Connections between the sounds that drums make and shapes of least perimeter

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Queen Dido of Carthage ∼815 BC?

- Dido is a ruler traveling through (modern-day) Tunisia with her subjects.
- Asks the Berber king if she can rent land.
- The king responds that she may rent the amount of land she can cover with an oxhide.
- Cuts the oxhide into thin strips and sews them together to create a very long and thin length of oxhide.
- Surrounds a large area of land with the length, founding Carthage on the interior.

From Virgil's *Aeneid*.

Queen Dido of Carthage ~815 BC?

Dido Purchases Land for the Foundation of Carthage. Engraving by Matthäus Merian the Elder, in *Historische Chronica*, Frankfurt a.M., 1630. Dido's people cut the hide of an ox into thin strips and try to enclose a maximal domain.

Natural Questions

Dido's idea to surround an enclosed shape with the oxhide leads to interesting mathematical questions:

1. What shape should Dido have enclosed so she could rent the largest area of land?
2. For a fixed area of land, what shape having that area has least perimeter?

For question 2, we can answer it mathematically by first showing a shape minimizing perimeter must exist, then applying a symmetrization argument.

Conclusion: Dido should create a circle with the oxhide!

What is the Laplace Differential Operator?

For a (twice differentiable) function $u(x, y)$, we have

$$\Delta u = \mathrm{div}\big(\mathrm{grad}(u)\big) = \nabla \cdot \nabla u$$

where div and grad can be explained physically.

- For (x, y), think of $\nabla u(x, y)$ as the vector in $\mathbf{R}^2$ which points in the direction of the greatest increase of u at (x, y) and having magnitude proportional to the increase. Over all of $\mathbf{R}^2$, ∇u defines a vector field (each point is associated to a vector).
- If $X(x, y)$ is a 2D vector field, then think of X as dictating the movement of particles. Then $\mathrm{div}(X)$ is the (pointwise) change of density of the particle. Particles moving away from a point (a source) are positive for divergence and particles moving into a point (a sink) are negative.

So Δu measures the change of density of particles moving along the greatest increase of u.

An Intuitive Example

Suppose we superimpose an aerial map of campus and the surrounding area onto the xy-plane. Here $u(x, y)$ will denote the altitude of each location in the plane.

Image file due to Wikipedia member Amerique.

Can you find locations in the picture where Δu appears to be positive and locations where Δu appears to be negative?

Δ considers CSUSB to be a positive place!

Examples of Models Using Δ

- Gravitational potential V from a continuous mass distribution ρ

$$\rho(\mathbf{x}) = \frac{1}{4\pi G}\Delta V(\mathbf{x})$$

(This is the application for which Laplace discovered it!)

- Function $u(\mathbf{x}, t)$ describing **heat** in space $\mathbf{x}$ and time t with $\alpha > 0$

$$\frac{\partial u}{\partial t} = \alpha \Delta_{\mathbf{x}} u(\mathbf{x}, t)$$

- Function $u(\mathbf{x}, t)$ describing mechanical displacement of **waves** in space $\mathbf{x}$ and time t with constant propagation speed c

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta_{\mathbf{x}} u(\mathbf{x}, t)$$

But that's not all....

Model a membrane or “drum” by Ω in the plane $\mathbf{R}^2$. The function $U : \Omega \to \mathbf{R}$ represents the (peak) amplitude and the spatial frequency is given by ω in the equation

$$-\frac{1}{2}\Delta U = \omega^2 U$$
$$U = 0 \text{ on the boundary of } \Omega.$$

Kac’s question was whether the values of ω for solutions U to the equation uniquely determine Ω.

In 1992, Gordon, Webb, and Wolpert gave two different planar domains having the same frequencies stating "one cannot hear the shape of a drum."

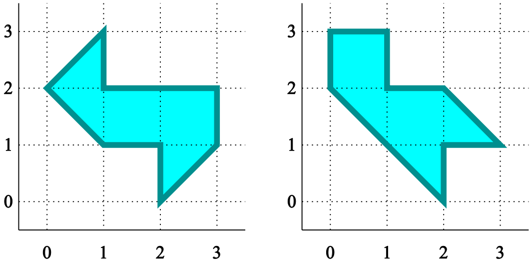


Image file due to Jitse Niesen.

In 2000, Zelditch gave really, really restrictive conditions where the answer is yes! (Drum must be convex with additional symmetry and its boundary must be real-analytic.) The problem is still open for smooth boundary!

Additional Questions

How do we make an instrument which produces really low tones?

When engineering drums out of a "fixed" material on Earth, can we build them to produce arbitrarily low tones?

Eigenvalues and Eigenfunctions of Δ

To simplify our discussion, we will focus on the related **eigenvalue equation**,

$$-\Delta u = \lambda u \tag{1}$$

$$u = 0 \text{ on the boundary of } \Omega. \tag{2}$$

We call u and $\lambda (= 2\omega^2)$ which satisfy Equations 1 and 2 **eigenfunctions** and **eigenvalues** respectively.

We number the eigenvalues in order of size with multiplicity

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

The Faber-Krahn Inequality

If Ω is a region (or drum$\cdots$or 2D shape), we will denote $|\Omega|$ as the **area** of Ω.

There is an explicit constant C, so that

$$\lambda_1(\Omega) \geq \frac{C}{|\Omega|}.$$

Further, equality holds when Ω is a (metric) ball, **the same shape that minimizes ratio of perimeter to enclosed area!**

Bound conjectured in $\sim$1894 by Lord Rayleigh. Planar (2D) case proved in 1923 by Faber and an analog was proved for higher dimensions by Krahn in 1925.

A similar inequality is true for the diameter of Ω.

For $\lambda_1(\Omega) \to 0$, we must have that $|\Omega| \to \infty$.

What are Riemannian Manifolds?

- A Riemannian 2-manifold, sometimes called a surface, can be thought of as small portions of $\mathbf{R}^2$ cut out and "molded" together to create a larger object.
- The term Riemannian indicates that we still have measurements such as lengths, angles, and volumes, just as in $\mathbf{R}^2$.
- For simplicity, we will only consider (complete) Riemannian 2-manifolds which are **closed**, meaning that they don't have a boundary and that they do not "go off to infinity."

For a closed (compact and without boundary) manifold M we can consider the same eigenvalue problem (with no boundary conditions):

$$-\Delta u = \lambda u$$

for $u : M \to \mathbf{R}$.

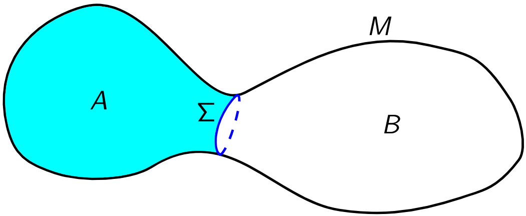
Index the eigenvalues by

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots .$$

Cheeger's Observation

- There are manifolds with arbitrarily small λ_1 whose volume and diameter do not got to infinity (or zero). They have small "bottlenecks".
- **Cheeger's constant** considers the ways of cutting the manifold into two pieces A and B along a collection of curves Σ. Specifically, we have

$$h(M) = \min_{\Sigma} \frac{|\Sigma|}{\min\{|A|, |B|\}}.$$



Geometric Control Over λ_1

Recall that we use M to denote a 2-dimensional Riemannian manifold:

- Cheeger '71 proved that

$$\lambda_1(M) \geq \frac{h(M)^2}{4}.$$

- Buser '82 proved that when M has Ricci curvature bounded from below by $-\delta^2$ with $\delta \geq 0$, then

$$\lambda_1(M) \leq 10h^2(M) + 2\delta h(M).$$

- Cheeger + Buser:

$$h(M)^2/4 \leq \lambda_1(M) \leq 10h^2(M) + 2\delta h(M).$$

- B. '15: An analog of Buser for the higher eigenvalues $\lambda_k(M)$, giving a bound from above which depends only on $h(M), \delta$.

Thank You!

Thank you very much for the invitation and for your interest and attention.