UCR Math Dept Topology Qualifying Exam

September 2019

• There are three parts in this exam and each part has three problems. You should complete two (and only two) problems of your choice in each part. Each problem is worth 10 points.
• Support each answer with a complete argument. State completely any definitions and basic theorems that you use.
• This is a closed book test. You may only use the test, something to write with, and the paper provided. All other material is prohibited. Write each of your solutions on separate sheets. Write your ID number on every sheet you use.
• The time for the exam is 3 hours.

Part I

1. Let X be a topological space. An equivalence relation $\sim$ on X is called *open* if the associated quotient map

$$\pi : X \to X/\!\sim$$

is open, when $X/\!\sim$ is endowed with the quotient topology. Let $\sim$ be an open equivalence relation.

Show that $X/\!\sim$ is Hausdorff if and only if

$$R := \{(x, y) \mid x \sim y\} \subset X \times X$$

is closed in the product topology of $X \times X$.

2. Let X be a topological space, and $C_\alpha \subset X, \alpha \in A$, be a locally finite family of closed sets.

(a) Show that $\bigcup_{\alpha \in A} C_\alpha \subset X$ is closed.

(b) Show that this property may fail without the assumption of local finiteness.

3. A topological space X is *normal* if given any two disjoint closed subsets $C, D \subset X$ there exist two disjoint open sets $U \supset C, V \supset D$. Prove that a compact Hausdorff space is normal.

Part II

4. Let X be a nonempty space which is Hausdorff, connected, simply connected, and locally path connected. Prove that every continuous mapping $f : X \to S^1$ is homotopic to a constant map.

5. Suppose that Q is a solid square in $\mathbb{R}^2$ with vertices A, B, C, D, let E denote the center point of Q, and consider the graph X whose edges are the boundary of Q plus the segments joining the vertices of Q to E. If $Y \subset X$ is the subgraph given by the boundary of Q, show that Y is not a deformation retract of X. [*Hint:* What are the fundamental groups?]

6. If E is a topological space with $e \in E$, then the local homology groups of E at e are defined by $H_k(E, E - \{e\})$. Show that if $f : X \to Y$ is a homeomorphism and $x \in X$, then f induces an isomorphism from $H_k(X, X - \{x\})$ to $H_k(Y, Y - \{f(x)\})$ for every integer k. [*Hint:* What is $f[X - \{x\}]$?]

Part III

7. Give an example of a smooth map $F : M \longrightarrow N$ and a smooth vector field X on M that is not F–related to any vector field on N.

8. Suppose $M \subset \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subset T\mathbb{R}^n$ be the set of all unit vectors tangent to M :

$$UM := \{ (x, v) \in T\mathbb{R}^n \mid x \in M, v \in T_x M, \ |v| = 1 \} .$$

It is called the **unit tangent bundle** of M. Prove that UM is an embedded $(2m - 1)$–dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$.

9. Show that $S^n \times \mathbb{R}$ is parallelizable for all $n \geq 1$.