ALGEBRA QUALIFYING EXAMINATION, PART A

SEPTEMBER 23, 2019

Solve any four problems out of five; indicate which ones should be graded. Each problem is worth 10 points. The maximal possible score is 40 points. ***All answers must be justified!*** In particular, for questions "Is it true that...?" you should provide either a proof or a counterexample.

1. Let M be an abelian monoid written multiplicatively (and so $M \times M$ is an abelian monoid in a natural way). Define a relation $\equiv$ on $M \times M$ by $(\alpha, \beta) \equiv (\alpha', \beta')$, $\alpha, \alpha', \beta, \beta' \in M$ if and only if $\alpha\beta'\gamma = \alpha'\beta\gamma$ for some $\gamma \in M$.

(a) Show that $\equiv$ is a congruence relation and prove that the set $G(M)$ of equivalence classes for this relation is a group. Why is it important that M is abelian?

(b) Identify $G(M)$ when $M = \mathbb{Z}_{\geq 0}$ (the *additive* monoid of non-negative integers) and when $M = \mathbb{Z}_{>0}$ (the *multiplicative* monoid of positive integers);

(c) Define a natural homomorphism of monoids $i : M \to G(M)$. What property should M have to guarantee that i is injective?

2. Find, without listing all of them, the number of elements of order 2, the number of conjugacy classes and the number of Sylow 5-subgroups in S_5.

3. Let G be a finite group and let $p \mid |G|$ be a prime. Let H be the intersection of all Sylow p-subgroups of G. Prove or find a counterexample to the following statements.

(a) H is normal in G;

(b) Every subgroup of H is normal in G;

(c) Every normal p-subgroup of G is contained in H.

4. Let $R = \mathbb{Z}[\sqrt{-1}] = \{x + y\sqrt{-1} : x, y \in \mathbb{Z}\} \subset \mathbb{C}$ (the ring of Gaussian integers).

(a) Find a prime ideal in R;

(b) Find an ideal I in R such that $I \cap \mathbb{Z}$ is prime but I is not prime. Here we identify $\mathbb{Z}$ with a subring of R in a natural way.

You may assume to be known that R is a Euclidean domain with $\varphi : R \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ defined by $\varphi(x + y\sqrt{-1}) = x^2 + y^2$, $x, y \in \mathbb{Z}$.

5. Let $p \neq q$ be primes and $k > 0$. Let R be the ring $\mathbb{Z}_{p^k q}$ and let S be the image of $\{p^i : i > 0\} \subset \mathbb{Z}$ in R under the canonical projection $\mathbb{Z} \to R$.

(a) Is the natural homomorphism $R \to S^{-1}R$ injective?

(b) Find $S^{-1}R$. What are ideals in $S^{-1}R$?

Qualifying Exam, 2019, Algebra Part B.

Answer any two of the following questions. Each is worth 20 points.

1. Suppose that V is a complex vector space.

(i) Define the dual vector space V^* and prove that $\dim V = \dim V^*$ if $\dim V < \infty$. Next define the canonical map $V \to (V^*)^*$ and show that it is injective. Prove that if $\dim V = \infty$ then this map is not surjective. (In fact show that the identity map on $(V^*)^*$ will not be in the image).

(ii) Suppose that $T : V \to V$ is a linear transformation. Define the corresponding transformation $T^* : V^* \to V^*$.

(iii) Let V be a three dimensional complex vector space and $T : \mathbf{C}^3 \to \mathbf{C}^3$ be the linear transformation given by the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Determine the matrix of T^*.

2.(a) Consider the polynomial ring $k[x, y]$ in two variables with coefficients in a field k. Let I be the ideal generated by x and J be the ideal generated by y. Prove that $k[x, y]/I$ is a module for $k[x, y]$ and show that

$$k[x, y]/I \otimes_{k[x,y]} k[x, y]/J \cong k.$$

You must prove that maps are well defined and give all details of proof.

(b) Let I be as in part (a). Prove that I is a free $k[x, y]$-module. Determine whether there exists a submodule M of $k[x, y]$ such that $k[x, y] \cong I \oplus M$.

3. (a) Suppose that $R = \mathbf{Z}_2[x]$. Prove that R is a principal ideal domain. You must show that if I is an ideal in R then it must be principal. You may not quote a theorem for your answer.

(b) Consider the module $A = R/(1 + x^2)(1 + x^3)$ where R is as in part (a). Find the invariant factors and elementary divisors of A.

2019 Algebra Qual - Part C

Solve any 4 out of the following 5 problems. Indicate which ones should be graded.

(1) Suppose F/K is a field extension of degree n and $f \in K[x]$ is a polynomial of degree d. Prove that if f is irreducible over K and $\gcd(n,d) = 1$, then f is irreducible over F.

(2) Suppose F/K is a normal algebraic extension, M/K is any algebraic extension, and $\sigma : F \to M$ is a K-homomorphism. Prove that if $\sigma' : F \to M$ is any K-homomorphism, then the image of σ' is equal to the image of σ.

(3) Suppose F/K is a separable finite extension and M/K is an algebraic closure of K containing F. Suppose $u \in F$ is an element such that $\sigma(u) = u$ for every K-homomorphism $\sigma : F \to M$. Prove that $u \in K$.

(4) List all the intermediate fields K of $\mathbf{F}_{2^{100}}/\mathbf{F}_{2^{10}}$ and indicate in a diagram their inclusion relations. List the corresponding Galois groups $\mathrm{Aut}_K(\mathbf{F}_{2^{100}})$.

(5) Let $F = K(x_1, x_2)$ where K is a field and x_1, x_2 are indeterminates. Explain whether $\{x_1^2, x_2^2\}$ is a transcendence basis of F/K.