Math 206A Qualifying Exam

September 2019

Please choose any 3 of the following 5 problems.

1. Let X be a non-negative random variable.

Part a: (5 points) Show that

$$\mathbf{E}(X) = \int_0^\infty \mathbf{P}(X \geq t)\, dt$$

Part b: (5 points) Suppose that $X_1, X_2, \ldots, X_n, \ldots$ are independent variables, all having the same distribution as X. Show that with probability 1 there are only finitely many n for which $X_n \geq n$.

2. Let $\{X_n\}$ be a sequence of random variables.

Part a: (5 points) Define what it means for X_n to converge to X in probability, and what it means for X_n to converge to X almost surely.

Part b: (5 points) Suppose that $X_n \to X$ almost surely. Must $\mathbf{E}(X_n) \to \mathbf{E}(X)$? Prove or give a counterexample with justification.

3. Recall that a Normal Variable with mean μ and variance σ^2 (also known as an $N(\mu, \sigma^2)$ variable) has density given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Part a: (2 points) Set up an integral that gives the characteristic function $\varphi(t)$ of such a normal variable. **You do not need to evaluate this integral!**.

It turns out that the integral from part a evaluates to

$$\varphi(t) = e^{it\mu - \frac{1}{2}\sigma^2 t^2}$$

Part b: (4 points) Suppose that $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent. Determine, with justification, the distribution of $X + Y$.

Part c: (4 points) Suppose that $X_n \to X$ in distribution, each X_n has a distribution of the form $N(0, \sigma_n^2)$. Show that the σ_n^2 are converging to some σ^2, and that $X \sim N(0, \sigma^2)$.

4. Let $X_1, X_2, \dots$ be independent variables, with $\mathbf{E}(X_n) = 1$ and $\mathbf{Var}(X_n) = n^{1.5} - (n-1)^{1.5}$. Define the **Sample Mean** $\overline{X_n}$ by

$$\overline{X_n} = \frac{1}{n}\left(X_1 + \cdots + X_n\right)$$

Part a (5 points) Determine the mean and variance of $\overline{X_n}$. (Hint: Telescope!)
Part b (5 points) Show that $\overline{X_n} \to 1$ in probability.

5. Let $X_1, \dots, X_n$ be variables, and suppose that they have a joint density $f(x_1, x_2, \dots, x_n)$ which factors as

$$f(x_1, x_2, \dots, x_n) = f_1(x_1) f_2(x_2) \dots f_n(x_n)$$

as the product of n functions depending individually only on a single x_i (the $f_i(x_i)$ are not necessarily density functions). Show that $(X_1, \dots, X_n)$ are independent.

Qualify Exam on Numerical Analysis, Fall 2019

Please choose any three problems and submit the full solutions.

Problem 1. Consider an ODE of the form

$$y' = f(x, y),\ y(x_0) = y_0,\ x, x_0 \in [a, b]$$

whose solution is to be approximated using a general one-step method of the form

$$y_{i+1} = y_i + h\Phi(x_i, y_i, h) \tag{1}$$

where $h = \frac{b-a}{N}$ is the step size and $\Phi(x, y, h)$ is Lipschitz continuous on y with constant λ. Assume that the method is of order p so that the local truncation error gives

$$y(x_{i+1}) = y(x_i) + h\Phi(x_i, y(x_i), h) + \tau_i h^{p+1}$$

where $y(x)$ is the exact solution of the IVP and τ_i is a constant that depends on derivatives of this solution.

(a) Derive an error estimate for the obtained approximation of the form

$$|e_i| \leq C_1|e_0| + C_2 h^p,\ i = 0, 1, ..., N$$

where C_1 and C_2 are constants.

(b) For a second order Runge-Kutta method of your choice, give the explicit representation of $\Phi(x_i, y(x_i), h)$ that arises when the method is expressed in form (1).

Problem 2. Consider the multistep method

$$y_i - y_{i-4} = \frac{h}{3}[8f_{i-1} - 4f_{i-2} + 8f_{i-3}]$$

applied to the ODE $y'(x) = f(x, y)$, where $f_i = f(x_i, y_i)$.
What is the order of the local truncation error of the method?

Problem 3. Prove that

$$||X^{(k)} - X|| \leq ||T||^k ||X^{(0)} - X||$$

and

$$||X^{(k)} - X|| \leq \frac{||T||^k}{1 - ||T||} ||X^{(1)} - X^{(0)}||$$

where T is an $n \times n$ matrix with $||T|| < 1$ and

$$X^{(k)} = TX^{(k-1)} + C, \quad k = 1, 2, \ldots$$

with $X^{(0)}$ arbitrary, $C \in R^n$, and $X = TX + C$.

Problem 4. Derive an $O(h^4)$ five-point formula to approximate $f'(x_0)$ that uses $f(x_0 - h)$, $f(x_0)$, $f(x_0 + h)$, $f(x_0 + 2h)$ and $f(x_0 + 3h)$.

Problem 5. (Construction of Gaussian Quadrature) Let $\{p_0(x), p_1(x), ...\}$ be Legendre polynomials satisfying

- For each n, $p_n(x)$ is a monic polynomial of degree n
- $\int_{-1}^{1} p(x)p_n(x)\,dx = 0$ for any $p(x)$ of degree $\leq n$.

Prove that

(1) All m zeros of a Legendre polynomial $p_m(x)$ reside in the interval $[-1, 1]$ and they are simple.

(2) Let $x_1, x_2, ..., x_n$ are the n roots of the n^{th} Legendre polynomial $p_n(x)$ and define

$$c_i = \int_{-1}^{1} \prod_{j=1, j\neq i}^{j=n} \frac{x - x_j}{x_i - x_j}\,dx,$$

then for any polynomial $p(x)$ of degree less than $2n$,

$$\int_{-1}^{1} p(x)\,dx = \sum_{i=1}^{i=n} c_i p(x_i).$$

Problem 1. Variational problems.

a) Prove fundamental lemma of the calculus of variations.
b) A spring hanging vertically under gravity. Consider a mass *m* on the end of a spring of natural length *l* and spring constant *k*. Let y be the vertical coordinate of the mass as measured from the top of the spring. Assume the mass can only move up and down in the vertical direction. The Lagrangian for this problem is as follows:

$$L = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k(y-\ell)^2 + mgy.$$

Explain terms in this Lagrangian, corresponding variational problem and determine and solve the corresponding Euler- Lagrange equation of motion. (Lagrangian is the function in the functional of the variational problem.)

Problem 2.

a. The Brachistochrone problem. An experimenter lets a bead slide with zero initial velocity down a wire that connects two fixed points with zero initial. Determine the shape of the wire that the bead slides from one end to the other in minimal time.
b. Describe functional for the Brachistochrone problem with nonzero initial velocity. Do not solve variational problem.

Problem 3. Chaotic dynamical systems.

a. Give a definition of a chaotic dynamical system.
b. Give definition of the tent map and demonstrate sensitive dependence on initial conditions for the tent map.
c. Describe bifurcation diagram of the logistic map:

$$x_{n+1} = r \cdot x_n(1 - x_n)$$

and explain period doubling.

d. Describe in detail behavior of trajectories of the logistic map when parameter *r* reaches bifurcation point at 4.

Problem 4. Bifurcations of dynamical systems.

a. Give detailed definition of a bifurcation of a dynamical system.
b. Find equilibrium points and determine their types for different values of parameter μ of the following dynamical system:

$dx_1/dt = \mu x_1 + x_1^3$
$dx_2/dt = -x_2.$

c. Explain what happens with the equilibrium points and their stability as μ goes through zero in this dynamical system. Sketch bifurcation diagram.

d. Give a definition of a normal form and explain in detail what normal forms are used for.

Problem 5. Variational problems.

a. Prove that if y(x) is an extremal of the functional J:

$$J[y(x)] = \int_{x_0}^{x_1} F(x, y, y')dx.$$

then it satisfies the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

b. Give definitions of the Hamilton-Jacobi equation and eikonal. Explain relation between eikonals and extremals.