Complex Analysis – Qualifying Examination
September 2019

Part A and B

Answer all questions. Provide as much details as you could.

Problem 1 *Determine explicitly the largest disk about the origin where image under the mapping* $f(z) = z^2 - 2z$ *is one-to-one.Justify your answer.*

Problem 2 *Does there exist an analytic function* $f : D \to D$ *with* $f(\frac{1}{2}) = \frac{3}{4}$ *and* $f'(\frac{1}{2}) = \frac{2}{3}$*? Here* D *is the open unit disk.*

Problem 3 *Let* G *be a region and* $a \in G$*. Suppose that* f *is continuous on* G *and analytic on* $G \backslash \{a\}$*. Prove that* f *is analytic at* a*.*

Problem 4 *Given a Möbius transformation* $T(z) = \frac{az+b}{cz+d}$ *, determine necessary and sufficient conditions on* a, b, c, d *so that* T *map the domain* $D = \{z : \operatorname{Re} z > 0\}$ *onto* $G = \{z : \operatorname{Re} z < 0\}$*.*

Problem 5 *Prove Vitali's Theorem:* $H(G)$ *is the set of holomorphic functions on* G*. If* G *is a region and* $f_n \in H(G)$ *is locally bounded and* $f \in H(G)$ *has the property that* $A = \{z \in G : \lim_n f_n(z) = f(z)\}$ *has a limit point in* G*, then* f_n *converges in* f *in* $H(G)$*.*

Problem 6 *Let* G *be a region and* u *a non-constant harmonic functions on* G*. Show that* u *is an open map.*

Part C

Answer all questions. Provide as much details as you could.

Problem 7 *(a) Show that any compact Riemann surface admits a branched cover over the Riemann sphere* S^2*. (b) Let* M *be a compact Riemann surface of genus 5. Suppose* F *is a branched cover from* M *to* S^2*. Prove that* $R = 2(4+r)$*, where* R*=the total ramification index of* F*,* r*=number of sheets of the cover.*

Problem 8 *Prove that the Dolbeault cohomolgy group* $H^{1,1}(X) = \frac{\Omega^{1,1}}{\bar{\partial}\Omega^{1,0}}$ *is isomorphic to* $\mathbb{C}$ *for any compact Riemann surface* X*, and* $\mathbb{C}$*= complex number field.*

Problem 9 *Prove that the sum of residues of all the poles of a meromorphic one-form on a compact Riemann surface is equal to zero.*

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