Real Analysis Qualifier — 2019

Undergraduate Problems

Choose one problem from 1–2 and one problem from 3–4. Please show all work. Unsupported claims will not receive credit.

1. Find a function $f\colon \mathbb{R} \to \mathbb{R}$ that is differentiable everywhere but whose derivative is not continuous everywhere. Prove it has both these properties.

2. Find a function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at all irrational numbers and discontinuous at all rational numbers. Prove it has both these properties.

3. Prove straight from the definition of Riemann integral that this function $f\colon [0,1] \to \mathbb{R}$ is Riemann integrable:
$$f(x) = \begin{cases} 0 & \text{if} \quad x \leq 1/2 \\ 1 & \text{if} \quad x > 1/2 \end{cases}$$

4. Prove straight from the definition of Riemann integral that this function $f\colon [0,1] \to \mathbb{R}$ is not Riemann integrable:
$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \\ 1 & \text{if} \quad x \notin \mathbb{Q} \end{cases}$$

209A

Choose one problem from 1–2, one from 3–4 and one from 5–6. Please show all work. Unsupported claims will not receive credit. The notion of measure or measurable is in the sense of Lebesgue, unless stated otherwise.

1. State the definition that a set in $\mathbb{R}$ is Lebesgue measurable. Prove that every countable set in $\mathbb{R}$ is Lebesgue measurable.

2. Prove that every set $S \subseteq \mathbb{R}$ with positive outer measure contains a nonmeasurable set.

3. Construct a bounded open set $O \subset \mathbb{R}$ such that the measure of O is strictly less than the measure of its closure $\bar{O}$.

4. Let $\{f_n\}$ be a sequence of measurable functions on $[0, 1]$. Prove that the set of points $x \in [0, 1]$ where $f_n(x)$ does not converge is measurable.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Prove that $|f|$ is also measurable. Is the converse true? Why?

6. State Egoroff's Theorem. Use it to prove the Dominated Convergence Theorem for measurable functions on the interval $[0, 1]$ with Lebesgue measure.

209B

Choose one problem from 1–2, one from 3–4, and one from from 5–6. Please show all work. Unsupported claims will not receive credit.

1. Find a sequence of measurable functions $f_n\colon [0,1] \to \mathbb{R}$ that converges to zero in measure but not almost everywhere. Prove it has both these properties.

2. Find a sequence of functions $f_n\colon [0,1] \to \mathbb{R}$ that converges to zero pointwise but has

$$\lim_{n\to\infty} \int_{[0,1]} f_n \, dx = +\infty.$$

   Prove it has both these properties.

3. Let ℓ^1 be the space of real-valued sequences $(c_i)_{i=1}^\infty$ with $\sum_{i=1}^\infty |c_i| < \infty$, made into a Banach space with the norm

$$\|c\| = \sum_{i=1}^\infty |c_i|.$$

   Find an infinite-dimensional closed subspace of ℓ^1 that is not all of ℓ^1.

4. Let $C[0,1]$ be the vector space of continuous real-valued functions $f\colon [0,1] \to \mathbb{R}$, made into a normed vector space with the norm

$$\|f\| = \int_{[0,1]} |f(x)| \, dx.$$

   Show that $C[0,1]$ is not a Banach space with this norm.

5. Suppose that V is a Banach space and $v_i \in V$ is a sequence with

$$\sum_{i=1}^\infty \|v_i\| < \infty.$$

   Prove that the sum

$$\sum_{i=1}^\infty v_i$$

   converges in the norm topology on V.

6. Using the Baire category theorem, prove this version of the uniform boundedness principle: if V and W are Banach spaces and $S \subseteq L(V,W)$ is a set of bounded operators from V to W such that

$$\sup_{T\in S} \|Tv\| < \infty \text{ for all } x \in V,$$

   then

$$\sup_{T\in S} \|T\| < \infty.$$

209C

Choose one problem from 1–2 and one from 3-4. Please show all work. Unsupported claims will not receive credit. In what follows we use Lebesgue measure on $[0, 1]$ and $\mathbb{R}$.

1. For any $f \in C[0, 1]$, the space of continuous functions on $[0, 1]$ with the sup norm, define a linear functional

$$Tf = \int_0^{1/4} xf(x)dx.$$

   (a) Prove that T is a bounded linear functional on $C[0, 1]$.
   (b) Find the norm of T.

2. Show that the dual space of $L^\infty[0, 1]$ strictly contains $L^1[0, 1]$.

3. (a) State the definition of $\mathcal{S}(\mathbb{R})$, the space of Schwartz functions on $\mathbb{R}$.
   (b) Prove that $f(x) = e^{-x^2}$ is in $\mathcal{S}(\mathbb{R})$ and find its Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

4. (a) State the definition of $\mathcal{S}'(\mathbb{R})$, the space of tempered distributions on $\mathbb{R}$.
   (b) Prove that the Heaviside function

$$h(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

   can be viewed as a tempered distribution and find its derivative.