Qualify Exam on Differential Equations, Fall 2025

Please choose any TWO problems in this part.

Part A

Problem 1. Suppose $f(t,y)$ and $g(t,y)$ be continuous functions on the domain D and they satisfy

$$f(t,y) < g(t,y), \forall (t,y) \in D.$$

Let $\phi(t)$ be the solution of

$$y'(t) = f(t, y(t)), y(t_0) = y_0$$

and $\psi(t)$ be the solution of

$$y'(t) = g(t, y(t)), y(t_0) = y_0$$

on (a,b), where $(t_0, y_0) \in D$. Show that for $t_0 < t < b$, $\phi(t) < \psi(t)$; for $a < t < t_0$, $\phi(t) > \psi(t)$.

Problem 2. Consider the following system

$$x' = x(b - x - \frac{y}{1+x}), y' = y(\frac{x}{1+x} - ay)$$

where $x, y \geq 0$ and $a, b > 0$ are parameters.
(a) Sketch the nullclines and discuss the bifurcations that occur as b varies.
(b) Show that a positive fixed point exists for $a, b > 0$.
(c) Show that a Hopf bifurcation occurs at the positive fixed point if

$$a = a_c = \frac{4(b-2)}{b^2(b+2)} \text{ and } b > 2.$$

Problem 3. Let $x \in R^1$, $g(x)$ is continuous and $xg(x) > 0$ if $x \neq 0$. Prove that the zero solution of the equation

$$x'' + g(x) = 0$$

is stable, but not asymptotically stable.

Qualify Exam- MATH 207B, Fall 2025

Choose 2 problems to answer.

Problem 1. Prove that for each connected set $V \subset\subset U$, there exists a positive constant C, depending on V, such that

$$\sup_V u \leq C \inf_V$$

for all nonnegative harmonic functions u in U.

Problem 2. (Backwards Uniqueness for the heat equation) Let $u_1, u_2 \in C^2(\overline{U}_T)$ solve the heat equation $u_t - \Delta u = 0$ with

$$u_1 = u_2 = g \ \text{ on } \ \partial U \times [0, T]$$

and $u_1(x, T) = u_2(x, T)$ for $x \in U$. Prove that

$$u_1 \equiv u_2 \quad \text{within} \ \ U_T.$$

Problem 3. (Finite propagation speed of waves)Let u solve the wave equation $u_{tt} - \Delta u = 0$. If $u = u_t = 0$ on $B(x_0, t_0) \times \{t = 0\}$ with $t_0 > 0$, then $u = 0$ on $C(x_0, t_0)$, where

$$C(x_0, t_0) = \{(x, t) : 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t\}.$$

Prove this result.

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Written Qualifying Exams on Differential Equation - Part III (C)

Problem 1.

(a) State the Gagliardo–Nirenberg–Sobolev inequality and the Rellich–Kondrachov Compactness Theorem.

(b) Let $\{u_m\}_{m=1}^{\infty}$ be a sequence of functions defined on a bounded open set $U \subset \mathbb{R}^6$ with $\partial U \in C^2$. Assume that $\{u_m\}$ is bounded in the Sobolev space $H^2(U)$. Use the Gagliardo–Nirenberg–Sobolev inequality and the Rellich–Kondrachov Compactness Theorem to show that $\{u_m\}$ is relatively compact in $L^5(U)$; that is, $\{u_m\}$ contains a sub-sequence that converges in $L^5(U)$.

Problem 2.

(a) State the Lax–Milgram Theorem.

(b) Use the Lax–Milgram Theorem to prove the existence of a weak solution $u(\boldsymbol{x}) = u(x_1, x_2)$, $u \in H_0^1(U)$, to the following elliptic boundary value problem in the unit disk $U = \{|\boldsymbol{x}| < 1\} \subset \mathbb{R}^2$, for an arbitrary function $f \in L^2(U)$:

$$\begin{cases} -\Delta u + x_1^2\, \partial_{x_2} u + u &= f(\boldsymbol{x}) \text{ in } U, \\ u &= 0 \text{ on } \partial U. \end{cases}$$

Problem 3.

(a) State the weak maximum principle for functions satisfying $\Delta u \geq 0$.

(b) Let $U \subset \mathbb{R}^n$ be a bounded open set, and let $u \in C^2(\overline{U})$ solve the following boundary value problem:

$$\begin{cases} \Delta u &= u \text{ in } U, \\ u &= 1 \text{ on } \partial U. \end{cases}$$

Prove that $0 \leq u(\boldsymbol{x}) \leq 1$ for all $\boldsymbol{x} \in \overline{U}$.