Math 206A Qualifying Exam

2020

Please answer any **3** of the following 5 problems.

1. Consider a sequence of (not necessarily independent) random variables such that X_n has distribution

$$\mathbf{P}\left(X_n = \frac{1}{n}\right) = 1 - \frac{1}{n^2}, \ \mathbf{P}\left(X_n = n^2\right) = \frac{1}{n^2}$$

Part a: (5 points) Does X_n converge to 0 in probability? Explain.
Part b: (5 points) Must $X_n \to 0$ almost surely? Prove or give a counterexample.

2. Let a sequence X_n be defined as follows: $X_0 = 1$. For $n \geq 1$,

$$\mathbf{P}(X_n = \frac{3}{2}X_{n-1}) = \mathbf{P}(X_n = \frac{1}{2}X_{n-1}) = \frac{1}{2}$$

Assume that the choice of $\frac{3}{2}$ vs. $\frac{1}{2}$ is independent for each n.

Part a: (4 points) Compute $\mathbf{E}(X_n)$ as a function of n.
Part b: (6 points) Let $Y_n = (X_n)^{1/n}$. Show that there is a constant c such that $Y_n \to c$ almost surely, and determine that c (You may wish to consider logarithms here).

3. Let $X_1, \ldots, X_n$ be independent, identically distributed variables each having characteristic function

$$\varphi(t) = \frac{1}{1+t^2}$$

Part a: (5 points) Determine the mean and variance of X_1.
Part b: (5 points) Let $\overline{X_n} = \frac{1}{n}(X_1 + \cdots + X_n)$. Determine the mean, variance, and characteristic function of $\overline{X_n}$. (Note: If you did not solve part a, you may let μ and σ^2 be the mean and variance of X_1, and give your answers in terms of μ and σ^2).

4. Let X be a random number between 0 and 1. We can think of X as

$$X = 0.a_0a_1a_2a_3\dots$$

where each a_i is equally likely to be any digit between 0 and 9.

Let A_1 be the event that X contains every 2 digit sequence at least once (i.e. that for each i and j we can find an n such that $(a_n, a_{n+1}) = (i, j)$). Let A_2 be the event that X contains every 2 digit sequence infinitely often.

Part a: (4 points) Determine whether each of A_1 and A_2 is a tail event, in the sense of Kolmogorov's $0-1$ law.
Part b: (6 points) show that $\mathbf{P}(A_2) = 1$.

5. (10 points) Let $X_1, X_2 \dots X_n$ be independent, identically distributed random variables, and let

$$S_n = X_1 + \cdots + X_n$$

Let $c > 0$ be an arbitrary constant. Show that for any t we have

$$\mathbf{P}\left(S_n \geq t\right) \leq \frac{\left[E(e^{cX_1})\right]^n}{e^{ct}}$$

Qualify Exam on Numerical Analysis, Fall 2020

Please choose any three problems and submit the full solutions.

Problem 1. Consider the multistep method

$$y_i - y_{i-3} = \frac{h}{8}[3f_i + 9f_{i-1} + 9f_{i-2} + 3f_{i-3}]$$

applied to the ODE $y'(x) = f(x, y)$, where $f_i = f(x_i, y_i)$.
(a) What is the order of the local truncation error of the method?
(b) Determine whether the method is convergent or not.

Problem 2. Define

$$T_n(\cos\theta) = \cos n\theta,\ n = 0, 1, 2, ...,\ -\pi \le \theta \le \pi.$$

(a) Show that each T_n is a polynomial of degree n and that the T_n satisfy the three-term recurrence relation

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t),\ n = 1, 2, ...$$

(b) Prove that T_n is an nth orthogonal polynomial with respect to the weight function $\omega(t) = (1 - t^2)^{-\frac{1}{2}}$, $-1 < t < 1$.
(c) Find the explicit values of the zeros of T_n, thereby verify that all the zeros reside in the open support of the weight function.
(d) Find b_1, b_2, c_1, c_2 such that the order of the following quadrature is four.

$$\int_{-1}^{1} f(\tau)\frac{d\tau}{\sqrt{1-\tau^2}} \approx b_1 f(c_1) + b_2 f(c_2).$$

Problem 3. The sequence $\{x_n\}$ is defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}},\ \text{for } n \ge 1,\ A \text{ is a positive number.}$$

Show that x_n converges to $\sqrt{A}$ whenever $x_0 > 0$. What happens if $x_0 < 0$?

Problem 4. Determine a quadratic spline s that interpolates the data $f(0) = 0$, $f(1) = 1$, $f(2) = 2$ and satisfies $s'(0) = 2$.

Problem 5. Picard's method for solving the initial-value problem

$$y' = f(t, y),\ a \le t \le b,\ y(a) = y_0,$$

is described as follows: Let $y_0(t) = y_0$ for each t in $[a, b]$. Define a sequence of functions $\{y_k(t)\}$ by

$$y_k(t) = y_0 + \int_a^t f(\tau, y_{k-1}(\tau))d\tau,\ k = 1, 2, \ldots$$

(a) Integrate $y' = f(t, y)$ and use the initial condition to derive Picard's method.
(b) Generate $y_0(t)$, $y_1(t)$, $y_2(t)$ and $y_3(t)$ for the initial-value problem

$$y' = -y + t + 1,\ 0 \le t \le 1,\ y(0) = 1.$$

(c) Solve the analytical solution of the initial value problem in **(b)** and compare the results in **(b)** to the Maclaurin series of the analytical solution.

Math 206C Qualifying Exam, Fall 2020

Please provide detailed answers and solutions for any 3 of the following 5 problems.

Problem 1. (10 points)

a) (2 points) Give definition of a continuous functional at a point.
b) (2 points) Give definition of a liner functional and give an example of such functional.
c) (2 points) Give definition of a variational derivative of a functional and give an example.
d) (4 points) Prove fundamental lemma of the calculus of variations.

Problem 2. (10 points)

a) (2 points) Give definition of an extremum of a functional.
b) (4 points) Prove the following theorem: If y(x) is an extremum of the functional J[y(x)], then y(x) satisfies the Euler-Lagrange equation.
c) (4 points) Use Euler-Lagrange equation to find the curve with the shortest length connecting: 1) two points in two dimensional Euclidian space, 2) two points on a sphere.

Problem 3. (10 points)

d) (5 points) Describe and explain in detail functionals for the Brachistochrone problem with 1) zero and 2) nonzero initial velocity. Do not solve variational problems.
e) (3 points) Define general variational problem and explain connection of solution of such problem with solution of the Hamilton-Jacobi equation.
f) (2 points) Give a definition of a chaotic discrete dynamical system and provide an example of such system.

Problem 4. (10 points)

a) (2 points) Give definitions of a fixed point of continuous and discrete dynamical systems.
b) (2 points) Give definition of stability of a fixed point of a discrete dynamical system.
c) (4 points) Determine stability of all fixed points of discrete dynamical systems:
1) $x_{n+1} = 6(x_n)^2$, 2) logistic system: $x_{n+1} = r\, x_n (1 - x_n)$ for $1<r<3$.
d) (2 points) Explain period doubling in the bifurcation diagram of the logistic discrete dynamical system as parameter r changes from 0 to 4.

Problem 5. (10 points)

The basic variables identifying the state of the population in the epidemiological SIR model are as follows:

- *S*(*t*) is the number of susceptibles at time *t*;
- *I*(*t*) is the number of infectives at time *t*;
- *R*(*t*) is the number of immune at time *t*.

The occurrence of a disease imparting immunity is described by the following SIR system:

$$
\begin{aligned}
\frac{dS}{dt} &= -\frac{\beta SI}{N}, \\
\frac{dI}{dt} &= \frac{\beta SI}{N} - \nu I, \\
\frac{dR}{dt} &= \nu I.
\end{aligned}
$$

with the assumption that: $S + I + R = N = const.$

a) (3 points) Explain meaning of each term and each parameter in the equations.
b) (2 points) Show that I(t) approaches 0 as t approaches infinity. Hint: N is fixed.
c) (3 points) Compute the Jacobian matrix for the system consisting of first two equations above and calculate eigenvalues of this matrix. Use expressions for the eigenvalues to determine ranges of parameter values for which the equilibrium of the system is stable.
d) (2 points) Give definition of an epidemic and explain why an epidemic occurs only if:

$$R = \frac{\beta S(0)}{\nu N} > 1.$$

Hint: Two first equations are decoupled from the third equation. Divide second equation by the first equation. Solve resulting first order equation and analyze its solution.

Math 206A Qualifying Exam

September 2019

Please choose any 3 of the following 5 problems.

1. Let X be a non-negative random variable.

Part a: (5 points) Show that

$$\mathbf{E}(X) = \int_0^\infty \mathbf{P}(X \geq t)\, dt$$

Part b: (5 points) Suppose that $X_1, X_2, \ldots, X_n, \ldots$ are independent variables, all having the same distribution as X. Show that with probability 1 there are only finitely many n for which $X_n \geq n$.

2. Let $\{X_n\}$ be a sequence of random variables.

Part a: (5 points) Define what it means for X_n to converge to X in probability, and what it means for X_n to converge to X almost surely.
Part b: (5 points) Suppose that $X_n \to X$ almost surely. Must $\mathbf{E}(X_n) \to \mathbf{E}(X)$? Prove or give a counterexample with justification.

3. Recall that a Normal Variable with mean μ and variance σ^2 (also known as an $N(\mu, \sigma^2)$ variable) has density given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Part a: (2 points) Set up an integral that gives the characteristic function $\varphi(t)$ of such a normal variable. **You do not need to evaluate this integral!**.

It turns out that the integral from part a evaluates to

$$\varphi(t) = e^{it\mu - \frac{1}{2}\sigma^2 t^2}$$

Part b: (4 points) Suppose that $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent. Determine, with justification, the distribution of $X + Y$.
Part c: (4 points) Suppose that $X_n \to X$ in distribution, each X_n has a distribution of the form $N(0, \sigma_n^2)$. Show that the σ_n^2 are converging to some σ^2, and that $X \sim N(0, \sigma^2)$.

4. Let $X_1, X_2, \dots$ be independent variables, with $\mathbf{E}(X_n) = 1$ and $\mathbf{Var}(X_n) = n^{1.5} - (n-1)^{1.5}$. Define the **Sample Mean** $\overline{X_n}$ by

$$\overline{X_n} = \frac{1}{n}(X_1 + \cdots + X_n)$$

Part a (5 points) Determine the mean and variance of $\overline{X_n}$. (Hint: Telescope!)
Part b (5 points) Show that $\overline{X_n} \to 1$ in probability.

5. Let $X_1, \dots, X_n$ be variables, and suppose that they have a joint density $f(x_1, x_2, \dots, x_n)$ which factors as

$$f(x_1, x_2, \dots, x_n) = f_1(x_1) f_2(x_2) \dots f_n(x_n)$$

as the product of n functions depending individually only on a single x_i (the $f_i(x_i)$ are not necessarily density functions). Show that $(X_1, \dots, X_n)$ are independent.

Qualify Exam on Numerical Analysis, Fall 2019

Please choose any three problems and submit the full solutions.

Problem 1. Consider an ODE of the form

$$y' = f(x, y),\ y(x_0) = y_0,\ x, x_0 \in [a, b]$$

whose solution is to be approximated using a general one-step method of the form

$$y_{i+1} = y_i + h\Phi(x_i, y_i, h) \tag{1}$$

where $h = \frac{b-a}{N}$ is the step size and $\Phi(x, y, h)$ is Lipschitz continuous on y with constant λ. Assume that the method is of order p so that the local truncation error gives

$$y(x_{i+1}) = y(x_i) + h\Phi(x_i, y(x_i), h) + \tau_i h^{p+1}$$

where $y(x)$ is the exact solution of the IVP and τ_i is a constant that depends on derivatives of this solution.

(a) Derive an error estimate for the obtained approximation of the form

$$|e_i| \le C_1|e_0| + C_2h^p,\ i = 0, 1, ..., N$$

where C_1 and C_2 are constants.

(b) For a second order Runge-Kutta method of your choice, give the explicit representation of $\Phi(x_i, y(x_i), h)$ that arises when the method is expressed in form (1).

Problem 2. Consider the multistep method

$$y_i - y_{i-4} = \frac{h}{3}[8f_{i-1} - 4f_{i-2} + 8f_{i-3}]$$

applied to the ODE $y'(x) = f(x, y)$, where $f_i = f(x_i, y_i)$.
What is the order of the local truncation error of the method?

Problem 3. Prove that

$$||X^{(k)} - X|| \le ||T||^k||X^{(0)} - X||$$

and

$$||X^{(k)} - X|| \le \frac{||T||^k}{1 - ||T||}||X^{(1)} - X^{(0)}||$$

where T is an $n \times n$ matrix with $||T|| < 1$ and

$$X^{(k)} = TX^{(k-1)} + C, \quad k = 1, 2, \ldots$$

with $X^{(0)}$ arbitrary, $C \in R^n$, and $X = TX + C$.

Problem 4. Derive an $O(h^4)$ five-point formula to approximate $f'(x_0)$ that uses $f(x_0 - h)$, $f(x_0)$, $f(x_0 + h)$, $f(x_0 + 2h)$ and $f(x_0 + 3h)$.

Problem 5. (Construction of Gaussian Quadrature) Let $\{p_0(x), p_1(x), ...\}$ be Legendre polynomials satisfying

- For each n, $p_n(x)$ is a monic polynomial of degree n
- $\int_{-1}^{1} p(x)p_n(x)\,dx = 0$ for any $p(x)$ of degree $\leq n$.

Prove that

(1) All m zeros of a Legendre polynomial $p_m(x)$ reside in the interval $[-1, 1]$ and they are simple.

(2) Let $x_1, x_2, ..., x_n$ are the n roots of the n^{th} Legendre polynomial $p_n(x)$ and define

$$c_i = \int_{-1}^{1} \prod_{j=1, j\neq i}^{j=n} \frac{x - x_j}{x_i - x_j}\,dx,$$

then for any polynomial $p(x)$ of degree less than $2n$,

$$\int_{-1}^{1} p(x)\,dx = \sum_{i=1}^{i=n} c_i p(x_i).$$

Problem 1. Variational problems.

a) Prove fundamental lemma of the calculus of variations.
b) A spring hanging vertically under gravity. Consider a mass *m* on the end of a spring of natural length *l* and spring constant *k*. Let y be the vertical coordinate of the mass as measured from the top of the spring. Assume the mass can only move up and down in the vertical direction. The Lagrangian for this problem is as follows:

$$L = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k(y-\ell)^2 + mgy.$$

Explain terms in this Lagrangian, corresponding variational problem and determine and solve the corresponding Euler- Lagrange equation of motion. (Lagrangian is the function in the functional of the variational problem.)

Problem 2.

a. The Brachistochrone problem. An experimenter lets a bead slide with zero initial velocity down a wire that connects two fixed points with zero initial. Determine the shape of the wire that the bead slides from one end to the other in minimal time.
b. Describe functional for the Brachistochrone problem with nonzero initial velocity. Do not solve variational problem.

Problem 3. Chaotic dynamical systems.

a. Give a definition of a chaotic dynamical system.
b. Give definition of the tent map and demonstrate sensitive dependence on initial conditions for the tent map.
c. Describe bifurcation diagram of the logistic map:

$$x_{n+1} = r \cdot x_n(1 - x_n)$$

and explain period doubling.

d. Describe in detail behavior of trajectories of the logistic map when parameter *r* reaches bifurcation point at 4.

Problem 4. Bifurcations of dynamical systems.

a. Give detailed definition of a bifurcation of a dynamical system.
b. Find equilibrium points and determine their types for different values of parameter μ of the following dynamical system:

$$dx_1/dt = \mu\, x_1 + x_1^3$$
$$dx_2/dt = -x_2.$$

c. Explain what happens with the equilibrium points and their stability as μ goes through zero in this dynamical system. Sketch bifurcation diagram.

d. Give a definition of a normal form and explain in detail what normal forms are used for.

Problem 5. Variational problems.

a. Prove that if y(x) is an extremal of the functional J:

$$J[y(x)] = \int_{x_0}^{x_1} F(x, y, y')dx.$$

then it satisfies the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

b. Give definitions of the Hamilton-Jacobi equation and eikonal. Explain relation between eikonals and extremals.