Real Analysis Qualifier — 2019

Undergraduate Problems

Choose one problem from 1–2 and one problem from 3–4. Please show all work. Unsupported claims will not receive credit.

1. Find a function $f\colon \mathbb{R} \to \mathbb{R}$ that is differentiable everywhere but whose derivative is not continuous everywhere. Prove it has both these properties.

2. Find a function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at all irrational numbers and discontinuous at all rational numbers. Prove it has both these properties.

3. Prove straight from the definition of Riemann integral that this function $f\colon [0,1] \to \mathbb{R}$ is Riemann integrable:
$$f(x) = \begin{cases} 0 & \text{if} \quad x \leq 1/2 \\ 1 & \text{if} \quad x > 1/2 \end{cases}$$

4. Prove straight from the definition of Riemann integral that this function $f\colon [0,1] \to \mathbb{R}$ is not Riemann integrable:
$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \\ 1 & \text{if} \quad x \notin \mathbb{Q} \end{cases}$$

209A

Choose one problem from 1–2, one from 3–4 and one from 5–6. Please show all work. Unsupported claims will not receive credit. The notion of measure or measurable is in the sense of Lebesgue, unless stated otherwise.

1. State the definition that a set in $\mathbb{R}$ is Lebesgue measurable. Prove that every countable set in $\mathbb{R}$ is Lebesgue measurable.

2. Prove that every set $S \subseteq \mathbb{R}$ with positive outer measure contains a nonmeasurable set.

3. Construct a bounded open set $O \subset \mathbb{R}$ such that the measure of O is strictly less than the measure of its closure $\bar{O}$.

4. Let $\{f_n\}$ be a sequence of measurable functions on $[0, 1]$. Prove that the set of points $x \in [0, 1]$ where $f_n(x)$ does not converge is measurable.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Prove that $|f|$ is also measurable. Is the converse true? Why?

6. State Egoroff's Theorem. Use it to prove the Dominated Convergence Theorem for measurable functions on the interval $[0, 1]$ with Lebesgue measure.

209B

Choose one problem from 1–2, one from 3–4, and one from from 5–6. Please show all work. Unsupported claims will not receive credit.

1. Find a sequence of measurable functions $f_n \colon [0,1] \to \mathbb{R}$ that converges to zero in measure but not almost everywhere. Prove it has both these properties.

2. Find a sequence of functions $f_n \colon [0,1] \to \mathbb{R}$ that converges to zero pointwise but has

$$\lim_{n\to\infty} \int_{[0,1]} f_n \, dx = +\infty.$$

Prove it has both these properties.

3. Let ℓ^1 be the space of real-valued sequences $(c_i)_{i=1}^\infty$ with $\sum_{i=1}^\infty |c_i| < \infty$, made into a Banach space with the norm

$$\|c\| = \sum_{i=1}^\infty |c_i|.$$

Find an infinite-dimensional closed subspace of ℓ^1 that is not all of ℓ^1.

4. Let $C[0,1]$ be the vector space of continuous real-valued functions $f \colon [0,1] \to \mathbb{R}$, made into a normed vector space with the norm

$$\|f\| = \int_{[0,1]} |f(x)| \, dx.$$

Show that $C[0,1]$ is not a Banach space with this norm.

5. Suppose that V is a Banach space and $v_i \in V$ is a sequence with

$$\sum_{i=1}^\infty \|v_i\| < \infty.$$

Prove that the sum

$$\sum_{i=1}^\infty v_i$$

converges in the norm topology on V.

6. Using the Baire category theorem, prove this version of the uniform boundedness principle: if V and W are Banach spaces and $S \subseteq L(V,W)$ is a set of bounded operators from V to W such that

$$\sup_{T\in S} \|Tv\| < \infty \text{ for all } x \in V,$$

then

$$\sup_{T\in S} \|T\| < \infty.$$

209C

Choose one problem from 1–2 and one from 3-4. Please show all work. Unsupported claims will not receive credit. In what follows we use Lebesgue measure on $[0, 1]$ and $\mathbb{R}$.

1. For any $f \in C[0, 1]$, the space of continuous functions on $[0, 1]$ with the sup norm, define a linear functional
$$Tf = \int_0^{1/4} xf(x)dx.$$
(a) Prove that T is a bounded linear functional on $C[0, 1]$.
(b) Find the norm of T.

2. Show that the dual space of $L^\infty[0, 1]$ strictly contains $L^1[0, 1]$.

3. (a) State the definition of $\mathcal{S}(\mathbb{R})$, the space of Schwartz functions on $\mathbb{R}$.
(b) Prove that $f(x) = e^{-x^2}$ is in $\mathcal{S}(\mathbb{R})$ and find its Fourier transform
$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

4. (a) State the definition of $\mathcal{S}'(\mathbb{R})$, the space of tempered distributions on $\mathbb{R}$.
(b) Prove that the Heaviside function
$$h(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$
can be viewed as a tempered distribution and find its derivative.

Real Analysis Qualifier — 2018

Undergraduate Problems

Choose one problem from 1–2 and one problem from 3–4. Please show all work. Unsupported claims will not receive credit.

1. Describe a function $f\colon (0,1) \to \mathbb{R}$ that is continuous but not uniformly continuous. Prove that it has these properties.

2. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ has a continuous derivative f'. Prove that f is uniformly continuous.

3. Prove or disprove this statement: if $f\colon \mathbb{R} \to \mathbb{R}$ is bounded and uniformly continuous, then f^2 is uniformly continuous.

4. Describe a function $f\colon [0,1] \to \mathbb{R}$ that is not continuous but is Riemann integrable. Prove that it has these properties.

209A

Choose one problem from 1–2, one from 3–4 and one from 5–6. Please show all work. Unsupported claims will not receive credit.

1. Let f_n be a sequence of measurable real valued functions on $\mathbb{R}$. Show that

$$A = \{x \in \mathbb{R} | \lim_{n\to\infty} f_n(x) \text{ exists}\}$$

is measurable.

2. Show that $[0,1]$ has the same cardinality as the power set of the set of all positive integers.

3. Prove that the product of two measurable real valued functions on $\mathbb{R}$ is measurable. (Hint: show that if f is measurable, then f^2 is measurable.)

4. State the Dominated Convergence Theorem (DCT) and Fatou's Lemma. Show that DCT follows from Fatou's Lemma.

5. State Egoroff's Theorem. Use it to prove the Dominated Convergence Theorem for measurable functions on the interval $[0,1]$ with Lebesgue measure.

6. Let f be an integrable function on $\mathbb{R}$. Prove that

$$\lim_{h\to 0} \int |f(x+h) - f(x)| dx = 0.$$

209B

Choose one problem from 1–2, one from 3–4, and one from from 5–6. Please show all work. Unsupported claims will not receive credit.

1. Describe a function $f\colon [0,1] \to [0,1]$ that is continuous, differentiable a.e., but not of bounded variation. Prove that it is not of bounded variation.

2. Describe a set $U \subseteq [0,1]$ that is open, contains all the rational numbers in this interval, and has Lebesgue measure $\leq 1/5$. Prove that it has all these properties.

3. Show that if $V \subseteq C[0,1]$ is a dense linear subspace then V separates points.

4. Let X and Y be topological spaces. Prove that $f\colon X \to Y$ is continuous iff for every net $(x_\lambda)_{\lambda\in\Lambda}$ in X,
$$x_\lambda \to x \implies f(x_\lambda) \to f(x).$$

5. Suppose that V is a normed real vector space. Show that V^*, the real vector space of all bounded linear maps $\ell\colon V \to \mathbb{R}$ with the norm
$$\|\ell\| = \sup_{0\neq v\in V} \frac{|\ell(v)|}{\|v\|},$$
is complete. (Don't use a big theorem of which this is a special case: do it 'by hand'.)

6. If V is a real Banach space, prove that there is a norm-preserving linear map $i\colon V \to V^{**}$.

Choose one problem from 1–2 and one from 3-4. Please show all work. Unsupported claims will not receive credit. In what follows we use Lebesgue measure on $[0,1]$ and $\mathbb{R}$.

1. For any $f \in C[0,1]$, the space of continuous functions on $[0,1]$ with the sup norm, define a linear functional $Tf = \int_0^{1/4} f(x)dx$.
   (a) Prove that T is a bounded linear functional on $C[0,1]$.
   (b) Find the norm of T.

2. Show that the dual space of $L^\infty[0,1]$ strictly contains $L^1[0,1]$.

3. (a) State the definition of $S(\mathbb{R})$, the space of Schwartz functions on $\mathbb{R}$.
   (b) Prove that $f \in S(\mathbb{R})$ if and only if the Fourier transform $\hat{f} \in S(\mathbb{R})$.

4. Let T be a tempered distribution on $S(\mathbb{R})$. The Fourier transform of T is defined by

$$\int \hat{T}\phi = \int T\hat{\phi}$$

and its derivative is by

$$\int T'\phi = -\int T\phi',$$

for all $\phi \in S(\mathbb{R})$. Use these to find the Fourier transform of the first and second derivative of δ_y, the Dirac delta supported at a point $y \in \mathbb{R}$, given by $\delta_y(x) = \delta(x-y)$.

Real Analysis Qualifier — 2017

Undergraduate Problems

Choose one problem from 1–2 and one problem from 3–4. Please show all work. Unsupported claims will not receive credit.

1. Prove or disprove this statement: if $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of continuous functions and $f_n \to f$ uniformly, then f is continuous.

2. Prove or disprove this statement: if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, then their product fg is continuous.

3. Prove or disprove this statement: if $f, g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous, then their product fg is uniformly continuous.

4. Give a function $f : [0, 1] \to \mathbb{R}$ that is not Riemann integrable, and prove that it is not.

209A

Choose one problem from 1–3 and one problem from 4–6. Please show all work. Unsupported claims will not receive credit.

1. Suppose that $\mathcal{M}$ is any σ-algebra of subsets of a set X, and μ is a measure on this σ-algebra. Prove or disprove this statement: there is always a σ-algebra $\overline{\mathcal{M}}$ containing $\mathcal{M}$ and a complete measure $\overline{\mu}$ extending μ.

2. A collection $\mathcal{A}$ of subsets of X is **closed under countable increasing unions** if whenever $A_i \in \mathcal{A}$ is a sequence of sets with $A_i \subseteq A_{i+1}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. Prove or disprove this statement: if $\mathcal{A}$ is an algebra closed under countable increasing unions, then $\mathcal{A}$ is a σ-algebra.

3. Prove or disprove this statement: if the functions $f_n\colon [0,1] \to \mathbb{R}$ are continuous and for every $x \in [0,1]$ we have $\lim_{n\to\infty} f_n(x) = 0$, then
$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0.$$

4. A sequence of measurable functions $f_n\colon X \to \mathbb{R}$ converges to zero **in measure** if for any $\epsilon > 0$,
$$\lim_{n\to\infty} \mu\big(\{x \in X\colon |f_n(x)| > \epsilon\}\big) = 0.$$
Prove that if f_n converges to zero in measure and μ is a finite measure then
$$\lim_{n\to\infty} \int_X \frac{f_n}{1+|f_n|}\,d\mu = 0.$$

5. Let f be a nonnegative function on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} f(x)^p dx < \infty$. Prove that
$$\int_{\mathbb{R}^n} f(x)^p dx = \int_0^\infty p t^{p-1} m\{x \;:\; f(x) > t\}\,dt.$$
where $m(S)$ is the Lebesgue measure of the set S. Hint: use the Fubini–Tonelli theorem.

6. Let f be an integrable function on $\mathbb{R}$. Prove that
$$\lim_{h\to 0} \int |f(x+h) - f(x)| dx = 0.$$

209B

Choose one problem from 1–2 and two problems from 3–6. Please show all work. Unsupported claims will not receive credit.

1. Prove or disprove: finite linear combinations of the functions $\{e^{-nx}\}_{n\in\mathbb{N}}$ are dense in $C[0,1]$ with its usual sup norm topology.

2. Prove that if μ, ν, λ are measures on the measurable space $(X, \mathcal{M})$ and $\mu \ll \lambda$, $\nu \ll \lambda$, then $\mu + \nu \ll \lambda$.

3. Let $f = f(x)$ be an absolutely continuous function on $[0,1]$ such that f' is in $L^2[0,1]$ and that $f(0) = 0$. Prove that
$$\lim_{x\to 0^+} \frac{f(x)}{x^{1/2}} = 0.$$
Hint: use the Fundamental Theorem of Calculus on $f(x) - f(0)$.

4. Describe a set $S \subseteq [0,1]$ that is nowhere dense yet has positive Lebesgue measure. Prove that it has these properties.

5. Suppose V is a real Banach space. Prove that a linear functional $f\colon V \to \mathbb{R}$ bounded if and only it is continuous.

6. Suppose $f \in L^2[0,1]$ and $f^2 \in L^2[0,1]$. Show that $f + 1 \in L^3[0,1]$.

209C

Choose one problem from 1–2 and one from 3–6. Please show all work. Unsupported claims will not receive credit. In what follows we use Lebesgue measure on $[0,1]$ and $\mathbb{R}$.

1. Prove that if T is a linear map from a real Hilbert space H to itself that preserves the norm, then T also preserves angles.

2. Show that $L^\infty[0,1]$ is nonseparable, i.e., it does not have a countable dense subset.

3. If $f \in L^p \cap L^\infty$ for some $p < \infty$ and $f \in L^q$ for all $q > p$, then
$$||f||_\infty = \lim_{q\to\infty} ||f||_q$$

4. Suppose $f \in L^2(\mathbb{R})$. Then the L^2 derivative f' exists iff $\xi\widehat{f} \in L^2$, in which case $\widehat{f}'(\xi) = 2\pi i\xi\widehat{f}(\xi)$.

5. Suppose that f is continuously differentiable on $\mathbb{R}$ except at $x_1, \ldots, x_m$, where f has jump discontinuities, and that its pointwise derivative df/dx (defined except at the the points x_j) is in $L^1_{\text{loc}}(\mathbb{R})$. Then the distribution derivative f' of f is given by:
$$f' = (df/dx) + \sum_{j=1}^{m}[f(x_j+) - f(x_j-)]\tau_{x_j}\delta$$
where τ_x is the operation that translates a distribution by x.

Real Analysis Qualifier

Part I

Solve one problem out of (1)–(2), one out of (3)–(4) and one out of (5)–(6). In what follows, let $(X, \mathcal{M}, \mu)$ be a measure space.

(1) Prove or disprove this statement: if $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of continuous functions and $f_n \to f$ uniformly, then f is continuous.

(2) Prove or disprove this statement: if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, then their product fg is continuous.

(3) Prove or disprove this statement: there is a σ-algebra $\overline{\mathcal{M}}$ containing $\mathcal{M}$ and a complete measure $\overline{\mu}$ extending μ.

(4) A collection $\mathcal{A}$ of subsets of X is **closed under countable increasing unions** if whenever $A_i \in \mathcal{A}$ is a sequence of sets with $A_i \subseteq A_{i+1}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. Prove or disprove this statement: if $\mathcal{A}$ is an algebra closed under countable increasing unions, then $\mathcal{A}$ is a σ-algebra.

(5) A sequence of measurable functions $f_n : X \to \mathbb{R}$ converges to zero **in measure** if for any $\epsilon > 0$,

$$\lim_{n\to\infty} \mu\big(\{x \in X : |f_n(x)| > \epsilon\}\big) = 0.$$

Prove that if f_n converges to zero in measure then

$$\lim_{n\to\infty} \int_X \frac{f_n}{1 + |f_n|}\, d\mu = 0.$$

(6) Prove or disprove this statement: if the functions $f_n : [0, 1] \to \mathbb{R}$ are continuous and for every $x \in [0, 1]$ we have $\lim_{n\to\infty} f_n(x) = 0$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0.$$

Part II

Solve 3 out of the 5 problems below.

(1) State the Hölder and Minkowski inequalities. Use the former to prove the later.

(2) Let $f \in L^1(\mathbf{R}^n)$ and let

$$M(f) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

where $B(x,r)$ is the ball of radius r centered at x. Prove that there exists a constant $C > 0$ such that for all $\alpha > 0$:

$$|\{x : \ Mf(x) > \alpha\}| \leq \frac{C}{\alpha} \int |f(y)|dy.$$

You may use this fact: Let C be a collection of open balls in $\mathbb{R}^n$ that covers a set U of finite measure. Then there exist finitely many disjoint balls $B_1, \ldots, B_k$ in C such that the sum of the volume of these balls is greater than $3^{-n}a$, where a is any number less than the measure of U.

(3) (a) State the definition of weak and strong convergence of sequences in a Banach space.

(b) Does weak convergence imply strong convergence? Explain why it does or does not.

(c) Show that every weakly convergent sequence in a Banach space is bounded with respect to the norm of the Banach space. (You may assume the uniform boundedness principle.)

(4) Prove that a linear functional f on a normed vector space is bounded if and only if $f^{-1}(\{0\})$ is a closed subspace of X.

(5) Show that the Banach space $X = L^1[0,1]$ is not reflexive, namely X is a proper subset of X^{**}.

Part III

Solve 3 out of the 5 problems below.

(1) Prove that every closed convex set in a Hilbert space has a unique element of minimal norm.

(2) If $f \in L^p \cap L^\infty$ for some $p < \infty$ and $f \in L^q$ for all $q > p$, then

$$\|f\|_\infty = \lim_{q \to \infty} \|f\|_q$$

(3) Let $f(x) = \frac{1}{2} - x$ on the interval $[0, 1)$. Extend f to be periodic function on $\mathbb{R}$. Use Fourier series of f to show that

$$\sum_{k \geq 1} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

(4) If $\psi \in C^\infty(\mathbb{R})$, show that

$$\psi \delta^{(k)} = \sum_{j=0}^{k} (-1)^j \frac{k!}{j!(k-j)!} \psi^{(j)}(0) \delta^{(k-j)}$$

where $\delta^{(k)}$ is the k-th derivative of delta function.

(5) Show that $L^\infty[0, 1]$ is not separable, i.e., it does not have a countable dense subset.

Real Analysis Qualifier — 2016

Undergraduate Problems

Choose 2 problems from the following. Please show all work. Unsupported claims will not receive credit.

1. Show that a continuous function on a compact interval has a maximum and minimum value.

2. Let $f = \chi_{\mathbb{Q}}$ be the characteristic function of the set of rational numbers. True or false:

   (a) f is Riemann integrable on [0,1].

   (b) f is Lebesgue integrable on [0,1].

Justify your answers. How would your answers be modified if $\mathbb{Q}$ were replaced by the set of irrational numbers?

3. State the inverse function theorem for a function of two variables and give a sketch of the proof of that theorem.

4. Show that if $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence of real numbers then either $x_n \to x$ for some $x \in \mathbb{R}$ or for every $M \in \mathbb{R}$ there exists n with $x_n > M$.

5. Find a sequence of continuous functions $f_n\colon [0,1] \to \mathbb{R}$ that converges pointwise but not uniformly. Prove that this is the case.

6. Give an infinitely differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ for which f and all its derivatives $f', f'', f''', \ldots$ vanish at the origin, yet f is nonzero. Prove that this is the case.

Part A

Choose 2 problems from the following. Please show all work. Unsupported claims will not receive credit.

1. Let $\{f_n\}$ be a sequence of real-valued measurable functions on a measure space $(X, \mathcal{M}, \mu)$.

Show that the sets

(a) $A := \{x \in X : f_n(x) \to +\infty\}$,

(b) $B := \{x \in X : f_n(x) \to -\infty\}$, and

(c) $C := \{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$

are all measurable.

2. State precisely the Lebesgue dominated convergence theorem and give a sketch of its proof.

[*Advice*: You may use, without proof, one of the other classic convergence theorems.]

3. (a) Let $(X, \mathcal{M}, \mu)$ be a measure space and f be a μ-integrable function such that $f(x) > 0$ for μ-almost every $x \in X$. Show that if $\int_A f d\mu = 0$, where $A \in \mathcal{M}$, then $\mu(A) = 0$.

(b) Let λ denote Lebesgue measure on $\mathbb{R}$. Show that if $f : [0, +\infty) \to \mathbb{R}$ is a Lebesgue integrable function such that $\displaystyle\int_0^t f(x) d\lambda(x) = 0$ for each $t \geq 0$, then $f(x) = 0$, for every $x \geq 0$.

4. Show that

(a) $\displaystyle\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} = 1.$

(b) If

$$F(t) := \int_0^{+\infty} \frac{e^{-xt}}{1 + x^2}\, dx,$$

for each $t > 0$, then F is well defined, as well as twice continuously differentiable on $(0, +\infty)$ and satisfies the differential equation

$$F''(t) + F(t) = \frac{1}{t}, \text{ for all } t > 0.$$

5. Let $[a, b]$ be a compact interval. Show that if $f\colon [a, b] \to \mathbb{R}$ is an absolutely continuous function, then

$$\int_b^b |f'(x)| dx = V_f,$$

the total variation of f on $[a, b]$.

Part B

Choose 3 problems from the following. Please show all work. Unsupported claims will not receive credit.

1. State and prove Hölder's inequality assuming the elementary inequality $ab \leq a^p/p + b^q/q$ for positive numbers a, b, p, q such that $1/p + 1/q = 1$.

2. Construct a sequence of L^2 functions on $[0, 1]$ which converges to the 0 function weakly , but which does not converge to 0 strongly in L^2 norm.

3. State the definition of an absolutely continuous function on $\mathbb{R}$. Let g be a Lipschitz function on $[0, 1]$ and f be an absolutely continuous function from $[0, 1]$ to $[0, 1]$. Prove that the composite $g \circ f$ is also absolutely continuous.

4. State the open mapping and closed graph theorems. Use the former to prove the latter.

5. Let F the the linear functional on $C[-1, 1]$ defined by

$$F(x) = \int_{-1}^{0} x(t)dt - \int_{0}^{1} x(t)dt, \quad \forall x \in C[-1, 1].$$

Prove that the norm of F is equal to 2.

Part C

Choose 3 problems from the following. Please show all work. Unsupported claims will not receive credit. In what follows we use Lebesgue measure on $[0,1]$ and $\mathbb{R}$.

1. Prove that if H is a Hilbert space, $L \subseteq H$ is a closed linear subspace, and $v \in H$ then there exists a point $x \in L$ achieving the minimum distance to v. In other words, if $y \in H$ then $\|y-v\| \geq \|x-v\|$.

2. Do all of these:

(a) Find a function $f\colon \mathbb{R} \to \mathbb{R}$ that is in $L^{10}(\mathbb{R})$ but not $L^{12}(\mathbb{R})$. Prove this is the case.

(b) Find a function $f\colon \mathbb{R} \to \mathbb{R}$ that is in $L^{12}(\mathbb{R})$ but not $L^{10}(\mathbb{R})$. Prove this is the case.

(c) Prove that if $f \in L^{10}(\mathbb{R})$ and $f \in L^{12}(\mathbb{R})$ then $f \in L^{11}(\mathbb{R})$.

3. Prove, from scratch, that if $f \in L^1(\mathbb{R}^n)$ then $\widehat{f} \in L^\infty(\mathbb{R}^n)$.

4. Using the fact stated in Problem 3, prove that if $f \in L^1(\mathbb{R}^n)$ then its Fourier transform $\widehat{f}$ is continuous and approaches zero at infinity.

5. Prove that there is a tempered distribution $T \in \mathcal{S}(\mathbb{R})^*$ given by

$$T(f) = \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}-[-\epsilon,\epsilon]} \frac{f(x)}{x}\, dx$$

[*Advice*: It may help to show that $T(f) = \lim_{\epsilon \downarrow 0} \int_\epsilon^{+\infty} \frac{f(x) - f(-x)}{x}\, dx$.]

Real Analysis Qualifier

Part I

Solve one problem out of (1)–(2), one out of (3)–(4) and one out of (5)–(6). In what follows, let $(X, \mathcal{M}, \mu)$ be a measure space.

(1) Prove or disprove this statement: if $f_n \colon \mathbb{R} \to \mathbb{R}$ is a sequence of continuous functions and $f_n \to f$ uniformly, then f is continuous.

(2) Prove or disprove this statement: if $f, g \colon \mathbb{R} \to \mathbb{R}$ are continuous, then their product fg is continuous.

(3) Prove or disprove this statement: there is a σ-algebra $\overline{\mathcal{M}}$ containing $\mathcal{M}$ and a complete measure $\overline{\mu}$ extending μ.

(4) A collection $\mathcal{A}$ of subsets of X is **closed under countable increasing unions** if whenever $A_i \in \mathcal{A}$ is a sequence of sets with $A_i \subseteq A_{i+1}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. Prove or disprove this statement: if $\mathcal{A}$ is an algebra closed under countable increasing unions, then $\mathcal{A}$ is a σ-algebra.

(5) A sequence of measurable functions $f_n \colon X \to \mathbb{R}$ converges to zero **in measure** if for any $\epsilon > 0$,

$$\lim_{n\to\infty} \mu\big(\{x \in X \colon |f_n(x)| > \epsilon\}\big) = 0.$$

Prove that if f_n converges to zero in measure then

$$\lim_{n\to\infty} \int_X \frac{f_n}{1 + |f_n|}\, d\mu = 0.$$

(6) Prove or disprove this statement: if the functions $f_n \colon [0, 1] \to \mathbb{R}$ are continuous and for every $x \in [0, 1]$ we have $\lim_{n\to\infty} f_n(x) = 0$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0.$$

Part II

Solve 3 out of the 5 problems below.

(1) State the Hölder and Minkowski inequalities. Use the former to prove the later.

(2) Let $f \in L^1(\mathbf{R}^n)$ and let

$$M(f) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

where $B(x,r)$ is the ball of radius r centered at x. Prove that there exists a constant $C > 0$ such that for all $\alpha > 0$:

$$|\{x : \ Mf(x) > \alpha\}| \leq \frac{C}{\alpha} \int |f(y)|dy.$$

You may use this fact: Let C be a collection of open balls in $\mathbb{R}^n$ that covers a set U of finite measure. Then there exist finitely many disjoint balls $B_1, \ldots, B_k$ in C such that the sum of the volume of these balls is greater than $3^{-n}a$, where a is any number less than the measure of U.

(3) (a) State the definition of weak and strong convergence of sequences in a Banach space.

(b) Does weak convergence imply strong convergence? Explain why it does or does not.

(c) Show that every weakly convergent sequence in a Banach space is bounded with respect to the norm of the Banach space. (You may assume the uniform boundedness principle.)

(4) Prove that a linear functional f on a normed vector space is bounded if and only if $f^{-1}(\{0\})$ is a closed subspace of X.

(5) Show that the Banach space $X = L^1[0,1]$ is not reflexive, namely X is a proper subset of X^{**}.

PART III

Solve 3 out of the 5 problems below.

(1) Prove that every closed convex set in a Hilbert space has a unique element of minimal norm.

(2) If $f \in L^p \cap L^\infty$ for some $p < \infty$ and $f \in L^q$ for all $q > p$, then

$$\|f\|_\infty = \lim_{q \to \infty} \|f\|_q$$

(3) Let $f(x) = \frac{1}{2} - x$ on the interval $[0,1)$. Extend f to be periodic function on $\mathbb{R}$. Use Fourier series of f to show that

$$\sum_{k \geq 1} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

(4) If $\psi \in C^\infty(\mathbb{R})$, show that

$$\psi \delta^{(k)} = \sum_{j=0}^{k} (-1)^j \frac{k!}{j!(k-j)!} \psi^{(j)}(0) \delta^{(k-j)}$$

where $\delta^{(k)}$ is the k-th derivative of delta function.

(5) Show that $L^\infty[0,1]$ is not separable, i.e., it does not have a countable dense subset.

2014 Real Analysis Qual Exam

Part I

Solve one problem out of (1)–(2), one problem out of (3)–(4), and one problem out of (5)–(6).

(1) Prove or disprove this statement: the subset $S \subseteq [0, 1]$ consisting of numbers without a 7 in their decimal expansion is a Borel set with Lebesgue measure zero. (If there is a choice, always use the expansion without an infinite repeating sequence of 9's.)

(2) Prove or disprove this statement: if $(X, \mathcal{M}, \mu)$ is a measure space and $f_n \colon X \to \mathbb{R}$ is a sequence of measurable functions such that $f_n \to f$ pointwise, then f is measurable.

(3) Prove or disprove this statement: if X is a metric space then the σ-algebra of Borel subsets of X is generated by the collection of closed balls in X.

(4) Prove or disprove this statement: the σ-algebra of Borel subsets of $\mathbb{R}$ is generated by intervals of the form $[a, a + 1]$ for $a \in \mathbb{R}$.

(5) Prove or disprove this statement: if $f_n \colon \mathbb{R} \to \mathbb{R}$ are integrable functions with $f_n \to 0$ pointwise and

$$|f_n(x)| \leq \frac{1}{|x| + 1}$$

for all n, x, then

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dx = 0.$$

(6) Prove or disprove this statement: if $f_n \colon \mathbb{R} \to \mathbb{R}$ are integrable functions such that $f_n \to 0$ in measure, then $f_n \to 0$ in L^1.

PART II

Solve one problem out of (1)–(2), one problem out of (3)–(5), and one problem out of (6)–(7).

(1) Let $f(x) := \int_0^{+\infty} e^{-xt}\, t^x\, dt$, for $x > 0$. Show that f is well defined and C^1 (continuously differentiable) on $(0, +\infty)$, and compute its derivative.

(2) Let p_1, p_2 be such that $1 \leq p_1 \leq p_2 < \infty$ and let $f \in L^{P_1} \cap L^{P_2}$. Show that the map $p \mapsto \|f\|_p$ is well defined and continuous on $[p_1, p_2]$. [Hint: first state and prove a suitable inequality involving $|f|^p$.]

(3) Show that the sequence of functions

$$f_n(x) := \left(\frac{2}{\pi}\right)^{1/2} \cdot \sin(nx), \text{ for } n = 1, 2, 3, \ldots$$

is an orthonormal basis of $L^2[0, \pi]$ but not of $L^2[0, 2\pi]$, even though it is an orthonormal sequence in $L^2[0, 2\pi]$.

(4) Prove that if $f\colon [0, +\infty) \to \mathbb{R}$ is a continuous function tending to zero at infinity and such that

$$\int_0^{+\infty} f(x) e^{-nx}\, dx = 0 \text{ for } n = 0, 1, 2, \ldots,$$

then f is the zero function.

(5) State the duality theorem for L^p spaces and give a sketch of its proof when $1 < p < \infty$. Briefly explain what happens when $p = 1$ or $p = \infty$.

(6) Show that a uniform limit of continuous functions on $[0, 1]$ is continuous on $[0, 1]$. Is this true if $[0, 1]$ is replaced by any metric space?

(7) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Show that $df \equiv 0$ if and only if f is constant, and that df is constant if and only if f is an affine function.

Part III

Solve one problem out of (1)–(2) and two problems out of (3)–(6).

(1) Prove that if $f\colon [0,2\pi] \to \mathbb{R}$ is continuous, then

$$\lim_{n\to\infty} \int_0^{2\pi} f(x)\sin(nx)\,dx = 0.$$

(2) What is the power series expansion of the function $\int_0^x \exp(-t^2)dt$? Prove the power series converges to this function for all $x \in \mathbb{R}$.

(3) State and prove Hölder's inequality for functions on $\mathbb{R}$.

(4) Let f be a smooth function on $\mathbb{R}$ with compact support. Show that if the Fourier transform of f also has compact support, then f is identically zero.

(5) Show that $L^1[0,1]$ is not the dual of $L^\infty[0,1]$.

(6) Show that any normed vector space can be embedded into a Banach space.

Real Analysis 2013

Part I

Choose 3 problems from the following; however, you are not allowed to choose both (1) and (2). A measure space is always a general $(X, \mathcal{M}, \mu)$. A measure on $\mathbb{R}$ is Lebesgue measure, unless otherwise is specified.

(1) Show that $[0, 1]$ is uncountable.

(2) Let f_n be a sequence of measurable real valued functions on $\mathbb{R}$. Show that $A = \{x \in \mathbb{R} | \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable.

(3) Let $f \in L^1(\mathbb{R}, dx)$ and $F(x) = \int_{-\infty}^{x} f(t)dt$. Show that $F(x)$ is uniformly continuous.

(4) Prove that the product of two measurable real valued functions on $\mathbb{R}$ is measurable. (**Hint**: show that if f is measurable, then f^2 is measurable.)

(5) Evaluate $\int_0^\infty e^{-sx}x^{-1}\sin^2 x\, dx$ for $s > 0$ by integrating $e^{-sx}\sin(2xy)$ in a domain in $\mathbb{R}^2$. Exchange of iterated integrals needs to be justified.

PART II

Solve any 3 out of the following 6 problems.

(1) Prove that for all $1 \le p < \infty$, the L^p norm and sup norm on $C[0,1]$ are not equivalent, and also $C[0,1]$ is not complete in $L^p[0,1]$.

(2) Let $(X, \|\cdot\|)$ be a normed space. A set $E \subset X$ is called weakly bounded if $\sup_{x\in E} \|x\|_*$ is finite. Here $\|x\|_*$ is the weak norm. A set $E \subset X$ is called strongly bounded if $\sup_{x\in E} \|x\|$ is finite. Prove that E is weakly bounded if and only if it is strongly bounded.

(3) Let X and Y be compact Hausdorff spaces. Show that the algebra generated by functions of the form $f(x,y) = g(x)h(y)$ where $g \in C(X)$ and $h \in C(Y)$ is dense in $C(X \times Y)$.

(4) Let f be integrable over $(-\infty, \infty)$ and $g \in L^\infty(-\infty, \infty)$. Prove:

$$\lim_{t\to 0} \int_{-\infty}^{\infty} |g(x)[f(x) - f(x+t)]|dx = 0.$$

(5) Construct a function on $[0,1]$ which is continuous, monotone but not absolutely continuous.

(6) Suppose $f \in L^p([0,1])$ for all $p > 0$. Prove that

$$\lim_{p\to 0} \|f\|_p = \exp(\int_0^1 \ln |f|).$$

Part III

Solve any 3 out of the following 5 problems.

(1) Prove that if $f \in L^p(\mathbb{R})$ and $f \in L^q(\mathbb{R})$ with $1 \leq p < q < \infty$, then $f \in L^r(\mathbb{R})$ for all r with $p \leq r \leq q$.

(2) Starting from the definition of a Hilbert space, prove that if H is a complex Hilbert space and $v, w \in H$, then

$$|\langle v, w \rangle| \leq \|v\| \, \|w\|.$$

(3) Suppose f is in the Schwartz space $\mathcal{S}(\mathbb{R})$, and define the Fourier transform of f by

$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} \, dx$$

Prove that

$$\widehat{f'}(k) = -2\pi i k \, f(k).$$

Note: passing derivatives through integrals needs to be rigorously justified.

(4) Prove that there exists a nonzero polynomial in n variables, $P \colon \mathbb{R}^n \to \mathbb{R}$, such that if $f \colon \mathbb{R}^n \to \mathbb{C}$ is measurable and

$$|f(x)| \leq \frac{1}{P(x)}$$

for all $x \in \mathbb{R}^n$, then $f \in L^1(\mathbb{R}^n)$.

(5) Find a distribution T on $\mathbb{R}$ whose Fourier transform is the Dirac delta supported at the number 2. Rigorously prove that

$$\widehat{T}(k) = \delta(k-2).$$

Printed Name: ________________ **Signature:** ________________

2012 (reused)

Real Analysis Qualifying Exam ~~13~~ November ~~2010~~

Instructions:

- Work problems 1 through 3 and 6 of the remaining 9 problems.
- Show all your work and always justify your answers.

Hint: The length of a problem has little to do with its difficulty.

(1) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Show that f is uniformly continuous on any compact set $K \subseteq \mathbb{R}^n$.

(2) Let $f : [0,1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Use the definition of the Riemann integral to show that f is not Riemann integrable.

(3) (a) Let $f\colon [0,1] \to \mathbb{R}$ be continuous with the property that

$$\int_0^1 f(x)x^n\,dx = 0$$

for all $n = 0, 1, 2, \ldots$. Show that f is identically zero.

(b) Let (X, d) be a compact metric space. Show that the metric space, $C(X)$, equipped with the sup norm, is separable.

Hint: think of the distance function.

For grader's use only

1		7	
2		8	
3		9	
4		10	
5		11	
6		12	

(4) (a) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative measurable functions. Use the Monotone Convergence Theorem to prove that

$$\int_X (\liminf_n f_n) d\mu \le \liminf_n \int_X f_n d\mu.$$

(b) Give an example of a sequence of nonnegative Borel measurable functions on the real line for which the inequality in Fatou's Lemma is strict—and prove that the inequality is strict.

(5) Let μ be a finite measure on $\mathcal{B}(\mathbb{R})$. The goal of this problem is to show that μ is regular, that is for all $A \in \mathcal{B}(\mathbb{R})$ and all $\epsilon > 0$, there exists an open set O and a closed set F such that $F \subseteq A \subseteq O$ and $\mu(O \backslash F) \le \epsilon$. In order to do this we define the family,

$$\mathcal{C} = \{A \in \mathcal{B}(\mathbb{R}) : \forall\, \epsilon > 0,\ \exists\, O \text{ open and } F \text{ closed} \\ \text{for which } F \subseteq A \subseteq O \text{ and } \mu(O\backslash F) \le \epsilon\}.$$

(a) Show that $(a, b) \in \mathcal{C}$ for all $-\infty < a < b < +\infty$.

(b) Show that $\mathcal{C}$ is a σ-algebra.
Hint: The infinite union of closed set is not necessarily closed; however, you can remedy this problem by using the fact that a measure is continuous from above.

(c) Conclude from (a) and (b) that μ is regular.

(6) Let f be a positive function in $L^1 \cap L^\infty(X, \mathcal{M}, \mu)$ with $\|f\|_{L^\infty} \le 1$, where μ is a finite measure. Show that

$$\lim_{t\to 0^+} \frac{1}{t} \int_X (f^t - 1)\, d\mu = \int_X \log f\, d\mu$$

when $\log f$ is in $L^1(X, \mathcal{M}, \mu)$.

Hint: You may use, without proof, the inequality $\log x < x - 1 < 0$ for all $0 < x < 1$.

(7) (a) Prove the Cauchy-Schwarz inequality for a real Hilbert space.

(b) Let $K = K(x, y)$ be a continuous function on $[0, 1] \times [0, 1]$ and define $T\colon L^2([0, 1]) \to L^2([0, 1])$ for almost all x in $[0, 1]$ by

$$Tf(x) = \int_0^1 K(x, y) f(y)\, dy.$$

Show that T is well-defined and is a bounded linear operator that satisfies

$$\|T\| \le \|K\|_{L^2([0,1]\times[0,1])}.$$

(8) Let $(X, \mathcal{A}, \mu)$ be a measure space and set $L^p = L^p(X, \mathcal{A}, \mu)$. Show that given $f \in L^1 \cap L^2$ we have the following properties:
  (a) $f \in L^p$ for each $1 \leq p \leq 2$.
  (b) $\lim_{p \to 1^+} \|f\|_{L^p} = \|f\|_{L^1}$.

Hint: In this problem, if you use a non-standard inequality, other than the Holder or the Minkowski inequality, you must both state it and then prove it.

(9) Let $l^\infty = l^\infty_{\mathbb{R}}$ be the space of all bounded sequences in $\mathbb{R}$ and

$$c = \{x = (x_n)_{n=1}^\infty \in l^\infty \colon \lim_{n\to\infty} x_n \text{ exists and is finite}\}.$$

Equip c with the supremum norm, $\|x\|_c = \|x\|_{l^\infty} = \sup_{n\geq 1}|x_n|$.
  (a) Show that c is a Banach space.
  Hint: Prove that c is closed in l^∞.
  (b) Set

$$L(x) = \lim_{n\to\infty} x_n$$

  for any $x \in c$. Show that L is a bounded linear functional on c.
  (c) Define $p\colon l^\infty \to \mathbb{R}$ by

$$p(x) = \limsup_{n\to\infty} x_n.$$

  Show that p is a sublinear functional on l^∞ and that $p(x) = L(x)$ for all $x \in c$.
  (d) Show that L has a linear extension (still denoted by L) from c to l^∞ such that $L(x) \leq p(x)$ for all $x \in l^\infty$ and:
    (i) $\liminf_{n\to\infty} x_n \leq L(x) \leq \limsup_{n\to\infty} x_n$ for all $x \in l^\infty$.
    (ii) $L(x) \geq 0$ for all x in l^∞ such that $x \geq 0$.
    (iii) L is bounded with $\|L\| = 1$.

(10) (a) Show that $\{(2\pi)^{-1/2}e^{inx}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2([0, 2\pi])$ (or, more precisely, for $L^2_{\mathbb{C}}([0, 2\pi])$, the space of all square-integrable, complex-valued measurable functions on $[0, 2\pi]$).
  (b) Show that for any 2π-periodic, square-integrable function, f, on $\mathbb{R}$, we have the Fourier series expansion,

$$f = \sum_{n\in\mathbb{Z}} c_n e^{inx}$$

  having the property that

$$\|f\|^2_{L^2} = \sum_{n\in\mathbb{Z}} |c_n|^2,$$

  and calculate the Fourier coefficients, c_n, in terms of f.

(c) Find the Fourier expansion of the 2π-periodic function,

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{\pi}{2}, \\ 0 & \text{if } \frac{\pi}{2} \le x < 2\pi. \end{cases}$$

(11) Let X be a locally compact Hausdorff space (LCH). Recall that a Borel measure, μ, on X is a *Radon measure* if it is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets.

(a) State one form of the Riesz representation theorem as it applies to Radon measures.

(b) For each of the following, determine whether or not they are Radon measures and explain why:

(i) The Dirac delta function (also called the Dirac measure) on $\mathbb{R}$.

(ii) Counting measure on $\mathbb{R}^n$.

(iii) Lebesgue measure on $\mathbb{R}^n$.

(12) Assume that f lies in $L^1(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$, $d \ge 1$, and that ∇f lies in $L^1(\mathbb{R}^d)$. Let $\widehat{f}$ be the Fourier transform of f. Show that $(1 + |\xi|^2)^{\frac{1}{2}} \widehat{f}(\xi)$ lies in $L^2(\mathbb{R}^d)$ if and only if both f and ∇f lie in $L^2(\mathbb{R}^d)$.

Real Analysis Qualifying Exam, 2011

Name **Score**

Please show all work. Unsupported claims will not receive credit.

Part I.

Answer three of the following problems.

1. Let C be a collection of open sets of real numbers. Show that there is a countable subcollection O_i of C such that $\cup_{O\in C} O = \cup_{i=1}^{\infty} O_i$.
2. If $f \geq 0$, $\int f d\mu < \infty$, then prove that for every $\epsilon > 0$ there exists a measurable set E such that $\mu(E) < \infty$, $\int_E f d\mu > \int f - \epsilon$.
3. Suppose that ν_j is a sequence of positive measures. Prove the following. If $\nu_j \perp \mu, \forall j$, then $\sum_{j=1}^{\infty} \nu_j \perp \mu$, and if $\nu_{j=1}^{\infty} \ll \mu, \forall j$, then $\sum_{j=1}^{\infty} \nu_j \ll \mu$.
4. Let E be a Lebesgue measurable set in $\mathbb{R}$, whose measure is positive. Prove that E contains a subset which is not Lebesgue measurable. You may use without proof the standard non-measurable set in $[0, 1]$
5. Let $f \in L^1(dx)$ and $F(x) = \int_{-\infty}^{x} f(t)dt$. Show that $F(x)$ is continuous.

Part II.

Answer three of the following problems.

1. State and prove a version of the Vitali covering lemma on $\mathbb{R}^n$.
2. State a version of the Fubini theorem on double integrals. Give a counter example when the absolute value sign is dropped from the integrand in the condition of the theorem.
3. Show that every weakly convergent sequence in a Banach space is bounded with respect to the norm of the Banach space.
4. State the open mapping and closed graph theorem. Assuming the open mapping theorem, prove the closed graph theorem.
5. Let H be an infinite dimensional Hilbert space. Show that the unit sphere $S = \{x \in H | ||x|| = 1\}$ is weakly dense in the unit ball $B = \{x \in H | ||x|| \leq 1\}$.

Part III.

Answer three of the following problems.

1. Show that set $\Sigma := \{f = \Sigma_1^n a_j \chi_{E_j} | n \in \mathbb{N}, a_j \in \mathbb{C}, m(E_j) < \infty\}$ is dense in L^p for any $p \in [1, \infty)$.

2. State and prove Hölder's inequality in $\mathbb{R}^1$.

3. Let $f \in L^p(X) \cap L^\infty(X)$. (recall that this means $f \in L^q(X)$ for all $q > p$.) Show that $\lim_{q\to\infty} ||f||_q = ||f||_\infty$.

4. Let $f_n(x) = \frac{n}{2}\chi_{[-1/n,1/n]}$. Show (directly, instead of citing a theorem that immediately implies this) that for any $g \in L^1(\mathbb{R})$, $\lim_{n\to\infty} ||f_n \star g||_1 = 0$.

5. (a) Compute the Fourier Transform of $\chi_{[-1,1]}$.
   (b) Compute the Fourier Transform of $\frac{\sin^2 2\pi x}{x^2}$.
   (c) Are there any two non-zero elements f, g of $L^1(\mathbb{R})$ such that $f \star g = 0$ a.e. ? (Hint: $\tau_a \hat{h}$?)

Part I

Choose 3 problems from the following; however, you are not allowed to choose both (1) and (2). A measure space is always a general $(X, \mathcal{M}, \mu)$. A measure on $\mathbb{R}$ is Lebesgue measure, unless otherwise is specified.

(1) Let X be a compact metric space. Show that if O_n is an open and dense subset of X for $n = 1, 2, \ldots$ then $O = \cap_n O_n$ is not empty.

Hint: Create a shrinking sequence of closed sets F_n so that $F_n \subset O_n$.

(2) Let $\{a_n\}$ be a sequence in $\mathbb{R}$ and $\lim_{n\to\infty} a_n = a$. Show that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a.$$

(3) Let f be a positive function in $L^1 \cap L^\infty(X, \mathcal{M}, \mu)$, where X is a finite measure space and assume that $\|f\|_{L^\infty} \le 1$. Show that

$$\lim_{t\to 0^+} \frac{1}{t} \int_X (f^t - 1)\, d\mu = \int_X \log f \, d\mu$$

when $\log f$ is in $L^1(X, \mathcal{M}, \mu)$.

Hint: First show that $\log x < x - 1 < 0$ for all $x < 1$.

(4) We know that there exist a non-measurable subset of $[0, 1]$. Prove that $E \subset [0, 1]$ with $m^*(E) > 0$ has a non-measurable subset.

(5) (a) Let f, g be measurable functions. Show that fg is also measurable. (**Hint**: show that f^2 is measurable.)

(b) Let $\{f_n\}$ be a sequence of measurable functions and $f_n \to f$ a.e. as $n \to \infty$. Show that f is also measurable.

(6) For an integrable function, f, show that $\int |f| d\mu = 0$ if and only if $f = 0$ a.e.

Part II

Solve any 3 out of the following 5 problems.

(1) Suppose $1 \leq p < \infty$. If $f_n, f \in L^p[0,1]$ and $f_n \to f$ a.e. in $[0,1]$, prove:
$\|f_n - f\|_p \to 0$ if and only if $\|f_n\| \to \|f\|_p$.
Give a counterexample if the condition $\|f_n\| \to \|f\|_p$ is dropped.

(2) Let $f : [0,1] \to \mathbf{R}$ be an absolutely continuous function. Suppose $f'(x) = 0$ a.e. in $[0,1]$. Prove f is a constant.

(3) State without proof a version of the Fubini-Tonelli theorem on double integrals of nonnegative functions. Use a counterexample to show that the nonnegativity is necessary.

(4) Let $Y = C([0,1])$ and $X = C^1([0,1])$ both of which are equipped with the L^∞ norm. Show
(a) X is not complete.
(b) The map $\frac{d}{dx} : X \to Y$ is closed but not bounded.
(c) Is statement (b) a contradiction of the closed graph theorem? Why?

(5) Prove that a linear functional f on a normed vector space X is bounded if and only if $f^{-1}(\{0\})$ is closed.

Part III

Solve any 3 out of the following 5 problems.

(1) Let $f\colon X \to \mathbb{R}$ be μ-measurable on X with

$$(*) \qquad \int_{\operatorname{supp} f} e^{|f|}\, d\mu = 1.$$

Prove that for all f in $L^p(X)$. For $X = \mathbb{R}$ and μ being Lebesgue measure, give an example of f satisfying $(*)$ with f *not* lying in L^∞.

(2) Let f lie in $L^2(\mathbb{R})$ and for any $y > 0$ define

$$g_y(x) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x-t)^2 + y^2} f(t)\, dt.$$

Show that:
(a) For each $y > 0$, g_y lie in $L^2(\mathbb{R})$.
(b) For each $y > 0$, define the map, L_y, on $L^2(\mathbb{R})$ by $L_y(f) = g_y$. Show that L_y is a bounded linear operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.
(c) As $y \to 0$, $g_y \to f$ in $L^2(\mathbb{R})$.

Hint: Think in terms of convolutions.

(3) Let f lie in $L^1(\mathbb{R})$ and recall that the Fourier transform of f is the function $\widehat{f}\colon \mathbb{R} \to \mathbb{C}$ defined by

$$\widehat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i x y}\, dx.$$

Prove that $\widehat{f}$ is uniformly continuous.

(4) Let f be in $L^2([0,\infty))$. Prove or (by producing a counterexample) disprove each of the following:
(a) If f is also continuous then $\lim_{x\to\infty} f(x) = 0$.
(b) $\displaystyle \lim_{n\to\infty} \int_n^{n+1} |f(t)|\, dt = 0.$

(5) Suppose that f lies in $L^p([0,\infty))$ for $1 < p < \infty$. Prove that
(a) $\displaystyle \left| \int_0^x f(t)\, dt \right| \le \|f\|_{L^p}\, x^{1-\frac{1}{p}}$ for all $x > 0$,
(b) $\displaystyle \lim_{x\to\infty} x^{\frac{1}{p}-1} \int_0^x f(t)\, dt = 0.$

Qualifying Exam, Analysis, 11/14/2009. Asaeda

Part I

Choose 3 problems from the following and work on them, but choose only one out of (1)–(3).

(1)

Prove the mean value theorem: if a function f is continuous in $[a, b]$, differentiable in (a, b), then there is $\xi \in (a, b)$ so that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi)$$

Undergrad

holds.

(2) Show that a continuous function on $[a, b]$ is uniformly continuous.

(3) Prove the root test: let $\sum a_n$ be a series and $\alpha = \limsup \sqrt[n]{|a_n|}$. Show that if $\alpha < 1$, then $\sum a_n$ converges, and if $\alpha > 1$, then $\sum a_n$ diverges.

(4) Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on $\mathbb{R}$. Show that $\sup_n f_n$ is also measurable.

(5) Let ν be a signed measure on measurable space $(X, \mathcal{B})$. Show that Hahn decomposition is unique except for null set.

(6) (a) Let $(X, \mathcal{B}, \mu)$ be a measure space and g non-negative measurable function on X. Set $\nu E := \int_E g d\mu$. Show that ν is a measure on $\mathcal{B}$.

(b) Let f be a non-negative measurable function on X. Then show that

$$\int f d\nu = \int f g d\mu.$$

(hint: first do it for a simple function.)

(7) Show that almost all real numbers with respect to Lebesgue measure have the digit 7 in their decimal expansion. (Hint: Consider something similar to Cantor set.)

Part II   Zhang

Choose 3 problems from the following and work on them.

WII Final #4

(1). Prove that a topological space X is compact if and only if it has the finite intersection property, i.e. every family of closed sets has nonempty intersection if every finite subfamily has nonempty intersection.

Chunghoon #[illegible]

(2). Let T be a linear map from the Banach space X to itself. Suppose $\|T\| < 1$. Prove that $I + T$ is invertible. Give a counter example if $\|T\| = 1$.

Chunghoon #6

(3). Let X be a complete metric space and $\{U_n\}$ a sequence of dense open sets in X, $n = 1, 2, \ldots$. Prove $\cap_{n=1}^{\infty} U_n$ is dense in X.

WII Final #7

(4). Show that every weakly convergent sequence in a Banach space is bounded with respect to the norm of the Banach space.

Chunghoon #9 Folland

(5). Prove that a linear map from a vector space X to another vector space Y is bounded if and only if it is continuous. State (without proof) the Hahn-Banach theorem. Folland

Part III X u

Choose 3 problems from the following.

(1) If f and its Fourier transform are both smooth functions on R^n with compact support, show that f is identically zero. (HW)

(2) Suppose that $f \in L^1(R^n)$. Show that the Fourier transform of f is a continuous function which vanishes at infinity.

(3) Show that the product rule for derivatives is valid for products of smooth functions and distributions. Folland

(4) Let $f(x) = \frac{1}{2} - x$ on the interval $[0, 1)$, and extend f to be periodic on R. Calculate the Fourier transform of f. Use the Parseval identity to show that $\sum_{k\geq 1} k^{-2} = \pi^2/6$. ✓ (209C HW)

Qualifying Exam, Analysis, 11/1/2008.

Write the solutions for each part on separate bunches of paper, because we grade them separately. Please do not mix.

Part I

Solve 3 problems from the following.

1). Let $f : [0,1] \to \mathbf{R}$ be a Riemann integrable function which is greater than 1. Show that $1/f$ is also Riemann integrable on $[0,1]$.

2). Let f be a continuous function on $[0, 2\pi]$. Prove that

$$\lim_{i\to\infty} \int_0^{2\pi} f(x)\sin(ix)dx = 0.$$

Here $i = 1, 2, 3, ...$

3). Construct a strictly monotone function from $[0,1]$ to $\mathbf{R}$ whose derivative is zero a.e.

4). Suppose f is in $L^p(-\infty,\infty)$ and $L^q(-\infty,\infty)$, where $1 < p < q < \infty$. Is it true that f is in $L^r(-\infty,\infty)$ for all $r \in [p,q]$? Why?

5). State the Hölder and Minkowski inequalities. Use the former to prove the later.

Part II

Solve 3 problems from the following.

(1) Use the open mapping theorem to show that if the vector space V is complete in both of the norms $||\cdot||_1$ and $||\cdot||_2$, and that there is a constant $a > 0$ such that $a||v||_1 \leq ||v||_2$ for all $v \in V$, then the two norms are equivalent. State the closed graph theorem and use the above result to prove it.

(2) State and prove some version of the Hahn-Banach theorem.

(3) Show that a vector subspace V in another vector space W is weakly closed if and only if it is strongly closed.

(4) prove or disprove the following:

(i) The dual space of $L^1([0,1])$ is $L^\infty([0,1])$.

(ii) The dual space of $L^\infty([0,1])$ is $L^1([0,1])$.

Part III

Solve 3 problems from the following.

1) Determine the Hausdorff dimension of the Cantor ternary set.

2) let X be a normed linear space, and Y a Banach space. Show that $B(X, Y)$, the space of bounded linear functions from X to Y, is a Banach space.

3) Let H be a Hilbert space, and T a bounded linear operator from H to H. Show that if $TT^* = T^*T$, then

$$||T|| = \lim_{n\to\infty} \sqrt[n]{||T^n||}.$$

4) Let X, Y be Banach spaces and T a bounded linear operator from X to Y. Show that there is a constant $c > 0$ such that $||Tx|| \geq c||x||$ for any $x \in X$ iff $\mathrm{Ker}T = \{0\}$ and the range of T is closed.

Real Analysis Qualifier, November 2007.

Student ID number (Not name): Score

Please show all work. Unsupported result will not get credit. Each problem is worth 10 points. Note there are three parts in the exam. The problems are listed on p1, p5, p9 respectively. Please use the space provided for you answer.

Part I.

Choose 3 problems from the following.

1. Let **C** be a collection of open sets of real numbers. Then there is a countable subcollection $\{O_i\}$ of C such that

$$\cup_{O\in \mathbf{C}} O = \cup_{i=1}^{\infty} O_i.$$

2. Let f be a continuous real valued function on $[a, b]$ and suppose that $f(a) \leq r \leq f(b)$. Show that there is a $c \in [a, b]$ such that $f(c) = r$.

3. Use Vitali's lemma to show that if f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ a.e., then f is a constant.

4. Let f be a bounded measurable function on $[0, 1]$. Prove that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Part II

Choose 3 problems from the following.

1. Use the open mapping theorem to show that if the vector space V is complete in both of the norms $||\cdot||_1$ and $||\cdot||_2$, then there are constants $a, b > 0$ such that

$$a||v||_1 \leq ||v||_2 \leq b||v||_1$$

for all $v \in V$. State the closed graph theorem and use the above result to prove it.

2. State and prove some version of the Hahn-Banach theorem.

3. Show that $L^\infty[0,1]$ is not separable.

4. Show that $L^1[0,1]$ is not a dual of any normed linear space.

Part III

Choose 3 problems from the following.

1. Determine the extreme points of the unit sphere in $C[0,1]$.

2. State (without proof) the Tonelli theorem for nonnegative functions in a product space of σ finite measures. Give a counter example showing the σ finiteness is necessary in general.

3. Determine the Hausdorff dimension of the Cantor ternary set.

4. Let f and g be $L^2(\mathbf{R}^1)$ and $L^1(\mathbf{R}^1)$ functions. Consider the function

$$h(x) = \int_{\mathbf{R}^1} f(x-y)g(y)dy.$$

Prove that h is a $L^2(\mathbf{R}^1)$ function. Moreover

$$\|h\|_2 \leq \|f\|_2 \, \|g\|_1.$$

Qualifying Exam, Analysis, 11/18/2006.

Part I

Choose 3 problems from the following, however you are not allowed to choose both (1) and (2).

(1) prove the ratio test: let Σa_n be a series, where $a_n \neq 0$ for all n, and

$$\alpha := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Show that if $\alpha < 1$, then Σa_n converges, and if $\alpha > 1$, Σa_n diverges.

(2) Find the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

(3) State the Monotone Convergence Theorem (MCL) and Fatou's lemma (FL). Prove the MCT using FL.

(4) Prove Lebesgues's decomposition theorem: Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and ν a σ-finite measure defined on $\mathcal{B}$. Then there are measures ν_0 and ν_1 so that $\nu = \nu_0 + \nu_1$, $\nu_0 \perp \mu$, and $\nu_1 << \mu$. (Hint: use Radon-Nikodym, consider $\lambda = \mu + \nu$.)

(5) We know that there exist a non-measurable subset of $[0.1]$. Prove that $E \subset [0,1]$ with $m^*(E) > 0$ has a non-measurable subset.

(6) Let $\{f_n\}$ be a sequence of real-valued measurable functions on a measure space $(X, \mathfrak{B}, \mu)$. Show that the sets

- $A = \{x \in X | f_n(x) \to +\infty\}$
- $B = \{x \in X | \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}\}$

are measurable.

Part II

Solve 2 problems.

(1). Let μ denote the Lebesgue measure on R^1, let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued, integrable functions on R^1. All the following are on R^1 w.r.t the Lebesgue measure.

Determine if the following statements are true or false. If false, give a counter-example. You don't have to provide proofs for the true statements.

(i) If $f_n \to f$ a.e, then $f_n \to f$ in measure.

(ii)If $f_n \to f$ in measure, then $f_n \to f$ a.e.

(iii) If$f_n \to f$ in measure then, there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e. as $k \to \infty$.

(iv) If$f_n \to f$ in measure, then $\int_{R^1} |f_n(x) - f(x)| d\mu \to 0$.

(v) If $f_n \to f$ in measure and $f_n \geq 0$, then $\int_{R^1} \underline{\lim}_{n\to\infty} f_n \geq \int f$.

(2.) Consider the operator

$$h(x) = \int_R f(x-y)g(y)dy.$$

(a) If $f, g \in L^1(R)$, show that $h \in L^1(R)$ with

$$\|h\|_{L^1(R)} \leq \|f\|_{L^1(R)} \|g\|_{L^1(R)}.$$

(b) If $f \in L^\infty(R)$ and $g \in L^1(R)$, show that $h \in L^\infty(R)$ with

$$\|h\|_{L^\infty(R)} \leq \|f\|_{L^\infty(R)} \|g\|_{L^1(R)}.$$

(c) If f is a continuous function with compact support and g is integrable, show that $h(x)$ is uniformly continuous.

(3). Let $f \geq 1$ be a $L^1[0,1]$ function. Prove that $\ln f$ is a $L^p[0,1]$ function for all $1 \leq p < \infty$.

(4). State the Tonelli theorem for double integrals (cf Royden Chapter 12) and give an example showing the nonnegativity condition in the theorem is necessary.

Part III

Solve 2 problems, including (1).

(1)(a)What are the extreme points of the unit sphere in $C(X)$, where X is a compact Hausdorff space ?
(b) Show that $C([0,1])$ is not the dual of any normed linear space.

(2) Let X, Y be Banach spaces and T a bounded linear operator from X and Y. Show that there is a constant $c > 0$ such that $||Tx|| \geq c||x||$ for any $x \in X$ if and only if $\mathrm{Ker}T = \{0\}$ and the range of T is closed.

(3) Let X be a normed linear space, and X^* is its dual space, i.e. the space of real-valued linear functionals on X. Show that X^* is a Banach space.

Ph.D. Qualifying Examination (Real Analysis)

Spring 2005

Rules: There are three sections on this examination. Group I covers Undergraduate material (Advanced Calculus), while Groups II and III cover Graduate material (Measure Theory and Integration, along with Functional Analysis). The test consists of **6 problems**, 2 from each group. (Each problem is weighted equally. No extra-credit will be given for any additional problem attempted beyond these 6 problems.) *Please always justify your answers. Also, please indicate at the beginning of the test which problems you have selected in each group.*

Good luck!

GROUP I

1. Let X be a complete metric space with metric d, and A be a subset of X. Prove that A is a closed subset of X if and only if A is complete with respect to the metric d.

2. Let K be a compact subset of $\mathbb{R}^n$ and $\{B_j\}_{j=1}^{\infty}$ a sequence of open balls that covers K. Prove that there is a positive number ε such that each ε-ball centered at a point of K is contained in one of the balls B_j.

3. Prove the Root Test: let $\sum a_n$ be a series, and

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

Show that if $\alpha < 1$, then $\sum a_n$ converges, and if $\alpha > 1$, then $\sum a_n$ diverges.

4. Find the integral

$$\int_0^\infty \frac{dx}{(x^2+1)^2}.$$

5. Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ such that

(a) f is discontinuous everywhere.
(b) f is continuous at exactly one point.
(c) f is continuous on a dense set, and discontinuous on a dense set.

Group II

6. Let Σ be a σ-algebra of measurable subsets of a measurable space X.

(a) Define what it means for a (finite) real valued function f on X to be measurable.

(b) Prove from your definition that if f and g are measurable, so is $f+g$.

7. Suppose that $f(x) = \sum_{k=1}^{\infty} a_k x^k$, where the a_k's are real and $\sum_{k=1}^{\infty} |a_k| < \infty$. Show that f is of bounded variation on the interval $[-1, 1]$. *Hint:* Treat first the case in which all of the a_k's are non-negative.

8.

(a) Let X be a Banach space, and T a bounded linear operator on X, $T \in \mathcal{B}(X, X)$. Define $\|T\|$.

(b) Let $X = \mathbb{R}^2$, with the usual norm $\|(x, y)\| = \sqrt{x^2 + y^2}$. Let T be the linear operator on X which, in the standard basis $\{e_1 = (1, 0), e_2 = (0, 1)\}$, has the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

What is $\|T\|$?

9. Define what it means for two norms on the same Banach space to be equivalent. For $1 \le p < \infty$ and $x = (x_1, x_2 \ldots, x_n) \in \mathbb{R}^n$, define $\|x\|_p = [\sum_{i=1}^{n} |x_i|^p]^{1/p}$. Define also $\|x\|_\infty = \sup\{|x_i| : 1 \le i \le n\}$. Show that all of these norms are equivalent. You do not need to show that $\mathbb{R}^n$ with such a norm is a Banach space.

Group III

10. Evaluate

$$\lim_{N \to \infty} N \int_0^N \frac{1}{s} \log(1 + (s/N)) \frac{ds}{1 + s^2}.$$

Justify your calculations.

11. Let ϕ be a monotone increasing function on $[0, 1]$ with $\phi(0) = 0$.

(a) What can be said about ϕ' (the derivative of ϕ)?

(b) What can be said about $\Phi(t) = \int_0^t \phi'(s)\, ds$?

(c) What can be said about Φ'?

(d) What can be said about $\Psi(t) = \int_0^t \Phi'(s)\, ds$?

12. Let X be a normed linear space, and X^* its dual space of real-valued linear functionals. Show that X^* is a Banach space. Note: X is not assumed to be a Banach space.

13. Let X and Y be Banach spaces and let T be a bounded linear operator from X to Y, $T \in \mathcal{B}(X, Y)$. Show that there is a constant $c > 0$ such that $\|Tx\| \ge c\,\|x\|$ for all $x \in X$ if and only if $\ker T = \{0\}$ and the range of T is closed.

Real Analysis Qualifier 05, Part I

I. Determine for which values of the real parameters α and β the following series converges:

$$\sum_{n=2}^{\infty} \frac{1}{n^{\alpha}(\log n)^{\beta}}.$$

Justify precisely your answer.

II. Show directly that if $f : [0,1] \to \mathbf{R}$ is continuous, then it is also uniformly continuous. Is this still true if $[0,1]$ is replaced by either $(0,1)$ or $[0,\infty)$? Explain your answer.

III. 1) Given $f : [a,b] \to \mathbf{R}$ a Riemann integrable function, we define $F : [a,b] \to \mathbf{R}$ by

$$F(x) = \int_a^x f(t)dt,$$

for each $x \in [a,b]$. Show directly that if f is continuous at some point x_o of [a,b], then F is differentiable at x_o and $F'(x_o) = f(x_o)$.

2) Give an example of a Riemann integrable function f whose associated function F is differentiable on $[a,b]$ but satisfies $F' \neq f$.

IV. Show that if $f : [0,1] \to \mathbf{C}$ is continuous and satisfies, for all $n \in \mathbf{Z}$

$$\int_0^1 f(t)e^{i2n\pi t}dt = 0,$$

for all $n \in \mathbf{Z}$, then $f = 0$ on $[0,1]$.

Does this conclusion still hold if $\mathbf{Z}$ (the set of all integers) is replaced by $\{0,1,2,...\}$?

V. Let $[a,b]$ be a compact interval and $f_n, f, g : [a,b] \to \mathbf{R}$, for $n = 1,2,....$ In each case, explain whether the given statement is true or false (by quoting an appropriate theorem or giving a counterexample):

(a) If each f_n is C^1 (i.e., continuously differentiable), $f_n \to f$ uniformly and $f_n' \to g$ pointwise, then f is C^1 and $f' = g$.

(b) If $f_n \to f$ pointwise and each f_n is continuous, then f is continuous.

(c) If there exists a finite constant M such that each f_n is C^1 and satisfies $|f_n(x)| \leq M$ and $|f_n'(x)| \leq M$, for all $x \in [a,b]$ and $n \geq 1$, then the sequence $\{f_n\}$ has a uniformly convergent subsequence on $[a,b]$.

VI. Show that the equation $x_1^2+x_1^2+...+x_n^2 = 1$ defines the unit sphere as a smooth (i.e. continuously differentiable) $(n-1)$-dimensional surface in $\mathbf{R}^n$, and calculate its tangent plane at every point.

Real Analysis Qualifier 05, Part II

1. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and u be a L^1 function in Ω. Prove

$$\lim_{p\to 0} [\frac{1}{|\Omega|}\int_\Omega |u|^p dx]^{1/p} = \exp [\frac{1}{|\Omega|}\int_\Omega \ln |u|\ dx].$$

Here $|\Omega|$ is volume of Ω.

2. Prove or disprove: any monotone function in some interval $(a, b) \subset \mathbf{R}$ is measurable.

3. Let $f \geq 0$ be a L^p function on $[a, b] \subset \mathbf{R}$, $p > 0$. Prove that

$$\int_a^b f^p dx = p\int_0^\infty t^{p-1}|\{x \mid f(x) > t\}|dt.$$

Here $|\{x \mid f(x) > t\}|$ is the Lebesgue measure of the set.

4. Let g be a bounded measurable function and $f \in L^1(-\infty, \infty)$. Prove

$$\lim_{t\to 0}\int_{-\infty}^\infty |g(x)[f(x) - f(x+t)]|dx = 0.$$

5. Let f_n be a sequence of integrable functions on $[a, b] \subset \mathbf{R}$, converging almost everywhere to an integrable function f. Prove that

$$\int_a^b |f - f_n|dx \to 0$$

if and only if

$$\int_a^b |f_n|dx \to \int_a^b |f|dx$$

when $n \to \infty$.

Real Analysis Qualifying Exam
November 19, 2005

Part III

1. (a) Let X be a Banach space, and T a bounded linear operator on X, $T \in B(X,X)$. Define $\|T\|$.

   (b) Let X=$\mathbf{R}^2$, with the norm $\|(x,y)\| = \sqrt{x^2+y^2}$. Let T be the linear operator on X which, in the standard basis $\{e_1=(1,0),\ e_2=(0,1)\}$, has the matrix
$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$
   What is $\|T\|$?

2. Define what it means for two norms on the same Banach space to be equivalent. For $1 \leq p < \infty$ and $x=(x_1,x_2...,x_n) \in \mathbf{R}^n$, define $\|x\|_p = \left[\Sigma_{i=1}^n |x_i|^p\right]^{1/p}$. Define also $\|x\|_\infty = \sup\{\,|x_i| : 1 \leq i \leq n\}$. Show that all of these norms are equivalent. You do not need to show that $\mathbf{R}^n$ with such a norm is a Banach space.

3. Let X be a normed linear space, and X^* its dual space of real-valued continuous linear functionals. Show that X^* is a Banach space. Note: X is not assumed to be a Banach space.

4. Let X and Y be Banach spaces and let T be a bounded linear operator from X to Y, $T \in B(X,Y)$. Show that there is a constant c>0 such that $\|Tx\| \geq c\|x\|$ for all $x \in X$ if and only if $\ker T = \{0\}$ and the range of T is closed.

5. Show that the dual of $L^\infty[0,1]$ is NOT $L^1[0,1]$.

Ph.D. Qualifying Examination (Real Analysis)

Fall 2004

Rules: There are three sections on this examination. Group I covers Undergraduate material (Advanced Calculus), while Groups II and III cover Graduate material (Measure Theory and Integration, along with Functional Analysis). The test consists of **6 problems**, 2 from each group. (Each problem is weighted equally. No extra-credit will be given for any additional problem attempted beyond these 6 problems.) *Please always justify your answers. Also, please indicate at the beginning of the test which problems you have selected in each group.*

Good luck!

GROUP I

1. Let f be a continuous function on $[0,1]$. Show that f assumes its maximum and minimum.

2. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{n!\Gamma(\alpha)} x^n, \qquad -1 < x < 1.$$

Show that the series converges absolutely and defines a continuous function f when $\alpha > 0$ is an irrational number. (Note that $\Gamma(x+1) = x\Gamma(x)$, $x > 0$.) Verify that

$$(1-x)f'(x) = \alpha f(x)$$

and deduce that

$$f(x) = (1-x)^{-\alpha},$$

giving the binomial expansion for general powers α.

3. Give a precise statement of L'Hôpital's rule. Give a careful proof of one of the cases.

4. Show that

$$\int_0^{\infty} \frac{2\sin^4 x}{x^2}\,dx = \int_0^{\infty} \frac{2\sin^2 x \cos^2 x}{x^2}\,dx = \int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

5. Show that the set $[0,1]$ is uncountable.

Ph.D. Qualifying Examination (Real Analysis)

Fall 2004

Prelude: There are three sections in this examination. Group I covers Undergraduate material (Advanced Calculus), while Groups II and III cover Graduate material (Measure Theory and Integration, along with Functional Analysis). The test consists of a total of 10 problems to be selected as follows:

Please do 3 problems from Group I, 4 problems from Group II, and 3 problems from Group III. (Each problem is weighted equally. No extra-credit will be given for any additional problem attempted beyond these 10 problems.) *Please always justify your answers. Also, please indicate at the beginning of the test which problems you have selected in each group.* Good luck!

GROUP I

1. Find a power series expansion for $\int_0^x e^{-t^2}\,dt$.
   Justify rigorously your answer.

2. Let $f_n(x) = \frac{\sin nx}{1+nx}$, for $x > 0$. Discuss the pointwise and the uniform convergence of $\{f_n\}_{n=1}^{\infty}$ on $(0,\infty)$ and on intervals of the form $[a,\infty)$, with $a > 0$.

3. Let $f(x) = \int_{x^2}^{x} e^{-t^2}\,dt$, for $x \in \mathbb{R}$.
   Show that f is differentiable and calculate $f'(x)$.

4. Let $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ if $x \in \mathbb{R}\backslash\mathbb{Q}$. Is f Riemann integrable on $[0,1]$? If so, calculate $\int_0^1 f(x)dx$.
   Justify your answers.

5. Let K_1 and K_2 be disjoint nonempty compact subsets of $\mathbb{R}$.
   Show that there exists $x_j \in K_j (j = 1,2)$ such that

$$0 < |x_1 - x_2| = \inf\{|y_1 - y_2| : y_1 \in K_1, y_2 \in K_2\}.$$

Would that still be true if K_1 and K_2 were only closed subsets of $\mathbb{R}$?

6. Suppose $\lim_{n\to\infty} a_n = A$, where $a_n \in \mathbb{C}$ for all $n \geq 1$. Prove that

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = A.$$

7. Show that the gravitational potential $V(x, y, z) = \frac{c}{r}$ (where c is a constant and $r = \sqrt{x^2 + y^2 + z^2}$) is a solution of the 3-dimensional Laplace equation (away from the origin):

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

GROUP II

8. Show that the identity

$$\int_0^\infty \frac{e^{-x} - e^{-xt}}{x} dx = \log t$$

holds for all $t > 0$.

Justify your answer.

9. If $f : [0,1] \to \mathbb{R}$ is absolutely continuous, show that f is a constant function if and only if $f' = 0$ a.e. on $[0,1]$. Is this statement still true if f is only assumed to be differentiable (Lebesgue) almost everywhere on $[0,1]$? Justify your answer by either providing a proof or a counterexample.

10. Assume that $\varphi \in C^1([0,1])$, with $\varphi(0) = \varphi(1) = 0$. Show that

$$\int_0^1 \frac{1}{x^2}\varphi^2(x)dx \leq 4\int_0^1 \varphi'(x)^2 dx.$$

11. Let $\{f_n\}$ be a sequence of Lebesgue integrable functions on a measure space $(X, \mathcal{A}, \mu)$ such that $f_n \to f$ μ-a.e., with f μ-integrable.

Prove that

$$\int_X |f - f_n| d\mu \to 0 \Leftrightarrow \int_X |f_n| \to \int_X |f| d\mu.$$

12. Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, with $1 \leq p \leq \infty$. Show that

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p,$$

where $(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy$ denotes the convolution of f and g.

13. Let $f \geq 0$ be a Lebesgue integrable function on $[0,1]$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)|\sin(nx)|dx = \frac{2}{\pi}\int_0^1 f(x)dx.$$

14. If $E \subset \mathbb{R}$ is a set such that $m^*(E) > 0$ (where m^* denotes Lebesgue outer measure on $\mathbb{R}$), show that E contains a non-measurable subset.

15. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and 2π-periodic. Show that for every $\varepsilon > 0$, there exists a finite Fourier series

$$\varphi(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

such that $|f(x) - \varphi(x)| < \varepsilon$, for all $x \in \mathbb{R}$.

GROUP III

16. (a) State (without proof) the Radon–Nikodym Theorem.

(b) Verify the uniqueness of the Radon–Nikodym derivative by proving the following lemma: If $(X, \mathcal{A}, \mu)$ is a measure space and $f \in L^1(\mu)$ satisfies $\int_A f d\mu = 0$ for all $A \in \mathcal{A}$, then $f = 0 \quad \mu$-a.e.

17. Let $K \in L^2([0,1] \times [0,1])$. Show that $T : L^2[0,1] \to L^2[0,1]$, defined (for $f \in L^2[0,1]$ and a.e. $x \in [0,1]$) by

$$(Tf)(x) = \int_0^1 K(x,y) f(y) dy,$$

is a bounded linear operator on $L^2[0,1]$ and estimate its norm.

18. Let H be a Hilbert space, with inner product $(\cdot\,,\,\cdot)$.
Assume that $T, T^* : H \to H$ are two functions on H satisfying

$$(Tx, y) = (x, T^*y)$$

for all $x, y \in H$.

Show that T and T^* are bounded linear operators on H satisfying $\|T\| = \|T^*\|$ and $\|TT^*\| = \|T\|^2$.

19. Which of the following statements is correct?
(a) $L^2[0,1]$ is separable.
(b) $L^\infty[0,1]$ is separable.
(c) The (Banach) dual of $L^2[0,1]$ is itself.
(d) The dual of $L^1[0,1]$ is $L^\infty[0,1]$, and conversely.
(e) The dual of $L^{4/3}[0,1]$ is $L^4[0,1]$, and conversely.

[**Advice**: In each case, justify briefly your answer.]

20. State the Hahn–Banach Theorem and give a *sketch* of its proof.

21. If X is a Banach space, show that there is a natural way of embedding X into its second dual $X^{**} := (X^*)^*$, and that this embedding is a norm-preserving operator. (Hence, X can be considered as a subspace of X^{**}.) Give an example where $X = X^{**}$ and another example where $X \neq X^{**}$.

22. Is the dual of $L^1[0,1]$ equal to $L^\infty[0,1]$?
(Carefully justify your answer.)

Ph.D. Qualifying Examination (Real Analysis)

Spring 2003

Prelude: There are three sections in this examination. Group I covers Undergraduate material (Advanced Calculus), while Groups II and III cover Graduate material (Measure Theory and Integration, along with Functional Analysis). The test consists of a total of 10 problems to be selected as follows:

Please do 3 problems from Group I, 4 problems from Group II, and 3 problems from Group III. (Each problem is weighted equally. No extra-credit will be given for any additional problem attempted beyond these 10 problems.) *Please always justify your answers, Also, please indicate at the beginning of the test which problems you have selected in each group.* Good luck!

GROUP I

1. Calculate the volume of the unit ball in $\mathbb{R}^3$. (Show your work.) Explain briefly how you would proceed to extend this result to $\mathbb{R}^n$, for any $n \geq 1$.

2. Show that the series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$ is convergent, and calculate its sum.

3. Let $\{x_n\}$ be a Cauchy sequence in a metric space. Show that if $\{x_n\}$ admits a convergent subsequence, then the entire sequence is convergent.

4. Find the following sums:

$$(a) \sum_{n=1}^{\infty} nx^n, |x| < 1. \qquad (b) \sum_{n=1}^{\infty} n^2 x^n, |x| < 1.$$

[**Advice**: Justify your answers.]

5. Let $\mathcal{A}$ be the real vector space spanned by the functions

$$1, \sin x, \sin^2 x, \dots, \sin^n x, \dots,$$

defined on $[0,1]$. Show that $\mathcal{A}$ is dense in $C[0,1]$, the space of real-valued continuous functions on $[0,1]$ equipped with the sup norm.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x,y) = (\cos(\sqrt{x^2+y^2}), \sin(\sqrt{x^2+y^2})).$$

Show that f is continuously differentiable on $\{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$ and determine both directly and by using a well-known theorem whether f is locally invertible near the point $(x_0, y_0) = (1,1)$.

GROUP II

7. Let $\{f_n\}$ be a sequence of real-valued measurable functions on a measure space $(X, \mathcal{A}, \mu)$. Show that the sets
(i) $A = \{x \in X : f_n(x) \to +\infty\}$,
(ii) $B = \{x \in X : f_n(x) \to -\infty\}$, and
(iii) $C = \{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}\}$
are all measurable.

8. Let X and Y be Banach spaces and let $B(X, Y)$ denote the space of bounded linear operators from X to Y, equipped with its standard operator norm. *Show* that $B(X, Y)$ is complete. Is the assumption that X be a Banach space (rather than a normed space) necessary?

9. (a) State Hölder's Inequality for L^p spaces.

(b) Show that for any two nonnegative numbers a and b,

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Use this to give a direct proof of Hölder's Inequality.

[0,1] [0,1] [0,1]

10. If $f \in L^q(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for some $q \in [1, \infty)$, show that $f \in L^p(\mathbb{R})$ for all $p > q$ and that

$$||f||_\infty = \lim_{p\to\infty} ||f||_p.$$

[**Hint** (for the first part): Show that $||f||_p \leq ||f||_q$.]

11. Let H be a real Hilbert space. State and prove the Cauchy-Schwarz Inequality in H. When does equality hold?

12. Let f be a bounded measurable function on $[0, +\infty)$. Show that

$$F(x) = \int_0^{+\infty} \frac{f(t)e^{-xt}}{\sqrt{t}} dt, \ x > 0,$$

is continuously differentiable on $(0, +\infty)$. Compute its derivative $F'(x)$, for $x > 0$.

GROUP III

13. Let X be a Banach space, $T : X \to X$ a bounded linear operator on X, and I the identity operator on X. If $\|T\| < 1$, then show that $I - T$ is bounded and invertible on X. Calculate its inverse.

14. State the Baire Category Theorem and give a *sketch* of its proof.

15. Let X be a normed linear space and f be a nonzero linear functional on X. Prove each of the following two statements:

(a) f is continuous if and only if its kernel is a closed subspace of X.

(b) f is discontinuous if and only if its kernel is dense in X.

[Recall that the kernel of f is equal to $f^{-1}(\{0\})$.]

16. Show that the Hilbert space $L^2 = L^2[0, +\infty)$ is separable (i.e., admits a countable dense subset).

[**Hint**: Consider a truncation of the power functions, for example.]

17. For $n \in \mathbb{Z}$, let $f_n : [0, 2\pi] \to \mathbb{C}$ be defined by $f_n(x) = e^{inx}$. Show that the sequence $\{f_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the complex Hilbert space $L^2([0, 2\pi])$.

Ph.D. Qualifying Examination (Real Analysis)

November 23, 2002

Prelude: There are three sections in this examination. Group I covers Undergraduate material (Advanced Calculus), while Groups II and III cover Graduate material (Measure Theory and Integration, along with Functional Analysis). The test consists of a total of 10 problems to be selected as follows:

Please do 3 problems from Group I, 4 problems from Group II, and 3 problems from Group III. (Each problem is weighted equally. No extra-credit will be given for any additional problem attempted beyond these 10 problems.) *Please always justify your answers, Also, please indicate at the beginning of the test which problems you have selected in each group.* Good luck!

GROUP I

1. Show that if $\{x_n\}$ is a convergent sequence in $\mathbb{R}^k$, with limit b, then

$$A := \{x_n : n \geq 1\} \cup \{b\}$$

is a compact subset of $\mathbb{R}^k$.

Would the result be true in an arbitrary metric space (rather than in $\mathbb{R}^k$)?

2. Let $f_n(x) = \cos(x + \frac{1}{n})$, for $x \in \mathbb{R}$ and $n \geq 1$.

Is the sequence $\{f_n\}_{n=1}^{\infty}$ uniformly convergent on $\mathbb{R}$? If so, what is its limit?

3. Let $\{f_n\}$ be a uniformly convergent sequence of continuous functions on $[a, b]$ and let $c \in [a, b]$. Prove directly that

$$\lim_{n\to\infty} \lim_{x\to c} f_n(x) = \lim_{x\to c} \lim_{n\to\infty} f_n(x).$$

[**Advice**: You should give a *direct* proof and not quote a known theorem.]

4. Suppose that f is a real-valued continuous function on $[a,b]$. Show that

$$\lim_{n\to\infty}\int_a^b f(x)\sin(nx)dx = 0$$

5. Let

$$\varphi(t) = \int_0^\infty e^{-tx^2}dx, \text{ for } t > 0.$$

Find $t_0 > 0$ such that $\varphi(t_0) = 1$. Is such a point t_0 unique?
[**Advice**: Justify your computation.]

6. (a) Give the definition of differentiability of a function $f : \mathbb{R}^n \to \mathbb{R}$, at the point $a \in \mathbb{R}^n$.

(b) Show that if f is differentiable at a, then it is continuous at a. Is the converse true? Explain your answer.

GROUP II

7. Given $f \in L^p(\mathbb{R})$, with $p \geq 1$, construct a sequence $\{f_n\}$ of functions in $L^\infty(\mathbb{R})$ such that $f_n \to f$ in $L^p(\mathbb{R})$.

[**Hint**: Truncate f.]

8. Show that a normed linear space E is complete if and only if every absolutely convergent series of elements of E is convergent in E.

9. Consider the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \text{for } x > 0.$$

Show that Γ is continuously differentiable on $(0, +\infty)$ and calculate its derivative.

10. Let $L^\infty = L^\infty[0,1]$ and $L^1 = L^1[0,1]$.
Show that

$$(L^\infty)' \neq L^1,$$

where $(L^\infty)'$ denotes the dual of L^1.

11. Construct a sequence of functions $\{f_n\}$ that converges in measure but does not converge almost everywhere.

12. Let $(A, \mathcal{A}, \mu)$ be a measure space. Let f be a bounded function defined on a subset of A with finite measure. Prove that f is Lebesgue integrable if and only if f is measurable.

13. Prove that for any set $E \subset [0,1]$ with positive Lebesgue outer measure (i.e., $m^*(E) > 0$), there exists a subset of E which is not measurable.

[**Advice**: You may assume, without proof, the existence of a non-measurable subset of $[0,1]$.)

GROUP III

14. (a) State (without proof) Fatou's Lemma.

(b) Construct a sequence of nonnegative functions on [0,1] so that the inequality in Fatou's Lemma is strict.

15. Let $L^p = L^p[0,1]$ be equipped with its usual norm. Give a *sketch* of the proof that L^p is complete, for $1 \leq p < \infty$.

[**Advice**: You may use, without proof, the result stated in problem 8.]

16. (a) State (without proof) the Hahn-Banach Theorem.

(b) Suppose $(E, \|\cdot\|)$ is a normed linear space, F is a closed subspace of E, $x_0 \notin F$, $x_0 \in E$, and $\delta = \inf\{\|x_0 - x\| : x \in F\}$. Show that there exists $\varphi \in E'$ such that $\varphi(x_0) = \delta$, $\varphi(x) = 0$ for $x \in F$ and $\|\varphi\| \leq 1$. (Here, E' denotes the dual of E.)

17. Let $g \in L^q[0,1]$ with $1 < q < \infty$ and let $p > 0$ be given by $\frac{1}{p} + \frac{1}{q} = 1$. Let F be the linear functional on $L^p[0,1]$ defined by $F(f) = \int_0^1 fg dx$. Show that F is continuous and that $\|F\| = \|g\|_q$.

18. Let g, h, k be functions on $\mathbb{R}^n$ of class L^1, L^2 and L^2, respectively. Show that the function $B(x, y) := h(x)k(x - y)g(y)$, with $x, y \in \mathbb{R}^n$, belongs to L^1 on $\mathbb{R}^{2n}$.

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Real Analysis Qualifying Examination

December 22, 2001

Instructions: Work 7 out of the following 9 problems!

1. Let $f : [0, \pi] \to \Re$ be a continuous function. Show that for every $\varepsilon > 0$, there is a trigonometric function T_n defined as

$$T_n(x) = \sum_{k=0}^{n} a_k \cos kx$$

such that $\sup_{0 \le x \le \pi} |f(x) - T_n(x)| < \varepsilon$, and explain why this conclusion no longer holds if the cosine function is replaced by the sine function in $T_n(x)$.

2. For every subset $M \subseteq \Re$ define $M + a = \{x + a : x \in M\}$.

   (a) Let $E \subseteq \Re$ be a Borel set. Show that $E + a$ is also a Borel set.

   (b) If $f : \Re \to \Re^+$ is a Borel function and g is defined as $g(x) = \sum_{n=1}^{\infty} f(x+n)$, $x \in \Re$, show that $\int_{\Re} g(x)\, d\mu(x) < \infty$ iff $f = 0$ μ-almost everywhere, where μ is the Lebesgue measure on $\Re$. Give an example to show that "almost everwhere" cannot be omitted here.

3. Let $f(x,y) = e^{-xy^2} \cos x$, $0 \le x < \alpha$, $0 \le y < \infty$. Using the elementary result $\int_0^\infty e^{-\beta t^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}$, $\beta > 0$, justify the steps and complete the following evaluation:

   (a) $\int_0^\alpha \int_0^\infty f(x,y)\, dy\, dx = \int_0^\infty \int_0^\alpha f(x,y)\, dx\, dy$, $\alpha > 0$, (dx and dy are Lebesgue measure).

   (b) $\int_0^\alpha \int_0^\infty f(x,y)\, dy\, dx = \frac{\sqrt{\pi}}{2} \int_0^\alpha \frac{\cos x}{\sqrt{x}} dx$.

   (c) Using integration by parts, show that $\int_0^\alpha f(x,y)\, dx = \frac{y^2}{1+y^4}\left(1 - e^{-\alpha y^2} \cos \alpha\right) + e^{-\alpha y^2} \frac{\sin \alpha}{1+y^4}$.

   (d) $\lim_{\alpha \to 0} \int_0^\infty \int_0^\alpha f(x,y)\, dx\, dy = \int_0^\infty \frac{y^2}{1+y^4} dy = \frac{1}{4}\pi\sqrt{2}$ (You may use the elementary fact that $\int \frac{y^2}{1+y^4} dy = \frac{1}{8}\sqrt{2} \ln \frac{1-\sqrt{2}y+y^2}{1+\sqrt{2}y+y^2} + \frac{1}{4}\sqrt{2} \arctan\left(\sqrt{2}y + 1\right) + \frac{1}{4}\sqrt{2} \arctan\left(\sqrt{2}y - 1\right) + C$).

(e) Deduce that $\int_0^\infty \frac{\cos x}{\sqrt{x}} dx = \frac{1}{2}\sqrt{2}\sqrt{\pi}$, even though $\frac{\cos x}{\sqrt{x}}$ is not Lebesgue integrable on $\Re^+$.

4. Give an example of a Lebesgue integrable function that is not Riemann integrable. Work out the details!

5. A function f is said to satisfy a Lipschitz condition on an interval $[a, b]$, if there is a constant M such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [a, b]$.

   (a) Show that a function satisfying a Lipschitz condition is absolutely continuous.

   (b) Show that an absolutely continuous function f satisfies a Lipschitz condition if and only if $|f'|$ is bounded.

   (c) Prove or give a counterexample: f satisfies a Lipschitz condition if one of its derivatives (say D^+) is bounded.

6. Let ℓ^∞ be the space of all bounded sequences of real numbers. For $(\xi_n)_{n\geq 1}$ define

$$\left\| (\xi)_{n\geq 1} \right\|_\infty = \sup_{n\geq 1} |\xi_n|$$

   Show directly from the definitions that ℓ^∞ is a Banach space.

7. Let

$$S = \{f \in L^\infty(\mathbf{R}) : \ |f(x)| \leq 1/(1+x^2)\}$$

   Which of the following statements is true? Prove your answers!

   (a) The closure of S is compact in $L^\infty(\mathbf{R})$ with its norm topology.

   (b) S is closed in the $L^\infty(\mathbf{R})$ with its norm topology.

   (c) The closure of S in the weak-$*$ topology is compact in that topology. (Hint: recall that $L^\infty(\mathbf{R}) = L^1(\mathbf{R})^*$.)

8. State and prove the Radon-Nikodym theorem.

9. Prove the Lebesgue Decomposition Theorem: if μ and ν are σ-finite measures on the measurable space $(X, \mathcal{A}, \mu)$, then we can find measures ν_0, ν_1 with $\nu = \nu_0 + \nu_1$, $\nu_0 \perp \mu$, and $\nu_1 \ll \mu$. (Hint: use the Radon-Nikodym theorem and consider the measure $\lambda = \mu + \nu$.)

Real Analysis

Syllabus for the Qualifying Examination

1. **Undergraduate material**
   - (a) $\mathbf{R}$ and $\mathbf{R}^n$
   - (b) Basic topology: compact and connected sets, convergent sequences, Cauchy sequences, metric space completion
   - (c) Sequences and series - numerical
   - (d) Continuity
   - (e) Differentiation
   - (f) Riemann integral
   - (g) Sequences and series of functions, uniform convergence
   - (h) Fourier series (chapter 8 in ref. (d))
   - (i) Several variables: differentiation, inverse and implicit function theorem, Stokes theorem
   - (j) Stone-Weierstrass theorem
   - (k) Arzela-Ascoli theorem

2. **Measure theory**
   - (a) Abstract measures, Borel measures, and their properties
   - (b) Measurable and μ-measurable functions
   - (c) Constructive measure theory: outer measures, Caratheodory extension, Lebesgue and Lebesgue-Stieltjes measure, completion and uniqueness, $\mathcal{B} \subset\neq \mathcal{L} \subset\neq \mathcal{P}(\mathbf{R})$
   - (d) Integration: abstract Lebesgue integral, convergence theorems - dominated convergence, monotone convergence, Fatou's lemma.
   - (e) Special examples: Cantor sets, Cantor functions, distribution functions
   - (f) Convergence relations and examples: ae, unif, a.un, measure, mean convergences; Egoroff's theorem; Lusin's theorem; Littlewood's 3-principles
   - (g) Relation of Lebesgue integral to Riemann integral, characterization of Riemann integrable functions
   - (h) Fubini theorem (without proof)

3. **Functional Analysis**
   - (a) Lebesgue spaces: Jensen, Minkowski, Holder inequalities; completeness (Riesz-Fisher); density theorems
   - (b) Banach spaces: standard examples, bounded linear operators and functionals, duality
   - (c) The three basic principles: Hahn-Banach theorem, Baire category theorem, uniform

boundedness theorem, open mapping theorem/closed graph theorem

(d) Geometry: second dual, quotients, adjoints, direct sums, projections

(e) Radon-Nikodym theorem: decompositions - Hahn, Jordan, Lebesgue; Radon-Nikodym theorem; characterization of dual of L^p; fundamental theorem of calculus; change of variables formula

(f) Regular Borel measures and the dual of $C[a, b]$

(g) Hilbert spaces: characterization of inner product, orthonormal bases - Parseval and Bessel theorems, orthogonal projections, representation of the dual, Fourier series

4. **References**

(a) W. Rudin, Real and Complex Analysis

(b) H. Royden, Real Analysis

(c) E. Hewitt & K. Stromberg, Real and Abstract Analysis

(d) W. Rudin, Principles of Mathematical Analysis 3d. ed., (undergraduate material)

(e) T. Apostol, Mathematical Analysis (undergraduate material)

Ph.D. Qualifying Examination (Real Analysis)

November 18, 2000

Prelude: There are three sections in this examination. Group I covers Undergraduate material (Advanced Calculus), while Groups II and III cover Graduate material (Measure Theory and Integration, along with Functional Analysis). The test consists of a total of 10 problems to be selected as follows:

Please do 3 problems from Group I, 4 problems from Group II, and 3 problems from Group III. (Each problem is weighted equally. No extra-credit will be given for any additional problem attempted beyond these 10 problems.) *Please always justify your answers. Also, please indicate at the beginning of the text which problems you have selected in each group.* Good luck!

GROUP I

1. (a) Give the definition of a connected set in Euclidean space $\mathbb{R}^n$.

   (b) Show that if $C \subset \mathbb{R}^n$ is connected, then any continuous function $f : C \to \mathbb{Z}$ is constant. (Here, $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$ denotes the set of integers.) Is the converse true?

2. (a) Let $f(x) = \|x\|$, for $x \in \mathbb{R}^n$, where $\|x\| = (\sum_{k=1}^n x_k^2)^{1/2}$ denotes the Euclidean length of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Compute $f'(x)$, the differential of f at x, whenever it exists. If $f'(x)$ does not exist for some x, explain why.

   (b) Answer the same questions as in (a) for the function $f(x) = \|x\|^2$, for $x \in \mathbb{R}^n$.

3. Let $\zeta(x) = \sum_{n=1}^\infty n^{-x}$, for $x > 1$.

   (a) Show that this series of functions is uniformly convergent on $(a, +\infty)$, for every $a > 1$.

   (b) Show that ζ is continuous on $(1, +\infty)$.

4. (a) Give an example of a sequence of real-valued functions $\{f_n\}$ which converges pointwise but not uniformly on $[0, 1]$.

   (b) Give an example of a series of differentiable functions $\sum_{n=1}^\infty f_n$ which converges uniformly on $[0, 1]$ but such that there exists x_0 in $[0, 1]$ at which the series of derivatives $\sum_{n=1}^\infty f_n'(x_0)$ does not converge.

5. (a) State (without proof) the Stone-Weierstrass Theorem (for real-valued functions).

   (b) Show that the algebra (of real-valued functions) generated by the set $\{1, x^2\}$ is dense in $C[0, 1]$ but is not dense in $C[-1, 1]$. (Here, $C(I)$ denotes the space of real-valued continuous functions on I and is equipped with the topology of uniform convergence.)

GROUP II

6. Let E be a Banach space and let $L(E)$ denote the space of bounded linear operators from E to itself, equipped with its usual norm. Show that $L(E)$ is a Banach space (that is, show that $L(E)$ is *complete* for its natural metric).

7. Show that
$$F(t) = \int_{-\infty}^{\infty} \frac{\sin(x^2 t)}{1+x^2}\,dx$$
is well-defined and continuous on $\mathbb{R}$.

8. Show that every nondecreasing function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

9. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be a measurable function. Let $\theta \in [0,1]$ and p, q, s be three strictly positive numbers such that $\dfrac{1}{s} = \dfrac{\theta}{p} + \dfrac{1-\theta}{q}$.

   Show that
$$\|f\|_s \leq \|f\|_p^{\theta} \;\; \|f\|_q^{1-\theta}.$$
   (Here, for example, $\|f\|_p$ denotes the L^p-norm of f.)

   Please note: You are allowed to use a well-known inequality.

10. Show that the normed space $L^2[0,1]$ is separable but that $L^\infty[0,1]$ is not.

    (Recall that a metric space is *separable* if it contains a *dense* and at most *countable* subset. Further, here, for example, $L^2[0,1]$ is equipped with Lebesgue measure on $[0,1]$.)

11. Show that any Borel measure μ on $\mathbb{R}$ which is translation invariant and such that $\mu([0,1]) = 1$ must coincide with Lebesgue measure on $\mathbb{R}$.

    (Recall that μ is *translation invariant* if $\mu(a+B) = \mu(B)$, for every $a \in \mathbb{R}$ and Borel set $B \subset \mathbb{R}$, where $a + B = \{a + x : x \in B\}$.)

12. Let $(E, \|\cdot\|)$ be a normed linear space and let $T : E \to E$ be a linear map. Show that the following statements are equivalent:

    (i) T is continuous.

    (ii) T is continuous at 0.

    (iii) T is bounded; i.e., there exists $C > 0$ such that
$$\|Tx\| \leq C\|x\|, \quad \text{for all } x \in E.$$

GROUP III

13. State precisely and sketch the proof of the Baire Category Theorem.

14. State precisely and sketch the proof of the Hahn-Banach Theorem.

15. State precisely Lebesgue's Dominated Convergence Theorem (but do **not** sketch its proof). Give an example showing that if the dominating condition is not satisfied, then the conclusion of the theorem does not hold.

16. Given $1 \leq p < \infty$, show that $L^p(\mathbb{R})$, equipped with its usual metric (and with Lebesgue measure), is *complete*.

17. Let H be a real Hilbert space. Show that its dual H' is isomorphic to H. (Here, H' denotes the space of continuous linear functionals on H.)

    [*Advice*: Justify precisely your answer.]

18. (a) State (without proof) the Closed Graph Theorem and the Open Mapping Theorem.

    (b) Assuming the Open Mapping Theorem, deduce the Closed Graph Theorem.

F..l version

Ph.D. Qualifying Exam (Real Analysis)
September 25, 1999

Directions: Please do all 15 (equally weighted) problems.

1. Show that a (scalar-valued) continuous function on the interval $[0, 1]$ is necessarily uniformly continuous.

2. Give an example where the integral and sum of an infinite sequence of continuous functions cannot be interchanged. State (without proof) a theorem guaranteeing that this interchange can be carried out.

3. Describe the Cantor (ternary) set C via "removing middle thirds" and show that C is uncountable and has measure 0.

4. Prove or disprove the following statement:
A real-valued continuous function on the interval $[0, 1]$ that is differentiable with derivative $= 0$ everywhere except on a set of (one-dimensional) measure zero is necessarily constant.

5. Given a Banach space X, denote by $\mathcal{L}(X)$ the space of bounded linear operators from X to itself, equipped with its usual norm, and let G denote the set of invertible elements in $\mathcal{L}(X)$.
(a) Show that if $A \in \mathcal{L}(X)$ and $\|A\| < 1$, then $I - A \in G$; further, compute $(I - A)^{-1}$.
(b) Show that G is an open subset of $\mathcal{L}(X)$ and that the map $\phi : G \to G$ defined by $\phi(Q) = Q^{-1}$ is continuous.

6. Sketch the proof of the Hahn-Banach Theorem. [Advice: You should explain the basic steps involved in the proof, but you do not need to provide every technical detail.]

7. State and prove the Baire Category Theorem. Moreover, give at least one significant application of this theorem in real or in functional analysis.

8. If $\{f_n\}$ is a sequence of pointwise bounded functions on $[a, b]$, show that there exists a subsequence of $\{f_n\}$ which converges on a dense subset of $[a, b]$. (Assume that the functions are $\mathbb{R}^k$-valued.)

9. Let X be any compact subset of $\mathbb{R}$ containing an interval (of positive length). Is it possible that $\{f \in C(X) : |f(x)| \leq 1, \forall x \in X\}$ is a compact subset of $C(X)$? Prove your assertion.

10. State Fatou's Lemma. Prove the Lebesgue Dominated Convergence Theorem using Fatou's Lemma.

11. State the Fundamental Theorem of Calculus (FTOC) (relating a function F to the integral of its derivative) in its most general form, stating the necessary and sufficient condition (C) that F must satisfy in order that the theorem hold. Finally, consider $F(x) = |x|$ on $\mathbb{R}$. Illustrate the truth or falsity of the FTOC in this case; i.e., show that $F(x)$ either does or does not satisfy C and also that $F(x)$ does or does not satisfy the statement of the FTOC.

12. Let L^1 be the Banach space of Lebesgue integrable functions on $[0, 1]$. Let F be a bounded linear functional on L^1. Prove that there is a bounded measurable function g so that $F(f) = \int_0^1 f(x)g(x)dx$, $f \in L^1$. (You may use properties of absolutely continuous functions, density of step functions, etc. You may NOT use the fact that the dual space of L^1 is L^∞.)

13. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function that is homogeneous of order k (i.e., $f(\lambda x_1, ..., \lambda x_n) = \lambda^k f(x_1, ..., x_n)$, $\forall \lambda \in \mathbb{R}$). Show that $x_1 \frac{\partial f}{\partial x_1} + \cdots + x_n \frac{\partial f}{\partial x_n} = k\, f$ (i.e., $\vec{x} \cdot f'(\vec{x}) = k\, f(\vec{x})$).

14. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentable function. Define the directional derivative of f (at $\vec{x}$) in the direction of the unit vector $\vec{v} \in \mathbb{R}^n$. Show that this derivative has maximum modulus when v is in the direction of the gradient (f') of f.

15. Let $f(x)$ be a continuous function from $[a, b]$ into itself.

(a) Prove from basic principles that $f(x_0) = x_0$ for some $x_0 \in [a, b]$.

(b) Assume in addition that the derivative f' exists on (a, b) and that $|f'(x)| \leq \alpha$, for some $0 < \alpha < 1$. Prove that the fixed point x_0 is unique and state and prove an algorithm for finding x_0.

Real Analysis
Qualifying Examination
October 21, 1998

Prelude: there are three sections in this examination: Group I covers Advanced Calculus; Group II covers Measure Theory & Integration; Group III covers Functional Analysis. Please do 3 problems from Group I, 4 problems from Group II, and 3 problems from Group III.

GROUP I

1. Let X be a compact topological space. Suppose that $f : X \to \mathbf{R}$ has the property that $\{x | f(x) \geq a\}$ is closed for each $a \in \mathbf{R}$. Prove that f is bounded above and that it attains its least upper bound.

2. Let f be a continuous function on $[0,1]$ such that

$$\int_0^1 x^n f(x)dx = 0, \text{ for all } n = 0, 1, 2, ...$$

Show that $f(x) = 0$ for all $x \in [0,1]$.

3. Let

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\cos nx}{n}, \quad x \in \mathbf{R}$$

At which points is f well-defined? At which points is it continuous?

4. Let $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ be a continuously differentiable function. Define for $t \in [0,1]$ and $n = 1, 2, 3, ...$

$$\begin{aligned} y_0(t) &= e^{-t} \\ y_{n+1}(t) &= y_n(t) + \int_0^t f(t, y_n(t))dt \end{aligned}$$

Show that the sequence (y_n) converges uniformly to a continuous function on $[0,1]$. HINT: Use the Cauchy criterion.

5. State the Inverse Function Theorem for functions from $\mathbf{R}^n$ into $\mathbf{R}^n$. What conclusion can drawn from this theorem about the function $f(x,y) = (x+y, x^2 - y)$ near the point $(0,0)$ in $\mathbf{R}^2$?

6. Show that the series $S_1 = \sum_{n=1}^{\infty} nq^n$ and $S_2 = \sum_{n=1}^{\infty} n^2 q^n$ are uniformly convergent for $0 < q < 1$. Show that $S_1 = q(1-q)^{-2}$. Using this find a similar expression for S_2. compact subsets

7. Let X be a compact subset of $\mathbf{R}$ containing an interval of positive length.Is it possible that $\{f \in C(X) | \, |f(x)| \leq 1, \text{ for all } x \in X\}$ is a compact subset of $C(X)$? Prove your assertion. Here $C(X)$ is the space of continuous functions on X.

GROUP II

8. Prove or give a counterexample: Let (f_n) be a sequence of real-valued Lebesgue integrable functions on $[0,1]$ equipped with Lebesgue measure. If $f_n \geq 0$ and $\int_0^1 f_n(x)dx \geq 1$ for all $n = 1, 2, \ldots$ and if $f_n \to f$ almost everywhere, then $\int_0^1 f(x)dx \geq 1$.

9. Fix a measure space (Ω, Σ, μ). Assume that $f \in L^r(\Omega, \Sigma, \mu)$ for some $0 < r < \infty$. Show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$. (Note the right hand side may be finite or infinite.)

10. Suppose that the sequence of real-valued functions (f_n) converges to f in L^1 on $[0,1]$ equipped with Lebesgue measure and that f_n converges to g pointwise almost everywhere. Show that $f = g$ a.e.

11. Prove or disprove: Every Lebesgue integrable function on $\mathbf{R}$ tends pointwise a.e. to 0 at infinity.

12. Show that every complex measure has a unique decomposition as the sum of a discrete measure and a continuous measure.

13. Show that for all $r > 0$ it is true that

$$\int_0^\infty \left(\int_0^r e^{-xy^2} \sin x dx\right) dy = \int_0^r \left(\int_0^\infty e^{-xy^2} \sin x dy\right) dx$$

Use this to show that

$$\int_0^\infty \frac{\sin x}{\sqrt{x}} dx = \frac{\sqrt{2\pi}}{2}$$

Note: Some useful integrals are given below in the postscript.

14. Prove that if (X, Σ, μ) and (Y, Υ, ν) are measure spaces, then there exists a measure π on $\Sigma \times \Upsilon$ such that

$$\pi(A \times B) = \mu(A)\nu(B)$$

for $A \in \Sigma$ and $B \in \Upsilon$. Moreover, show that if both measure spaces are σ-finite then π is unique.

15. Find the linear function $f(x) = A + Bx$ with A, B real such that

$$\int_0^1 |x^2 - f(x)|^2 dx \leq \int_0^1 |x^2 - (a + bx)|^2 dx$$

for all real a, b.

16. Let $(\mathbf{R}^n, \mathcal{B}, \mu)$ be Lebesgue measure on the Borel sets of Euclidean n-space. If $T : \mathbf{R}^n \to \mathbf{R}^n$ is the shift operator defined by $(Tf)(x) = f(x + x_0)$ for a fixed $x_0 \in \mathbf{R}^n$, verify that, for each μ-integrable f, Tf is measurable and μ-integrable and that $\int_{\mathbf{R}^n} f d\mu = \int_{\mathbf{R}^n} (Tf) d\mu$.

GROUP III

17. Prove or disprove:
(i) The dual space of $L^1(\mathbf{R})$ is $L^\infty(\mathbf{R})$.
(ii) The dual space of $L^\infty(\mathbf{R})$ is $L^1(\mathbf{R})$.

18. Let (Ω, Σ, μ) be a finite measure space. Fix $1 < p < \infty$ and define the conjugate exponent of p to be q such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that the dual space of $L^p(\mu)$ is isometrically isomorphic to $L^q(\mu)$.

19. Show that the canonical injection of a Banach space X into its second dual X^{**} is an isometric isomorphism of X onto a subspace of X^{**}. Give two examples of infinite-dimensional Banach spaces: one for which the range of the injection is onto all of X^{**} and one for which it is not.

20. Let X be a linear space which is a Banach space under each of the norms $\|\cdot\|_1$, $\|\cdot\|_2$. If there is a positive constant A such that $\|x\|_1 \le A\|x\|_2$ for all $x \in X$, prove that there is a positive constant B such that $\|x\|_2 \le B\|x\|_1$ for all $x \in X$.

21. If $v_1, \ldots, v_n$ is a vector space basis of a subspace V of the Hilbert space H and if $f \in H$, then describe in terms of $f, v_1, \ldots, v_n$ the point in V nearest to f.

22. If $v_1, \ldots, v_n$ is a vector space basis of a subspace V of the Banach space X and if $f \in X$, describe in terms of $f, v_1, \ldots, v_n$ a point in V nearest to f (via a Hahn-Banach separating functional $h_{f,V} \in X^*$). In the special case that $X = L^p[0,1]$, $1 < p < \infty$, show the exact form of $h_{f,V}$.

23. Show that the space l^∞ of all bounded sequences of real numbers is a Banach space when equipped with supremum norm.

Postscript: here are three integrals (two indefinite and one definite) that may be of use in the second part of problem 13.

$$\int e^{-\alpha t} \sin t\,dt = -\frac{\alpha \sin t + \cos t}{1+\alpha^2} e^{-\alpha t}$$

$$\int \frac{dy}{1+y^4} = \frac{1}{4\sqrt{2}}\left[\ln\left(\frac{1+y\sqrt{2}+y^2}{1-y\sqrt{2}+y^2}\right) + 2\arctan\left(\frac{y\sqrt{2}}{1-y^2}\right)\right]$$

$$\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$$

Real Analysis Qualifying Exam
October 17, 1997

Note: Part A is advanced calculus, and students should attempt three questions from this part.

Part B and Part C are on measure theory and functional analysis. Students should attempt 3 problems from one and four problems from the other of Parts B and C. Thus the test is for 10 problems in all with a lot of choice.

GROUP I

1. Let $f : [0,1] \to \mathbb{R}$ be a continuous, but not necessarily differentiable, function. Show that it attains a maximum at some point of the closed unit interval.

2. If $a_n \to a$ as $n \to \infty, (a_n, a \in \mathbb{R})$ show that $\lim_{n\to\infty}(1+\frac{a_n}{n})^n = e^a$ by first establishing that $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$ and then using it.

3. State the Inverse Function Theorem for functions from $\mathbf{R}^n$ into $\mathbf{R}^n$. What conclusion can be drawn from this theorem about the function $f(x,y) = (x+y, x^2-y)$ near the point $(0,0)$ in $\mathbf{R}^2$?

4. Prove or give a counterexample: the set $\mathbf{R}$ of real numbers with its usual topology is a union of nowhere dense subsets.

5. Prove that if X and Y are topological spaces, $f\colon X \to Y$ is continuous, and $K \subseteq X$ is compact, then $f(K) \subseteq Y$ is compact.

GROUP II

6. Verify the truth of the following statements:

 a There is a σ-algebra of $[0,1]$ such that a continuous real valued function on $[0,1]$ is not measurable.
 b There is a σ-algebra of $[0,1]$ such that every real valued function on $[0,1]$ is measurable.

7. The class of all pointwise limits of sequences of real continuous functions on $\mathbb{R}$ is called the first Baire class. Show that each Baire class function is Borel measurable on $\mathbb{R}$.

8. J.E. Littlewood (1944) stated in intuitive language three principles for functions of a real variable:

(i) Every measurable set is nearly a finite union of intervals.
(ii) Every measurable function is nearly continuous.
(iii) Every convergent sequence of functions is nearly uniformly convergent.

a) For each of these principles state a theorem which makes the intuition precise.
b) Which of these theorems depend on the fact that it's really Lebesgue measure that is being discussed? Which depend on the fact that the measure space is finite?
c) Prove **one** of the theorems given in (a).

9. Show that every complex measure has a unique decomposition as the sum of a discrete measure and a continuous measure.

10. Prove or give a counterexample: if $E_i \subseteq X$ are measurable subsets of a measure space X and $E_i \subseteq E_{i+1}$ for all i then

$$\mu(\cup E_i) = \lim_{i\to\infty} \mu(E_i).$$

11. Prove the Lebesgue Decomposition Theorem: if μ and ν are σ-finite measures on the measurable space (X, A, μ), then we can find measures ν_0, ν_1 with $\nu = \nu_0 + \nu_1$, $\nu_0 \perp \mu$, and $\nu_1 \ll \mu$. (Hint: use the Radon-Nikodym theorem and consider the measure $\lambda = \mu + \nu$.)

GROUP III

12. If $L^p(0,1)$ is the Lebesgue space on the unit interval with Lebesgue measure, show that $L^2(0,1)$ is dense in $L^1(0,1)$. Using this and the Bessel inequality in $L^2(0,1)$ or otherwise, show that for any $f \in L^1(0,1)$ the numbers $a_n = \int_0^1 e^{int} f(t)dt \to 0$ as $n \to \infty$.

13. Let μ, ν be the Lebesgue measure and the counting measure (i.e., $\nu(A) = \#$ points in A) on $[0,1]$. Verify that $\mu << \nu$ (i.e., μ is absolutely continuous relative to ν) but not conversely. Give reasons why there does not exist a measurable function f such that $\mu(A) = \int_A f d\nu$ (i.e., $\frac{d\mu}{d\nu}$ does not exist).

14. Prove or disprove: (i) The dual space of $L^1(\mathbf{R})$ is $L^\infty(\mathbf{R})$. (ii) The dual space of $L^\infty(\mathbf{R})$ is $L^1(\mathbf{R})$.

15. Let X be a linear space which is a Banach space under each of the norms $\|\cdot\|_1$, $\|\cdot\|_2$. If there is a positive constant A such that $\|x\|_1 \leq A\|x\|_2$ for all $x \in X$, prove that there is a positive constant B such that $\|x\|_2 \leq B\|x\|_1$ for all $x \in X$.

16. Prove or give a counterexample: if X is a measure space and $f, g \in L^4(X)$ then $fg \in L^1(X)$.

17. Prove or disprove: finite linear combinations of the functions $\{e^{-nx^2}\}_{n\geq 0}$ are dense in $C[0,1]$ with its usual norm topology.

Real Analysis
Qualifying Examination
October 19, 1996

There are three groups of problems in this examination. Group I covers advanced calculus, Group II covers measure theory and integration, and Group III covers functional analysis. Please do 3 problems from Group I, 4 problems from Group II, and 3 problems from Group III.

GROUP I

1. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

2. Suppose that $f: \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x\to\pm\infty} f(x) = +\infty$. Using the fact that every Cauchy sequence in $\mathbb{R}$ converges, show that f attains a minimum value.

3. Suppose $f: \mathbb{R} \to \mathbb{R}$ is one-to-one and onto, and let f^{-1} denote the inverse function — not the reciprocal!

(a) If f is continuous, is f^{-1} necessarily continuous? Give a proof or counterexample.

(b) If f is differentiable, is f^{-1} necessarily differentiable? Give a proof or counterexample.

4. Determine the radius of convergence of the series $f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$. Prove that within the radius of convergence, $f''(z) = f(z)$.

5. Prove that there is no value of k such that the equation $x^3 - 3x + k = 0$ has two distinct roots in $[0, 1]$.

6. Find

$$\overline{\lim_{n\to\infty}} \sin(nx)$$

where $\overline{\lim}$ denotes the limit superior (or 'lim sup'). Prove your answer is true.

GROUP II

1. J. E. Littlewood stated in intuitive language three basic principles for functions of a real variable:

- i) Every measurable set is nearly a finite union of intervals.
- ii) Every measurable function is nearly continuous.
- iii) Every convergent sequence of functions is nearly uniformly convergent.

a) For each of these principles state a theorem that makes the intuition precise.
b) State *two more* theorems that make some of Littlewood's principles precise in *different ways*, again for functions of a real variable.

2. Suppose that $f_n: X \to \mathbb{R}$ is a sequence of measurable functions on a σ-finite measure space X, and suppose that $f_n \to 0$ in the L^2 and L^4 norms.
a) Does $f_n \to 0$ in the L^1 norm? Prove or give a counterexample.
b) Does $f_n \to 0$ in the L^3 norm? Prove or give a counterexample.
c) Does $f_n \to 0$ in the L^5 norm? Prove or give a counterexample.

3. Describe a function $f: [0,1] \to \mathbb{R}$ that is Lebesgue integrable but not Riemann integrable. Prove it is not Riemann integrable and compute its Lebesgue integral.

4. Define an uncountable measurable subset of $[0,1]$ having measure zero and prove it has both of these properties.

5. Suppose μ is a measure on a space X and suppose $f_n: X \to \mathbb{R}$ are nonnegative measurable functions with $f_{n+1} \leq f_n$. Prove the following claim or give a counterexample:

$$\int_X \underline{\lim} f_n \, d\mu = \underline{\lim} \int_X f_n d\mu.$$

6. State the Monotone Convergence Theorem and Fatou's Lemma. and use the second to prove the first.

7. Describe a measurable function $f: \mathbb{R}^2 \to R$ such that:
- for all x, $f(x, y)$ is an integrable function of y
- for all y, $f(x, y)$ is an integrable function of x
- the function $\int f(x, y) dy$ is an integrable function of x
- the function $\int f(x, y) dx$ is an integrable function of y
- $\int (\int f(x, y) dx) \, dy \neq \int (\int f(x, y) dy) \, dx$.

(All integrals here are from $-\infty$ to $+\infty$, with respect to Lebesgue measure.)

GROUP III

1. Let $L(X)$ be the space of bounded linear operators on the Banach space X, and suppose $T \in L(X)$ has $\|T\| < 1$.

(a) Show that $\sum_{n=0}^{\infty} T^n$ converges in the norm topology on $L(X)$ to some operator $S \in L(X)$. Show this operator satisfies $S(1-T) = (1-T)S = 1$, where 1 denotes the identity operator on X.

(b) Give another proof that $1 - T$ has a bounded inverse.

2. Suppose that $\mathcal{H}$ is a complex Hilbert space and $L \subset \mathcal{H}$ is an arbitrary subspace. Show that $(L^{\perp})^{\perp}$ is the closure of L.

3. Use the Open Mapping Theorem to show that if the vector space V is complete in both of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, then there are constants $c, C > 0$ such that

$$c\|v\|_1 \leq \|v\|_2 \leq C \leq \|v\|_1$$

for all $v \in V$. State the Closed Graph Theorem and use the above result to prove it.

4. Let $C_\infty(\mathbb{R})$ be the space of continuous functions $f: \mathbb{R} \to \mathbb{R}$ with $\lim_{x \to \pm\infty} f(x) = 0$.

(a) Show that $C_\infty(\mathbb{R})$ is a Banach space when equipped with the the sup norm.

(b) Let $L \subset C_\infty(\mathbb{R})$ be the subspace consisting of functions supported on a compact set. Is L dense in $C_\infty(\mathbb{R})$? Prove or disprove.

5. Let L be the space of finite real linear combinations of functions on $[0, 1]$ of the form $\sin(nx^2), \cos(nx^2)$ for $n \in \mathbb{Z}$. Is L dense in $C[0, 1]$? Prove or disprove.

6. State and prove some version of the Hahn-Banach theorem.

Real Analysis
Qualifying Examination
September 20, 1995

Prelude: there are three sections in this examination: Group I covers Advanced Calculus; Group II covers Measure Theory & Integration; Group III covers Functional Analysis. Please do 3 problems from Group I, 4 problems from Group II, and 3 problems from Group III.

GROUP I

1. Prove that a real-valued continuous function on a bounded closed real interval attains a maximum value.

2. Let f be a continuous function on $[0,1]$ such that

$$\int_0^1 x^n f(x)dx = 0, \text{ for all } n = 0, 1, 2, ...$$

Show that $f(x) = 0$ for all $x \in [0,1]$.

3. Let

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\cos nx}{n}, \qquad x \in \mathbf{R}$$

At which points is f well-defined? At which points is it continuous?

4. Let $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ be a continuously differentiable function. Define for $t \in [0,1]$ and $n = 1, 2, 3, ...$

$$\begin{aligned} y_0(t) &= e^{-t} \\ y_{n+1}(t) &= y_n(t) + \int_0^t f(t, y_n(t))dt \end{aligned}$$

Show that the sequence (y_n) converges uniformly to a continuous function on $[0,1]$. HINT: Use the Cauchy criterion.

5. State the Inverse Function Theorem for functions from $\mathbf{R}^n$ into $\mathbf{R}^n$. What conclusion can drawn from this theorem about the function $f(x,y) = (x+y, x^2-y)$ near the point $(0,0)$ in $\mathbf{R}^2$?

6. Show that the series $S_1 = \sum_{n=1}^{\infty} nq^n$ and $S_2 = \sum_{n=1}^{\infty} n^2 q^n$ are uniformly convergent for $0 < q < 1$. Show that $S_1 = q(1-q)^{-2}$. Using this find a similar expression for S_2.

7. State the Arzela-Ascoli Theorem. Give reasons why this theorem does, or does not, apply to three following collections of functions:
(i) $A_1 = \{f_n : \ f_n(x) = x - n, \ n \geq 1, \ x \in [0,1]\}$,

(ii) $A_2 = \{f_n : \ f_n(x) = x^n, \ n \geq 1, \ x \in [0,1]\}$,
(iii) $A_3 = \{f_n : \ f_n(x) = (1+(x+n)^2)^{-1}, \ n \geq 1, \ x \in [0,\infty)\}$.

GROUP II

8. J.E. Littlewood (1944) stated in intuitive language three principles for functions of a real variable:

- (i) Every measurable set is nearly a finite union of intervals.
- (ii) Every measurable function is nearly continuous.
- (iii) Every convergent sequence of functions is nearly uniformly convergent.

- a) For each of these principles state a theorem which makes the intuition precise.
- b) Which of these theorems depend on the fact that it's really Lebesgue measure that is being discussed? Which depend on the fact that the measure space is finite?
- c) Prove **one** of the theorems given in (a).

9. Fix a measure space (Ω, Σ, μ). Assume that $f \in L^r(\Omega, \Sigma, \mu)$ for some $0 < r < \infty$. Show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$. (Note the right hand side may be finite or infinite.)

10. Fix two positive measure spaces (Ω, Σ, μ) and (Ω', Σ', μ') . Let $f : \Omega \to \mathbf{C}, \ g : \Omega' \to \mathbf{C}$. Suppose that $f(\omega) = g(\omega')$ a.e.$[\mu \otimes \mu']$. Show that there is a constant $a \in \mathbf{C}$ such that $f(\omega) = a$ a.e.$[\mu]$ and $g(\omega') = a$ a.e.$[\mu']$.

11. Prove or disprove: Every Lebesgue integrable function on $\mathbf{R}$ tends pointwise to 0 at infinity.

12. Show that every complex measure has a unique decomposition as the sum of a discrete measure and a continuous measure.

13. Show that for all $r > 0$ it is true that

$$\int_0^\infty \left(\int_0^r e^{-xy^2} \sin x dx\right) dy = \int_0^r \left(\int_0^\infty e^{-xy^2} \sin x dy\right) dx$$

Use this to show that

$$\int_0^\infty \frac{\sin x}{\sqrt{x}} dx = \frac{\sqrt{2\pi}}{2}$$

Note: Some useful integrals are given below in the postscript.

14. Prove that if (X, Σ, μ) and (Y, Υ, ν) are measure spaces, then there exists a measure π on $\Sigma \times \Upsilon$ such that

$$\pi(A \times B) = \mu(A)\nu(B)$$

for $A \in \Sigma$ and $B \in \Upsilon$. Moreover, show that if both measure spaces are σ-finite then π is unique.

15. Construct a non-negative mass-one continuous measure μ on $[0, 1]$ such that the pointwise derivative $D\mu(x)$ has value 0 for λ-almost all x.

16. Let $(\mathbf{R}^n, \mathcal{B}, \mu)$ be Lebesgue measure on the Borel sets of Euclidean n-space. If $T : \mathbf{R}^n \to \mathbf{R}^n$ is the shift operator defined by $(Tf)(x) = f(x + x_0)$ for a fixed $x_0 \in \mathbf{R}^n$, verify that, for each μ-integrable f, Tf is measurable and μ-integrable and that $\int_{\mathbf{R}^n} f d\mu = \int_{\mathbf{R}^n} (Tf) d\mu$.

GROUP III

17. Prove or disprove:
(i) The dual space of $L^1(\mathbf{R})$ is $L^\infty(\mathbf{R})$.
(ii) The dual space of $L^\infty(\mathbf{R})$ is $L^1(\mathbf{R})$.

18. Let (Ω, Σ, μ) be a finite measure space. Fix $1 < p < \infty$ and define the conjugate exponent of p to be q such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that the dual space of $L^p(\mu)$ is isometrically isomorphic to $L^q(\mu)$.

19. Show that the canonical injection of a Banach space X into its second dual X^{**} is an isometric isomorphism of X onto a subspace of X^{**}. Give two examples of infinite-dimensional Banach spaces: one for which the range of the injection is onto all of X^{**} and one for which it is not.

20. Let X be a linear space which is a Banach space under each of the norms $\|\cdot\|_1$, $\|\cdot\|_2$. If there is a positive constant A such that $\|x\|_1 \leq A\|x\|_2$ for all $x \in X$, prove that there is a positive constant B such that $\|x\|_2 \leq B\|x\|_1$ for all $x \in X$.

21. Let $\mathcal{L}(\mathcal{X})$ be the space of bounded linear operators on the Banach space $\mathcal{X}$ and let $T \in \mathcal{L}(\mathcal{X})$.
(a) Show that if $\|T\| < 1$, then $I - T$ is invertible with bounded inverse.
(b) Deduce from (the proof of) part (a) that the group of invertible elements of $\mathcal{L}(\mathcal{X})$ is an open set in $\mathcal{L}(\mathcal{X})$ (equipped with the operator norm topology).

22. Let C be a closed convex set in an Hilbert space H. Show that, given any $x \in H$, there exists a unique closest element to x in C (i.e. a point $c \in C$ such that $\|x - c\| = \inf_{y \in C} \|x - y\|$).
HINT: Use the parallelogram inequality.

23. Show that the space l^∞ of all bounded sequences of real numbers is a Banach

space when equipped with supremum norm.

Postscript: here are three integrals (two indefinite and one definite) that may be of use in the second part of problem 13.

$$\int e^{-\alpha t} \sin t dt = -\frac{\alpha \sin t + \cos t}{1+\alpha^2} e^{-\alpha t}$$
$$\int \frac{dy}{1+y^4} = \frac{1}{4\sqrt{2}} \left[\ln \left(\frac{1+y\sqrt{2}+y^2}{1-y\sqrt{2}+y^2} \right) + 2 \arctan \left(\frac{y\sqrt{2}}{1-y^2} \right) \right]$$
$$\int_0^\infty e^{-u^2} du = \frac{\sqrt{\pi}}{2}$$

Real Analysis Qualifying Exam

March 12, 1994

Instructions. Do 3 problems from Group I (problems 1–5), do 4 problems from Group II (problems 6–11) and do 3 problems from Group III (problems 12–16).

Qualifying Exam in Real Analysis
September 1993

Do as many problems as you can. Problems (1) through (4) cover undergraduate material and problems, (5) through (12) cover graduate level material. The problems covering graduate level material will be given more weight.

(1) a) Prove that if f_n, $n = 1, 2, \ldots$, is a real continuous function on the interval $[0,1] = \{x : 0 \leq x \leq 1\}$ and the sequence $\{f_n\}$ converges uniformly to the function f on $[0,1]$, then f is continuous on $[0.1]$.

b) Show that the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges to a continuous function on $[0,1]$.

(2) Let $\mathcal{F}$ denote the family of functions on the interval $0 \leq x \leq 1$ of the form $f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ where a_n is real and $|a_n| \leq 1/n^3$. State a general theorem and then use the theorem to show that any sequence in $\mathcal{F}$ has a subsequence that converges uniformly on the interval $0 \leq x \leq 1$.

(3) a) State the inverse function Theorem for functions from $\mathbb{R}^n$ into $\mathbb{R}^n$.

b) State and verify a conclusion that can be drawn from this theorem about the function $f(x,y) = (x + y, x^2 - y)$ near the point $(0,0)$ in $\mathbb{R}^2$.

(4) a) Let f be a real valued function on the interval $a \leq x \leq b$. Prove the Riemann-Lebesgue lemma under the following assumptions:

i) f is continuous; ii) f is Lebesgue integrable. The Riemann-Lebesgue lemma states:

$$\lim_{n \to \infty} \int_a^b f(t) \cos(nt) dt = 0.$$

b) Give an application of the Riemann-Lebesgue lemma (and its obvious counterpart) to the theory of Fourier series.

(5) Suppose that A is a bounded linear operator on a Hilbert space such that

$$\|p(A)\| \leq C \sup\{|p(z)| : \ z \text{ complex}, \ |z| = 1\}$$

for all polynomials p with complex coefficients, where C is a constant independent of p. Show that to each pair x and y in the Hilbert space there corresponds a complex Borel measure μ on the circle, $\{z : |z| = 1\}$, such that

$$(A^n x, y) = \int z^n \, d\mu(z), \text{ for } n = 0, 1, 2, \ldots$$

Hint. Use the Hahn-Banach and Riesz representation Theorems.

(6) Suppose that μ is a finite positive measure, φ is a bounded linear functional on $L^p(\mu)$, $1 < p < \infty$, and φ is positive ($\varphi(f) \geq 0$, if $f \geq 0$). Show that φ is represented by a positive function in $L^q(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$. Hint. Use the Radon-Nikodym Theorem. Don't use a Riesz representation Theorem of which this is a special case.

(7) Let $Y = \{a + bx : \ a, b \text{ complex}\}$ and regard Y as a subspace of $L^2[0,1]$ (with Lebesgue measure). Find the function in Y which is nearest to the function $f(x) = x^2$ in the complex Hilbert space $L^2[0,1]$.

(8) Let $F(x) = \int_0^{+\infty} e^{-xt} \left(\frac{\sin(2t)}{1+t^2}\right) dt$, $x \geq 0$. Prove that F is infinitely differentiable on $0 \leq x < +\infty$ and compute its n-th derivative for every $n \geq 1$.

(9) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a Borel measurable function. Assume that for every ball B in $\mathbb{R}^3$, f is Lebesgue integrable on B and $\int_B f(x)dx = 0$. What can you deduce about f? Justify carefully your answer.

(10) Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ where $\frac{1}{p} + \frac{1}{q} = 1$, with $1 \leq p \leq \infty$. Define $f * g$ by:

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Prove that $f * g$ is uniformly continuous on $\mathbb{R}$.

(11) Let $f : [0,1] \to [0,1]$ be a continuous function. Show that the graph of f has zero (two-dimensional) Lebesgue measure.

(12) Prove that Lebesgue measure is the unique (prove only uniqueness) translation invariant positive measure on the Borel sets of the reals such that $\mu([0,1]) = 1$. (The measure μ is said to be translation-invariant if $\mu(A+x) = \mu(A)$ for all real numbers x and all Borel sets A, where $A + x$ denotes the set $\{a + x : \ a \in A\}$.)

Mathematics 209ABC: Real Analysis
Qualifying Examination 1991-92 June 10, 1992

Prelude: It is unlikely that you will get to do all of these problems in the allotted three hours. Consequently, you should choose the ones you do work on with some care.

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1. Define

- a) variation of a complex measure
- b) complete orthonormal set in a Hilbert space
- c) outer measure
- d) normed linear space
- e) orthogonal projection on a Hilbert space

2. J.E. Littlewood (1944) stated in intuitive language three principles for functions of a real variable:

- (i) Every measurable set is nearly a finite union of intervals.
- (ii) Every measurable function is nearly continuous.
- (iii) Every convergent sequence is nearly uniformly convergent.
- a) For each of these principles state a theorem which makes the intuition precise.
- b) Which of these theorems depend on the fact that it's really Lebesgue measure on the line that is being discussed? Which depend on the fact that the measure space is finite?
- c) Prove **one** of the theorems given in (a).

3. Let (Ω, Σ, μ) be a finite measure space. Fix $1 < p < \infty$ and define the conjugate exponent of p to be q such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that the dual space of $L^p(\mu)$ is isometrically isomorphic to $L^q(\mu)$.

4. Show that for all $r > 0$ it is true that

$$\int_0^\infty (\int_0^r e^{-xy^2} \sin x dx) dy = \int_0^r (\int_0^\infty e^{-xy^2} \sin x dy) dx$$

Use this to show that

$$\int_0^\infty \frac{\sin x}{\sqrt{x}} dx = \frac{\sqrt{2\pi}}{2}$$

Note: Some useful integrals are given below in the postscript.

Real Analysis Qualifying Exam
Thurs. June 20, 1:00 - 4:00
1991

1. Suppose that E is a Lebesgue measurable set of finite Lebesgue measure and $\varepsilon > 0$. Show that there is a compact set A contained in E such that the set $E - A$ has Lebesgue measure $< \varepsilon$. Give proof from basics.

2. Suppose that H is a complex separable Hilbert space. Show the following.

a) If $x, y \in H$ and x and y are perpendicular then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

b) If $x \in H$, $\{u_1, u_2, \dots, u_n\}$ is a finite orthonormal set in H and $x' = \sum_{j=1}^{n} (x, u_j) u_j$, then

$$\|x - x'\| \leq \|x - \sum_{j=1}^{n} c_j u_j\|$$

~~[illegible]~~ for any choice of complex numbers, $c_1, c_2, \dots, c_n$.

Hint. Use a).

3. a) Let H be a separable complex Hilbert space and $\{u_1, u_2, \dots\}$ an infinite (countable) orthonormal set in H (not necessarily an ON basis). Use 2 b) to show that if x is in the closure of the set of finite linear combination of elements in $\{u_1, u_2, \dots\}$, then $x = \sum_{j=1}^{\infty} (x, u_j) u_j$.

b) Show that if c_j is a complex number and $\sum_{j=1}^{\infty} |c_j|^2 < \infty$, then $\sum_{j=1}^{\infty} c_j u_j \in H$. (Prove from basics.)

4. Suppose that μ is a measure, $\varepsilon > 0$ and that h is measurable, ≥ 0 and $\int h \, d\mu < +\infty$. Show that there is a $\delta > 0$ such that

$$\int_A h \, d\mu < \varepsilon, \quad \text{whenever } \mu(A) < \delta.$$

5. Suppose that X is a normed linear space, $x \in X$ and $x \neq 0$. Show that there is a φ in the dual space of X such that $\varphi(x) \neq 0$.

6. Prove that $L^1(\mu)$ is complete.

7. Prove the uniform boundedness Principle.
Hint. Use the Baire category theorem.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space.
Suppose that $A_n \in \mathcal{M}$ for $n = 1, 2, \ldots$
such that $A_1 \subset A_2 \subset \cdots$. Let $A = \bigcup_{n=1}^{\infty} A_n$.
Show that $\lim_{n \to +\infty} \mu(A_n) = \mu(A)$.

9. Let μ be Lebesgue measure on $[-\pi, \pi]$.
For $f \in L^1(\mu)$, let $f^\wedge(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt$ and
$(S_n f)(x) = \sum_{k=-n}^{n} f^\wedge(k)\, e^{ikx}$ for $n = 1, 2, \ldots$.
Show that each S_n is a bounded projection on $L^p(\mu)$, $1 \le p < \infty$.

10. Let $(X, \mathcal{M}, \mu)$ be a finite measure space.
Suppose that h is measurable, ≥ 0 and $\int h\, d\mu < \infty$.
Suppose that $f_1, f_2, \ldots$ is a sequence of nonnegative measurable functions such that $f_1(x) \le f_2(x) \le \ldots$
for each $x \in X$ and $\lim_{n \to +\infty} f_n(x) > h(x)$ for each $x \in X$. Show that $\lim_{n \to +\infty} \int f_n\, d\mu > \int h\, d\mu$.
You may use #4 and properties of measures and integrals (except convergence theorems for integrals).

Ph.D. QUALIFIER EXAM IN REAL ANALYSIS, June 20, 1990, (1:00–4:00 p.m.)

[Maximum: 100]

1. Let (Ω, Σ) be a measurable space and $\mu : \Sigma \to \mathbb{R}^+$ be additive. If $A_i \in \Sigma$, $i = 1, 2, \ldots, n$, show that

$$\mu\Big(\bigcup_{i=1}^n A_i\Big) = \sum_{i=1}^n \mu(A_i) - \sum_{1 \le i < j \le n} \mu(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} \mu(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n-1} \mu\Big(\bigcap_{i=1}^n A_i\Big)$$

If $\mu(\bigcap_{i=1}^k A_i) = [(n-k)!\,]/n!$, $1 \le k \le n$, evaluate $\mu(\bigcup_{i=1}^n A_i)$. (20)

2. Let f, g be a pair of real functions on the line $\mathbb{R}$ such that f is continuous and $g(x) = x^2$ if x=integer, $= 0$ otherwise. Show that $f \circ g : \mathbb{R} \to \mathbb{R}$ is a Borel function. (15)

3. If (Ω, Σ, μ) is a finite measure space, $p \ge 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, show, with the uniform boundedness principle, that $fg \in L^1(\mu)$ for all $g \in L^q(\mu)$ implies $f \in L^p(\mu)$. [This is sometimes called the *inverse Hölder inequality.*] (20)

4. If (Ω, Σ, μ) is a finite measure space, indicate relationships between uniform convergence, almost uniform convergence, almost everywhere convergence, and convergence in $L^p(\mu)$- mean $(1 \le p < \infty)$. Provide a proof or a counterexample for the statement: almost everywhere convergence $\Rightarrow$ convergence in $L^1(\mu)$-mean. (20)

5. Let $\mathcal{X}$ be a Banach space, $\mathcal{X}^*, \mathcal{X}^{**}$ be its first and second adjoint spaces. Show that the canonical injection $\mathcal{X} \hookrightarrow \mathcal{X}^{**}$ given by $x \longmapsto x^{**}$ where $x^{**}(x^*) = x^*(x)$ for all $x^* \in \mathcal{X}^*$ is an isometry. (15)

6. Let (Ω, Σ, μ) be a measure space and $0 < f \in L^1(\mu)$. If $\nu_f(A) = \int\limits_A f d\mu$, $A \in \Sigma$, show that $\mu << \nu_f$ and find $\frac{d\mu}{d\nu_f}$ if it exists. (15)

7. Let $\Omega = \mathbb{R}$, $F : \Omega \to \mathbb{C}$ be a bounded continuous function. If μ is the Lebesgue measure on Ω, $f, g \in L^1(\mu)$, let

$$\tilde{f}(x) = \int_\Omega F(xy)f(y)d\mu(y) \ , \ \tilde{g}(x) = \int_\Omega F(xy)g(y)d\mu(y) \ .$$

Show that $\tilde{f}$ and $\tilde{g}$ exist and satisfy

$$\int_\Omega f\tilde{g} \ d\mu = \int_\Omega \tilde{f}g \ d\mu \ .$$

Justify your steps and give an example of an $F(\cdot)$ for which the above hypotheses hold. (20)

Real Analysis Qualifying Exam 6/17/89

Directions: Do 18 of the following 20 equivalued #'s. Make sure to indicate which 2 you do <u>not</u> want graded.

#1. Let X be a compact topological space and f be a real-valued function on X with the property that each set of the form $\{x : f(x) \geq a\}$ is closed in X. Prove that f is bounded from above and that it attains its least upper bound.

#2. a) Let g be an integrable function on $[0,1]$. Show that there is a bounded measurable function f such that $\|f\|_\infty \neq 0$ and $\int_0^1 fg\,dx = \|g\|_1 \|f\|_\infty$.

b) Let g be a bounded measurable function. Show that for each $\varepsilon > 0$ there is an integrable function f such that $\int_0^1 fg\,dx \geq (\|g\|_\infty - \varepsilon)\|f\|_1$.

#3. Let X be any compact subset of $\mathbb{R}$ containing an interval (of positive measure). Is it possible that $\{f \in C(X) : |f(x)| \leq 1, \text{ all } x \in X\}$ is a compact subset of $C(X)$, the space of continuous functions on X? Prove your assertion.

#4 Prove or disprove (i.e., provide a counterexample): If each f_n is non-negative, $\int_0^1 f_n(x)\,dx \geq 1$, all n, and $f_n \to f$ almost everywhere, then $\int_0^1 f(x)\,dx \geq 1$.

#5. Prove or disprove: Suppose that each f_n is continuously differentiable and satisfies $|f_n(x)| \le 1$ and $|f_n'(x)| \le 1$, all $x \in \mathbb{R}$. Then there is a subsequence of the f_n converging uniformly to a continuous function on $\mathbb{R}$.

#6. Define l^∞ and show that it is a complete (Banach) space.

#7. If F is continuous on $[0,1]$ and F' exists a.e. (Lebesgue measure) and $\left|\frac{F(x)-F(t)}{x-t}\right| \le M < \infty$ for $0 \le x, t \le 1$, prove that $F(1) - F(0) = \int_0^1 F'(x)\,dx$.

#8. a) If Λ is a finite set what is the smallest algebra of subsets of Λ containing all singletons (i.e., all sets of the form $\{\lambda\}$ for all $\lambda \in \Lambda$)?

b) What is the smallest σ-algebra of subsets of $\mathbb{R}$ containing all sets of the form $\{x\}$ for all $x \in \mathbb{R}$?

#9. State Fatou's Lemma. Prove the Lebesgue Dominated Convergence Theorem using Fatou's Lemma.

#10. Let f_n be a sequence of real-valued measurable functions defined on $[0,1]$. Show that $f_n \to f$ in measure if and only if $\int_0^1 \frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|}\,dx \to 0$.

#11. a) State the principle of uniform boundedness for a family $\mathcal{F}$ of linear operators

b) Show that if $x_n \to x$ weakly, then $\langle \|x_n\| \rangle$ is bounded.

#12. f is a continuous function on $\mathbb{R}$.

a) Show that there is a sequence $\langle p_n \rangle$ of polynomials having the following property: For every compact subset K of $\mathbb{R}$, $p_n \to f$ uniformly on K.

b) Assume in addition that f has a continuous derivative f'. Show that the sequence $\langle p_n \rangle$ of polynomials can be chosen so that for every compact subset K of $\mathbb{R}$, $p_n \to f$ and $p_n' \to f'$ uniformly on K.

You are allowed to quote the Weierstrass approximation theorem.

#13. Let $X = Y = [0,1]$, m = Lebesgue measure on X, n = counting measure on Y, and set

$$f(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}.$$

Prove that f is measurable on $X \times Y$ by first noting that $f = \chi_D$ where D is the diagonal of the unit square and then showing that $D \in \mathcal{R}_{\sigma\delta}$ (i.e. D is a countable intersection of countable unions of measurable rectangles).

#14. a) State precisely Tonelli's theorem.

b) Calculate the iterated integrals of the function $f(x,y)$ in #13, and note that they are different.

c) What hypothesis of Tonelli's theorem is violated in b)? Why? (Explain.)

#15. Set $F(x) = x$ and $G(x) = \tan^{-1}x$, $-\infty < x < \infty$. Let $\mathcal{C}$ = the set of all half-open intervals $(a,b]$, $a<b$, and define $\mu(a,b] = F(b) - F(a)$ and $\nu(a,b] = G(b) - G(a)$, and proceed to extend μ and ν to be measures on the smallest σ-algebra $\mathcal{A}$ containing $\mathcal{C}$. Determine the Radon-Nikodym derivative $\left[\frac{d\nu}{d\mu}\right]$. If $\nu E = \int_E f \, d\mu$, for all $E \in \mathcal{A}$, determine

#16. Let $f: E \to \overline{\mathbb{R}}$ be measurable. Exhibit a sequence of simple measurable functions $\langle \varphi_n \rangle$ on E such that $\varphi_n(x) \to f(x)$ for every $x \in E$.

#17. If $v_1, \ldots, v_n$ is a vector space basis of a subspace V of the Hilbert space H and $f \in H$, describe in terms of $f, v_1, \ldots, v_n$ the point nearest to f in

#18. Let L^1 be the Banach space of Lebesgue integrable functions on $[0,1]$. Let F be a bounded linear functional on L^1. Prove that there is a bounded measurable function g so that

$$F(f) = \int_0^1 f(x)\, g(x)\, dx, \quad f \in L^1.$$

(You may use properties of absolutely continuous functions, density of step functions, etc. You may NOT use the fact that the dual space of L^1 is L^∞.)

#19. For $\varepsilon \geq 0$, consider the function f on $[0,1]$ defined by

$$f(x) = \begin{cases} x^{1+\varepsilon} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

a) For which ε is f of bounded variation? Prove your answer.

b) For which ε is f absolutely continuous? Prove your answer, by representing f as an indefinite integral (show the integrand $\in L^1$).

#20. If $f \in L[a,b]$ then define I by $(If)(x) = \int_a^x f(t)\,dt$, and if g is differentiable at x, define D by $(Dg)(x) = g'(x)$.

a) In terms of the operators D and I, <u>state</u> precisely the Fundamental Theorem of Calculus (FTOC); include all hypothe

b) Give an elementary <u>proof</u> of FTOC if $f \in C[a,b]$.

c) In terms of the operators D and I, <u>state</u> precisely the characterization theorem for $f \in AC[a,b]$.

1. Using the notion of product measure, explain how one obtains two dimensional Lebesgue measure.

2. For each Lebesgue measurable subset E of $\mathbb{R}$, let

$$\sigma(E) = \{(x,y) : x-y \in E\}.$$

Show that $\sigma(E)$ is a measurable subset of $\mathbb{R}^2$.

3. Show that $\exists\, F \in \{L^\infty[0,1]\}^* \ni$ there is no $y \in L^1[0,1]$ for which

$$F(x) = \int_0^1 x(t)\,y(t)\,dt \qquad \forall x \in L^\infty[0,1].$$

4. Show the following:

H 1. $A \subset C(X)$ is an algebra where X is a compact top. sp.
 2. A separates the points of X.
 3. $\exists\, p \in X \ni f(p) = 0 \quad \forall f \in A$.

C $\quad g \in C(X)$ and $g(p) = 0 \Rightarrow g \in \bar{A}$.

5. Derive a representation for the bounded linear functionals on c_0.

6. Show the following:

H 1. $(X, \mathcal{B}, \mu)$ is a meas. sp. with $\mu(X) < \infty$.

2. ν is a measure defined on $\mathcal{B}$, $\nu \ll \mu$, and $\nu(X) < \infty$.

3. f is a nonnegative meas. fn.

C For every $E \in \mathcal{B}$,

$$\int_E f \, d\nu = \int_E f \left[\frac{d\nu}{d\mu}\right] d\mu.$$

7. Give an example of an $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ measure spaces with f measurable on $X \times Y \ni$

$$\int_X \left[\int_Y f \, d\nu\right] d\mu \neq \int_Y \left[\int_X f \, d\mu\right] d\nu.$$

8. Prove the following:

H f is an extreme point of the closed unit ball in $L^\infty[0,1]$

C $|f(t)| = 1$ a.e. on $[0,1]$.

9. Establish the following result in ℓ^4:

$\exists \langle x_n \rangle_{n=1}^{\infty} \ni x_n \to 0$ weakly but not strongly.

1. Using the notion of product measure, explain how one obtains two dimensional Lebesgue measure.

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$$\sigma(E) = \{(x,y) : x-y \in E\}.$$

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3. Show that $\exists\, F \in \{L^\infty[0,1]\}^* \ni$ there is no $y \in L^1[0,1]$ for which

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C $|f(t)| = 1$ a.e. on $[0,1]$.

9. Establish the following result in ℓ^4:

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Real Analysis Qualifying Exam 6/9/83

#1. Let X be a compact topological space and f be a real-valued function on X with the property that each set of the form $\{x : f(x) \geq a\}$ is closed in X. Prove that f is bounded from above and that it attains its least upper bound.

#2. a) Let g be an integrable function on $[0,1]$. Show that there is a bounded measurable function f such that $\|f\|_\infty \neq 0$ and $\int_0^1 fg\,dx = \|g\|_1 \|f\|_\infty$.

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#5. Prove or disprove: Suppose that each f_n is continuously differentiable and satisfies $|f_n(x)| \le 1$ and $|f_n'(x)| \le 1$, all $x \in \mathbb{R}$. Then there is a subsequence of the f_n converging <u>uniformly</u> to a continuous function on $\mathbb{R}$.

#6. Prove or disprove: Let f and g be two <u>absolutely continuous</u> functions defined on $[0,1]$. Suppose that $f'(x) = g'(x)$ for almost all x in $[0,1]$. Then there is a constant K so that $f(x) = g(x) + K$ for all x in $[0,1]$.

#7. If F is continuous on $[0,1]$ and F' exists a.e. (Lebesgue measure) and $\left|\dfrac{F(x)-F(t)}{x-t}\right| \le M < \infty$ for $0 \le x, t \le 1$, prove that $F(1) - F(0) = \int_0^1 F'(x)\,dx$.

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#11. a) State the principle of uniform boundedness for a family $\mathcal{F}$ of linear operators

b) Show that if $x_n \to x$ weakly, then $\langle \|x_n\| \rangle$ is bounded.

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You are allowed to quote the Weierstrass approximation theorem.

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c) What hypothesis of Tonelli's theorem is violated in b)? Why? (Explain.)

#15. Set $F(x) = x$ and $G(x) = \tan^{-1} x$, $-\infty < x < \infty$.
Let $\mathcal{C}$ = the set of all half-open intervals $(a,b]$, $a<b$, and define $\mu(a,b] = F(b) - F(a)$ and $\nu(a,b] = G(b) - G(a)$, and proceed to extend μ and ν to be measures on the smallest σ-algebra $\mathcal{A}$ containing $\mathcal{C}$. Determine the Radon-Nikodym derivative $\left[\frac{d\nu}{d\mu}\right]$. If $\nu E = \int_E f \, d\mu$, for all $E \in \mathcal{A}$, determine

#16. Let $f: E \to \overline{\mathbb{R}}$ be measurable. Exhibit a sequence of simple measurable functions $\langle \varphi_n \rangle$ on E such that $\varphi_n(x) \to f(x)$ for every $x \in E$.

#17. If $v_1, \ldots, v_n$ is a vector space basis of a subspace V of the Hilbert space H and $f \in H$, describe in terms of $f, v_1, \ldots, v_n$ the point nearest to f in V.

#18. Let L^1 be the Banach space of Lebesgue integrable functions on $[0, 1]$. Let F be a bounded linear functional on L^1. Prove that there is a bounded measurable function g so that

$$F(f) = \int_0^1 f(x)\, g(x)\, dx, \quad f \in L^1.$$

(You may use properties of absolutely continuous functions, density of step functions, etc. You may NOT use the fact that the dual space of L^1 is L^∞.)

Real Analysis Qualifying Exam 6/4/82

#1. Prove that if $x_n \geq 0$ and $y_n \geq 0$ for all n, then $\overline{\lim}\, x_n y_n \leq (\overline{\lim}\, x_n)(\overline{\lim}\, y_n)$ provided the product on the right is not of the form $0 \cdot \infty$ or $\infty \cdot 0$.

#2. Show that if f and g are measurable functions then fg is measurable.

#3. X is a compact topological space. f is a real-valued function on X with the property that each set of the form $\{x : x \in X,\ f(x) \leq a\}$ is closed in X. Prove that f is bounded from below and that it attains its greatest lower bour

#4. Let X be an *infinite* compact metric space. Let $C(X)$ be the space of continuous functions on X, provided with the usual metric. Is it possible that the ball $\{f : f \in C(X),\ |f(x)| \leq 1, \text{ all } x \in X\}$ is a compact subset of $C(X)$?

#5. Let A be the subset of the interval $[0,1]$ consisting of those numbers having a decimal expansion with no sevens. Show that A is a measurable set and find its measure. (Hint: analogously the Cantor ternary set is those numbers in $[0,1]$ with no 1's in their ternary expansions.)

#6. (A form of) Egoroff's theorem is given by the following statement: Let E be a measurable set of finite measure and $\langle f_n \rangle$ a sequence of measurable functions defined on E. Let f be a measurable real-valued function such that for each x in E we have $f_n(x) \longrightarrow f(x)$. Then given $\varepsilon > 0$ and $\delta > 0$, there is a measurable set $A \subset E$ with $mA < \delta$ and an integer N such that for all $x \notin A$ and all $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

a) Given the truth of the statement above, ~~prove the~~ state and prove the Bounded Convergence Theorem.

b) Using a), state and prove Fatou's Lemma.

#7. What is a <u>connected</u> topological space? Prove that if f is a continuous real-valued function on a connected space and f takes the values a and b then f assumes all values between a and b.

#8. Let X be a compact metric space and $\mathcal{F}$ an equicontinuous family of functions on X, and suppose $|f| \leq 1$ for each $f \in \mathcal{F}$. Prove that each sequence $\langle f_n \rangle$ from $\mathcal{F}$ has a uniformly convergent subsequence.

#9. a) State the Fubini Theorem.

b) If $\sum_{j=1}^{m} \sum_{k=1}^{n} |a_{jk}| \leq C < \infty$ for all m, n state why the Fubini Theorem implies that $\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} a_{jk} \right) = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{jk} \right)$.

#10. a) Show by example that if F is continuous on $[0,1]$ and F' exists a.e. (Lebesgue measure) then (*) may fail.

(*) $\quad F(1) - F(0) = \int_0^1 F'(x)\,dx$

b) If F' exists everywhere, does (*) hold? Why?

#11. Suppose that X is a real normed linear space, V is a subspace, f is a bounded linear functional on V, $x \in X$ such that $X = \{ v + \lambda x : v \in V, \lambda \in \mathbb{R} \}$. Show by construction that f has an extension F to X where F is linear and F has the same norm as f.

#12. a) ~~A topological space~~ State the Radon-Nikodym Theor

b) Give an example of two (positive) measures where neither is absolutely continuous with respect to the other.

#13. Use the definition of the Lebesgue integral and the Monotone Convergence Theorem to prove that $\int_0^1 x\,dx = \frac{1}{2}$.

#14. If $v_1, \ldots, v_n$ is a vector space basis of a subspace V of the Hilbert space H and $x \in H$, describe in terms of $x, v_1, \ldots,$ the point nearest to x in V. Prove your assertion.

#15. a) State Hölder's inequality.

b) State and prove (using a)) the Minkowski inequ

#16. If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are two measure spaces

a) define measurable rectangle;

b) show that the complement of a measurable rectangle is a disjoint union of measurable rectangles;

c) show that the intersection of two measurable rectangles is a measurable rectangle;

d) show that a countable union of measurable rectangles is a countable disjoint union of measurable rectangles.

#17. State the Riesz Representation Theorem

a) for $L^p(\mu)$, $1 \le p < \infty$, $(X, \mathcal{A}, \mu)$ is a σ-finite measure space;

b) for $C(X)$, X is a compact ~~Hausdorff~~ metric space;

c) If $F(g) = \int fg\, d\mu$ where $g \in C(X)$, $\mu X < \infty$, show that F is a bounded linear functional. What is $\|F\|$?

#18. Denote by l^∞ the space of all bounded sequences of real numbers and define $\| \langle x_\nu \rangle \| = \sup |x_\nu|$. Show that l^∞ is a Banach space.

Real Analysis Qualifying Exam 6/4/82

#1. Prove that if $x_n \geq 0$ and $y_n \geq 0$ for all n, then $\overline{\lim}\, x_n y_n \leq (\overline{\lim}\, x_n)(\overline{\lim}\, y_n)$ provided the product on the right is not of the form $0 \cdot \infty$ or $\infty \cdot 0$.

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Ph.D. Qualifying Examination Real Analysis 6-13-

Do eight out of ten.
Do one more for extra credit if you wish.
Do #11, for extra extra credit.

1. Prove that ℓ^p is complete, $1 < p < \infty$.

2. Give a representation for the bounded linear functionals on c_0.

3. For each Lebesgue measurable subset E of $\mathbb{R}$, let

$$\sigma(E) = \{(x,y) : x - y \in E\}.$$

Show that $\sigma(E)$ is a measurable subset of $\mathbb{R}^2$.

4. Using the notion of product measure, explain how one obtains two dimensional Lebesgue measure.

5. Show that $\exists\ F \in \{L^\infty[0,1]\}^* \ni$ there is no $y \in L^1[0,$ for which $F(x) = \int_0^1 x(t)y(t)\,dt \quad \forall\ x \in L^\infty[0,1]$.

6. Let H be a Hilbert space and let P be a subset of H. Show that $P^{\perp\perp}$ is the smallest closed linear subspace containing P.

Prove the following:

7. H. 1. f and $f_n \in L^3[0,1]$, $n = 1, 2, \ldots$

2. $f_n \to f$ a.e. on $[0,1]$.

3. $\exists M \ni \|f_n\| \leqq M \quad \forall n$.

4. $g \in L^{3/2}[0,1]$

C. $\lim_{n\to\infty} \int_0^1 f_n g = \int_0^1 fg$.

8. H. 1. $(X, \mathcal{B}, \mu)$ is a meas. sp. with $\mu(X) < \infty$.

2. ν is a measure on $\mathcal{B}$ with $\nu(X) < \infty$.

3. $\nu << \mu$

4. f is a nonneg. meas. fn.

C $\int_E f\,d\nu = \int_E f\left[\frac{d\nu}{d\mu}\right] d\mu \quad \forall E \in \mathcal{B}$.

9. H. f is a cont. r. v. fn. on $\mathbb{R}$ with $f(x+2\pi) = f(x) \quad \forall x \in \mathbb{R}$

C. Given $\varepsilon > 0$, $\exists$ a trig. poly. $\phi \ni |f(x) - \phi(x)| < \varepsilon \quad \forall x \in \mathbb{R}$.

10. H 1. H is a separable Hilbert sp.

2. $\langle \phi_n \rangle_{n=1}^{\infty}$ is a CONS in H.

3. For $x \in H$, set $x_n = \sum_{j=1}^{n} x^\wedge(j)\,\phi_j$.

C $\lim_{n\to\infty} \|x - x_n\| = 0 \quad \forall x \in H$.

Extra Extra Credit 11. Show that the analogue of problem 7 is false for f and $f_n \in L^1[0,1]$.