only if each factor  $X_i$  is a regular Hausdorff space.
3. Show that if  $f : X \rightarrow Y$  is a continuous map from a compact metric space  $X$  into a Hausdorff space  $Y$ , the  $f(X)$  is metrizable.  $f(X)$
4. Let  $f : S^2 \rightarrow S^2$  be a continuous map, where  $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ . Show that if  $f$  is not onto, then  $f$  is homotopic to a constant map.
5. (a) Prove that  $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$  is not a retract of the closed disk  $V = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ .  
(b) Assuming the truth of (1), prove that any continuous map  $f : V \rightarrow V$  has a fixed point.
6. Find the fundamental groups of each of the following spaces:
  - (a) The 2-dimensional torus  $T^2$ .
  - (b) The doubly-punctured plane  $\mathbb{R}^2 \setminus \{(1, 0), (-1, 0)\}$ .
  - (c) The pinched torus  $(S^1 \times S^1) / (S^1 \times \{1\})$ .
7. (a) State the definition of a topological manifold.  
(b) Is the figure  $\infty$  a topological manifold? Give the proof of your answer.  
(c) Let  $M$  be a topological manifold with fundamental group  $\pi_1(M) = \mathbb{Z}$  (the integers). Prove that the universal covering space  $\tilde{M}$  must be non-compact.
8. (a) State Whitney's embedding theorem.  
(b) Let  $M$  be an  $n$ -dimensional compact differentiable manifold without boundary. Prove that there exists an immersion  $f : M \rightarrow \mathbb{R}^k$  for sufficiently large  $k > n$ .
9. (a) Let  $w = a dx + b dy + c dz$  be a one-form in  $\mathbb{R}^3$ . What are necessary and sufficient conditions that  $w$  is closed?  
(b) Let  $w$  be an exact  $n$ -form on  $S^n$ . Compute  $\int_{S^n} w$ . Give reasons for your answer!

# Topology Qualifier

Department of Mathematics  
University of California, Riverside

September 3, 1994

Work seven out of the following nine problems.

$T_3 =$  regular Hausdorff

1. Let  $(X_i)_{i \in I}$  be a family of topological spaces. Show that the Cartesian product  $\prod_{i \in I} X_i$  is a  $T_3$ -space if and only if each  $X_i$  is a  $T_3$ -space.
2. True or false: Let  $X$  be a topological space, and let  $\alpha : X \rightarrow Y$  be a quotient map. Assume that for each  $y \in Y$  the preimage  $\alpha^{-1}(y)$  is homeomorphic to a topological space  $Z$ . Then  $X$  is homeomorphic to  $Y \times Z$ .
3. Quotients of locally compact spaces.
  1. The continuous image of a locally compact space need not be locally compact. (Let  $X$  be the plane, and let  $A$  be the x-axis. Then  $X/A$  is not locally compact.)
  2. The closed continuous image of a locally compact space is locally compact, provided that the preimage of each point is compact (so the non-compactness of  $A$  was needed in part 1)
  3. The condition of 2 is not necessary.
4. Determine the fundamental groups and the universal covers of the following spaces (no proof is required):
  1.  $S = \{(x, y) \in \mathbb{R}^2 : 4x^2 + y^2 = 1\}$
  2.  $Y = D \setminus \{\text{five distinct points}\}$ , where  $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
  3.  $M_3 =$  a compact orientable two dimensional manifold (without boundary) with genus equal to three.

*connected sum*

- 6 4.  $X = T^2 + S^2$ , where  $T^2 = S^1 \times S^1 =$  torus,  $S^2 =$  the two dimensional sphere,  $S^1 =$  unit circle.
- 6 5.  $P =$  the projective plane.
5. Let  $X = A \cup B$ , where  $A =$  a space homeomorphic to  $\mathbb{R}^3$ ,  $B =$  a space homeomorphic to  $S^1 \times \mathbb{R}^2$ , where  $S^1$  is the unit circle, and  $A \cap B = S^2$  (i.e. the two dimensional sphere). Determine the fundamental group of  $X$ . Present your proof (or argument) in detail.
6. 1. Prove that  $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$  is not a retract of the closed unit disc  $V = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ .
2. Assume the truth of (1), prove that any continuous map  $f : V \rightarrow V$  has a fixed point.
7. Prove that, besides a single exception, the real projective space is orientable if and only if its dimension is odd. What is the exception?
8. Let  $f : S^2 \rightarrow \mathbb{R}^2$  be a smooth map and  $\omega$  be a differential form of degree 2 on  $\mathbb{R}^2$ .
1. Prove that  $\int_{S^2} f^*\omega = 0$ .
2. Prove that exists a point  $x \in S^2$  where  $f^*\omega$  is equal to zero.
9. Prove that an immersion  $X \rightarrow Y$ , where  $X$  and  $Y$  are smooth closed connected manifolds of the same dimension, is a covering projection. Is the condition that the manifolds are closed necessary? If yes, give an example proving this. If no, prove the statement without using the condition.

QUALIFYING EXAMINATION  
TOPOLOGY 1993

1. Let  $U, V$  be subsets of a space  $X$  such that  $X = U \cup V$  and both  $X$  and  $U \cap V$  are connected. Prove that if  $U$  and  $V$  are open then they are connected. Find an example which shows that the latter condition is necessary.

2. Let  $A$  be a topological space. Fill the following table with pluses and minuses according to your answers to the corresponding questions.

If $A$ is:	connected	Hausdorff	non-Hausdorff	separable	compact	noncompact	second countable
Has $X$ the same property, if:							
$X \subset A$							
$X$ is an open subset of $A$							
$X$ is a closed subset of $A$							
$A$ is dense in $X$							
$X$ is a quotient space of $A$							
$X$ is an open subset of $\mathbb{R}^n$							

3. Find the fundamental group of each of the following spaces:

- (1) The 2-dimensional torus  $T^2$ .
- (2) The doubly-punctured plane  $\mathbb{R}^2 \setminus \{(1, 0), (-1, 0)\}$ .
- (3) The pinched torus  $(S^1 \times S^1)/S^1 \times \{1\}$ .

4. Classify three-point spaces up to homotopy equivalence. (What are the homotopy types of topological spaces consisting of 3 points?)

5. Prove:

- (1)  $A$  is a retract of  $X$  if and only if, for every space  $Y$ , each continuous function  $f : A \rightarrow Y$  admits a continuous extension  $F : X \rightarrow Y$ .
- (2) Let  $X$  be deformable into  $A$ . Then for every space  $Y$ , extensions  $F, G : A \rightarrow Y$  of homotopic continuous functions  $f, g : A \rightarrow Y$  are homotopic.

6. Let  $X$  be a space that is the union of two open simply-connected subspaces  $U$  and  $V$ , whose intersection is path-connected and has fundamental group  $G$ . Calculate the fundamental group of  $X$ .

7. Prove that there is no convex polyhedron all <sup>of</sup> whose faces are 6-gons.

8. Let  $X$  be a smooth compact oriented manifold of dimension  $n$ . Prove the following three statements.

(1) There exists a differential form  $\omega$  of degree  $n - 1$  on  $\partial X$  with  $\int_{\partial X} \omega \neq 0$ .

(2) For any smooth map  $\rho : X \rightarrow \partial X$  and differential form  $\omega$  of degree  $n - 1$  on  $\partial X$

$$d\rho^*\omega = 0.$$

(3) If there exists a smooth retraction  $\rho : X \rightarrow \partial X$  then  $\int_{\partial X} \omega = 0$  for any differential form  $\omega$  of degree  $n - 1$  on  $\partial X$ .

How are these three statements related to the Borsuk theorem?

**Qualifying Examination in Topology**  
September 18, 1992

Name:
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1	2	3	4	5	6	7	8	$\Sigma$	Result

1. Prove that if  $U, V$  are open sets and  $U \cup V, U \cap V$  are connected, then  $U$  and  $V$  are connected. Find an example which shows that for non-open sets this is not true.

2. Prove that there is no immersion of sphere  $S^2$  into plane  $\mathbb{R}^2$ .

3. Prove that projective line is not a retract of projective plane.



4. Consider sphere  $S^2$  (which is defined in  $\mathbb{R}^3$  by equation  $x^2 + y^2 + z^2 = 1$ ) and the unit upward vector at point  $(1, 0, 0)$ . Find coordinates of it in the coordinate system defined by the stereographic projection of  $S^2 \setminus (0, 0, 1)$  onto plane  $z = 0$ .

5. Let  $f$  be a map  $\mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by formula

$$f : (x, y, z) \mapsto (x^2 - y^2, 2xy, z^2).$$

Find

$$d_{(x_0, y_0, z_0)} f (x\partial/\partial x + y\partial/\partial y + xy\partial/\partial z)$$

and

$$(d_{(x_0, y_0, z_0)} f) * (xz dx \wedge dy).$$

6. Let  $\mathcal{F}$  be the set of all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  (not necessarily continuous). Give  $\mathcal{F}$  the topology induced by the basis consisting of all subsets  $B_{E, \epsilon}(f)$ , where  $f \in \mathcal{F}$ , where  $E \subset \mathbb{R}$  is a finite set, and where  $\epsilon > 0$ , and

$$B_{E, \epsilon} = \{g \in \mathcal{F} : |g(x) - f(x)| < \epsilon \forall x \in E\}.$$

Show that  $\mathcal{F}$  is a separable (i.e. has a countable dense subset) topological space.

7. Let  $(\mathbb{R}, \tau)$  be the set of real numbers, equipped with the usual topology, and let  $(\mathbb{R}, \sigma)$  be the Sorgenfrey line.

- a) Is the set of all rational numbers a retract of  $\mathbb{R}$ ?
- b) Is the set of all integers a retract of the set of all rational numbers?
- c) Is the Sorgenfrey line  $(\mathbb{R}, \sigma)$  a retract of  $(\mathbb{R}, \tau)$ ?
- d) Is  $(\mathbb{R}, \tau)$  a retract of the Sorgenfrey line?
- e) If  $A$  is a retract of  $X$ , is  $\beta A$  a retract of  $\beta X$ . ( $\beta A$  and  $\beta X$  are the Stone-Cech compactifications of  $A$  and  $X$ , respectively.)

8. If  $X$  is a compact  $T_2$  separable space. Is  $X$  a quotient of the Stone-Cech compactification of the rational numbers?

## Qualifying Examination

Topology  
Spring 1991

1. Let  $\mathcal{F}$  be the set of all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  (not necessarily continuous). Give  $\mathcal{F}$  the topology induced by the basis consisting of all subsets  $B_{E,\epsilon}(f)$ , where  $f \in \mathcal{F}$ , where  $E \subset \mathbb{R}$  is a finite set, and where  $\epsilon > 0$ , and

$$B_{E,\epsilon} = \{g \in \mathcal{F} : |g(x) - f(x)| < \epsilon \forall x \in E\}.$$

Show that  $\mathcal{F}$  is a separable (i.e. has a countable dense subset) topological space.

2. Let  $X = (\mathbb{R}, \sigma)$  be the Sorgenfrey line, i.e. the topology on  $X$  has the intervals of the form  $[r, s) = \{x \in \mathbb{R} : r \leq x < s\}$  as a basis. Show that  $X$  is a  $T_4$  topological space, but  $X \times X$  is not  $T_4$ . In  $X$ , let  $A$  be the union of all lines with slope 1 passing through the  $y$ -axis in points with irrational coordinates. Is  $A$   $T_4$ ?
3. Suppose that  $(A, \sigma)$  and  $(X, \tau)$  are topological spaces.  $A$  is said to be a retract of  $X$  if there is a pair of mappings  $i : A \rightarrow X$  and  $r : X \rightarrow A$  such that  $i$  is an embedding,  $r$  is a continuous map, and  $r \circ i = id_A$ . - Let  $(\mathbb{R}, \tau)$  be the set of real numbers, equipped with the usual topology, and let  $(\mathbb{R}, \sigma)$  be the Sorgenfrey line.
  - a. Is the set of all rational numbers a retract of  $\mathbb{R}$ ?
  - b. Is the set of all integers a retract of the set of all rational numbers?
  - c. Is the Sorgenfrey line  $(\mathbb{R}, \sigma)$  a retract of  $(\mathbb{R}, \tau)$ ?
  - d. Is  $(\mathbb{R}, \tau)$  a retract of the Sorgenfrey line?
  - e. If  $A$  is a retract of  $X$ , is  $\beta A$  a retract of  $\beta X$ ? ( $\beta A$  and  $\beta X$  are the Stone-Cech compactifications of  $A$  and  $X$ , respectively.)

4. If  $X$  is a compact  $T_2$  separable space. Is  $X$  a quotient of the Stone-Cech compactification of the rational numbers?
5. Let  $X$  be a metric space, and let  $\phi : X \rightarrow Y$  be a continuous, open map onto a Hausdorff space  $Y$ . Assume that  $\phi^{-1}(y)$  is compact for every  $y \in Y$ . Show that  $Y$  is a metrizable space.
6. Let  $X = [0, 1] \times [0, 1]$  the unit square, equipped with the usual topology. Define an equivalence relation  $\Theta$  on  $X$  by

$$\begin{aligned} ((x, r), (y, s)) \in \Theta &\Leftrightarrow (x, r) = (y, s) \text{ or} \\ &x = 0, y = 1, r = 1 - s \text{ or} \\ &x = 1, y = 0, r = 1 - s. \end{aligned}$$

Let  $M = X/\Theta$ , equipped with the quotient topology. (The space  $M$  is also called the Möbius band.)

- a. Show that  $M$  is a compact Hausdorff space.
- b. Show that  $M$  is homotopically equivalent to the circle  $S^1$ , and hence  $\pi_1(M) = \mathbb{Z}$ .
- c. Let  $\phi : S^1 \rightarrow M$  be given by

$$\phi(e^{2\pi i \phi}) = \begin{cases} [(2\phi, 1)]\Theta & 0 \leq \phi \leq 1/2; \\ [(2\phi - 1, -1)]\Theta & 1/2 \leq \phi \leq 1. \end{cases}$$

Show that  $\phi$  is a well-defined continuous map. Also, show that if  $\pi_1(S^1)$  and  $\pi_1(M)$  are identified with the integers, then  $\pi_1(\phi) : \pi_1(S^1) \rightarrow \pi_1(M)$  is given by  $\pi_1(\phi)(z) = u \cdot z$ , where  $u$  is equal to  $\pm 2$ .

7. As in (6), let  $M$  be the Möbius band, and let  $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$  be the closed disk. Let  $X = M \cup_{\partial M = \partial D} D$  be the space obtained by identifying the boundary circle of  $M$  with that of  $D$ . Compute the singular homology groups  $H_1(X)$  and  $H_2(X)$ .

Qualifying Examination

Topology, Spring 1989

1. Show that if  $X$  is a compact Hausdorff space and if  $x, y \in X$  are in different components of  $X$ , then there is a separation  $X = U \cup V$  with  $x \in U$  and  $y \in V$ .

Qualifying Examination

Topology, Spring 1989

2. Suppose that  $X$  is a subset of  $\mathbb{R}^2$  ( $\mathbb{R}^2$  has its usual topology).
- (i) Suppose that both projections  $\pi_1(X), \pi_2(X) \subseteq \mathbb{R}$  are connected. Need  $X$  be connected?
  - (ii) If  $X$  is connected, need  $\overline{X} \setminus X$  be connected, too?

Qualifying Examination

Topology, Spring 1989

3. (a) Define the terms sequence, subsequence, net, and subnet.
- (b) Show that not every subnet of a sequence needs to be a subsequence.
- (c) Prove that the Stone-Cech compactification of the natural numbers (with the discrete topology) has cardinality greater than  $c$ .

Qualifying Examination

Topology, Spring 1989

4. Let  $\mathbb{R}^2$  have its usual topology. A subset  $A \subseteq \mathbb{R}^2$  is said to have a *countable local base* if there is a countable family of open sets  $\{V_1, V_2, \dots\}$  such that if  $V \subseteq \mathbb{R}^2$  is an open set and  $A \subseteq V$ , then there is a positive integer  $i$  such that  $A \subseteq V_i \subseteq V$ .

Determine which of the following subsets of  $\mathbb{R}^2$  have countable local bases. This entails either exhibiting one or showing that one cannot be found. Let  $\mathbf{N} = \{1, 2, 3, \dots\}$ .

- (i) a compact subset of  $\mathbb{R}^2$ ;
- (ii)  $\mathbf{N} \times \{0\}$ ;
- (iii)  $X = \{(1/n, 0) : n \in \mathbf{N}\}$ .

Qualifying Examination

Topology, Spring 1989

5. Show: If  $f : X \rightarrow Y$  is a continuous map from a compact metric space  $X$  into a Hausdorff space  $Y$ , then  $f(X)$  is metrizable.

Qualifying Examination

Topology, Spring 1989

6. Prove or disprove: Let  $X$  be a compact Hausdorff space, and let  $U_n \subseteq X$  be a decreasing sequence of open subsets such that  $\bigcap_{0 \leq n} U_n = \{x_0\}$ . Then  $(U_n)_n$  is a neighborhood basis at  $x_0$ .

7. Let  $C$  be the Cantor set.

(a) Let  $X$  be a totally disconnected, compact Hausdorff space, and let  $A \subseteq X$  be a closed subspace of  $X$ . Show that if  $U \subseteq A$  is open and closed in  $A$ , then there is an open and closed subset  $V \subseteq X$  such that  $U = V \cap A$ .

(b) Show that for every totally disconnected, compact Hausdorff space  $X$ , every closed subset  $A \subseteq X$  and every continuous map  $\varphi : A \rightarrow C$  there is a continuous extension  $\psi : X \rightarrow C$  such that  $\psi|_A = \varphi$ .

(c) Conclude that if  $\iota : C \rightarrow X$  is an embedding of  $C$  into a totally disconnected compact Hausdorff space  $X$ , then there is a retraction  $\rho : X \rightarrow C$  such that  $\rho \circ \iota = \text{id}_C$ .

(d) Show that the dual of (b) is not true: If  $\rho : X \rightarrow C$  is a continuous surjection, then there is not necessarily an embedding  $\iota : C \rightarrow X$  such that  $\rho \circ \iota = \text{id}_C$ .

8. (a) Show that the unit interval  $[0, 1]$  has the fixed point property (i.e. for every continuous  $f : [0, 1] \rightarrow [0, 1]$  there is a point  $x \in [0, 1]$  with  $f(x) = x$ ).

(b) Show that the "letter T" has the fixed point property. The "letter T" is defined as  $T = [0, 1] \times \{0\} \cup \{1/2\} \times [0, 1] \subseteq \mathbb{R}^2$ .

9. Let  $f, g : X \rightarrow Y \times Z$  be two continuous maps, and let  $\pi_1 : Y \times Z \rightarrow Y$  and  $\pi_2 : Y \times Z \rightarrow Z$  be the projection maps. Show that  $f$  is homotopic to  $g$  if and only if  $\pi_i \circ f$  is homotopic to  $\pi_i \circ g$  for  $i = 1, 2$ .

10. Compute the first fundamental group of (a) the torus  $S_1 \times S_2$ , and (b) the Moebius band.

Qualifier: Topology (1984)

June 7, 1984

① Let  $X$  be a topological space. A set  $U \subseteq X$  is called a regular open set if  $U$  is equal to the interior of its closure.

Show:

(a) For every open set  $V \subseteq X$  there exists a smallest regular open set  $U$  such that  $V \subseteq U$ .

(b) Is the collection of all regular open sets closed under finite intersections? arbitrary intersections? finite unions? arbitrary unions? Give proofs or counterexamples!

② Show: (a) An additive subgroup of the reals which contains more than one member is either dense in  $\mathbb{R}$  or has a smallest positive element.  
(b) A closed subgroup of  $\mathbb{R}$  is either equal to  $\mathbb{R}$ ,  $\{0\}$ , or isomorphic to  $\mathbb{Z}$ .

③ Let  $Y$  be completely regular and let  $f: Y \rightarrow \mathbb{R}$  be lower semicontinuous (i.e. for each  $\varepsilon \in \mathbb{R}$  the set  $\{y: f(y) > \varepsilon\}$  is open.) Show that  $f$  is the pointwise supremum of continuous functions  $g: Y \rightarrow \mathbb{R}$ .

Conversely, show that if  $Y$  is any space such that each lower semicontinuous function is a supremum of continuous functions, then  $Y$  is completely regular.

④ (Connectedness) Let  $n \geq 2$

(a) If  $A = \{x \in \mathbb{R}^n: \text{all coordinates of } x \text{ are rational}\}$ , then  $\mathbb{R}^n \setminus A$  is connected.

(b) If  $B = \{x \in \mathbb{R}^n: \text{at least one coordinate of } x \text{ is rational}\}$  then  $\mathbb{R}^n \setminus B$  is not connected.

⑤ Show: If for every closed subset  $A \subseteq X$  of a topological space  $X$  and for every continuous map  $f: A \rightarrow \mathbb{R}$  there is a continuous extension  $\bar{f}: X \rightarrow \mathbb{R}$  of  $f$ , then  $X$  is normal (The converse is also true and known as Tietze's Extension Theorem)



⑥ Let  $I = [0, 1]$  be the unit interval. Which of the following subspaces of  $I^I$  is compact in the topology of pointwise convergence?

a)  $\{f \in I^I \mid f(0) = 0\}$

b)  $\{f \in I^I \mid f \text{ is continuous and } f(0) = 0\}$

c)  $\{f \in I^I \mid f \text{ is differentiable and } |f'(x)| \leq 1 \forall x \in I\}$

⑦ Let  $\beta\mathbb{N}$  denote the Stone-Čech compactification of the natural numbers. Show:

(a) Every compact metric space is a continuous image of  $\beta\mathbb{N}$

(b) Is  $\beta\mathbb{N}$  a continuous image of the Cantor set  $2^{\mathbb{N}}$ .

⑧ Find the homology groups of  $S^2 \times S^2$ , where  $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$