2024 Applied Mathematics Qualifying Exam Part A

Instructions:

• Choose any three of the following five questions (if you do more, the first three will be graded.)

• Do not use the theorems that are essentially what is asked to prove in the problems (e.g. Law of Large Numbers, Chebyshev inequality, Central Limit Theorem). Prove those from scratch. Show all appropriate work in order to receive full credit.

Problem 1: Let X_1, X_2, X_3, ... , and X be random variables.
(a) Suppose X_n are independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Give necessary and sufficient conditions on the p_n for the sequence X_n to $X_n \xrightarrow{p} 0$.
(b) Suppose that X_k is a sequence of independent random variables with mean zero, i.e., $\mathrm{E}(X_k) = 0$, and up to fourth moment is finite. Show that $\sum_{k=1}^{n} X_k/n$ converges to 0 almost surely.

Problem 2: A Poisson random variable $\{X_k\}$ with parameter λ_k has characteristic function

$$\mathbf{P}(X_k = m) = e^{-\lambda_k}\frac{\lambda_k^m}{m!}, \quad m = 0, 1, 2, ...,$$

(a) Suppose that X_1 and X_2 are Poisson random variables with (λ_1) and (λ_2). Determine the distribution of $X_1 + X_2$ with justification.
(b) Consider a sequence of i.i.d. Poisson random variables $\{X_k\}$ with parameter $\lambda = 1$. Show that

$$\frac{\frac{1}{n}\sum_{k=1}^{n} X_k - 1}{1/\sqrt{n}} \xrightarrow{d} N(0, 1).$$

Problem 3: Let X_1, X_2, X_3, ... , and X be random variables.
(a) Suppose that X_n converge to X in mean. Does X_n converge to X in probability? Prove or give a counter example with justification.
(b) Let X_1, X_2, X_3, ... be a sequence of random variables, and let X be a random variable. Define what it means for X_n converge to X in distribution, and for X_n converge to X almost surely.

Problem 4: Let X_n be independent identically distributed (i.i.d.) random variables each having the same distribution as X. Let $a > 0$, and suppose that $\mathrm{E}[e^{aX}]$ is finite. Let $S_n = X_1 + \cdots + X_n$. Show that

$$\mathbf{P}(S_n \geq t) \leq \frac{\left(\mathrm{E}[e^{aX}]\right)^n}{e^{at}}$$

*Do not use Chebyshev inequality. Prove it from scratch.

Problem 5:
(a) Consider a sequence of events, $E_1, E_2, \ldots$, on the probability space $(\Omega, \mathcal{F}, \mu)$. Show that, if $\sum_{n=1}^{\infty} \mathbf{P}(E_n) < \infty$, then $\mathbf{P}(E_n, \text{ i.o.}) = 0$.
(b) Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. For a given $A \in \mathcal{F}$ such that $\mu(A) \neq 0$, define $\mu_A : \mathcal{F} \to [0, \infty]$ by $\mu_A(E) \doteq \mu(E \cap A)/\mu(A)$. Show that μ_A is a probability measure on $(\Omega, \mathcal{F})$.
(c) Suppose that $\{X_n\}$ is a sequence of real valued random variables on $(\Omega, \mathcal{F}, \mu)$. Show that $\limsup_{n\to\infty} X_n$ is also a random variable.

Math 206B Qualifying Exam, Fall 2024

Please provide detailed answers and solutions for any 3 of the following 5 problems.

Problem 1. (10 points)

a. (2 points) Let $f \in C^2[a,b]$, consider using Newton's method to find a zero x^* of f, i.e., $f(x^*) = 0$. Write down the updating formula of Newton's method.

b. (4 points) Prove the following:
Theorem of Convergence of Newton's Method, Part I: Let $x^* \in [a,b]$ be a zero of $f \in C^2[a,b]$. If $f'(x^*) \neq 0$ and $f''(x^*) < \infty$, then $\exists \delta > 0$, s.t., $\forall$ initial guess $x_0 \in (x^* - \delta, x^* + \delta)$, the sequence $\{x_n\}$ generated by Newton's method converges to x^*.

c. (4 points) **Definition of Convergence with Order** p: If $\exists p > 1,\ C > 0$, s.t.,

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^p} = C$$

Then $\{x_n\}_{n=1}^{\infty}$ converges to x^* with order p.

Use the above Definition to prove:

Theorem of Convergence of Newton's Method, Part II: Let $x^* \in [a,b]$ be a zero of $f \in C^2[a,b]$. If $f'(x^*) \neq 0$ and $f''(x^*) < \infty$, then $\exists \delta > 0$, s.t., $\forall$ initial guess $x_0 \in (x^* - \delta, x^* + \delta)$, the sequence $\{x_n\}$ generated by Newton's method converges to x^* with order 2.

(You may use the result of part b, that is, even you didn't prove part b, you can still assume that the convergence is held, based on which you may continue to prove that the convergence is of order 2.)

Problem 2. (10 points) Perform a polynomial interpolation to the following data:

(1) (2 pt) Using Lagrange interpolation:

x	0	0.5	1
$f(x)$	0	1	0

(2) (2 pt) Using the Newton form of interpolation (either forward or backward difference):

x	0	0.5	1	1.5
$f(x)$	0	1	0	-1

Problem 3. (10 points)

(1) (5 pt) Prove the following theorem.
Theorem Let $q(x)$ be a nontrivial polynomial fo degree $n+1$ such that, for all

$0 \le k \le n$

$$\int_{-1}^{1} x^k q(x) dx = 0\,.$$

Let $x_0, ..., x_n$ be the zeros of q. Then the following quadrature rule

$$\int_{-1}^{1} f(x) dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

with $A_i = \int_{-1}^{1} \frac{\prod_{j\neq i}(t-x_j)}{\prod_{j\neq i}(x_i-x_j)} dt$ is exact for all polynomials of degree at most $2n+1$.

(2) (5 pt) Given that $q(x) = \frac{1}{2}(3x^2 - 1)$ is such that

$$\int_{-1}^{1} q(x) dx = 0 \quad \text{and} \quad \int_{-1}^{1} x q(x) dx = 0\,.$$

Apply the above theorem to show the quadrature rule

$$\int_{-1}^{1} f(x) dx \approx f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

is exact up to a polynomial of degree 3.

Problem 4. (10 points) Consider the following IVP:

$$\begin{cases} y'(t) = f\left(t, y(t)\right) & \text{on } [0,1] \\ y(0) = \alpha \end{cases}$$

and the following general difference equation to approximate the solution to the above IVP:

$$\begin{cases} y_{i+1} = y_i + h\Phi\left(t_i, y_i, h\right) \\ y_0 = \alpha \end{cases}$$

a. (2 points) Taylor's series method of the third order is defined with

$$\begin{aligned} \Phi(t, y, h) &= f(t, y) + \frac{h}{2} f'(t, y) + \frac{h^2}{6} f''(t, y) \\ &= f + \frac{h}{2}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f\right) + \frac{h^2}{6}\mathcal{F} \end{aligned}$$

Derive the expression of $\mathcal{F}$ with respect to f and its partial derivatives with respect to t and y.

b. (3 points) Consider the difference method

(1)
$$\begin{cases} y_0 = \alpha \\ y_{i+1} = y_i + a_1 f\left(t_i, y_i\right) + a_2 f\left(t_i + \alpha, y_i + \delta f(t_i, y_i)\right) \end{cases}$$

Find the general conditions of a_1, a_2, α, δ, such that

$$a_1 f\left(t, y\right) + a_2 f\left(t + \alpha, y + \delta f(t, y)\right) = f(t, y) + \frac{h}{2}\left(f_t(t, y) + f_y(t, y) f(t, y)\right) + O(h^2)$$

c. (2 points) One particular solution that satisfies the above condition is

$$a_1 = a_2 = \frac{1}{2}, \ \delta_2 = \alpha_2 = h$$

Suggest a Runge-Kutta method based on this particular solution, and write down its Butcher Tableau.

d. (3 points) Explain that no choice of the constants a_1, a_2, α, δ can make the difference method Eq. (1) having local truncation error $O\left(h^3\right)$. The local truncation error is defined as follows:

Definition: for an explicit one-step method:

$$y_{i+1} = y_i + h\Psi(t_i, y_i; f, h), \quad y_0 = \alpha$$

the *local truncation error* $\tau_i(h)$ at the ith step is:

$$\tau_i(h) = \frac{y(t_i) - (y(t_{i-1}) + h\Psi\left(t_{i-1}, y(t_{i-1}), h\right))}{h}$$

Problem 5. (10 points) Consider $A = (a_{ij})$ and its splitting $A = D - L - U$, where

$$D = \operatorname{diag}\left(a_{11}, a_{22}, \cdots, a_{nn}\right), \quad a_{ii} \neq 0$$

is the diagonal part of A, and

$$-L = \begin{bmatrix} 0 & & & \\ a_{21} & 0 & & \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}, \qquad -U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & \ddots & & \vdots \\ & & 0 & a_{n-1,n} \\ & & & 0 \end{bmatrix},$$

are the lower and upper triangular parts of A, respectively. Consider the following linear system:

$$\begin{cases} 2x_1 - x_2 = 1 \\ -x_1 + 2x_2 = 1 \end{cases} \tag{2}$$

a. (2.5 points) Jacobi method is defined as

$$\mathbf{x}^{(k+1)} = D^{-1}(L+U)\mathbf{x}^{(k)} + D^{-1}\mathbf{b}$$

or equivalently

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j\neq i}}^{n} \left(-a_{ij}x_j^{(k)}\right) + b_i \right]$$

For the linear system Eq. (2), starting with

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

find $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ using Jacobi method.

b. (2.5 points) Gauss-Seidal method is defined as

$$\mathbf{x}^{(k+1)} = (D-L)^{-1}U\mathbf{x}^{(k)} + (D-L)^{-1}\mathbf{b}$$

or equivalently

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[\sum_{j>i}\left(-a_{ij}x_j^{(k)}\right) + \sum_{j<i}\left(-a_{ij}x_j^{(k+1)}\right) + b_i\right]$$

For the linear system Eq. (2), starting with

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

find $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ using Gauss-Seidal method.

c. (5 points) Let

$$B_J = D^{-1}(L+U), \quad B_{GS} = (D-L)^{-1}U \tag{3}$$

from Jacobi and Gauss-Seidal methods, respectively.

Definition: For a square matrix $M = (m_{ij})$, its *spectrum* $\sigma(M)$ is defined as

$$\sigma(M) = \{\lambda | \lambda I - M \text{ singular.}\}$$

and M is *strictly diagonal dominant* if

$$|m_{ii}| > \sum_{j\neq i} |m_{ij}|$$

for all i.

Gershgorin Disk Theorem: For a square matrix $M = (m_{ij})$, $\forall\lambda \in \sigma(M)$, $\exists i$, s.t.

$$|m_{ii} - \lambda| \leq \sum_{j\neq i} |m_{ij}|$$

Use the above Gershgorin Disk Theorem to prove the following theorem:

Theorem: Suppose $A = (a_{ij})$ is strictly diagonal dominant. Let B_J and B_{GS} be the matrices of A from Jacobi and Gauss-Seidal methods, respectively (Eq. (3)). $\forall\lambda_J \in \sigma(B_J)$, $\forall\lambda_{GS} \in \sigma(B_{GS})$, we have $|\lambda_J| < 1$, $|\lambda_{GS}| < 1$.

Qualifying Exam Part C, Fall 2024

Please provide detailed answers for any 3 of the following 5 problems. If you do more, the first three will be graded.

Problem 1. (10 points)

a. (5 points) Let $g(x)$ be a continuous function on $[a, b]$ and assume

$$\int_a^b g(x)\eta'(x)dx = 0$$

for every $\eta(x) \in C^1(a, b)$ satisfying $\eta(a) = \eta(b) = 0$. Show that $g(x) \equiv c$ for all $x \in [a, b]$, where c is a constant.

b. (5 points) Show that sphere is the solid figure of revolution which for a given surface area having maximum volume enclosed.

Problem 2. (10 points)

a. (3 points) Find and solve the Euler-Lagrange equation for the problem of minimizing

$$I[y] = \int_{-1}^{1} \frac{1}{2}(y')^2 + xydx$$

among all smooth functions.

b. (3 points)Prove the following statement:
Let J be a functional of the form $J[y] = \int_{x_0}^{x_1} f(y, y')dx$ and define the function H by

$$H(y, y') = y'\frac{\partial f}{\partial y'} - f\,.$$

Then H is a constant along any extremal y.

c. (4 points) Prove the following statement:
Transversality is the same as orthogonality **if and only if** $F(x, y, y')$ has the form $F = g(x, y)\sqrt{1 + y'^2}$ with $g(x, y) \neq 0$ near the point of intersection.

Problem 3. (10 points)

a. (4 points) Given a functional $J[y] = \int_a^b (y')^2 dx$, write down the corresponding Hamilton-Jacobi equation. Solve the Euler-Lagrange equation for this functional and construct a solution to the Hamilton-Jacobi equation.

b. (3 points) Let S^1 denote the unit circle on the plane. For the map $f(\theta) = 2\theta$ defined on S^1, find all of the points that are periodic with period n.

c. (3 points) For the logistic map with $r = 4$, find the first four symbols in the itinerary starting from $x_0 = \frac{1}{4}$.

Problem 4. (10 points)

a. (2 points) Give the definition of topological conjugacy.

b. (4 points) Compute the Lyapunov exponent for the tent map

$$F(x) = \begin{cases} 2rx, & 0 \le x \le 1/2 \\ 2r(1-x), & 1/2 < x \le 1 \end{cases}, \quad \text{with } r > 0.$$

b. (4 points) Find and classify all of the fixed points of the following map

$$(x_{n+1}, y_{n+1}) = \begin{cases} (cx_n, 2y_n), & y_n \le 1/2 \\ (1 + c(x_n - 1), 1 + 2(y_n - 1)), & y_n > 1/2 \end{cases},$$

on $x \in [0,1]$ and $y \in [0,1]$, with $0 < c \le 1/2$.

Problem 5. (10 points)

a. (3 points) Find all bifurcation points of the following system. Then identify the types of bifurcation.

$$\dot{x} = r^2 + rx - x^3 .$$

b. (3 points) Show the following system is Hamiltonian and find $H(x, y)$

$$\frac{dx}{dt} = x^2 + 2y \, , \; \frac{dy}{dt} = -2xy \, .$$

c. (4 points) Consider the following 2D system in polar coordinates,

$$\dot{r} = r(1 - r^2) + \mu r \cos\theta$$
$$\dot{\theta} = 1 \, .$$

Show that there is a stable limit cycle when $\mu = 0$ and apply the Poincaré-Bendixson Theorem to show that a closed orbit still exists for $\mu > 0$, as long as μ is sufficiently small.

2023 Applied Mathematics Qualifying Exam Part A

Instructions:
• Choose any three of the following five questions (if you do more, the first three will be graded.)
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Problem 1: (a) Consider a sequence of events, E_1, E_2, ..., on the probability space $(\Omega, \mathcal{F}, \mu)$. Show that, if $\sum_{n=1}^{\infty} \mathbf{P}(E_n) < \infty$, then $\mathbf{P}(E_n, \text{ i.o. }) = 0$.
(b) Suppose that X_k is a sequence of independent random variables with mean zero, i.e., $\mathrm{E}(X_k) = 0$, and up to fourth moment is finite. Show that $\sum_{k=1}^{n} X_k/n$ converges to 0 almost surely.

Problem 2: A normal random variable X with mean μ and variance σ^2 has probability density

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

and the characteristic function of X is given by $\varphi(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$.
(a) Suppose that X_1 and X_2 are normal random variables with (μ_1, σ^2) and (μ_2, σ^2). Determine the distribution of $X_1 + X_2$ with justification.
(b) Suppose that X_k are normal random variables with (μ, σ^2). Show that

$$\frac{\sum_{k=1}^{n} X_k - n\mu}{\sqrt{n}\sigma} \xrightarrow{d} N(0,1).$$

Problem 3: Let X_1, X_2, X_3, ... , and X be random variables.
(a) Suppose X_n are independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Give necessary and sufficient conditions on the p_n for the sequence X_n to $X_n \xrightarrow{p} 0$.
(b) Suppose that X_k is a sequence of independent random variables with mean zero, i.e., $\mathrm{E}(X_k) = 0$, and up to fourth moment is finite. Show that $\sum_{k=1}^{n} X_k/n$ converges to 0 almost surely.

Problem 4: Let X_n be independent identically distributed (i.i.d.) random variables each having the same distribution as X. Let $a > 0$, and suppose that $\mathrm{E}[e^{aX}]$ is finite. Let $S_n = X_1 + \cdots + X_n$. Show that

$$\mathbf{P}(S_n \geq t) \leq \frac{\left(\mathrm{E}[e^{aX}]\right)^n}{e^{at}}$$

Problem 5:
(a) Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. For a given $A \in \mathcal{F}$ such that $\mu(A) \neq 0$, define $\mu_A : \mathcal{F} \to [0, \infty]$ by $\mu_A(E) \doteq \mu(E \cap A)/\mu(A)$. Show that μ_A is a probability measure on $(\Omega, \mathcal{F})$.
(b) Suppose that $\{X_n\}$ is a sequence of real valued random variables on $(\Omega, \mathcal{F}, \mu)$. Show that $\limsup_{n\to\infty} X_n$ is also a random variable.
(c) Let X_1, X_2, X_3, ... be a sequence of random variables, and let X be a random variable. Define what it means for X_n converge to X in distribution, and for X_n converge to X almost surely.

Qualifying Exam on Numerical Analysis, Fall 2023

Please choose any three problems and submit the full solutions.

Problem 1 Consider the following multistep method:

$$\frac{6-10a}{6} w_i - (1-4a)w_{i-1} - 3a w_{i-2} + \frac{2a}{3} w_{i-3} = h\left(4a^2+2a+1\right) f(t_i, w_i)$$

where $a \in \mathbb{R}$ is a parameter and $h > 0$ is the mesh size. Find all values of a such that the multistep method is consistent and evaluate the order of the scheme for each value a.

Problem 2 Consider the following eigenvalue problem for an $n \times n$ matrix A for $n > 2$:

$$Ax = \lambda x\,. \tag{1}$$

Let A be symmetric positive definite, and therefore there exists an orthonormal basis $\{v_i\}_{i=1}^n$ and their corresponding eigenvalues $\{\lambda_i\}_{i=1}^n$such that for any vector $x \in \mathbb{R}^n$, we can write:

$$x = \sum_{i=1}^n a_i v_i\,.$$

Now let $\lambda_1 > \lambda_2 > \lambda_i > 0$ for all $i \geq 3$.

(a) Show inductively that for all $k \in \mathbb{N}$

$$A^k x = \sum_{i=1}^n \lambda_i^k a_i v_i\,.$$

(b) Let $x^{(0)}$ be given. Define $\{x^{(i)}\}_{i=1}^\infty$ to be

$$x^{(i)} = Ax^{(i-1)}/||Ax^{(i-1)}||_2\,.$$

(i) Show that if $\langle x^{(0)}, v_1\rangle > 0$, then $x^i \to v_1$ as $i \to \infty$, and $Ax^i \to \lambda_1 v_1$ as $i \to \infty$.

(ii) Suppose $x^{(0)}$ is such that $\langle x^{(0)}, v_1\rangle = 0$ but $\langle x^{(0)}, v_2\rangle < 0$, will the sequence $\{x^i\}$ always converge as i goes to ∞? If so, please state what $\{x^i\}$ and $\{Ax^i\}$ will converge to as i goes to ∞, and prove your claim. If not, please provide a counterexample to showcase it may not converge?

Problem 3 Consider the following Part (a) and Part (b).

(a) Prove the following:
Theorem Let $\phi \in C^k([a,b])$. Assume that there exists $x^* \in (a,b)$ such that $\phi(x^*) = x^*$, and there exists k such that $\phi^{(i)}(x^*) = 0$ for all $i < k$. Then there exists $\varepsilon > 0$ such that if $x_0 \in (x^* - \varepsilon, x^* + \varepsilon)$, then the sequence

$$x_{n+1} = \phi(x_n)$$

converges to x^* at least with order k.

(b) Apply the above theorem (or otherwise) to show that there exists $\delta > 0$ such that the seqeunce p_n converges at last quadratically to 1 if $p_0 \in [-\delta, \delta]$, where p_n is given by the following fixed point iteration :

$$p_{n+1} = \phi_m(p_n)$$

with $m \in \mathbb{N}$, where ϕ_m is defined as

$$\phi_m(x) := \begin{cases} x - (m+2)\frac{f(x)}{f'(x)} & \text{when } x \neq 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and $f(x) := x^{m+2}(x + \pi)$.

Problem 4 Consider the following Part (a) and Part (b).

(a) Prove the following.
Theorem Suppose $x_0, x_1, ...x_n$ are distinct numbers in the interval $[a,b]$ and $f \in \mathcal{C}^{n+1}([a,b])$. Then for each $x \in [a,b]$, there exists an $\xi(x) \in [a,b]$ which depends on x continuously on each interval (x_i, x_{i+1}) such that

$$f(x) - h(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)...(x - x_n)\,,$$

where $h(x)$ is the Lagrange interpolation polynomial of degree n such that $f(x_k) = h(x_k)$ for all $k = 0, 1..., n$.

(b) Prove the following.
Theorem Let $w(x) \in \mathcal{C}^0([-1,1])$ and $w(x) > 0$ for all $x \in [-1,1]$. Suppose $q(x)$ be a nontrivial polynomial of degree $n+1$ such that, for all $0 \le k \le n$

$$\int_{-1}^{1} x^k q(x) w(x) dx = 0\,.$$

Let $x_0, ..., x_n$ be the zeros of q. Then the following quadrature rule

$$\int_{-1}^{1} f(x) w(x) dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

is exact for all polynomials of degree at most $2n+1$, where A_i are defined as

$$A_i := \int_{-1}^{1} \frac{\prod_{j \neq i}(t - x_j)}{\prod_{j \neq i}(x_i - x_j)} w(t) dt\,.$$

Problem 5 Consider the following initial value problem:

$$\begin{cases} x'(t) = f(t, x(t)) \text{ on } [0,1] \\ x(0) = x_0 \end{cases}$$

Consider the following fourth-order Runge-Kutta method (RK4):

$$w_{i+1} = w_i + \frac{1}{6}(F_1 + 4F_3 + F_4)$$

where

$$\begin{cases} F_1 = hf(t_i, w_i) \\ F_2 = hf(t_i + \frac{1}{2}h, w_i + \frac{1}{2}F_1) \\ F_3 = hf(t_i + \frac{1}{2}h, w_i + \frac{1}{4}F_1 + \frac{1}{4}F_2) \\ F_4 = hf(t_i + h, w_i - F_2 + 2F_3) \end{cases}$$

(a) Write down the Butcher Tableau of the above fourth-order Runge-Kutta method (RK4).

(b) Is this scheme implcit or explicit?

(c) Show the consistency of this Runge-Kutta method by applying a theorem.

(d) By direct expansion, show that the method agree with the Taylor series method of order 4 if $f(t, x) = x + t$.

Problem 1.

a) Find the curve with minimal length connecting two fixed points on a sphere of radius 5. Justify your answer.

b) Find Euler-Lagrange equations for the variational problems involving the following two functionals and justify your answer:

$$J[y] = \int_a^b (x - y)^2 \, dx$$

and

$$T[y] = \int_{x_0}^{x_1} \frac{\sqrt{1 + (y')^2}}{v(y)} dx$$

c) Provide a general form of a Hamilton-Jacobi equation and explain in detail the difference between Lagrangian and Hamilton-Jacobi descriptions.

Problem 2. For each of the following two maps F(x), plot the phase diagram for the trajectories $x_{n+1} = F(x_n)$, find all fixed points and demonstrate which of them are stable or unstable as well as degenerate or non-degenerate:

a) $F(x) = x^4 – 2x^2$
b) $F(x) = x – x^3$

Problem 3. Bifurcation is a change of the equilibrium points or periodic orbits of a dynamical system, or in their stability properties, as a parameter is varied. Provide bifurcation analysis for the following system.

1. Find the equilibrium points and their types for different values of parameter μ of the following system:

   $dx_1/dt = \mu – x_1^2$

   $dx_2/dt = −x_2.$

2. Explain what happens with the equilibrium points and their stability as μ goes through zero.

2022 Applied Mathematics Qualifying Exam Part A

Instructions: Choose any three of the following five questions (if you do more, only the first three will be graded.)

Problem 1: Let X_1, X_2, X_3, ... be a sequence of random variables, and let X be a random variable.
(a) Define what it means for X_n converge to X in distribution, and for X_n converge to X almost surely.
(b) Suppose that X_n converge to X in mean. Does X_n converge to X in probability? Prove or give a counter example with justification.

Problem 2: Consider a sequence of i.i.d. Poisson random variables $\{X_k\}$ with parameter $\lambda = 1$, that is,

$$\mathbf{P}(X_k = m) = e^{-\lambda}\frac{\lambda^m}{m!}, \quad m = 0,\, 1,\, 2, ...,$$

The characteristic function of Poisson random variable is $e^{\lambda(e^{it}-1)}$. Show that

$$\frac{\frac{1}{n}\sum_{k=1}^{n} X_k - 1}{1/\sqrt{n}} \xrightarrow{d} N(0,1).$$

Problem 3: (a) Consider a sequence of events, E_1, E_2, $\ldots$, on the probability space $(\Omega, \mathcal{F}, \mu)$. Show that, if $\sum_{n=1}^{\infty} \mathbf{P}(E_n) < \infty$, then $\mathbf{P}(E_n, \ \text{i.o.}) = 0$.
(b) Suppose that X_k is a sequence of independent random variables with mean zero, i.e., $\mathrm{E}(X_k) = 0$, $\mathrm{E}(|X_k|^2) = \sigma^2 < \infty$ and $\mathrm{E}(|X_k|^3) = \kappa < \infty$. Show that $\sum_{k=1}^{n} X_k/n$ converges to 0 almost surely.

Problem 4: A binomial random variable with parameters n and p is a distribution defined on non-negative integers from 0 to n with probability mass function given by

$$\mathbf{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

(a) The characteristic function of X binomial random variable with n and p is given by

$$\varphi(t) = (1 - p + pe^{it})^n.$$

Set up an equation that gives the characteristic function $\varphi(t)$ (no need to calculate) and compute the mean and variance of X using $\varphi(t)$.
(b) Suppose that X_1 and X_2 are binomial random variables with $(n_1,\, p)$ and $(n_2,\, p)$. Determine the distribution of $X_1 + X_2$ with justification.

Problem 5: Let X_n be independent identically distributed (i.i.d.) random variables each having the same distribution as X. Let $a > 0$, and suppose that $\mathrm{E}[e^{aX}]$ is finite. Let $S_n = X_1 + \cdots + X_n$. Show that

$$\mathbf{P}(S_n \geq t) \leq \frac{\left(\mathrm{E}[e^{aX}]\right)^n}{e^{at}}$$

Math 206B Qualifying Exam, Fall 2022

Please provide detailed answers and solutions for any 3 of the following 5 problems.

Problem 1. (10 points)

a. (2 points) Let $f \in C^2[a,b]$, consider using Newton's method to find a zero x^* of f, i.e., $f(x^*) = 0$. Write down the updating formula of Newton's method.

b. (4 points) Prove the following:

Theorem of Convergence of Newton's Method, Part I: Let $x^* \in [a,b]$ be a zero of $f \in C^2[a,b]$. If $f'(x^*) \neq 0$ and $f''(x^*) < \infty$, then $\exists \delta > 0$, s.t., $\forall$ initial guess $x_0 \in (x^* - \delta, x^* + \delta)$, the sequence $\{x_n\}$ generated by Newton's method converges to x^*.

c. (4 points) **Definition of Convergence with Order** p: If $\exists p > 1,\ C > 0$, s.t.,

$$\lim_{n \to \infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^p} = C$$

Then $\{x_n\}_{n=1}^{\infty}$ converges to x^* with order p.

Use the above Definition to prove:

Theorem of Convergence of Newton's Method, Part II: Let $x^* \in [a,b]$ be a zero of $f \in C^2[a,b]$. If $f'(x^*) \neq 0$ and $f''(x^*) < \infty$, then $\exists \delta > 0$, s.t., $\forall$ initial guess $x_0 \in (x^* - \delta, x^* + \delta)$, the sequence $\{x_n\}$ generated by Newton's method converges to x^* with order 2.

(You may use the result of part b, that is, even you didn't prove part b, you can still assume that the convergence is held, based on which you may continue to prove that the convergence is of order 2.)

Problem 2. (10 points)

a. (5 points) Let

$$f(x) = \frac{1}{x}, \quad x_0 = 2,\ x_1 = 2.75,\ x_2 = 4$$

Find the Lagrange interpolation polynomial of degree 2, $P_2(x)$ for f at the data points x_0, x_1, x_2.

b. (5 points) Let

$$g(x) = \frac{1}{x}, \quad x_0 = 1,\ x_1 = 2,\ x_2 = 4$$

Find the Newton interpolation polynomial of degree 2, $P_2(x)$ for g at the data points x_0, x_1, x_2.

Problem 3. (10 points) Consider an initial value problem (IVP)

$$y'(t) = f(t, y), \quad y(a) = b \tag{1}$$

A general m-step method can be written as

$$\sum_{j=0}^{m} a_j y_{k+j} = h \sum_{j=0}^{m} b_j f\left(t_{k+j}, y_{k+j}\right) \tag{2}$$

Definition: The *local truncation error* of a m-step method (Eq. 2) is

$$\tau_{k+m}(h) = \frac{1}{h} \sum_{j=0}^{m} a_j y\left(t_{k+j}\right) - \sum_{j=0}^{m} b_j f\left(t_{k+j}, y\left(t_{k+j}\right)\right)$$

where $y = y(t)$ is the analytic solution of the IVP. The m-step method is *consistent* if

$$\lim_{h \to 0} |\tau_{k+m}(h)| = 0$$

The method is *of order* p if p is the largest number that

$$\tau_{k+m}(h) = O(h^p)$$

Prove the following theorem:

Theorem: A m-step method (Eq. 2) is consistent if and only if

$$\sum_{j=0}^{m} a_j = 0, \quad \sum_{j=0}^{m} j a_j = \sum_{j=0}^{m} b_j$$

The method is of order $p \geq 1$ if and only if

$$\sum_{j=0}^{m} a_j = 0$$

$$\sum_{j=0}^{m} j^{\alpha} a_j = \alpha \sum_{j=0}^{m} j^{\alpha-1} b_j, \ \alpha = 1, 2, \cdots, p$$

$$\sum_{j=0}^{m} j^{p+1} a_j \neq (p+1) \sum_{j=0}^{m} j^{p} b_j$$

Problem 4. (10 points) Consider the following IVP:

$$\begin{cases} y'(t) = f\left(t, y(t)\right) & \text{on } [0,1] \\ y(0) = \alpha \end{cases}$$

and the following general difference equation to approximate the solution to the above IVP:

$$\begin{cases} y_{i+1} = y_i + h\Phi\left(t_i, y_i, h\right) \\ y_0 = \alpha \end{cases}$$

a. (2 points) Taylor's series method of the third order is defined with

$$\begin{aligned}\Phi(t,y,h) &= f(t,y)+\frac{h}{2}f'(t,y)+\frac{h^2}{6}f''(t,y)\\ &= f+\frac{h}{2}\left(\frac{\partial f}{\partial t}+\frac{\partial f}{\partial y}f\right)+\frac{h^2}{6}\mathcal{F}\end{aligned}$$

Derive the expression of $\mathcal{F}$ with respect to f and its partial derivatives with respect to t and y.

b. (3 points) Consider the difference method

$$\left\{\begin{array}{l} y_0=\alpha \\ y_{i+1}=y_i+a_1 f\left(t_i, y_i\right)+a_2 f\left(t_i+\alpha, y_i+\delta f\left(t_i, y_i\right)\right)\end{array}\right. \tag{3}$$

Find the general conditions of a_1, a_2, α, δ, such that

$$a_1 f\left(t,y\right)+a_2 f\left(t+\alpha, y+\delta f(t,y)\right)=f(t,y)+\frac{h}{2}\left(f_t(t,y)+f_y(t,y)f(t,y)\right)+O(h^2)$$

c. (2 points) One particular solution that satisfies the above condition is

$$a_1=a_2=\frac{1}{2},\ \delta_2=\alpha_2=h$$

Suggest a Runge-Kutta method based on this particular solution, and write down its Butcher Tableau.

d. (3 points) Explain that no choice of the constants a_1, a_2, α, δ can make the difference method Eq. (3) having local truncation error $O\left(h^3\right)$. The local truncation error is defined as follows:

Definition: for an explicit one-step method:

$$y_{i+1}=y_i+h\Psi(t_i,y_i;f,h), \quad y_0=\alpha$$

the *local truncation error* $\tau_i(h)$ at the ith step is:

$$\tau_i(h)=\frac{y(t_i)-\left(y(t_{i-1})+h\Psi\left(t_{i-1},y(t_{i-1}),h\right)\right)}{h}$$

Problem 5. (10 points) Consider $A=(a_{ij})$ and its splitting $A=D-L-U$, where

$$D=\operatorname{diag}\left(a_{11},a_{22},\cdots,a_{nn}\right), \quad a_{ii}\neq 0$$

is the diagonal part of A, and

$$-L=\begin{bmatrix}0 & & & \\ a_{21} & 0 & & \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & 0\end{bmatrix}, \qquad -U=\begin{bmatrix}0 & a_{12} & \cdots & a_{1n} \\ & \ddots & & \vdots \\ & & 0 & a_{n-1,n} \\ & & & 0\end{bmatrix},$$

are the lower and upper triangular parts of A, respectively. Consider the following linear system:

$$\begin{cases} 2x_1 - x_2 = 1 \\ -x_1 + 2x_2 = 1 \end{cases} \tag{4}$$

a. (2.5 points) Jacobi method is defined as

$$\mathbf{x}^{(k+1)} = D^{-1}(L+U)\mathbf{x}^{(k)} + D^{-1}\mathbf{b}$$

or equivalently

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1 \\ j\neq i}}^{n}\left(-a_{ij}x_j^{(k)}\right) + b_i\right]$$

For the linear system Eq. (4), starting with

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

find $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ using Jacobi method.

b. (2.5 points) Gauss-Seidal method is defined as

$$\mathbf{x}^{(k+1)} = (D-L)^{-1}U\mathbf{x}^{(k)} + (D-L)^{-1}\mathbf{b}$$

or equivalently

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[\sum_{j>i}\left(-a_{ij}x_j^{(k)}\right) + \sum_{j<i}\left(-a_{ij}x_j^{(k+1)}\right) + b_i\right]$$

For the linear system Eq. (4), starting with

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

find $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ using Gauss-Seidal method.

c. (5 points) Let

$$B_J = D^{-1}(L+U), \quad B_{GS} = (D-L)^{-1}U \tag{5}$$

from Jacobi and Gauss-Seidal methods, respectively.

Definition: For a square matrix $M = (m_{ij})$, its *spectrum* $\sigma(M)$ is defined as

$$\sigma(M) = \{\lambda | \lambda I - M \text{ singular.}\}$$

and M is *strictly diagonal dominant* if

$$|m_{ii}| > \sum_{j\neq i}|m_{ij}|$$

for all i.

Gershgorin Disk Theorem: For a square matrix $M = (m_{ij})$, $\forall \lambda \in \sigma(M)$, $\exists i$, s.t.

$$|m_{ii} - \lambda| \leq \sum_{j \neq i} |m_{ij}|$$

Use the above Gershgorin Disk Theorem to prove the following theorem:

Theorem: Suppose $A = (a_{ij})$ is strictly diagonal dominant. Let B_J and B_{GS} be the matrices of A from Jacobi and Gauss-Seidal methods, respectively (Eq. (5)). $\forall \lambda_J \in \sigma(B_J)$, $\forall \lambda_{GS} \in \sigma(B_{GS})$, we have $|\lambda_J| < 1$, $|\lambda_{GS}| < 1$.

Math 206C Qualifying Exam, Fall 2022

Please provide detailed answers and solutions for any 3 of the following 5 problems.

Problem 1. (10 points)

a. (4 points) Prove the fundamental lemma of calculus of variations.
b. (3 points) Describe the Brachistochrone problem and derive the functional of time in the Brachistochrone problem with nonzero initial velocity v_0 (DO NOT SOLVE THE E-L EQUATION).
c. (3 points) Give an example of linear functional and verify that it is linear.

Problem 2. (10 points)

a. (5 points) Derive the Beltrami identity from the Euler-Lagrange equation when $F = F(x, y, y')$ in the functional $J[y] = \int_a^b F dx$ does not depend on x explicitly.
b. (5 points) Find the extremal of the functional $J[y] = \int_0^{\pi/2} (-(y')^2 + y^2 + 2xy)dx$, where $y(0)$ and $y(\pi/2)$ are not specified.

Problem 3. (10 points)

a. (3 points) Write down the E-L equation for the following problem (DO NOT SOLVE THE E-L EQUATION)
"Determine the curve $y(x)$ from $(x, y) = (-1, 0)$ to $(1, 0)$ that has minimum length, but encloses an area $A = \frac{\pi}{2}$ with the x-axis."
b. (4 points) Given a functional $J[y] = \int_a^b f(y)\sqrt{1 + (y')^2}dx$, write down the corresponding Hamilton-Jacobi equation.
c. (3 points) Describe the definition of a non-degenerate critical point of a map and give an example.

Problem 4. (10 points)

a. (5 points) Find all the fixed points of the discrete dynamical system $x_{n+1} = F(x_n) = rx_n - x_n^3$ and determine for which r they exist. Determine the stability of the fixed points as a function of r.
b. (5 points) Consider the Hénon map $\mathbf{H}(x, y) = (a - x^2 + by, x)$. Find and classify all period-two orbits of the Hénon map with $a = 0.43$ and $b = 0.4$.
Hint: $(a-x^2)^2+(1-b)^3x-a(1-b)^2 = (x^2-(1-b)x-a+(1-b)^2)(x^2+(1-b)x-a)$.

Problem 5. (10 points)

a. (2 points) Give the definitions of the homoclinic and heteroclinic orbits.
b. (4 points) Consider a one-dimensional continuous dynamical system. Write down the normal form of a subcritical pitchfork bifurcation. Find the fixed points and determine their stability, and draw the bifurcation diagram.

c. (4 points) Consider the glycolysis oscillations,

$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$

where $x, y \geq 0$ and $a, b > 0$. Apply the Poincaré-Bendixson Theorem and find the Hopf bifurcation boundary on the parameter plane.

2021 Applied Mathematics Qualifying Exam Part A

Instructions: Choose any **three** of the following five questions (if you do more, only the first three will be graded. If you have questions how to interpret a problem, please ask!

Problem 1: Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables, and let X be a random variable.

Part a: (4 points) Define what it means for $X_n \to X$ in probability, and for $X_n \to X$ in distribution.
Part b: (3 points) Give (with justification) an example of a sequence of variables such that $X_n \to X$ in distribution, but not in probability.
Part c: (3 points) Give (with justification) an example of a sequence of variables such that $X_n \to X$ in distribution, but $\mathbf{E}(X_n)$ does not converge to $\mathbf{E}(X)$.

Problem 2: The **Poisson Distribution** with parameter λ is a distribution defined on the non-negative integers, with probabilities given by

$$\mathbf{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Part a: (5 points) Suppose that X has this distribution. Show that the characteristic function of X is given by

$$\varphi(t) = \exp\left(\lambda(e^{it} - 1)\right)$$

Part b: (5 points) Suppose that X_1 and X_2 are independent variables, having Poisson distributions with parameters λ_1 and λ_2 respectively. Determine, with justification, the distribution of $X_1 + X_2$.

You may assume the result of part a for part b even if you did not solve that part.

Problem 3. (10 points) Suppose that $X_1, X_2, \ldots$ are independent, identically distributed random variables satisfying

$$\mathbf{P}(0 \leq X_i \leq 1) = 1 \text{ and } \mathbf{P}(X_i = 1) < 1.$$

Show that there is a constant $c > 1$ (possibly depending on the common distribution of the X_i) such that

$$c^n \prod_{i=1}^{n} X_i \to 0 \text{ almost surely}$$

Problem 4: (10 points) Suppose that $X_1, X_2, \ldots$ are (not necessarily independent!) random variables satisfying the following three conditions:

- $\mathbf{E}(X_i) = 0$ for all i.
- There is an absolute constant C such that $\mathbf{Var}(X_i) \leq C$ for all i.
- $\mathbf{E}(X_i X_j) \leq 0$ for all $i \neq j$.

Show that the sample mean

$$\overline{X_n} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

converges to 0 in probability.

5. Let $A_1, A_2, \ldots$ be events satisfying $\mathbf{P}(A_i) \geq \frac{1}{2}$ for all i. Let A be the event that infinitely many of the A_i occur.

Part a: (2 point) Suppose that the A_i are independent. Why must $\mathbf{P}(A) = 1$? (Just citing a result is enough here).

Part b: (2 points) Show by example that it is possible to have $\mathbf{P}(A) < 1$ if the A_i are not independent.

Part c: (6 points) Show that $\mathbf{P}(A) \geq \frac{1}{2}$ regardless of whether or not the A_i are independent.

Part B

Qualifying Exam on Numerical Analysis, Fall 2021

Please choose any three problems and submit the full solutions.

Problem 1 Consider the following multistep method:

$$\frac{6-5a}{6}w_i - (1-2a)w_{i-1} - \frac{3a}{2}w_{i-2} + \frac{a}{3}w_{i-3} = h\left(a^2+a+1\right)f(t_i, w_i)$$

where $a \in \mathbb{R}$ is a parameter and $h > 0$ is the mesh size. Find all values of a such that the multistep method is consistent and evaluate the order of the scheme for each value a.

Problem 2 Prove the following.
Theorem Let $w(x) \in \mathcal{C}^0([-1,1])$ and $w(x) > 0$ for all $x \in [-1,1]$. Suppose $q(x)$ be a nontrivial polynomial of degree $n+1$ such that, for all $0 \le k \le n$

$$\int_{-1}^{1} x^k q(x) w(x) dx = 0\,.$$

Let $x_0, ..., x_n$ be the zeros of q. Then the following quadrature rule

$$\int_{-1}^{1} f(x)w(x)dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

is exact for all polynomials of degree at most $2n+1$, where A_i are defined as

$$A_i := \int_{-1}^{1} \frac{\prod_{j\neq i}(t-x_j)}{\prod_{j\neq i}(x_i-x_j)} w(t)dt\,.$$

Problem 3 Consider the following Part (a) and Part (b).

(a) Prove the following:
Theorem Let $\phi \in C^k([a,b])$. Assume that there exists $x^* \in (a,b)$ such that $\phi(x^*) = x^*$, and there exists k such that $\phi^{(i)}(x^*) = 0$ for all $i < k$. Then there exists $\varepsilon > 0$ such that if $x_0 \in (x^* - \varepsilon, x^* + \varepsilon)$, then the sequence

$$x_{n+1} = \phi(x_n)$$

converges to x^* at least with order k.

(b) Apply the above theorem (or otherwise) to show that there exists $\delta > 0$ such that the seqeunce p_n converges at last quadratically to 0 if $p_0 \in [-\delta, \delta]$, where p_n is given by the following fixed point iteration :

$$p_{n+1} = \phi_m(p_n)$$

with $m \in \mathbb{N}$ being a fixed integer, where ϕ_m is defined as

$$\phi_m(x) = \begin{cases} x - (m+2)\frac{f(x)}{f'(x)} & \text{when } x \neq 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and

$$f(x) = x^{m+2}(x+10)\,.$$

Problem 4 Consider the following Part (a) and Part (b).

(a) Prove the following:

Theorem Consider $A = P - R$ where P is invertible, and write $B := P^{-1}R$. Consider the iteration

$$x_{n+1} = Bx_n + P^{-1}b$$

Suppose $\|B\| < 1$ for some operator/induced norm $\|\cdot\|$ over $(\mathbb{R}^d, \|\cdot\|)$, i.e.

$$\|B\| = \sup_{x \in V} \frac{\|Bx\|}{\|x\|}$$

Then there exists a unique x^* such that for all $x_0 \in V$, x_n converges to x^*. In particular x^* satisfies

$$Ax^* = b\,.$$

Moreover,

$$\|x^* - x_n\| \leq \frac{\|B\|^n}{1 - \|B\|}\|x_1 - x_0\|\,.$$

(b) Let the $d \times d$ matrix $A = (a_{ij})_{i,j=1}^d$ be such that

$$\sum_{i=1,\ldots,d,\, i\neq j} \frac{|a_{ij}|}{|a_{ii}|} < 1 \quad \text{for all } 1 \leq j \leq d\,.$$

(Please be aware of how the index run through i and j.) Consider the above iteration in the theorem, with the choice

$$P = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & a_{(d-1)(d-1)} & 0 \\ 0 & \cdots & 0 & a_{dd} \end{bmatrix},$$

apply the previous theorem and choose an appropriate norm, or otherwise, to conclude that the Jacobi iteration converges.

Problem 5 Consider the following initial value problem:

$$\begin{cases} x'(t) = f(t, x(t)) & \text{on } [0,1] \\ x(0) = \alpha \end{cases}$$

for $x : [0,1] \to \mathbb{R}^d$, $d \in \mathbb{N}$, $d > 0$, and the following difference quation to approximate the solution to IVP above:

$$\begin{cases} w_{i+1} = w_i + h\Phi(t_i, w_i, w_{i+1}, h) \\ w_0 = \alpha \end{cases}$$

where $t_i = hi$. Write $x_i := x(t_i)$ and define $\tau(h)$ as the following function:

$$\tau(h) := \max_i \|\tau_i(h)\|$$

where $\tau_i(h)$ is the local truncation error of this one method defined as follows:

$$\tau_i(h) := \frac{x_{i+1} - x_i}{h} - \Phi(t_i, x_i, x_{i+1}, h)\,.$$

Prove the following theorem:

Theorem Given $h_0 > 0$, let $\Phi(t, x, p, h)$ satisfies the following Lipchitz condition, i.e.

$$\|\Phi(t, x, p, h) - \Phi(t, y, q, h)\| \leq L_1\|x - y\| + L_2\|p - q\|$$

for all $(t, x, p, h), (t, y, q, h) \in [0,1] \times \mathbb{R}^{2d} \times [0, h_0)$. Show that, when $h < \min(h_0, L_2^{-1})$, we have x_i and w_i satisfy the following inequality:

$$\|x_i - w_i\| \leq \frac{\tau(h)}{L_1 + L_2}\left(\exp\left(\frac{(L_1 + L_2)\, t_i}{1 - hL_2}\right) - 1\right)\,.$$

You may directly use the following lemma without proof:

Lemma Let $s, t > 0$ and $\{a_i\}_{i=0}^k$ satisfying that $a_0 \geq -t/s$ and $a_{i+1} \leq (1+s)a_i + t$ for $i = 0, 1, ..., k-1$, then

$$a_i \leq \exp(is)\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}\,.$$

Math 206C Qualifying Exam, Fall 2021

Please provide detailed answers and solutions for any 3 of the following 5 problems.

Problem 1. (10 points)

a. (2 points) Describe the fundamental lemma of the calculus of variations.
b. (3 points) Describe the Brachistochrone problem and derive the functional for it when the initial velocity is zero. Do not solve the variational problem.
c. (5 points) Show that if y(x) is an extremum of the functional $J[y(x)] = \int_a^b F(x, y, y')dx$, and if the endpoints $y(a)$ and $y(b)$ are not specified, then y(x) must satisfy the Euler-Lagrange equation and the natural boundary conditions.

Problem 2. (10 points)

a. (2 points) Describe the definition of the action function.
b. (3 points) Define the canonical variable, and write down the Legendre transform and the canonical form of Euler-Lagrange equation.
c. (5 points) Find the extremal of the functional $J[r] = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (r')^2}d\theta$ where $r = r(\theta)$.

Problem 3. (10 points)

a. (3 points) Explain the connection of the solution of the Euler-Lagrange equation with solution of the Hamilton-Jacobi equation.
b. (4 points) Given the functional $J[y(x)] = \int_{x_0}^{x_1} F(x, y, y')dx$. Prove that the transversality is the same as orthogonality if and only if $F(x, y, y')$ has the form $F(x, y, y') = g(x, y)\sqrt{1 + (y')^2}$ with $g(x, y) \neq 0$ near the point of intersection.
c. (3 points) Give definitions of the fixed points of continuous and discrete dynamical systems, and describe how to determine whether the fixed points are hyperbolic or not.

Problem 4. (10 points)

a. (2 points) Describe the definition of topological conjugacy.
b. (2 points) Give a definition of a chaotic discrete dynamical system.
c. (4 points) Determine the stability of fixed points and the period-2 cycle of the discrete dynamical system $x_{n+1} = rx_n(1 - x_n)$ when $r \in (3, 1 + \sqrt{6})$.
d. (2 points) Determine the first four elements in the itinerary with $x_0 = 1/8$ for the logistic map with $r = 4$.

Problem 5. (10 points)

a. (2 points) For a two-dimensional continuous dynamical system, give the definition of nullclines, and describe how it can help plot the phase portrait of the system.
b. (4 points) Consider the dynamical system $\dot{x} = rx + 4x^3$, find the bifurcation points and identify the type of bifurcation.

c. (4 points) For the following system,

$$\dot{x} = x(3 - x - y)$$
$$\dot{y} = y(2 - x - y)$$

find and classify all of the fixed points. Determine whether the relation between the two species x and y is competition or cooperation, and explain.

Math 206A Qualifying Exam

2020

Please answer any **3** of the following 5 problems.

1. Consider a sequence of (not necessarily independent) random variables such that X_n has distribution

$$\mathbf{P}\left(X_n = \frac{1}{n}\right) = 1 - \frac{1}{n^2}, \ \mathbf{P}\left(X_n = n^2\right) = \frac{1}{n^2}$$

Part a: (5 points) Does X_n converge to 0 in probability? Explain.
Part b: (5 points) Must $X_n \to 0$ almost surely? Prove or give a counterexample.

2. Let a sequence X_n be defined as follows: $X_0 = 1$. For $n \geq 1$,

$$\mathbf{P}(X_n = \frac{3}{2}X_{n-1}) = \mathbf{P}(X_n = \frac{1}{2}X_{n-1}) = \frac{1}{2}$$

Assume that the choice of $\frac{3}{2}$ vs. $\frac{1}{2}$ is independent for each n.

Part a: (4 points) Compute $\mathbf{E}(X_n)$ as a function of n.
Part b: (6 points) Let $Y_n = (X_n)^{1/n}$. Show that there is a constant c such that $Y_n \to c$ almost surely, and determine that c (You may wish to consider logarithms here).

3. Let $X_1, \ldots, X_n$ be independent, identically distributed variables each having characteristic function

$$\varphi(t) = \frac{1}{1+t^2}$$

Part a: (5 points) Determine the mean and variance of X_1.
Part b: (5 points) Let $\overline{X_n} = \frac{1}{n}(X_1 + \cdots + X_n)$. Determine the mean, variance, and characteristic function of $\overline{X_n}$. (Note: If you did not solve part a, you may let μ and σ^2 be the mean and variance of X_1, and give your answers in terms of μ and σ^2).

4. Let X be a random number between 0 and 1. We can think of X as

$$X = 0.a_0 a_1 a_2 a_3 \ldots$$

where each a_i is equally likely to be any digit between 0 and 9.

Let A_1 be the event that X contains every 2 digit sequence at least once (i.e. that for each i and j we can find an n such that $(a_n, a_{n+1}) = (i, j)$). Let A_2 be the event that X contains every 2 digit sequence infinitely often.

Part a: (4 points) Determine whether each of A_1 and A_2 is a tail event, in the sense of Kolmogorov's $0-1$ law.
Part b: (6 points) show that $\mathbf{P}(A_2) = 1$.

5. (10 points) Let $X_1, X_2 \ldots X_n$ be independent, identically distributed random variables, and let

$$S_n = X_1 + \cdots + X_n$$

Let $c > 0$ be an arbitrary constant. Show that for any t we have

$$\mathbf{P}\left(S_n \geq t\right) \leq \frac{\left[E(e^{cX_1})\right]^n}{e^{ct}}$$

Qualify Exam on Numerical Analysis, Fall 2020

Please choose any three problems and submit the full solutions.

Problem 1. Consider the multistep method

$$y_i - y_{i-3} = \frac{h}{8}[3f_i + 9f_{i-1} + 9f_{i-2} + 3f_{i-3}]$$

applied to the ODE $y'(x) = f(x, y)$, where $f_i = f(x_i, y_i)$.
(a) What is the order of the local truncation error of the method?
(b) Determine whether the method is convergent or not.

Problem 2. Define

$$T_n(\cos\theta) = \cos n\theta,\ n = 0, 1, 2, ...,\ -\pi \le \theta \le \pi.$$

(a) Show that each T_n is a polynomial of degree n and that the T_n satisfy the three-term recurrence relation

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t),\ n = 1, 2, ...$$

(b) Prove that T_n is an nth orthogonal polynomial with respect to the weight function $\omega(t) = (1 - t^2)^{-\frac{1}{2}}$, $-1 < t < 1$.
(c) Find the explicit values of the zeros of T_n, thereby verify that all the zeros reside in the open support of the weight function.
(d) Find b_1, b_2, c_1, c_2 such that the order of the following quadrature is four.

$$\int_{-1}^{1} f(\tau)\frac{d\tau}{\sqrt{1-\tau^2}} \approx b_1 f(c_1) + b_2 f(c_2).$$

Problem 3. The sequence $\{x_n\}$ is defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}},\ \text{for } n \ge 1,\ A \text{ is a positive number.}$$

Show that x_n converges to $\sqrt{A}$ whenever $x_0 > 0$. What happens if $x_0 < 0$?

Problem 4. Determine a quadratic spline s that interpolates the data $f(0) = 0$, $f(1) = 1$, $f(2) = 2$ and satisfies $s'(0) = 2$.

Problem 5. Picard's method for solving the initial-value problem

$$y' = f(t, y),\ a \le t \le b,\ y(a) = y_0,$$

is described as follows: Let $y_0(t) = y_0$ for each t in $[a, b]$. Define a sequence of functions $\{y_k(t)\}$ by

$$y_k(t) = y_0 + \int_a^t f(\tau, y_{k-1}(\tau))d\tau,\ k = 1, 2, \ldots$$

(a) Integrate $y' = f(t, y)$ and use the initial condition to derive Picard's method.
(b) Generate $y_0(t)$, $y_1(t)$, $y_2(t)$ and $y_3(t)$ for the initial-value problem

$$y' = -y + t + 1, \ 0 \leq t \leq 1, \ y(0) = 1.$$

(c) Solve the analytical solution of the initial value problem in **(b)** and compare the results in **(b)** to the Maclaurin series of the analytical solution.

Math 206C Qualifying Exam, Fall 2020

Please provide detailed answers and solutions for any 3 of the following 5 problems.

Problem 1. (10 points)

a) (2 points) Give definition of a continuous functional at a point.
b) (2 points) Give definition of a liner functional and give an example of such functional.
c) (2 points) Give definition of a variational derivative of a functional and give an example.
d) (4 points) Prove fundamental lemma of the calculus of variations.

Problem 2. (10 points)

a) (2 points) Give definition of an extremum of a functional.
b) (4 points) Prove the following theorem: If y(x) is an extremum of the functional J[y(x)], then y(x) satisfies the Euler-Lagrange equation.
c) (4 points) Use Euler-Lagrange equation to find the curve with the shortest length connecting: 1) two points in two dimensional Euclidian space, 2) two points on a sphere.

Problem 3. (10 points)

d) (5 points) Describe and explain in detail functionals for the Brachistochrone problem with 1) zero and 2) nonzero initial velocity. Do not solve variational problems.
e) (3 points) Define general variational problem and explain connection of solution of such problem with solution of the Hamilton-Jacobi equation.
f) (2 points) Give a definition of a chaotic discrete dynamical system and provide an example of such system.

Problem 4. (10 points)

a) (2 points) Give definitions of a fixed point of continuous and discrete dynamical systems.
b) (2 points) Give definition of stability of a fixed point of a discrete dynamical system.
c) (4 points) Determine stability of all fixed points of discrete dynamical systems:
1) $x_{n+1} = 6(x_n)^2$, 2) logistic system: $x_{n+1} = r\, x_n (1 - x_n)$ for $1<r<3$.
d) (2 points) Explain period doubling in the bifurcation diagram of the logistic discrete dynamical system as parameter r changes from 0 to 4.

Problem 5. (10 points)

The basic variables identifying the state of the population in the epidemiological SIR model are as follows:

- $S(t)$ is the number of susceptibles at time t;
- $I(t)$ is the number of infectives at time t;
- $R(t)$ is the number of immune at time t.

The occurrence of a disease imparting immunity is described by the following SIR system:

$$\frac{dS}{dt} = -\frac{\beta SI}{N},$$
$$\frac{dI}{dt} = \frac{\beta SI}{N} - \nu I,$$
$$\frac{dR}{dt} = \nu I.$$

with the assumption that: *S* + *I* + *R* = *N* = *const.*

a) (3 points) Explain meaning of each term and each parameter in the equations.
b) (2 points) Show that I(t) approaches 0 as t approaches infinity. Hint: N is fixed.
c) (3 points) Compute the Jacobian matrix for the system consisting of first two equations above and calculate eigenvalues of this matrix. Use expressions for the eigenvalues to determine ranges of parameter values for which the equilibrium of the system is stable.
d) (2 points) Give definition of an epidemic and explain why an epidemic occurs only if:

$$R = \frac{\beta S(0)}{\nu N} > 1.$$

Hint: Two first equations are decoupled from the third equation. Divide second equation by the first equation. Solve resulting first order equation and analyze its solution.

Math 206A Qualifying Exam

September 2019

Please choose any 3 of the following 5 problems.

1. Let X be a non-negative random variable.

Part a: (5 points) Show that

$$\mathbf{E}(X) = \int_0^\infty \mathbf{P}(X \geq t)\, dt$$

Part b: (5 points) Suppose that $X_1, X_2, \ldots, X_n, \ldots$ are independent variables, all having the same distribution as X. Show that with probability 1 there are only finitely many n for which $X_n \geq n$.

2. Let $\{X_n\}$ be a sequence of random variables.

Part a: (5 points) Define what it means for X_n to converge to X in probability, and what it means for X_n to converge to X almost surely.

Part b: (5 points) Suppose that $X_n \to X$ almost surely. Must $\mathbf{E}(X_n) \to \mathbf{E}(X)$? Prove or give a counterexample with justification.

3. Recall that a Normal Variable with mean μ and variance σ^2 (also known as an $N(\mu, \sigma^2)$ variable) has density given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Part a: (2 points) Set up an integral that gives the characteristic function $\varphi(t)$ of such a normal variable. **You do not need to evaluate this integral!**.

It turns out that the integral from part a evaluates to

$$\varphi(t) = e^{it\mu - \frac{1}{2}\sigma^2 t^2}$$

Part b: (4 points) Suppose that $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent. Determine, with justification, the distribution of $X + Y$.

Part c: (4 points) Suppose that $X_n \to X$ in distribution, each X_n has a distribution of the form $N(0, \sigma_n^2)$. Show that the σ_n^2 are converging to some σ^2, and that $X \sim N(0, \sigma^2)$.

4. Let $X_1, X_2, \ldots$ be independent variables, with $\mathbf{E}(X_n) = 1$ and $\mathbf{Var}(X_n) = n^{1.5} - (n-1)^{1.5}$. Define the **Sample Mean** $\overline{X_n}$ by

$$\overline{X_n} = \frac{1}{n}\left(X_1 + \cdots + X_n\right)$$

Part a (5 points) Determine the mean and variance of $\overline{X_n}$. (Hint: Telescope!)
Part b (5 points) Show that $\overline{X_n} \to 1$ in probability.

5. Let $X_1, \ldots, X_n$ be variables, and suppose that they have a joint density $f(x_1, x_2, \ldots, x_n)$ which factors as

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2) \ldots f_n(x_n)$$

as the product of n functions depending individually only on a single x_i (the $f_i(x_i)$ are not necessarily density functions). Show that $(X_1, \ldots, X_n)$ are independent.

Qualify Exam on Numerical Analysis, Fall 2019

Please choose any three problems and submit the full solutions.

Problem 1. Consider an ODE of the form

$$y' = f(x, y),\ y(x_0) = y_0,\ x, x_0 \in [a, b]$$

whose solution is to be approximated using a general one-step method of the form

$$y_{i+1} = y_i + h\Phi(x_i, y_i, h) \tag{1}$$

where $h = \frac{b-a}{N}$ is the step size and $\Phi(x, y, h)$ is Lipschitz continuous on y with constant λ. Assume that the method is of order p so that the local truncation error gives

$$y(x_{i+1}) = y(x_i) + h\Phi(x_i, y(x_i), h) + \tau_i h^{p+1}$$

where $y(x)$ is the exact solution of the IVP and τ_i is a constant that depends on derivatives of this solution.

(a) Derive an error estimate for the obtained approximation of the form

$$|e_i| \leq C_1|e_0| + C_2 h^p,\ i = 0, 1, ..., N$$

where C_1 and C_2 are constants.

(b) For a second order Runge-Kutta method of your choice, give the explicit representation of $\Phi(x_i, y(x_i), h)$ that arises when the method is expressed in form (1).

Problem 2. Consider the multistep method

$$y_i - y_{i-4} = \frac{h}{3}[8f_{i-1} - 4f_{i-2} + 8f_{i-3}]$$

applied to the ODE $y'(x) = f(x, y)$, where $f_i = f(x_i, y_i)$.
What is the order of the local truncation error of the method?

Problem 3. Prove that

$$||X^{(k)} - X|| \leq ||T||^k ||X^{(0)} - X||$$

and

$$||X^{(k)} - X|| \leq \frac{||T||^k}{1 - ||T||} ||X^{(1)} - X^{(0)}||$$

where T is an $n \times n$ matrix with $||T|| < 1$ and

$$X^{(k)} = TX^{(k-1)} + C, \quad k = 1, 2, \ldots$$

with $X^{(0)}$ arbitrary, $C \in R^n$, and $X = TX + C$.

Problem 4. Derive an $O(h^4)$ five-point formula to approximate $f'(x_0)$ that uses $f(x_0 - h)$, $f(x_0)$, $f(x_0 + h)$, $f(x_0 + 2h)$ and $f(x_0 + 3h)$.

Problem 5. (Construction of Gaussian Quadrature) Let $\{p_0(x), p_1(x), ...\}$ be Legendre polynomials satisfying

- For each n, $p_n(x)$ is a monic polynomial of degree n
- $\int_{-1}^{1} p(x)p_n(x)\,dx = 0$ for any $p(x)$ of degree $\leq n$.

Prove that

(1) All m zeros of a Legendre polynomial $p_m(x)$ reside in the interval $[-1, 1]$ and they are simple.

(2) Let $x_1, x_2, ..., x_n$ are the n roots of the n^{th} Legendre polynomial $p_n(x)$ and define

$$c_i = \int_{-1}^{1} \prod_{j=1, j\neq i}^{j=n} \frac{x - x_j}{x_i - x_j}\,dx,$$

then for any polynomial $p(x)$ of degree less than $2n$,

$$\int_{-1}^{1} p(x)\,dx = \sum_{i=1}^{i=n} c_i p(x_i).$$

Problem 1. Variational problems.

a) Prove fundamental lemma of the calculus of variations.
b) A spring hanging vertically under gravity. Consider a mass *m* on the end of a spring of natural length *l* and spring constant *k*. Let y be the vertical coordinate of the mass as measured from the top of the spring. Assume the mass can only move up and down in the vertical direction. The Lagrangian for this problem is as follows:

$$L = \frac{1}{2} m \dot{y}^2 - \frac{1}{2} k (y - \ell)^2 + mgy.$$

Explain terms in this Lagrangian, corresponding variational problem and determine and solve the corresponding Euler- Lagrange equation of motion. (Lagrangian is the function in the functional of the variational problem.)

Problem 2.

a. The Brachistochrone problem. An experimenter lets a bead slide with zero initial velocity down a wire that connects two fixed points with zero initial. Determine the shape of the wire that the bead slides from one end to the other in minimal time.
b. Describe functional for the Brachistochrone problem with nonzero initial velocity. Do not solve variational problem.

Problem 3. Chaotic dynamical systems.

a. Give a definition of a chaotic dynamical system.
b. Give definition of the tent map and demonstrate sensitive dependence on initial conditions for the tent map.
c. Describe bifurcation diagram of the logistic map:

$$x_{n+1} = r \cdot x_n (1 - x_n)$$

and explain period doubling.

d. Describe in detail behavior of trajectories of the logistic map when parameter *r* reaches bifurcation point at 4.

Problem 4. Bifurcations of dynamical systems.

a. Give detailed definition of a bifurcation of a dynamical system.
b. Find equilibrium points and determine their types for different values of parameter μ of the following dynamical system:

$dx_1/dt = \mu x_1 + x_1^3$
$dx_2/dt = -x_2.$

c. Explain what happens with the equilibrium points and their stability as μ goes through zero in this dynamical system. Sketch bifurcation diagram.

d. Give a definition of a normal form and explain in detail what normal forms are used for.

Problem 5. Variational problems.

a. Prove that if y(x) is an extremal of the functional J:

$$J[y(x)] = \int_{x_0}^{x_1} F(x, y, y')dx.$$

then it satisfies the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

b. Give definitions of the Hamilton-Jacobi equation and eikonal. Explain relation between eikonals and extremals.