2016 Algebra Qual - Part A

Choose 4 out of the following 5 problems.

(1) Let G be a group. Let $N \lhd G$ and $H < G$. Suppose that $[G : N]$ and $|H|$ are finite. Prove that if $[G : N]$ and $|H|$ are relatively prime, then $H < N$.

(2) Let G be a finite group. Suppose that $H < G$ and $H \neq G$.
(i) Prove that H has at most $[G : H]$ conjugates in G.
(ii) Prove that there exists a conjugacy class S in G such that $H \cap S = \emptyset$.

(3) How many elements of order 7 are there in a *simple* group of order 168?

(4) Let R be a commutative ring with identity. Let M be a *maximal* ideal of R.
(i) Prove that if R is a local ring, then $1 + x$ is a unit for every $x \in M$.
(ii) Prove that if $1 + x$ is a unit for every $x \in M$, then R is a local ring.

(5) Let R be a unique factorization domain and F its field of fractions. Let $f \in R[x]$ be a *monic* polynomial. Prove that if $c \in F$ is a root of f, then $c \in R$.

Algebra Qualifying Examination, Fall 2016, Part b

Answer any four of the following questions. All questions are worth 10 points

1. Let R be a commutative ring with identity and let a be a non–zero element in R. Suppose that P is a prime ideal properly contained in the principal ideal generated by a. Prove that $P = aP$. Suppose now that P is also principal. Prove that there exists $b \in R$ with $(1-ab)P = 0$ .What can you conclude about P if R is an integral domain and a is not a unit.

2. (a) Let R be a commutative ring with identity and regard R as a module for itself via left multiplication. Prove that this module is simple iff R is a field.
(b) Define a free module for a ring R. Suppose that R is a commutative ring with identity and satisfies the following condition: any submodule of a free module is free. Prove that R is a principal ideal domain.

3. Give examples to show that the following can happen for a ring R and modules M, N,
(i) $M \otimes_R N \ncong M \otimes_{\mathbb{Z}} N$, where $\mathbb{Z}$ is the ring of integers.
(ii) $u \in M \otimes_R N$ but $u \neq m \otimes n$ for any $m \in M$ and $n \in N$.
(iii) $u \otimes v = 0$ but $u, v \neq 0$.

4. Suppose that E is a three dimensional vector space over a field F and $f : E \to E$ is a non–zero linear transformation. Prove that there exists bases B_1 and B_2 of E such that the matrix of f is exactly one of the following.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5. Suppose that $D = (d_1, \cdots, d_n)$ is a diagonal matrix where the d_i, $1 \le i \le n$ are not necessarily distinct. What are the elementary and invariant factors of D? Suppose that A is similar to D. What can you say about its elementary divisors and invariant factors?

2016 Algebra Qual - Part C

(1) True/False: If E and F are extensions of $\mathbb{Q}$ such that $E \neq F$ and $[E : \mathbb{Q}] = [F : \mathbb{Q}] = 3$, then $[EF : \mathbb{Q}] = 9$. Prove or give a counterexample.

(2) Let p be a prime number and $a \in \mathbb{F}_p^*$.
(i) Prove that $x^p - x + a$ is irreducible and separable over $\mathbb{F}_p$.
(ii) What is the cardinality of the splitting field of $x^p - x + a$ over $\mathbb{F}_p$?

(3) Let n be a positive integer. Show that $\mathbb{C}(t^n) \subset \mathbb{C}(t)$ is a Galois extension and calculate its Galois group.

(4) Let ζ be a primitive 9th root of unity. Describe all the intermediate fields of the cyclotomic extension $\mathbb{Q} \subset \mathbb{Q}(\zeta)$. For each such intermediate field, give an explicit primitive element of the field as an extension of $\mathbb{Q}$.

Algebra Qualifier, Part A

September 22, 2017

Do four out of the five problems. Cross out the number of the problem that you don't want me to grade.

1. Let G be a finite abelian group of order m and let p be a prime integer dividing m.

(a) Prove that if G has exponent n, that is $x^n = 1_G$ for each $x \in G$, then $(G : 1_G)$ divides n^k for some positive integer k.

(b) Using induction on $(G : 1_G)$, show that G has a subgroup of order p.

2. How many elements of order 7 are there in a simple group of order 168? Prove your answer.

3. (a) Define the characteristic of a ring.

(b) Give an example of a commutative ring R, of characteristic zero having a unique maximal ideal and a non-maximal prime ideal P such that the characteristic of R/P is **not** zero.

4. Let P be a p-Sylow subgroup of a finite group G and let H be a p-subgroup of G with $H \subseteq N_P$, the normalizer of P. Show that $H \subseteq P$.

5. Let G be a finite group of order $p^n q$, p and q primes with $p > q$. Show that G is not simple.

ALGEBRA QUALIFYING EXAMINATION, PART B

Solve 4 questions out of five. Every question is worth 10 points. The total possible score is 40 points. All answers must be justified.

All rings are assumed to be unital and all modules are assumed to be unitary and left unless specified otherwise. Given an R-module M and $m \in M$, denote $\operatorname{Ann}_R m = \{r \in R : rm = 0\}$.

1. Let R be a commutative ring and M be an R-module. Let $M^* = \operatorname{Hom}_R(M, R)$.

(a) Show that the assignments $\xi \otimes m \mapsto (m' \mapsto \xi(m')m)$, $m, m' \in M$, $\xi \in M^*$ define a homomorphism of R-modules $\psi : M^* \otimes_R M \to \operatorname{End}_R M$. Why do we need R to be commutative?

(b) Suppose that M is a free. Show that ψ is injective.

(c) Suppose that M is free of finite rank. Show that ψ is an isomorphism.

2. Let M be a cyclic R-module generated by some $m \in M$ and let N be an R-module. Let $I = \operatorname{Ann}_R m$.

(a) Prove that $\operatorname{Hom}_R(M, N) \cong \{n \in N : I \subset \operatorname{Ann}_R n\}$ as an abelian group.

(b) Assuming that R is commutative, prove that $\operatorname{End}_R M \cong R/I$ as a ring.

3. Let R be an integral domain and M be an R-module. Let $\tau(M) = \{m \in M : \operatorname{Ann}_R m \neq 0\}$. Prove that if $0 \to M \xrightarrow{f} M' \xrightarrow{g} M'' \to 0$ is an exact sequence of R-modules then $0 \to \tau(M) \xrightarrow{f|_{\tau(M)}} \tau(M') \xrightarrow{g|_{\tau(M')}} \tau(M'')$ is a left exact sequence of R-modules. Explain why we need to assume that R is an integral domain.

4. Let M, M' be right R-modules and N, N' be left R-modules and let $f \in \operatorname{Hom}_R(M, M')$, $g \in \operatorname{Hom}_R(N, N')$.

(a) Is it always true that $\ker(f \otimes g) = \ker f \otimes_R N + M \otimes_R \ker g$? Prove or provide a counterexample.

(b) Give an example of an injective f such that $f \otimes 1 : M \otimes_R N \to M' \otimes_R N$ is not injective.

5. Let R be a commutative ring.

(a) Let M be a finitely generated R-module and let $\mathfrak{m} \subset R$ be a maximal ideal such that $\mathfrak{m}M = 0$. Show that M admits a natural structure of a finite dimensional vector space.

(b) Let K be a field. Describe isomorphism classes of $K[x]$-modules M such that $\dim_K M = 5$ and $x^r M = 0$ for some $r > 0$.

Algebra Qualifying Examination, Part C

Questions 1 through 4 are worth 10 points each. Question 5 is worth 20 points. You may do either all of questions one through four or Question 5 and two of questions 1 through four. The total possible score is 40 points. All answers must be fully justified.

1. (a) Let L be a field extension of K. Let X be a subset of L consisting of algebraic elements such that $L = K(X)$. Prove that L is algberaic over K.
(b) Let x be transcendental over K and let $u \in K(x)$ satisfy $u = (x^2 + 1)^{-1}$. Prove that x is algebraic over $K(u)$ and that u is transcendental over K.

2. Suppose that F is a field extension of K.
(a) Find the unique maximal subfield E with $F \supset E \supset K$ with $\mathrm{Aut}_E F = \mathrm{Aut}_K F$.
(b) Suppose that F is Galois over K and assume that L is an intermediate field of the extension $F \supset K$. Give an example of F, L, K such that $F \supset K$ is Galois but the extension $L \supset K$ is not Galois.

3. Let $\mathbf{Q}$ be the field of rational numbers and let ζ be a primitive cube root of unity.
(a) Prove that $\mathbf{Q}(\sqrt[3]{2} + \zeta) = \mathbf{Q}(\sqrt[3]{2},\ \zeta)$.
(b) Determine the Galois group of the extensions $\mathbf{Q}(\sqrt[3]{2}) \supset \mathbf{Q}$, $\mathbf{Q}(\zeta) \supset \mathbf{Q}$ and $\mathbf{Q}(\sqrt[3]{2}, \zeta) \supset \mathbf{Q}$.

4. Let F be the splitting field over $\mathbf{Q}$ of $x^4 - 5$. Explain why F is a Galois extension of $\mathbf{Q}$. Detemine all the intermediate fields L of this extension.

5. Consider the polynomial $f(x) = x^6 + x^3 + 1 \in \mathbb{Z}_2[x]$ and let $F \supset \mathbb{Z}_2$ be the splitting field of this polynomial.
(a) Explain why F is a finite field.
(b) Prove that if r is a root of this polynomial then $r^9 = 1$ and $r^m \neq 1$ for $m = 3, 6, 7$.
(c) Recall the theorem that every element of a finite field must satisfy the polynomial $x^{p^n} - x$ for some positive integer n and some prime p. Use this theorem and parts (a) and (b) of the problem to determine the cardinality of F.
(d) What is the Galois group of F over $\mathbb{Z}_2$ and determine all the intermediate fields of the extension $F \supset \mathbb{Z}_2$.

Algebra Qualifier, Part A

September 24, 2015

Do four out of the five problems. Cross out the number of the problem that you don't want me to grade.

1. Let G be a finite abelian group of order m and let p be a prime integer dividing m.

(a) Prove that if G has exponent n, that is $x^n = 1_G$ for each $x \in G$, then $(G : 1_G)$ divides n^k for some positive integer k.

(b) Using induction on $(G : 1_G)$, show that G has a subgroup of order p.

2. How many elements of order 7 are there in a simple group of order 168? Prove your answer.

3. (a) Define the characteristic of a ring.

(b) Give an example of a commutative ring R, of characteristic zero having a unique maximal ideal and a non-maximal prime ideal P such that the characteristic of R/P is not zero.

4. Let P be a p-Sylow subgroup of a finite group G and let H be a p-subgroup of G with $H \subseteq N_P$, the normalizer of P. Show that $H \subseteq P$.

5. Let G be a finite group of order $p^n q$, p and q primes with $p > q$. Show that G is not simple.

ALGEBRA QUALIFYING EXAMINATION, PART B

Solve any four problems out of five. Each problem is worth 10 points. The maximal possible score is 40 points. ***All answers must be justified!*** In particular, for questions "Is it true that...?" you should provide either a proof or a counterexample.

All rings are assumed to be unital and all modules are assumed to be unitary and left unless specified otherwise.

1. Let R be a ring and let M be an R-module. Then $\operatorname{End}_R M := \operatorname{Hom}_R(M, M)$ is a unital ring (usually non-commutative) and M is naturally a unitary $\operatorname{End}_R M$-module via $f.m = f(m)$, $m \in M$, $f \in \operatorname{End}_R M$.

(a) Suppose that M is such that $\mathrm{id}_M - f$ is invertible for any non-invertible $f \in \operatorname{End}_R M$. Prove that M is indecomposable as an R-module.

(b) If R is a division ring, prove that M is simple as an $\operatorname{End}_R M$-module. Is this statement still true if M is free but R is not a division ring?

2. Let R be a commutative ring and let E, F be free R-modules of the same finite rank. Prove that if $\psi \in \operatorname{Hom}_R(E, F)$ is surjective then it is an isomorphism. Is it true, under the same assumptions on E and F, that if ψ is injective then it is an isomorphism? Why do we need to assume that R is commutative?

3. Let M, M' be right R-modules and N, N' be left R-modules and let $f \in \operatorname{Hom}_R(M, M')$, $g \in \operatorname{Hom}_R(N, N')$.

(a) Is it always true that $\ker(f \otimes g) = \ker f \otimes_R N + M \otimes_R \ker g$?

(b) Give an example of an injective f such that $f \otimes 1 : M \otimes_R N \to M' \otimes_R N$ is not injective.

4. Let K be a field.

(a) Determine whether $K(x)$ is projective as a $K[x]$-module.

(b) Describe the $K[x]$-dual module of $K(x)$.

(c) Describe all isomorphism classes of $K[x]$-modules on which x acts nilpotently and which are 4-dimensional as K-vector spaces.

5. Let A and B be invertible $n \times n$-matrices over a field K. Prove that $A + \lambda B$ is invertible for all but finitely many $\lambda \in K$. Is this statement true if K is a commutative ring?

2018 ALGEBRA C QUAL

Name:

1. Let $K \to F \to L$ be field extensions, **not** assumed algebraic (much less finite-dimensional). For each of the following assertions, provide a proof or a counterexample.
   (i) If $K \to F$ and $F \to L$ are Galois and finite dimensional, then $K \to L$ is Galois.
   (ii) If $K \to L$ and $K \to F$ are Galois, then so is $F \to L$.
   (iii) If $K \to L$ is Galois, and $[F : K]$ is finite, then $F \to L$ is Galois.

2. Let $K \to F$ be a splitting field for a polynomial $f \in K[x]$ of degree $n \geq 1$. Prove that $[F : K]$ divides $n!$

3. Let L be an algebraic closure of the field $\mathbb{Z}_p$ where p is a prime number.
   (i) Prove that $\phi(x) = x^p$ defines a field automorphism of L, fixing $\mathbb{Z}_p$.
   (ii) Let n be a natural number. Prove that the fixed field F of ϕ^n, i.e.
   $F = \{x \in L : \phi^n(x) = x\}$ is a field with p^n elements.
   (iii) For F as above, prove that $\mathbb{Z}_p \to F$ is a Galois extension of degree n.

4. Let $K \to F$ be a Galois extension of degree 28. Prove that there exists a subfield E of F such that $[E : K] = 7$.