Qualify Exam on Differential Equations, Fall 2023

Please choose any TWO problems in each part.

Part A

Problem 1. For the following equation, sketch all the qualitatively different vector fields that occur as r is varied. Show that a bifurcation occurs at a critical value of r and identify the bifurcation type. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

$$x' = rx - \frac{x}{1+x^2}$$

Problem 2. Let $A(t)$ be a continuous function from t in R to the space of real-valued $d \times d$ matrices, $d \geq 1$.
(a) Show that for every solution of the linear system, $X'(t) = A(t)X(t)$, we have

$$||X(t)|| \leq ||X(0)||e^{\int_0^t ||A(s)||ds},$$

where $||A(s)||$ is the operator norm and $||X(t)||$ is the usual Euclidean norm.
(b) Show that if $\int_0^t ||A(s)||ds < \infty$, then every solution $X(t)$ has a finite limit as $t \to \infty$.

Problem 3. (a) Let $f(t)$ be a nonnegative function satisfying the inequality

$$f(t) \leq K_1 + \epsilon(t-\alpha) + K_2 \int_\alpha^t f(s)\,ds,$$

on an interval $\alpha \leq t \leq \beta$, where ϵ, K_1, K_2 are given positive constants. Show that

$$f(t) \leq K_1 exp[K_2(t-\alpha)] + \frac{\epsilon}{K_2}(exp[K_2(t-\alpha)] - 1).$$

(b) Let $f(t,y)$ and $g(t,y)$ be continuous and satisfy a Lipschitz condition with respect to y in a region D. Suppose $|f(t,y) - g(t,y)| < \epsilon$ in D for some $\epsilon > 0$. Let $\phi_1(t)$ be a solution of $y' = f(t,y)$ and let $\phi_2(t)$ be a solution of $y' = g(t,y)$ such that $|\phi_2(t_0) - \phi_1(t_0)| < \delta$ for some t_0 and some $\delta > 0$. Apply the inequality proved in (a) to find an upper bound of $|\phi_2(t) - \phi_1(t)|$ for all t such that $\phi_1(t)$ and $\phi_2(t)$ both exist.

Part B

Problem 1. Suppose $u \in C^2$ solves the wave equation $u_{tt} - \Delta u = 0$ on $\mathbb{R}^n \times (0, \infty)$. Fix $x \in \mathbb{R}^n$, $t > 0$, $r > 0$, and define

$$U(x; r, t) := \partial B(x, r) u(y, t) dS(y).$$

Prove that U satisfied

$$U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 \quad \text{in} \quad \mathbb{R}_+ \times (0, \infty).$$

(You can use known results from section 2.2 obtained on the Laplace equation without providing any proof.)

Problem 2. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume that $u \in C^2(\bar{\Omega})$ solves

$$\Delta u = u^{2k+1} \qquad \text{in} \quad \Omega,$$

with $u = 0$ on $\partial\Omega$. Here k is a positive integer. Show that $u \equiv 0$ in Ω.

Problem 3. Assume $n = 1$ and $u(x, t) = v(\frac{x^2}{t})$.

(a) Prove that

$$u_t = u_{xx}$$

if and only if

$$(*) \qquad 4zv''(z) + (2 + z)v'(z) = 0, \qquad z > 0.$$

(b) Show that the general solution of (*) is given by

$$v(z) = c \int_0^z s^{-\frac{s}{4}} s^{-\frac{1}{2}} ds + d.$$

(c) Use part (b) and differentiate $v(\frac{x^2}{t})$ with respect to x, and select a constant c properly to obtain the fundamental solution of the hear equation in dimension $n = 1$.

Part C

Question 1

Suppose u is a smooth solution of

$$u_t - \Delta u - u = 0$$

in U_T with

$$u = 0$$

on $\partial U \times [0, T]$ and

$$u(x, 0) = g(x) \geq 0.$$

where c is bounded but not necessarily nonnegative. Show that $u \geq 0$.

Question 2

Let U be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $f \in L^2(U)$ and $\mu > 0$ be a constant. Consider the Dirichlet problem

$$\begin{aligned} -\Delta u + \mu u = f \qquad &\text{in} \qquad U, \\ u = 0 \qquad &\text{on} \qquad \partial U. \end{aligned}$$

(1) Define what is means for $u \in H_0^1(U)$ to be a weak solution.
(2) Show that a weak solution exist.

Question 3

Consider the function

$$f(x) = \begin{cases} x & -1 < x < 0 \\ \sin x & 0 \leq x < 1 \end{cases}$$

(1) Determine the weak derivative $f'(x)$.
(2) Does $f''(x)$ exist in the weak sense?

Qualify Exam on Differential Equation, Fall 2022

Please choose any TWO problems in each part.

Part A

Problem 1. Consider the system $x'(t) = rx + x^3 - x^5$.
(a) Find algebraic expressions for all the fixed points as r varies.
(b) Sketch the vector fields as r varies. Be sure to indicate all the fixed points and their stability.
(c) Plot the bifurcation diagram for $-\infty < r < \infty$. Find values of r at which bifurcations occur and classify all the bifurcations.

Problem 2. A *fundamental solution* to the autonomous linear system, $X'(t) = AX$, is a nonsingular matrix-valued function, $\Phi : \mathbb{R} \to M^{d\times d}$, with $\Phi'(t) = A\Phi(t)$.
(a) Show that $\Psi(t) = e^{At}$ is a fundamental solution satisfying $\Psi(0) = I$, the identity matrix.
(b) Show that $X(t) = \Phi(t)\Phi(0)^{-1}X_0$ is a solution to the IVP, $X'(t) = AX$, $X(0) = X_0$.
(c) Show that any fundamental solution is of the form, $\Phi(t) = e^{At}M$, for some nonsingular matrix M.
(d) Consider the nonhomogeneous linear system,

$$X' = AX + f(t)$$

where f is continuous in time. Show that

$$X(t) = \Phi(t)\Phi(0)^{-1}X_0 + \int_0^t \Phi(t)\Phi^{-1}(s)f(s)ds$$

is a solution to the initial value problem, $X' = AX + f(t)$, $X(0) = X_0$.

Problem 3. Consider the following first-order 2D system of ODEs:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} (1+x)\sin y \\ 1 - x - \cos y \end{pmatrix}$$

(a) Determine all the fixed points.
(b) Determine the corresponding linear system near each fixed point.
(c) Find the eigenvalues of each linear system. What conclusions can you draw about the stability of the linear system?
(d) Draw a phase portrait of the nonlinear system to confirm your conclusions, or to extend them in those cases where the linear system does not provide definite information about the nonlinear system.

Part B

Problem 1. Let $u \in C^2(\Omega)$ be a harmonic function. Prove that

$$u(x) = \fint_{\partial B(x,r)} u dS = \fint_{B(x,r)} u dy,$$

for every $B(x,r) \subset \Omega$.

Problem 2. (Backwards Uniqueness for the heat equation) Let $u_1, u_2 \in C^2(\overline{U}_T)$ solve the heat equation $u_t - \Delta u = 0$ with

$$u_1 = u_2 = g \quad \text{on} \quad \partial U \times [0,T]$$

and $u_1(x,T) = u_2(x,T)$ for $x \in U$. Prove that

$$u_1 \equiv u_2 \quad \text{within} \quad U_T.$$

Problem 3. (Finite propagation speed of waves)Let u solve the wave equation $u_{tt} - \Delta u = 0$. If $u = u_t = 0$ on $B(x_0, t_0) \times \{t = 0\}$ with $t_0 > 0$, then $u = 0$ on $C(x_0, t_0)$, where

$$C(x_0, t_0) = \{(x,t) : 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t\}.$$

Part C

Question 1

Assume U is connected. A function $u \in H^1(U)$ is a weak solution of Newmann's problem

$$-\Delta u = f \qquad \text{in } U$$

and

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{in } \partial U$$

provided

$$\int_U Du \cdot Dv dx = \int_U fv dx$$

for all $v \in H^1(U)$. Prove that the equation has a weak solution if and only if

$$\int_U f dx = 0.$$

Question 2

Let U be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $f \in L^2(U)$ and $\mu > 0$ be a constant. Consider the Dirichlet problem

$$\begin{aligned} -\Delta u + \mu u = f \qquad &\text{in} \qquad U, \\ u = 0 \qquad &\text{on} \qquad \partial U. \end{aligned}$$

(1) Define what is means for $u \in H_0^1(U)$ to be a weak solution.
(2) Show that a weak solution exist.

Question 3

Suppose U is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0$$

a.e. in U. Prove u is constant a.e. in U.

Printed Name: ________________ **Signature:** ________________

Differential Equations Qualifying Exam, Fall 2021

Instructions: Work 2 out of 3 problems in each of the 3 parts for a total of 6 problems.

Part 1

Problem 1. Consider the system $x'(t) = rx(t) - \sin x(t)$.
(a) For the case $r = 0$, find and classify all the fixed points, and sketch the vector field.
(b) Show that when $r > 1$, there is only one fixed point. What kind of fixed point is it?
(c) As r decreases from ∞ to 0, classify all the bifurcations that occur.
(d) For $0 < r << 1$, find an approximate formula for values of r at which bifurcations occur.
(e) Classify all the bifurcations that occur as r decreases from 0 to $-\infty$.
(f) Plot the bifurcation diagram for$-\infty < r < \infty$, and indicate the stability of the various branches of fixed points.

Problem 2. Here is an iteration scheme of Tonelli, which can replace the iteration scheme we have been using in all of the proofs of existence we have seen: Fix $T > 0$ and for $n = 1, 2, ...$, define

$$x_n(t) = \begin{cases} x_0 & 0 \leq t \leq \frac{T}{n} \\ x_0 + \int_0^{t-T/n} f(s, x_n(s))ds & \frac{T}{n} \leq t \leq T \end{cases}$$

for an initial value problem $x'(t) = f(t, x(t))$ and $x(0) = x_0$. Use this iteration scheme as an alternative in the proof of solution existence for the IVP. State clearly the theorem you are proving including the conditions f needs to satisfy, and then prove it.

Problem 3. Let $A(t)$ be a continuous function from t in R to the space of real-valued $d \times d$ matrices, $d \geq 1$.
(a) Show that for every solution of the linear system, $X'(t) = A(t)X(t)$, we have

$$||X(t)|| \leq ||X(0)||e^{\int_0^t ||A(s)||ds},$$

where $||A(s)||$ is the operator norm and $||X(t)||$ is the usual Euclidean norm.
(b) Show that if $\int_0^t ||A(s)||ds < \infty$, then every solution $X(t)$ has a finite limit as $t \to \infty$.

Part 2

Problem 1. Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$ and define the cone

$$C = \{(x,t) : \mathbb{R}^n \times \mathbb{R} : 0 \le t \le t_0, |x - x_0| < t_0 - t\}.$$

Suppose $u \in C^2$ solves the wave equation $u_{tt} - \Delta u = 0$ with $u \equiv u_t \equiv 0$ on $B(x_0, t_0) \times \{t = 0\}$. Prove that $u \equiv 0$ withine the cone C.

Problem 2. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume that $u \in C^2(\bar{\Omega})$ solves

$$\Delta u = u^7 + 2u^5 + 3u \qquad \text{in } \ \Omega,$$

with $u = 0$ on $\partial\Omega$. Show that $u \equiv 0$ in Ω.

Problem 3. Assume $n = 1$ and $u(x,t) = v(\frac{x^2}{t})$.

(a) Prove that

$$u_t = u_x x$$

if and only if

$$(*) \qquad 4zv''(z) + (2+z)v'(z) = 0, \qquad z > 0.$$

(b) Show that the general solution of (*) is given by

$$v(z) = c\int_0^z s^{-\frac{s}{4}} s^{-\frac{1}{2}} ds + d.$$

(c) Use part (b) and differentiate $v(\frac{x^2}{t})$ with respect to x, and select a constant c properly to obtain the fundamental solution of the hear equation in dimension $n = 1$.

Part 3

Question 1

Assume U is connected. A function $u \in H^1(U)$ is a weak solution of Newmann's problem

$$-\Delta u = f \qquad \text{in } U$$

and

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{in } \partial U$$

provided

$$\int_U Du \cdot Dv dx = \int_U f v dx$$

for all $v \in H^1(U)$. Prove that the equation has a weak solution if and only if

$$\int_U f dx = 0.$$

Question 2

Let U be the interval $(0,1)$ in $\mathbb{R}$. Show that if u is a smooth solution to

$$u_{tt} - u_{xx} = 0$$

in $U \times (0,T]$ with

$$u = 0$$

on $\partial U \times [0,T]$ and

$$u(x,0) = u_t(x,0) = 0.$$

on $U \times \{t = 0\}$, then u is identically zero.

Question 3

Suppose U is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0$$

a.e. in U. Prove u is constant a.e. in U.

Qualify Exam on Differential Equation, Fall 2020

Choose 2 problems from each part to answer.

Part I

Problem 1. Consider the conservation equation in the form,

$$\begin{cases} \partial_t u + \partial_x(F(u)) = 0, \ (t,x) \in [0,\infty) \times R, \\ u(0) = u_0 \end{cases}$$

where we assume that F is twice continuously differentiable on R and strictly convex. We will suppose that u is a solution up to time $T > 0$.
(a) Let $t \to x(t)$ be a curve that satisfies $\dot{x}(t) = F'(u(t, x(t)))$ for $0 \leq t < T$. Show that u is constant along such a curve.
(b) Conclude from (a) that the curve $t \to x(t)$ must be a straight line.
(c) Let $\lambda = F'(u)$ and show that λ solves Burgers equation with $\lambda(0) = F'(u_0)$; that is

$$\begin{cases} \partial_t \lambda + \lambda \partial_x \lambda = 0, \ (t,x) \in [0,\infty) \times R \\ \lambda(0) = \lambda_0 \end{cases}$$

Problem 2. Consider the following first-order 2D system of ODEs:

$$\dot{x} = ((1+x)\sin y, 1 - x - \cos y).$$

(a) Determine all the fixed points.
(b) Determine the corresponding linear system near each fixed point.
(c) Find the eigenvalues of each linear system. What conclusions can you draw about the stability of the linear system?

Problem 3. Derive the characteristic ODEs and solve the solution for

$$\begin{cases} xu_y - yu_x = u, \ \text{in } \{(x,y) | x > 0, 0 < y < x\} \\ u(x,x) = x^2, \text{for } x \geq 0 \end{cases}$$

Part II

Problem 1. Prove that for each connected set $V \subset\subset U$, there exists a positive constant C, depending on V, such that

$$\sup_V u \leq C \sup_V$$

for all nonnegative harmonic functions u in U.

Problem 2. (Backwards Uniqueness for the heat equation) Let $u_1, u_2 \in C^2(\overline{U}_T)$ solve the heat equation $u_t - \Delta u = 0$ with

$$u_1 = u_2 = g \ \text{ on } \ \partial U \times [0,T]$$

and $u_1(x,T) = u_2(x,T)$ for $x \in U$. Prove that

$$u_1 \equiv u_2 \quad \text{within } \ U_T.$$

Problem 3. Let u solve the wave equation in dimension one i,e. $u_{tt} - \Delta u = 0$ in $\mathbb{R} \times [0, \infty)$ with

$$u = g, \quad u_t = h \quad \text{on} \quad \mathbb{R} \times \{t = 0\},$$

where g and h have compact support. Prove that for large enough time t we have

$$k(t) = p(t),$$

where $k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx$ in the kinetic energy and $p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx$.

Part III

Question 1

Assume U is connected. A function $u \in H^1(U)$ is a weak solution of Newmann's problem

$$-\Delta u = f \qquad \text{in } U$$

and

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{in } \partial U$$

provided

$$\int_U Du \cdot Dv dx = \int_U f v dx$$

for all $v \in H^1(U)$. Prove that the equation has a weak solution if and only if

$$\int_U f dx = 0.$$

Question 2

Let $u \in H^1(\mathbb{R}^n)$ have compact support and u is a weak solution of the semilinear PDE

$$-\Delta u + c(u) = f$$

where $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \to \mathbb{R}$ is smooth with $c(0) = 0$ and $c' \geq 0$. Assume $c(u) \in L^2(\mathbb{R}^n)$.

Mimicing the proof of Theorem 1 in Ch 6.3.1 (without the cut-ff function), derive the estimate

$$||D^2 u||_{L^2} \leq C||f||_{L^2}$$

Question 3

Let U be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume that $u \in C^2(\bar{U}) \cap H_0^1(U)$ be a strong solution to

$$\begin{aligned} \Delta u = u^5 + 2u^3 + 3u \qquad &\text{in} \qquad U, \\ u = 0 \qquad &\text{on} \qquad \partial U. \end{aligned}$$

Show that $u \equiv 0$ is the only solution.

PDE Written Qualifying Exam

Sep 24, 2019

NAME (please print):

1. Please answer each part of the exam according to the instruction.

2. The exam will be 180 minutes

Part 1	10 pts.	
Part 2	10 pts.	
Part 3	10 pts.	
Total	30 pts.	

Part 1

Problem 1. In this problem, we will seek a solution on a portion of the plane (to be specified in part (c)) to

$$\begin{cases} 2y\partial_x u + 2x\partial_y u = -u, \\ u(x,0) = \psi(x), \end{cases}$$

where $\psi: \mathbb{R} \to \mathbb{R}$ is continuous.

1. What are the characteristic equations for this PDE?
2. Solve the characteristic equations.
3. Using the solution in (b) to the characteristic equations, obtain an explicit solution to the PDE in the form $u = u(x,y)$ on the domain $\{(x,y) \in \mathbb{R}^2 : x > y > 0\}$.

   Hint: You may find the identity, $\cosh^2 z - \sinh^2 z = 1$, useful

Problem 2. Consider the 2D-system, $\dot{\mathbf{x}} = (x_2, -9\sin x_1 - \frac{1}{5}x_2)$, with $\mathbf{x} = \mathbf{x}(t) = (x_1(t), x_2(t))$.

1. What are the equilibrium (fixed) points of this system?
2. Show that the equilibrium points consist of stable foci (inward spirals) or saddle points.
3. Sketch the phase portrait [that is, a group of representative trajectories] of this system, including at least four equilibrium points in your sketch. You might find it useful to plot the direction field [arrows representing the direction of the underlying velocity field] at a few points to help fill in your sketch. (And this is one way to find, for instance, the direction of the spirals.)

Problem 3. Let $A(t)$ be a continuous function from t in $\mathbb{R}$ to the space of real-valued $d \times d$ matrices, $d \geq 1$.

1. State, precisely, what the phrase, "$A(t)$ is a continuous function from t in $\mathbb{R}$ to the space of real-valued $d \times d$ matrices, $d \geq 1$," means.
2. Show that for every solution $\mathbf{x}$ of the (non-autonomous) linear system, $\dot{\mathbf{x}} = A(t)\mathbf{x}$, we have

   $$\|\mathbf{x}(t)\| \leq \|\mathbf{x}(0)\| e^{\int_0^t \|A(s)\|\, ds},$$

   where $\|A(s)\|$ is the operator norm and $\|\mathbf{x}(t)\|$ is the usual Euclidean norm. **Note**: You do not need to prove existence of a solution, you may take that as given.
3. Show that if $\int_0^t \|A(s)\|\, ds < \infty$ then every solution has a finite limit as $t \to \infty$; that is, $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_\infty$ for some vector $\mathbf{x}_\infty \in \mathbb{R}^d$.

Part 2

Problem 1. Let $u \in C(\Omega)$ satisfy the mean value property

$$u(x) = \fint_{\partial B(x,r)} dS = \fint_{B(x,r)} u dy$$

for each ball $B(x,r) \subset \Omega$. Prove that $u \in C^\infty(\Omega)$.

Problem 2. Let $T > 0$, $c \in C^0(\overline{\Omega})$, and $u \in C_1^2(\Omega_T) \cap C^0(\overline{\Omega})$ satisfy

$$\begin{cases} u_t - \Delta u + c(x,t)u = 0 & \text{in } \Omega_T \\ u \le 0 & \text{on } \partial\Gamma_T, \end{cases}$$

where Γ_T is the parabolic boundary of $\Omega_T = \Omega \times (0,T)$. Prove that $u \le 0$ in $\overline{\Omega}_T$.

Problem 3. Fix $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$ and let

$$C = \{(x,t) | 0 \le t \le t_0, \quad |x - x_0| \le t_0 - t\}.$$

Prove that if $u \equiv u_t \equiv 0$ on $B(x_0, t_0) \times \{t = 0\}$, then $u \equiv 0$ within the cone C (Finite Propagation speed for the Wave Equation).

Part 3

Problem 1. Fix $\alpha > 0$ and let $U = B(0,1)$. Show that there exists a constant C depending only on n and α such that

$$\int_U u^2 dx \leq C \int_U |Du|^2 dx$$

provided $u \in H^1(U)$ and

$$|\{x \in U | u(x) = 0\}| \geq \alpha$$

Problem 2. Let U be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $f \in L^2(U)$ and $\mu > 0$ be a constant. Consider the Dirichlet problem

$$\begin{aligned} -\Delta u + \mu u = f \qquad & \text{in} \qquad U, \\ u = 0 \qquad & \text{on} \qquad \partial U. \end{aligned}$$

1. Define what is means for $u \in H_0^1(U)$ to be a weak solution.
2. Show that a weak solution exist
3. If f is smooth. What can we conclude about the weak solution.

Problem 3. Let U be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume that $u \in C^2(\bar{U}) \cap H_0^1(U)$ be a strong solution to

$$\begin{aligned} \Delta u = u^5 + 2u^3 + 3u \qquad & \text{in} \qquad U, \\ u = 0 \qquad & \text{on} \qquad \partial U. \end{aligned}$$

Show that $u \equiv 0$ is the only solution.

Printed Name: ________________ **Signature:** ________________

Applied Math Qualifying Exam Fall 2017

Instructions: Work 2 out of 3 problems in each of the 3 parts for a total of 6 problems.

Part 1

(1) Let $(u_n)_{n=1}^\infty$ be a sequence of harmonic functions defined on an open bounded subset U of $\mathbb{R}^d$, $d \ge 2$, with each $u_n \in C^2(U)$. Assume that $u_n \to u$ uniformly on U. Prove that u is harmonic on U.

(2) Consider the transport equation,

$$\begin{cases} \partial_t f_j + \mathbf{u} \cdot \nabla f_j = 0 & \text{on } \mathbb{R} \times U, \\ f_j(0,x) = f_{0,j}(x) & \text{on } U, \end{cases}$$

for $j = 1, 2$. Here,

- U is a bounded open subset of $\mathbb{R}^d$, $d \ge 2$, having C^∞ boundary;
- $\mathbf{u}$ is a given time-independent vector field in $C^\infty(\overline{U})$ with $\mathbf{u} \cdot \boldsymbol{n} = 0$ on ∂U;
- $f_j = f_j(t,x)$, $j = 1, 2$, is a scalar-valued function of time and space;
- $f_{0,j}$, $j = 1, 2$, lie in $C(\overline{U})$;

You may assume the existence and uniqueness of solutions and the existence and uniqueness of a flow map for $\mathbf{u}$ without proof. (Both solutions and the flow map will be continuous in time and space.)

(a) Use an energy argument to prove that for all $t \ge 0$,

$$\|f_1(t) - f_2(t)\|_{L^2}^2 \le \|f_{0,1} - f_{0,2}\|_{L^2}^2 \exp \int_0^t \|\operatorname{div} \mathbf{u}(s)\|_{L^\infty} \, ds.$$

Here, the L^2-norm is defined by

$$\|h\|_{L^2}^2 = \int_U h(x)^2 \, dx.$$

(b) Using the flow map for $\mathbf{u}$ (or any other method you can come up with) prove that for all $t \ge 0$,

$$\|f_1(t) - f_2(t)\|_{L^\infty} \le \|f_{0,1} - f_{0,2}\|_{L^\infty}.$$

(3) Let $\mathbf{v}\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a time-varying vector field. Assume that for some $M_1 > 0$, $\|\mathbf{v}(t)\|_{L^\infty} \leq M_1$ for all $t \in \mathbb{R}$ and for some $M_2 > 0$, $\mathbf{v}(t)$ has a Lipschitz constant no larger than M_2 for all $t \in \mathbb{R}$.

(a) Show that for any $(t_0, \mathbf{x}_0) \in \mathbb{R} \times \mathbb{R}^d$, solutions to

$$\begin{cases} \mathbf{x}'(t) = \mathbf{v}(t, \mathbf{x}(t)), \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

are unique. (You do not need to prove existence.)

(b) Define $\mathbf{Y}\colon \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ by

$$\mathbf{Y}(t_0, \mathbf{x}_0, t) = \mathbf{x}(t),$$

where $\mathbf{x}$ is the solution from (a). Prove that $\mathbf{Y}$ is continuous.

Part 2

(1) Let U be a bounded open set with smooth boundary ∂U. Consider the initial boundary value problem for $u(x,t)$:

$$\begin{cases} u_t - \Delta u + bu = f, & x \in U, t > 0, \\ u(x,0) = g(x), & x \in U, \\ u_t + \frac{\partial u}{\partial n} + u = 0, & x \in \partial U, t > 0, \end{cases}$$

where $\frac{\partial u}{\partial n}$ is the exterior normal derivative [and b is a constant]. Show that smooth solutions of this problem are unique.

(2) (a): Find an explicit solution to the problem:

$$\begin{cases} u_t - u_{xx} = \cos x, & x \in [0, 2\pi], t > 0, \\ u_x(0,t) = u_x(2\pi, t) = 0, & t > 0, \\ u(x,0) = \cos x + \cos 2x, & x \in [0, 2\pi]. \end{cases}$$

(Hint: consider $v(x,t) = u(x,t) - \cos x$, and employ the separation of variables to solve for v.)
(b): Does there exist a steady state solution to the equation in (a) with the boundary condition

$$u_x(0) = 1, \qquad u_x(2\pi) = 0?$$

Explain your answer.

(3) Find the solution of the partial differential equation

$$u_x + x^2 y u_y = -u,$$

with the condition $u(x = 0, y) = y^2$ using the method of characteristics.

Part 3

(1) Let U be a bounded domain in $\mathbb{R}^d$ with a C^∞ boundary, let $f \in L^2(U)$, and let $\mu > 0$ be a constant. Consider the Dirichlet problem,

$$\begin{cases} -\Delta u + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

(a) Define what it means for $u \in H_0^1(U)$ to be a weak solution to this Dirichlet problem.

(b) Show that a weak solution exists.

(2) Let U be a bounded domain in $\mathbb{R}^d$ with a C^∞ boundary. Assume that $u \in C^2(\overline{U}) \cap H_0^1(U)$ is a strong solution to

$$\begin{cases} \Delta u = u^3 + u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Note that $u \equiv 0$ is clearly a solution, but this is a nonlinear problem, so we have no general uniqueness theorem that covers it.

(a) Use the weak maximum principle to show that $u \equiv 0$ is the only solution.

(b) Show the same thing using an energy method.

(3) (a) Prove that for any $u \in C^1(\mathbb{R}^d)$ and any $p \in (1, \infty)$,

$$\partial_j |u|^p = p|u|^{p-1} \partial_j u \operatorname{sgn}(u).$$

Here, the derivative is a *classical* derivative. Also, $\operatorname{sgn}\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\operatorname{sgn}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

(b) Prove that for any $u \in H^1(\mathbb{R}^d)$ having the property that $|u| > \epsilon$ for some $\epsilon > 0$,

$$\partial_j |u|^2 = 2|u| \partial_j u \operatorname{sgn}(u),$$

where now we mean the *weak* derivative. (This is the weak derivative version of part (a) specialized to $p = 2$.)

Comment: The assumption that $|u(x)| > \epsilon$ is not necessary, but may help you in dealing with the sgn function, should you choose to employ a sequence of smooth approximating functions and use the result in part (a) for that sequence.

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APPLIED MATH QUALIFYING EXAM 11 OCTOBER 2014

Instructions: Work 2 out of 3 problems in each of the 3 parts for a total of 6 problems.

Part 1

(1) Let Ω be an open subset of $\mathbb{R}^d$ and let

$$C(\Omega) = \{f : \Omega \to \mathbb{R} \,\big|\, f \text{ is continuous}\}$$

with the norm,

$$\|f\|_{C(\Omega)} = \sup_{x \in \Omega} |f(x)|.$$

Prove that $C(\Omega)$ is a Banach space.

(2) Let Ω be a bounded domain in $\mathbb{R}^2$, $d \geq 1$, with smooth boundary.

(a) Use the divergence theorem to derive Green's identity,

$$\int_\Omega \Delta u \, v = -\int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} (\nabla u \cdot \mathbf{n}) v,$$

where u and v are smooth scalar-valued functions on $\overline{\Omega}$, and $\mathbf{n}$ is the outward unit normal vector.

(b) Consider the Cauchy problem,

$$\begin{cases} \partial_t u = \Delta u + cu & \text{for } (t, x) \in (0, \infty) \times \Omega, \\ u(t, x) = 0 & \text{for } (t, x) \in (0, \infty) \times \partial\Omega, \\ u(0, x) = g(x) & \text{for } x \in \Omega, \end{cases}$$

on a bounded domain $\Omega \subseteq \mathbb{R}^d$ having a smooth boundary. Here, c is a positive constant. Suppose u_1 and u_2 are two smooth solutions of the above Cauchy problem with different initial conditions g_1 and g_2. Show that if g_1 and g_2 are "close" in $L^2(\Omega)$ then the solutions u_1 and u_2 are also close in $L^2(\Omega)$ at any later time $t > 0$. Derive an estimate of how close. (Green's identity and Gronwall's inequality will be useful here.)

(3) Let $A(t)$ be a continuous function from t in $\mathbb{R}$ to the space of square, real-valued matrices.

(a) Show that for every solution of the (non-autonomous) linear system, $\dot{\mathbf{x}} = A(t)\mathbf{x}$, we have

$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}(0)\| e^{\int_0^t \|A(s)\| \, ds},$$

where $\|A(s)\|$ is the operator norm and $\|\mathbf{x}(t)\|$ is the usual Euclidean norm.

(b) Show that if $\int_0^t \|A(s)\| \, ds < \infty$ then every solution, $\mathbf{x}(t)$, has a finite limit as $t \to \infty$.

PART 2

(1) (a) Find the entropy solution to the Burgers' equation $u_t + uu_x = 0$ with the initial datum

$$g(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 1 - x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x \geq 1. \end{cases}$$

(b) Consider the Burgers' equation with source term 1 with the initial datum x:

$$u_t + uu_x = 1, \quad u(t = 0) = x.$$

Find the equation for the characteristics and also find an explicit formula for the solution of this initial value problem.

(2) Let $f \in C_c^2(\mathbb{R}^3)$ be given. Define for $x \in \mathbb{R}^3$

$$u(x) = \int_{\mathbb{R}^3} \Phi(x - y) f(y) dy$$

where $\Phi(x) = \frac{1}{4\pi|x|}$. Prove that $-\Delta u = f$ in $\mathbb{R}^3$. You can use the fact $u \in C^2(\mathbb{R}^3)$ without a proof.

(3) Let u be a classical solution of the following initial boundary value problem:

$$\begin{aligned} &u_t = u_{xx}, \qquad \text{in } (a, b) \times (0, T) \\ &u(a, t) = u(b, t) = 0 \\ &u(x, 0) = u_0(x) \end{aligned}$$

where u_0 is a continuous function.

(a) Show that the solutions are unique.

(b) Show that there exists a constant $\alpha > 0$ such that

$$\|u(\cdot, t)\|_{L^2}^2 \leq e^{-\alpha t} \|u_0\|_{L^2}^2.$$

Part 3

(1) Let U be the open unit ball in $\mathbb{R}^d$.

(a) Let

$$u(x) = |x|^{-\alpha}.$$

For which values of $\alpha > 0$, $d \geq 1$, and $p > 1$ does u belong to $W^{1,p}(U)$?

(b) Show that

$$u(x) = \log\log\left(1 + |x|^{-1}\right)$$

belongs to $W^{1,2}(U)$ but does not belong to $L^\infty(U)$.

(2) Let $U = (0,1)^2$, the unit square in $\mathbb{R}^2$. Can the Lax-Milgram theorem be applied to the bilinear form, $B[u,v]\colon H_0^1(U) \times H_0^1(U) \to \mathbb{R}$, defined by

$$B[u,v] = \int_0^1 \int_0^1 \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2} - \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}?$$

(3) Suppose $u \in C^2(U) \cap C(\overline{U})$ and let

$$Lu = \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j},$$

where the coefficient, a^{ij}, are continuous and satisfy the uniform ellipticity condition. Prove the weak maximum principle; namely, that if $Lu \leq 0$ then

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

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Applied Math Qualifying Exam 5 October 2013

Instructions: Work 2 out of 3 problems in each of the 3 parts for a total of 6 problems.

Part 1

(1) A *fundamental solution* to the autonomous linear system, $\dot{\mathbf{x}} = A\mathbf{x}$, is a nonsingular matrix-valued function, $\Phi\colon \mathbb{R} \to M^{d\times d}$, with $\Phi'(t) = A\Phi(t)$.

(a) Show that $\Psi(t) = e^{At}$ is a fundamental solution satisfying $\Psi(0) = I$, the identity matrix. (You may use standard facts about e^{At} without proof.)

(b) Show that $\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}_0$ is a solution to the IVP, $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$.

(c) Show that any fundamental solution is of the form, $\Phi(t) = e^{At}M$, for some non-singular matrix M.

(d) Consider the nonhomogeneous linear system,

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t),$$

where $\mathbf{b}$ is continuous in time. (So $\mathbf{b}$ can vary with time, but A cannot.) Show that

$$\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}_0 + \int_0^t \Phi(t)\Phi^{-1}(s)\mathbf{b}(s)\,ds$$

is a solution to the IVP, $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$.

(2) (a) Consider the linear system of ODEs,

$$\dot{y}_1 = -y_1, \quad \dot{y}_2 = 2y_2,$$

which has the origin as the only equilibrium point. Determine the explicit solution to this system given the initial condition, $\mathbf{y}(0) = \mathbf{a} = (a_1, a_2)$. What are the stable and unstable manifolds for this system? (One or both might be empty.)

(b) Now consider the perturbed, nonlinear system,

$$\dot{x}_1 = -x_1, \quad \dot{x}_2 = 2x_2 - 5\epsilon x_1^3,$$

which also has the origin as the only equilibrium point. Determine the explicit solution to this system given the initial condition, $\mathbf{x}(0) = \mathbf{a} = (a_1, a_2)$. (One method: let y_1, y_2 be the solution to the linear system in (a) with initial condition, $(y_1, y_2) = (1, 1)$, assume that $x_2 = c_1 y_2 + c_2 y_1^3$, and then determine c_1 and c_2.)

(c) What is the stable manifold for the system in (b)?

(3) Consider the system of equations,

$$\begin{cases} \dot{x_1} = x_2 - x_1 f(x_1, x_2), \\ \dot{x_2} = -x_1 - x_2 f(x_1, x_2), \end{cases}$$

where f lies in $C^1(\mathbb{R}^2)$.

(a) Show that if f is positive in some neighborhood of the origin then the origin is an asymptotically stable equilibrium point.

(b) Show that if f is negative in some neighborhood of the origin then the origin is an unstable equilibrium point.

Hint for both parts: Construct a Lyapunov function.

PART 2

(1) Let g be a bounded, continuous function on $\mathbb{R}^n$. For $(x,t) \in \mathbb{R}^n \times (0,+\infty)$ define

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy,$$

where Φ is the fundamental solution of the heat equation,

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}.$$

Let $x_0 \in \mathbb{R}^n$. Prove that

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0).$$

Hint: You can use the fact that $\int_{\mathbb{R}^n} \Phi(x,t)dx = 1$ for every $t > 0$ without proving it. You can also use without proving it the fact that for every $r_0 > 0$,

$$\lim_{(x,t)\to(x_0,0)} \int_{|y-x_0|>r_0} \Phi(x-y,t)dy = 0.$$

In other words, $\Phi(\cdot,t)$ has mass one and as $(x,t) \to (x_0,0)$ all the mass concentrate around the the point x_0.

(2) Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary and define the energy

$$E(w) = \frac{1}{2}\int_{\Omega} |\nabla w|^2 dx - \int_{\partial\Omega} hw,$$

where h is a smooth functions defined on the boundary of Ω. Suppose $u \in C^2(\overline{\Omega})$ satisfies

$$E(u) \le E(w) \text{ for all } w \in C^2(\overline{\Omega}).$$

What PDE is u satisfying? What are the boundary conditions? Prove it.

Hint: Start by considering perturbation $u + \epsilon v$ where $v \in C^2_c(\Omega)$. This will give you the PDE. Then consider perturbation $u + \epsilon v$ where $v \in C^2(\overline{\Omega})$ to get the boundary condition.

(3) Let u and v belong to $C^2_1(U_T) \cap C(\overline{U_T})$ and satisfy

$$u_t = \Delta u + f$$
$$v_t = \Delta v + g.$$

Show that if $u \ge v$ on the parabolic boundary Γ_T and $f \ge g$ in U_T then $u \ge v$ in all of $\overline{U_T}$. This is called a comparison principle.

PART 3

(1) (a) Prove or disprove the following:

Let U be a bounded, open subset of $\mathbb{R}^2$. If $u \in W^{1,2}(U)$, then $u \in L^\infty(U)$ with the estimate

$$\|u\|_{L^\infty(U)} \leq C\|u\|_{W^{1,2}(U)}$$

where C does not depend on u.

(b) Let U be a bounded, open set in $\mathbb{R}^n$ with smooth boundary. Show that

$$\|Du\|^2_{L^2(U)} \leq C\|u\|_{L^2(U)}\|D^2u\|_{L^2(U)}$$

for all $u \in H^1_0(U) \cap H^2(U)$ where C does not depend on u.

(2) Consider the following Dirichlet problem

$$-\Delta u + \mu u = f \text{ in } U$$
$$u = 0 \text{ on } \partial U$$

where μ is a given constant. U is a bounded, open subset of $\mathbb{R}^n$.

(a) Show the existence of a weak solution $u \in H^1_0(U)$ of the above problem for $\mu > 0$.

(b) Show the existence of a weak solution $u \in H^1_0(U)$ of the above problem for $\mu = 0$.

(c) Discuss the problem when $\mu < 0$.

(3) Consider the Poisson equation with Dirichlet boundary condition:

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

where U is a bounded, open subset of $\mathbb{R}^n$ and $f \in L^2(U)$. We know there exists a weak solution $u \in H^1_0(U)$. Prove that $u \in H^2_{loc}(U)$.