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Applied Math Qualifying Exam Fall 2017

Instructions: Work 2 out of 3 problems in each of the 3 parts for a total of 6 problems.

Part 1

(1) Let $(u_n)_{n=1}^\infty$ be a sequence of harmonic functions defined on an open bounded subset U of $\mathbb{R}^d$, $d \geq 2$, with each $u_n \in C^2(U)$. Assume that $u_n \to u$ uniformly on U. Prove that u is harmonic on U.

(2) Consider the transport equation,

$$\begin{cases} \partial_t f_j + \mathbf{u} \cdot \nabla f_j = 0 & \text{on } \mathbb{R} \times U, \\ f_j(0, x) = f_{0,j}(x) & \text{on } U, \end{cases}$$

for $j = 1, 2$. Here,

- U is a bounded open subset of $\mathbb{R}^d$, $d \geq 2$, having C^∞ boundary;
- $\mathbf{u}$ is a given time-independent vector field in $C^\infty(\overline{U})$ with $\mathbf{u} \cdot \boldsymbol{n} = 0$ on ∂U;
- $f_j = f_j(t, x)$, $j = 1, 2$, is a scalar-valued function of time and space;
- $f_{0,j}$, $j = 1, 2$, lie in $C(\overline{U})$;

You may assume the existence and uniqueness of solutions and the existence and uniqueness of a flow map for $\mathbf{u}$ without proof. (Both solutions and the flow map will be continuous in time and space.)

(a) Use an energy argument to prove that for all $t \geq 0$,

$$\|f_1(t) - f_2(t)\|_{L^2}^2 \\ \leq \|f_{0,1} - f_{0,2}\|_{L^2}^2 \exp \int_0^t \|\operatorname{div} \mathbf{u}(s)\|_{L^\infty} \, ds.$$

Here, the L^2-norm is defined by

$$\|h\|_{L^2}^2 = \int_U h(x)^2 \, dx.$$

(b) Using the flow map for $\mathbf{u}$ (or any other method you can come up with) prove that for all $t \geq 0$,

$$\|f_1(t) - f_2(t)\|_{L^\infty} \leq \|f_{0,1} - f_{0,2}\|_{L^\infty}.$$

(3) Let $\mathbf{v} \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a time-varying vector field. Assume that for some $M_1 > 0$, $\|\mathbf{v}(t)\|_{L^\infty} \leq M_1$ for all $t \in \mathbb{R}$ and for some $M_2 > 0$, $\mathbf{v}(t)$ has a Lipschitz constant no larger than M_2 for all $t \in \mathbb{R}$.

(a) Show that for any $(t_0, \mathbf{x}_0) \in \mathbb{R} \times \mathbb{R}^d$, solutions to

$$\begin{cases} \mathbf{x}'(t) = \mathbf{v}(t, \mathbf{x}(t)), \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

are unique. (You do not need to prove existence.)

(b) Define $\mathbf{Y} \colon \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ by

$$\mathbf{Y}(t_0, \mathbf{x}_0, t) = \mathbf{x}(t),$$

where $\mathbf{x}$ is the solution from (a). Prove that $\mathbf{Y}$ is continuous.

Part 2

(1) Let U be a bounded open set with smooth boundary ∂U. Consider the initial boundary value problem for $u(x,t)$:

$$\begin{cases} u_t - \Delta u + bu = f, & x \in U, t > 0, \\ u(x,0) = g(x), & x \in U, \\ u_t + \frac{\partial u}{\partial n} + u = 0, & x \in \partial U, t > 0, \end{cases}$$

where $\frac{\partial u}{\partial n}$ is the exterior normal derivative [and b is a constant]. Show that smooth solutions of this problem are unique.

(2) (a): Find an explicit solution to the problem:

$$\begin{cases} u_t - u_{xx} = \cos x, & x \in [0, 2\pi], t > 0, \\ u_x(0,t) = u_x(2\pi, t) = 0, & t > 0, \\ u(x,0) = \cos x + \cos 2x, & x \in [0, 2\pi]. \end{cases}$$

(Hint: consider $v(x,t) = u(x,t) - \cos x$, and employ the separation of variables to solve for v.)
(b): Does there exist a steady state solution to the equation in (a) with the boundary condition

$$u_x(0) = 1, \qquad u_x(2\pi) = 0?$$

Explain your answer.

(3) Find the solution of the partial differential equation

$$u_x + x^2 y u_y = -u,$$

with the condition $u(x = 0, y) = y^2$ using the method of characteristics.

Part 3

(1) Let U be a bounded domain in $\mathbb{R}^d$ with a C^∞ boundary, let $f \in L^2(U)$, and let $\mu > 0$ be a constant. Consider the Dirichlet problem,

$$\begin{cases} -\Delta u + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

(a) Define what it means for $u \in H_0^1(U)$ to be a weak solution to this Dirichlet problem.

(b) Show that a weak solution exists.

(2) Let U be a bounded domain in $\mathbb{R}^d$ with a C^∞ boundary. Assume that $u \in C^2(\overline{U}) \cap H_0^1(U)$ is a strong solution to

$$\begin{cases} \Delta u = u^3 + u & \text{in } U, \\ \qquad u = 0 & \text{on } \partial U. \end{cases}$$

Note that $u \equiv 0$ is clearly a solution, but this is a nonlinear problem, so we have no general uniqueness theorem that covers it.

(a) Use the weak maximum principle to show that $u \equiv 0$ is the only solution.

(b) Show the same thing using an energy method.

(3) (a) Prove that for any $u \in C^1(\mathbb{R}^d)$ and any $p \in (1, \infty)$,

$$\partial_j |u|^p = p|u|^{p-1} \partial_j u \operatorname{sgn}(u).$$

Here, the derivative is a *classical* derivative. Also, $\operatorname{sgn}\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\operatorname{sgn}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

(b) Prove that for any $u \in H^1(\mathbb{R}^d)$ having the property that $|u| > \epsilon$ for some $\epsilon > 0$,

$$\partial_j |u|^2 = 2|u| \partial_j u \operatorname{sgn}(u),$$

where now we mean the *weak* derivative. (This is the weak derivative version of part (a) specialized to $p = 2$.)

Comment: The assumption that $|u(x)| > \epsilon$ is not necessary, but may help you in dealing with the sgn function, should you choose to employ a sequence of smooth approximating functions and use the result in part (a) for that sequence.

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Applied Math Qualifying Exam 11 October 2014

Instructions: Work 2 out of 3 problems in each of the 3 parts for a total of 6 problems.

Part 1

(1) Let Ω be an open subset of $\mathbb{R}^d$ and let
$$C(\Omega) = \{f\colon \Omega \to \mathbb{R} \,\big|\, f \text{ is continuous}\}$$
with the norm,
$$\|f\|_{C(\Omega)} = \sup_{x\in\Omega} |f(x)|.$$
Prove that $C(\Omega)$ is a Banach space.

(2) Let Ω be a bounded domain in $\mathbb{R}^2$, $d \geq 1$, with smooth boundary.

(a) Use the divergence theorem to derive Green's identity,
$$\int_\Omega \Delta u\, v = -\int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} (\nabla u \cdot \mathbf{n}) v,$$
where u and v are smooth scalar-valued functions on $\overline{\Omega}$, and $\mathbf{n}$ is the outward unit normal vector.

(b) Consider the Cauchy problem,
$$\begin{cases} \partial_t u = \Delta u + cu & \text{for } (t,x) \in (0,\infty) \times \Omega, \\ u(t,x) = 0 & \text{for } (t,x) \in (0,\infty) \times \partial\Omega, \\ u(0,x) = g(x) & \text{for } x \in \Omega, \end{cases}$$
on a bounded domain $\Omega \subseteq \mathbb{R}^d$ having a smooth boundary. Here, c is a positive constant. Suppose u_1 and u_2 are two smooth solutions of the above Cauchy problem with different initial conditions g_1 and g_2. Show that if g_1 and g_2 are "close" in $L^2(\Omega)$ then the solutions u_1 and u_2 are also close in $L^2(\Omega)$ at any later time $t > 0$. Derive an estimate of how close. (Green's identity and Gronwall's inequality will be useful here.)

(3) Let $A(t)$ be a continuous function from t in $\mathbb{R}$ to the space of square, real-valued matrices.

(a) Show that for every solution of the (non-autonomous) linear system, $\dot{\mathbf{x}} = A(t)\mathbf{x}$, we have
$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}(0)\| e^{\int_0^t \|A(s)\|\, ds},$$
where $\|A(s)\|$ is the operator norm and $\|\mathbf{x}(t)\|$ is the usual Euclidean norm.

(b) Show that if $\int_0^t \|A(s)\|\, ds < \infty$ then every solution, $\mathbf{x}(t)$, has a finite limit as $t \to \infty$.

Part 2

(1) (a) Find the entropy solution to the Burgers' equation $u_t + uu_x = 0$ with the initial datum

$$g(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 1 - x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x \geq 1. \end{cases}$$

(b) Consider the Burgers' equation with source term 1 with the initial datum x:

$$u_t + uu_x = 1, \quad u(t = 0) = x.$$

Find the equation for the characteristics and also find an explicit formula for the solution of this initial value problem.

(2) Let $f \in C_c^2(\mathbb{R}^3)$ be given. Define for $x \in \mathbb{R}^3$

$$u(x) = \int_{\mathbb{R}^3} \Phi(x - y) f(y) dy$$

where $\Phi(x) = \frac{1}{4\pi|x|}$. Prove that $-\Delta u = f$ in $\mathbb{R}^3$. You can use the fact $u \in C^2(\mathbb{R}^3)$ without a proof.

(3) Let u be a classical solution of the following initial boundary value problem:

$$\begin{aligned} &u_t = u_{xx}, \qquad \text{in } (a, b) \times (0, T) \\ &u(a, t) = u(b, t) = 0 \\ &u(x, 0) = u_0(x) \end{aligned}$$

where u_0 is a continuous function.

(a) Show that the solutions are unique.

(b) Show that there exists a constant $\alpha > 0$ such that

$$\|u(\cdot, t)\|_{L^2}^2 \leq e^{-\alpha t} \|u_0\|_{L^2}^2.$$

Part 3

(1) Let U be the open unit ball in $\mathbb{R}^d$.
 (a) Let
$$u(x) = |x|^{-\alpha}.$$
 For which values of $\alpha > 0$, $d \geq 1$, and $p > 1$ does u belong to $W^{1,p}(U)$?
 (b) Show that
$$u(x) = \log\log\left(1 + |x|^{-1}\right)$$
 belongs to $W^{1,2}(U)$ but does not belong to $L^\infty(U)$.

(2) Let $U = (0,1)^2$, the unit square in $\mathbb{R}^2$. Can the Lax-Milgram theorem be applied to the bilinear form, $B[u,v]\colon H_0^1(U) \times H_0^1(U) \to \mathbb{R}$, defined by
$$B[u,v] = \int_0^1 \int_0^1 \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2} - \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}?$$

(3) Suppose $u \in C^2(U) \cap C(\overline{U})$ and let
$$Lu = \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j},$$
where the coefficient, a^{ij}, are continuous and satisfy the uniform ellipticity condition. Prove the weak maximum principle; namely, that if $Lu \leq 0$ then
$$\max_{\overline{U}} u = \max_{\partial U} u.$$

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APPLIED MATH QUALIFYING EXAM 5 OCTOBER 2013

Instructions: Work 2 out of 3 problems in each of the 3 parts for a total of 6 problems.

PART 1

(1) A *fundamental solution* to the autonomous linear system, $\dot{\mathbf{x}} = A\mathbf{x}$, is a nonsingular matrix-valued function, $\Phi\colon \mathbb{R} \to M^{d\times d}$, with $\Phi'(t) = A\Phi(t)$.

(a) Show that $\Psi(t) = e^{At}$ is a fundamental solution satisfying $\Psi(0) = I$, the identity matrix. (You may use standard facts about e^{At} without proof.)

(b) Show that $\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}_0$ is a solution to the IVP, $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$.

(c) Show that any fundamental solution is of the form, $\Phi(t) = e^{At}M$, for some non-singular matrix M.

(d) Consider the nonhomogeneous linear system,

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t),$$

where $\mathbf{b}$ is continuous in time. (So $\mathbf{b}$ can vary with time, but A cannot.) Show that

$$\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}_0 + \int_0^t \Phi(t)\Phi^{-1}(s)\mathbf{b}(s)\,ds$$

is a solution to the IVP, $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$.

(2) (a) Consider the linear system of ODEs,

$$\dot{y}_1 = -y_1, \quad \dot{y}_2 = 2y_2,$$

which has the origin as the only equilibrium point. Determine the explicit solution to this system given the initial condition, $\mathbf{y}(0) = \mathbf{a} = (a_1, a_2)$. What are the stable and unstable manifolds for this system? (One or both might be empty.)

(b) Now consider the perturbed, nonlinear system,

$$\dot{x}_1 = -x_1, \quad \dot{x}_2 = 2x_2 - 5\epsilon x_1^3,$$

which also has the origin as the only equilibrium point. Determine the explicit solution to this system given the initial condition, $\mathbf{x}(0) = \mathbf{a} = (a_1, a_2)$. (One method: let y_1, y_2 be the solution to the linear system in (a) with initial condition, $(y_1, y_2) = (1, 1)$, assume that $x_2 = c_1 y_2 + c_2 y_1^3$, and then determine c_1 and c_2.)

(c) What is the stable manifold for the system in (b)?

(3) Consider the system of equations,

$$\begin{cases} \dot{x_1} = x_2 - x_1 f(x_1, x_2), \\ \dot{x_2} = -x_1 - x_2 f(x_1, x_2), \end{cases}$$

where f lies in $C^1(\mathbb{R}^2)$.

(a) Show that if f is positive in some neighborhood of the origin then the origin is an asymptotically stable equilibrium point.

(b) Show that if f is negative in some neighborhood of the origin then the origin is an unstable equilibrium point.

Hint for both parts: Construct a Lyapunov function.

Part 2

(1) Let g be a bounded, continuous function on $\mathbb{R}^n$. For $(x,t) \in \mathbb{R}^n \times (0, +\infty)$ define

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy,$$

where Φ is the fundamental solution of the heat equation,

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}.$$

Let $x_0 \in \mathbb{R}^n$. Prove that

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0).$$

Hint: You can use the fact that $\int_{\mathbb{R}^n} \Phi(x,t)dx = 1$ for every $t > 0$ without proving it. You can also use without proving it the fact that for every $r_0 > 0$,

$$\lim_{(x,t)\to(x_0,0)} \int_{|y-x_0|>r_0} \Phi(x-y,t)dy = 0.$$

In other words, $\Phi(\cdot, t)$ has mass one and as $(x,t) \to (x_0, 0)$ all the mass concentrate around the the point x_0.

(2) Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary and define the energy

$$E(w) = \frac{1}{2}\int_{\Omega} |\nabla w|^2 dx - \int_{\partial\Omega} hw,$$

where h is a smooth functions defined on the boundary of Ω. Suppose $u \in C^2(\overline{\Omega})$ satisfies

$$E(u) \le E(w) \text{ for all } w \in C^2(\overline{\Omega}).$$

What PDE is u satisfying? What are the boundary conditions? Prove it.

Hint: Start by considering perturbation $u + \epsilon v$ where $v \in C_c^2(\Omega)$. This will give you the PDE. Then consider perturbation $u + \epsilon v$ where $v \in C^2(\overline{\Omega})$ to get the boundary condition.

(3) Let u and v belong to $C_1^2(U_T) \cap C(\overline{U_T})$ and satisfy

$$u_t = \Delta u + f$$
$$v_t = \Delta v + g.$$

Show that if $u \ge v$ on the parabolic boundary Γ_T and $f \ge g$ in U_T then $u \ge v$ in all of $\overline{U_T}$. This is called a comparison principle.

Part 3

(1) (a) Prove or disprove the following:

Let U be a bounded, open subset of $\mathbb{R}^2$. If $u \in W^{1,2}(U)$, then $u \in L^\infty(U)$ with the estimate

$$\|u\|_{L^\infty(U)} \leq C\|u\|_{W^{1,2}(U)}$$

where C does not depend on u.

(b) Let U be a bounded, open set in $\mathbb{R}^n$ with smooth boundary. Show that

$$\|Du\|^2_{L^2(U)} \leq C\|u\|_{L^2(U)}\|D^2u\|_{L^2(U)}$$

for all $u \in H^1_0(U) \cap H^2(U)$ where C does not depend on u.

(2) Consider the following Dirichlet problem

$$-\Delta u + \mu u = f \ \text{ in } \ U$$
$$u = 0 \ \text{ on } \ \partial U$$

where μ is a given constant. U is a bounded, open subset of $\mathbb{R}^n$.

(a) Show the existence of a weak solution $u \in H^1_0(U)$ of the above problem for $\mu > 0$.

(b) Show the existence of a weak solution $u \in H^1_0(U)$ of the above problem for $\mu = 0$.

(c) Discuss the problem when $\mu < 0$.

(3) Consider the Poisson equation with Dirichlet boundary condition:

$$\begin{cases} -\Delta u = f & \text{in } U \\ \quad u = 0 & \text{on } \partial U \end{cases}$$

where U is a bounded, open subset of $\mathbb{R}^n$ and $f \in L^2(U)$. We know there exists a weak solution $u \in H^1_0(U)$. Prove that $u \in H^2_{loc}(U)$.