SIAM J. MATH. ANAL.
Vol. 38, No. 1, pp. 210–232

NAVIER–STOKES EQUATIONS WITH NAVIER BOUNDARY CONDITIONS FOR A BOUNDED DOMAIN IN THE PLANE*

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Abstract. We consider solutions to the Navier–Stokes equations with Navier boundary conditions in a bounded domain Ω in $\mathbb{R}^2$ with a C^2-boundary Γ. Navier boundary conditions can be expressed in the form $\omega(v) = (2\kappa - \alpha)v \cdot \boldsymbol{\tau}$ and $v \cdot \mathbf{n} = 0$ on Γ, where v is the velocity, $\omega(v)$ the vorticity, $\mathbf{n}$ a unit normal vector, $\boldsymbol{\tau}$ a unit tangent vector, and α is in $L^\infty(\Gamma)$. These boundary conditions were studied in the special case where $\alpha = 2\kappa$ by J.-L. Lions and P.-L. Lions. We establish the existence, uniqueness, and regularity of such solutions, extending the work of Clopeau, Mikelić, and Robert and of Lopes Filho, Nussenzveig Lopes, and Planas, which was restricted to simply connected domains and nonnegative α.

Assuming a particular bound on the growth of the L^p-norms of the initial vorticity with p (*Yudovich vorticity*), and also assuming additional smoothness on Γ and α, we obtain a uniform-in-time bound on the rate of convergence in $L^2(\Omega)$ of solutions to the Navier–Stokes equations with Navier boundary conditions to the solution to the Euler equations in the vanishing viscosity limit. We also show that for smoother initial velocities, the solutions to the Navier–Stokes equations with Navier boundary conditions converge uniformly in time in $L^2(\Omega)$, and L^2 in time in $\dot{H}^1(\Omega)$, to the solution to the Navier–Stokes equations with the usual no-slip boundary conditions as we let α grow large uniformly on the boundary.

Key words. Navier–Stokes equations, vanishing viscosity limit

AMS subject classifications. 76D05, 76B99

DOI. 10.1137/040612336

1. Introduction.

Let Ω be a bounded domain in $\mathbb{R}^2$ with a C^2-boundary Γ consisting of a finite number of connected components, and let $\mathbf{n}$ and $\boldsymbol{\tau}$ be unit normal and tangent vectors, respectively, to Γ. We follow the convention that $\mathbf{n}$ is an outward normal vector and that the ordered pair $(\mathbf{n}, \boldsymbol{\tau})$ gives the standard orientation to $\mathbb{R}^2$. Define the rate-of-strain tensor,

$$D(v) = \frac{1}{2}\left[\nabla v + (\nabla v)^T\right].$$

We consider the existence, uniqueness, and regularity of solutions to the Navier–Stokes equations with *Navier boundary conditions*. These boundary conditions, introduced by Navier in [19] and derived by Maxwell in [18] from the kinetic theory of gases (see [12]), assume that the tangential "slip" velocity, rather than being zero, is proportional to the tangential stress. With a factor of proportionality a in $L^\infty(\Gamma)$, we can express Navier boundary conditions for a sufficiently regular vector field v as

$$v \cdot \mathbf{n} = 0 \text{ and } 2\nu(\mathbf{n} \cdot D(v)) \cdot \boldsymbol{\tau} + av \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma. \tag{1.1}$$

We will find it more convenient, however, to let $\alpha = a/\nu$, and write these boundary conditions in the form

$$v \cdot \mathbf{n} = 0 \text{ and } 2(\mathbf{n} \cdot D(v)) \cdot \boldsymbol{\tau} + \alpha v \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma. \tag{1.2}$$

*Received by the editors July 27, 2004; accepted for publication (in revised form) December 8, 2005; published electronically April 12, 2006. The author was supported in part by a Hubert S. Wall Memorial Fellowship in Mathematics at the University of Texas at Austin during the course of this work.

http://www.siam.org/journals/sima/38-1/61233.html

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(We give an equivalent form of Navier boundary conditions in Corollary 4.2.)

The reason for preferring the second form for the boundary conditions is that, in the vanishing viscosity limit, we will hold α fixed as we let ν approach zero, and we will show that the solution to the Navier–Stokes equations with Navier boundary conditions converges to a solution to the Euler equations. (See, however, the comment at the end of section 8.)

J.-L. Lions in [15, pp. 87–98] and P.-L. Lions in [16, pp. 129–131] consider the following boundary conditions, which we call *Lions* boundary conditions:

$$v \cdot \mathbf{n} = 0 \text{ and } \omega(v) = 0 \text{ on } \Gamma,$$

where $\omega(v) = \partial_1 v^2 - \partial_2 v^1$ is the vorticity of v. Lions boundary conditions are the special case of Navier boundary conditions in which $\alpha = 2\kappa$, as we show in Corollary 4.3.

J.-L. Lions, in Theorem 6.10 on page 88 of [15], proves existence and uniqueness of a solution to the Navier–Stokes equations in the special case of Lions boundary conditions but includes the assumption that the initial vorticity is bounded. With the same assumption of bounded initial vorticity, the existence and uniqueness are established in Theorem 4.1 of [5] for Navier boundary conditions, under the restriction that α is nonnegative (and in $C^2(\Gamma)$). This is the usual restriction, which is imposed to ensure the conservation of energy. Mathematically, negative values of α present no real difficulty, so we do not make that restriction (except in section 9). The only clear gain from removing the restriction, however, is that it allows us to view Lions boundary conditions as a special case of Navier boundary conditions for more than just convex domains (nonnegative curvature).

P.-L. Lions establishes an energy inequality on page 130 of [16] that can be used in place of the usual one for no-slip boundary conditions. He argues that existence and uniqueness can then be established for initial velocity in $L^2(\Omega)$—and no additional assumption on the initial vorticity—exactly as was done for no-slip boundary conditions in the earlier sections of his text. As we will show, P.-L. Lions's energy inequality applies to Navier boundary conditions in general, which gives us the same existence and uniqueness theorem as for no-slip boundary conditions. (P.-L. Lions's comment on the regularity of $\partial_t u$ does not follow as in [16], though, because (4.18) of [16] is not valid for general Navier boundary conditions.) We include a proof of existence and uniqueness in section 6 that closely parallels the classical proofs due to Leray as they appear in [15] and [22]. In section 7, we extend the existence, uniqueness, regularity, and convergence results of [5] and [17] to multiply connected domains.

It is shown in [17] that if the initial vorticity is in $L^p(\Omega)$ for some $p > 2$, then after extracting a subsequence, solutions to the Navier–Stokes equations with Navier boundary conditions converge in $L^\infty([0,T]; L^2(\Omega))$ to a solution to the Euler equations (with the usual boundary condition of tangential velocity on the boundary) as $\nu \to 0$. This extends a result in [5] for initial vorticity in $L^\infty(\Omega)$, and because the solution to the Euler equations is unique in this case, it follows that the convergence is strong in $L^\infty([0,T]; L^2(\Omega))$—that is, does not require the extraction of a subsequence.

The convergence in [17] also generalizes the similar convergence established for the special case of Lions boundary conditions on page 131 of [16] (though not including the case $p = 2$). The main difficulty faced in making this generalization is establishing a bound on the L^p-norms of the vorticity, a task that is much easier for Lions boundary conditions (see pages 91–92 of [15] or page 131 of [16]). In contrast, nearly all of [5] and [17], including the structure of the existence proofs, is directed toward establishing an analogous bound.

The methods of proof in [5] and [17] do not yield a bound on the rate of convergence. With the assumptions in [17], such a bound is probably not possible. We can, however, make an assumption that is weaker than that of [5] but stronger than that of [17] and achieve a bound on the rate of convergence. Specifically, we assume, as in [14], that the L^p-norms of the initial vorticity grow sufficiently slowly with p (Definition 8.2) and establish the bound given in Theorem 8.4. To achieve this result, we also assume additional regularity on α and Γ.

The bound on the convergence rate in $L^\infty([0,T]; L^2(\Omega))$ in Theorem 8.4 is the same as that obtained for $\Omega = \mathbb{R}^2$ in [14]. In particular, when α is nonnegative, it gives a bound on the rate of convergence for initial vorticity in $L^\infty(\Omega)$ that is proportional to

$$(\nu t)^{\frac{1}{2}\exp\left(-C\|\omega^0\|_{L^2\cap L^\infty} t\right)},$$

where C is a constant depending on Ω and α, and ω^0 is the initial vorticity. This is essentially the same bound on the convergence rate as that for $\Omega = \mathbb{R}^2$ appearing in [3].

Another interesting question is whether solutions to the Navier–Stokes equations with Navier boundary conditions converge to a solution to the Navier–Stokes equations with the usual no-slip boundary conditions if we let the function α grow large. We show in section 9 that such convergence does take place for initial velocity in $H^3(\Omega)$ and Γ in C^3 when we let α approach $+\infty$ uniformly on Γ. This type of convergence is, in a sense, an inverse of the derivation of the Navier boundary conditions from no-slip boundary conditions for rough boundaries discussed in [10] and [11].

In [13], Kato gives necessary and sufficient conditions for the vanishing viscosity limit of solutions to the Navier–Stokes equations with no-slip boundary conditions to converge to a strong solution to the Euler equations. In particular, he shows that the vanishing viscosity limit holds if and only if the L^2-norm of the gradient of the velocity in a boundary layer of width proportional to the viscosity vanishes sufficiently rapidly as the viscosity goes to zero. For Navier boundary conditions, it is easy to show that this norm on the boundary layer converges sufficiently rapidly, and because we have established the vanishing viscosity limit, it follows that Kato's conditions all hold, thus completing, in a sense, Kato's program for Navier boundary conditions. We describe this in more detail in section 10.

We follow the convention that C is always an unspecified constant that may vary from expression to expression, even across an inequality (but not across an equality). When we wish to emphasize that a constant depends, at least in part, upon the parameters $x_1, \dots, x_n$, we write $C(x_1, \dots, x_n)$. To distinguish between unspecified constants, we use C and C'.

For vectors u and v in $\mathbb{R}^2$, by $u \cdot \nabla v$ we mean the vector whose jth component is $u^i \partial_i v^j$. For 2×2 matrices A and B we define $A \cdot B = A^{ij} B^{ij}$, so $\nabla u \cdot \nabla v = \partial_j u^i \partial_j v^i$. Here, as everywhere in this paper, we follow the common convention that repeated indices are summed—whether or not one is a superscript and one is a subscript.

For the vector u and the scalar function ψ we define

$$u^\perp = (-u^2, u^1), \qquad \nabla^\perp \psi = (-\partial_2 \psi, \partial_1 \psi), \qquad \omega(u) = \partial_1 u^2 - \partial_2 u^1.$$

If X is a function space and k a positive integer, we define $(X)^k$ to be

$$\{(f_1, \dots, f_k) : f_1 \in X, \dots, f_k \in X\}.$$

For instance, $(H^1(\Omega))^2$ is the set of all vector fields, each of whose components lies in $H^1(\Omega)$. To avoid excess notation, however, we always suppress the superscript k when it is clear from the context whether we are dealing with scalar-, vector-, or matrix-valued functions.

We will make use of the following generalization of Gronwall's lemma. The succinct form of the proof is due to Tehranchi [21].

LEMMA 1.1 (Osgood's lemma). *Let L be a measurable nonnegative function and γ a nonnegative locally integrable function, each defined on the domain $[t_0, t_1]$. Let $\mu\colon [0,\infty) \to [0,\infty)$ be a continuous nondecreasing function, with $\mu(0) = 0$. Let $a \geq 0$, and assume that for all t in $[t_0, t_1]$,*

$$L(t) \leq a + \int_{t_0}^{t} \gamma(s)\mu(L(s))\, ds. \tag{1.3}$$

If $a > 0$, then

$$\int_a^{L(t)} \frac{ds}{\mu(s)} \leq \int_{t_0}^{t} \gamma(s)\, ds.$$

If $a = 0$ and $\int_0^\infty ds/\mu(s) = \infty$, then $L \equiv 0$.

Proof. We have

$$\begin{aligned}\int_a^{L(t)} \frac{dx}{\mu(x)} &\leq \int_a^{a+\int_{t_0}^{t}\gamma(u)\mu(L(u))\,du} \frac{dx}{\mu(x)}\\ &\leq \int_{t_0}^{t} \frac{\gamma(s)\mu(L(s))\,ds}{\mu(a+\int_{t_0}^{s}\gamma(u)\mu(L(u))\,du)} \leq \int_{t_0}^{t}\gamma(s)\,ds.\end{aligned}$$

The last inequality follows from (1.3), since μ is nondecreasing. ∎

We have stated Lemma 1.1 in the form that it appears on page 92 of [4]. This lemma is equivalent to a theorem of Bihari [2], though with an assumption only of measurability of μ rather than continuity; see, for example, Theorem 5.1 on pages 40–41 of [1].[1] An early form of the inequality appears in the work of Osgood [20], who assumes that $a = 0$, $\gamma \equiv 1$, and the bound is on $|L(t)|$ in (1.3); because of this, Lemma 1.1 is often referred to as Osgood's lemma. See also the historical discussion in section 2.14 of [6].

2. Function spaces. Let

$$E(\Omega) = \left\{ v \in (L^2(\Omega))^2 : \operatorname{div} v \in L^2(\Omega) \right\}, \tag{2.1}$$

as in [22], with the inner product

$$(u, v)_{E(\Omega)} = (u, v) + (\operatorname{div} u, \operatorname{div} v).$$

We will use the following theorem, which is Theorem 1.2 on page 7 of [22], several times.

LEMMA 2.1. *There exists a continuous linear operator $\gamma_{\mathbf{n}}$ mapping $E(\Omega)$ into $H^{-1/2}(\Gamma)$ such that*

$$\gamma_{\mathbf{n}} v = \text{ the restriction of } v \cdot \mathbf{n} \text{ to } \Gamma \text{ for every } v \text{ in } (\mathcal{D}(\overline{\Omega}))^2.$$

[1] The inequality in equation (5.2) of Theorem 5.1 of [1] should be $\leq$ instead of $\geq$.

Also, the following form of the divergence theorem holds for all vector fields v in $E(\Omega)$ and scalar functions h in $H^1(\Omega)$:

$$\int_\Omega v \cdot \nabla h + \int_\Omega (\operatorname{div} v) h = \int_\Gamma \gamma_{\mathbf{n}} v \cdot \gamma_0 h.$$

We always suppress the trace function γ_0 in our expressions, and we write $v \cdot \mathbf{n}$ in place of $\gamma_{\mathbf{n}} v$.

Define the following function spaces as in [5]:

$$\begin{aligned} H &= \left\{ v \in (L^2(\Omega))^2 : \operatorname{div} v = 0 \text{ in } \Omega \text{ and } v \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\}, \\ V &= \left\{ v \in (H^1(\Omega))^2 : \operatorname{div} v = 0 \text{ in } \Omega \text{ and } v \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\}, \\ \mathcal{W} &= \left\{ v \in V \cap H^2(\Omega) : v \text{ satisfies } (1.2) \right\}. \end{aligned} \tag{2.2}$$

We give $\mathcal{W}$ the H^2-norm, H the L^2-inner product and norm, which we symbolize by $(\cdot, \cdot)$ and $\|\cdot\|_{L^2(\Omega)}$, and V the H^1-inner product,

$$(u, v)_V = \sum_i (\partial_i u, \partial_i v),$$

and associated norm. This norm is equivalent to the H^1-norm, because Poincaré's inequality,

$$\|v\|_{L^p(\Omega)} \le C(\Omega, p) \|\nabla v\|_{L^p(\Omega)} \tag{2.3}$$

for all p in $[1, \infty]$, holds for all v in V.

Ladyzhenskaya's inequality,

$$\|v\|_{L^4(\Omega)} \le C(\Omega) \|v\|_{L^2(\Omega)}^{1/2} \|\nabla v\|_{L^2(\Omega)}^{1/2}, \tag{2.4}$$

also holds for all v in V, though the constant in the inequality is domain dependent, unlike the constant for the classical version of the space V.

We will also frequently use the following inequality, which follows from the standard trace theorem, Sobolev interpolation, and Poincaré's inequality:

$$\|v\|_{L^2(\Gamma)} \le C(\Omega) \|v\|_{L^2(\Omega)}^{1/2} \|\nabla v\|_{L^2(\Omega)}^{1/2} \le C(\Omega) \|v\|_V \tag{2.5}$$

for all v in V.

3. Hodge decomposition of H. Only simply connected domains are considered in [5] and [17]. To handle multiply connected domains we will need a portion of the Hodge decomposition of $L^2(\Omega)$. We briefly summarize the pertinent facts, drawing mostly from Appendix I of [22].

We assume that Ω is connected, for if it has multiple components we perform the decomposition separately on each component. Let $\Gamma_1, \dots, \Gamma_{N+1}$ be the components of the boundary Γ with Γ_{N+1} bounding the unbounded component of Ω^C. Let $\Sigma_1, \dots, \Sigma_N$ be one-manifolds with boundary that generate $H_1(\Omega, \Gamma; \mathbb{R})$, the one-dimensional real homology class of Ω relative to its boundary Γ.

We can decompose the space H into two subspaces, $H = H_0 \oplus H_c$, where

$$\begin{aligned} H_0 &= \{v \in H : \text{all internal fluxes are zero}\}, \\ H_c &= \{v \in H : \omega(v) = 0\}. \end{aligned}$$

An internal flux is a value of $\int_{\Sigma_i} v \cdot \mathbf{n}$. Then $H_0 = H_c^{\perp}$.

Define ψ_i, $i = 1, \dots, N$, to be the solution to $\Delta\psi_i = 0$ on Ω, $\psi_i = C_i$ on Γ_i, and $\psi_i = 0$ on all other components of Γ, where C_i is a nonzero constant. By elliptic regularity, ψ_i is in $H^2(\Omega)$ (apply, for instance, Theorem 8.12 on page 176 of [8]). Thus, $h_i := \nabla^{\perp}\psi_i$ is in $H^1(\Omega)$ and is divergence-free since $\operatorname{div}\nabla^{\perp} = 0$, and $h_i \cdot \mathbf{n} = 0$ since ψ_i is locally constant along Γ; that is, h_i is in V. The vectors $(h_1, \dots, h_N)$ form an orthogonal basis for $H_c \subseteq V$, which we can assume is orthonormal by choosing (C_i) appropriately.

If v is in V then v is also in H so there exist a unique u in H_0 and h in H_c such that $v = u + h$; also, $(u, h)_H = 0$. But h is in V; hence, u also lies in V. This shows that $V = (V \cap H_0) \oplus H_c$, though this is not an orthogonal decomposition of V.

Given v in H we construct an associated stream function ψ in $H^1(\Omega)$ as follows. Fix a point a on $\partial\Omega$. For any x in $\overline{\Omega}$, and let γ be a smooth curve in $\overline{\Omega}$ from a to x. Along the curve γ let $\boldsymbol{\tau}$ be a unit tangent vector in the direction of γ and $\mathbf{n}$ be the unit normal vector for which $(\mathbf{n}, \boldsymbol{\tau})$ gives the standard orientation to $\mathbb{R}^2$. Then one can show that the function ψ defined by

$$\psi(x) = -\int_{\gamma} v \cdot \mathbf{n}\, ds \tag{3.1}$$

is independent of the choice of γ and of the set of generators, and that $v = \nabla^{\perp}\psi$. (The salient fact is that $v \cdot \mathbf{n}$ integrates to zero along any generator of the first (nonrelative) homology because $v \cdot \mathbf{n} = 0$ along $\partial\Omega$.)

On the boundary component containing a, ψ is zero, because $v \cdot \mathbf{n} = 0$ on Γ. On the other boundary components, ψ is constant, because the internal fluxes are independent of the path. In the special case where v is in H_0, all the internal fluxes are zero, so ψ is zero on all of Γ. From the way that we defined the basis (h_k) for H_c, it is clear that the projection into H_c of a vector lying in H is uniquely determined by the value of its stream function on the boundary.

The following is due to Yudovich.

LEMMA 3.1. *For any p in $[2, \infty)$ and any v in H_0 with $\omega(v)$ in $L^p(\Omega)$,*

$$\|\nabla v\|_{L^p(\Omega)} \le C(\Omega) p \|\omega(v)\|_{L^p(\Omega)}.$$

Proof. Let v be in H_0 with $\omega(v)$ in $L^p(\Omega)$. Then, as noted above, the associated stream function ψ vanishes on Γ. Applying Corollary 1 of [24] with the operator $L = \Delta$ and $r = 0$ gives

$$\|\nabla v\|_{L^p(\Omega)} \le \|\psi\|_{H^{2,p}(\Omega)} \le C(\Omega) p \|\Delta\psi\|_{L^p(\Omega)} = C(\Omega) p \|\omega(v)\|_{L^p(\Omega)}. \qquad \square$$

For Ω simply connected, $H = H_0$, and Lemma 3.1 applies to all of H. The critical feature of Lemma 3.1 is that the dependence of the inequality on p is made explicit, a fact we will exploit in the proof of Theorem 8.4.

With the assumption of additional regularity on Γ, we have the following result for velocity fields in H.

COROLLARY 3.2. *Assume that Γ is $C^{2,\epsilon}$ for some $\epsilon > 0$. Then for any p in $[2, \infty)$ and any v in H with $\omega(v)$ in $L^p(\Omega)$,*

$$\|\nabla v\|_{L^p(\Omega)} \le C(\Omega) p \|\omega(v)\|_{L^p(\Omega)} + C'(\Omega) \|v\|_{L^2(\Omega)},$$

the constants $C(\Omega)$ and $C'(\Omega)$ being independent of p.

Proof. Because Γ is $C^{2,\epsilon}$, it follows from elliptic regularity theory that each ψ_i is in $C^{2,\epsilon}(\overline{\Omega})$ (apply, for instance, Theorem 6.14 on page 101 of [8]). Thus, each basis element $h_i = \nabla^\perp \psi_i$ for H_c is in $C^{1,\epsilon}(\overline{\Omega})$ and so ∇h_i is in $L^\infty(\Omega)$.

Let v be in H with $\omega(v)$ in $L^p(\Omega)$, and let $v = u + h$, where u is in H_0 and h is in H_c. Let $h = \sum_{i=1}^N c_i h_i$ and $r = \|h\|_{L^2(\Omega)} = (\sum_i c_i^2)^{1/2}$. Then

$$\begin{aligned}\|\nabla h\|_{L^p(\Omega)} &\le \sum_{i=1}^N |c_i| \, \|\nabla h_i\|_{L^p(\Omega)} \le \sum_{i=1}^N r|\Omega|^{1/p} \, \|\nabla h_i\|_{L^\infty(\Omega)} \\ &\le r \max\left\{1, |\Omega|^{1/2}\right\} \sum_{i=1}^N \|\nabla h_i\|_{L^\infty(\Omega)} \le C \, \|h\|_{L^2(\Omega)} .\end{aligned}$$

But $H_0 = H_c^\perp$, so $\|v\|_{L^2(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|h\|_{L^2(\Omega)}^2$, and thus $\|h\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)}$. Therefore,

$$\begin{aligned}\|\nabla v\|_{L^p(\Omega)} &\le \|\nabla u\|_{L^p(\Omega)} + \|\nabla h\|_{L^p(\Omega)} \\ &\le C(\Omega) p \, \|\omega(v)\|_{L^p(\Omega)} + C'(\Omega) \, \|v\|_{L^2(\Omega)}\end{aligned}$$

by virtue of Lemma 3.1. □

4. Vorticity on the boundary. Let κ be the curvature of Γ. Then κ is continuous because Γ is C^2, and if we parameterize each component of Γ by arc length, s, it follows that

$$\frac{\partial \mathbf{n}}{\partial \boldsymbol{\tau}} := \frac{d\mathbf{n}}{ds} = \kappa \boldsymbol{\tau}.$$

LEMMA 4.1. *If u and v are in $(H^2(\Omega))^2$ with $u \cdot \mathbf{n} = v \cdot \mathbf{n} = 0$ on Γ, then*

$$(v \cdot \nabla u) \cdot \mathbf{n} = -\kappa u \cdot v, \tag{4.1}$$

$$(\mathbf{n} \cdot \nabla v) \cdot \boldsymbol{\tau} = \omega(v) + (\boldsymbol{\tau} \cdot \nabla v) \cdot \mathbf{n} = \omega(v) - \kappa v \cdot \boldsymbol{\tau}, \tag{4.2}$$

$$(\mathbf{n} \cdot D(v)) \cdot \boldsymbol{\tau} = \frac{1}{2}\omega(v) - \kappa v \cdot \boldsymbol{\tau}. \tag{4.3}$$

Proof. Because $u \cdot \mathbf{n}$ has a constant value (of zero) along Γ,

$$0 = \frac{\partial}{\partial \boldsymbol{\tau}}(u \cdot \mathbf{n}) = \frac{\partial u}{\partial \boldsymbol{\tau}} \cdot \mathbf{n} + u \cdot \frac{\partial \mathbf{n}}{\partial \boldsymbol{\tau}} = (\boldsymbol{\tau} \cdot \nabla u) \cdot \mathbf{n} + \kappa u \cdot \boldsymbol{\tau},$$

so $(\boldsymbol{\tau} \cdot \nabla u) \cdot \mathbf{n} = -\kappa u \cdot \boldsymbol{\tau}$. But v is parallel to $\boldsymbol{\tau}$, so (4.1) follows by linearity. The identity in (4.3) is Lemma 2.1 of [5], and (4.2) is established similarly. □

COROLLARY 4.2. *A vector v in $V \cap H^2(\Omega)$ satisfies Navier boundary conditions (that is, lies in $\mathcal{W}$) if and only if*

$$\omega(v) = (2\kappa - \alpha) v \cdot \boldsymbol{\tau} \text{ and } v \cdot \mathbf{n} = 0 \text{ on } \Gamma. \tag{4.4}$$

Also, for all v in $\mathcal{W}$ and u in V,

$$(\mathbf{n} \cdot \nabla v) \cdot u = (\kappa - \alpha) v \cdot u \text{ on } \Gamma. \tag{4.5}$$

Proof. Let v be in $V \cap H^2(\Omega)$. Then from (4.3),

$$2(\mathbf{n} \cdot D(v)) \cdot \boldsymbol{\tau} + 2\kappa(v \cdot \boldsymbol{\tau}) = \omega(v). \tag{4.6}$$

If v satisfies Navier boundary conditions, then (4.4) follows by subtracting $2(\mathbf{n} \cdot D(v)) \cdot \boldsymbol{\tau} + \alpha v \cdot \boldsymbol{\tau} = 0$ from (4.6). Conversely, substituting the expression for $\omega(v)$ in (4.4) into (4.6) gives $2(\mathbf{n} \cdot D(v)) \cdot \boldsymbol{\tau} + \alpha v \cdot \boldsymbol{\tau} = 0$.

If v is in $\mathcal{W}$, then from (4.2),

$$(\mathbf{n} \cdot \nabla v) \cdot \boldsymbol{\tau} = \omega(v) - \kappa v \cdot \boldsymbol{\tau} = (2\kappa - \alpha) v \cdot \boldsymbol{\tau} - \kappa v \cdot \boldsymbol{\tau} = (\kappa - \alpha) v \cdot \boldsymbol{\tau},$$

and (4.5) follows from this, since u is parallel to τ on Γ. □

COROLLARY 4.3. *For initial velocity in $H^2(\Omega)$, Lions boundary conditions are the special case of Navier boundary conditions in which*

$$\alpha = 2\kappa.$$

That is, any solution of (NS) with Navier boundary conditions where $\alpha = 2\kappa$ is also a solution to (NS) with Lions boundary conditions.

5. Weak formulations. We give two equivalent formulations of a weak solution to the Navier–Stokes equations with Navier boundary conditions, in analogy with Problems 3.1 and 3.2 on pages 190–191 of [22].

For all u in $\mathcal{W}$ and v in V,

$$\begin{aligned} \int_\Omega \Delta u \cdot v &= \int_\Omega (\operatorname{div} \nabla u^i) v^i = \int_\Gamma (\nabla u^i \cdot \mathbf{n}) v^i - \int_\Omega \nabla u^i \cdot \nabla v^i \\ &= \int_\Gamma (\mathbf{n} \cdot \nabla u) \cdot v - \int_\Omega \nabla u \cdot \nabla v = \int_\Gamma (\kappa - \alpha) u \cdot v - \int_\Omega \nabla u \cdot \nabla v, \end{aligned} \tag{5.1}$$

where we used (4.5) of Corollary 4.2. This motivates our first formulation of a weak solution.

DEFINITION 5.1. *Given viscosity $\nu > 0$ and initial velocity u^0 in H, u in $L^2([0,T]; V)$ is a weak solution to the Navier–Stokes equations (without forcing) if $u(0) = u^0$ and*

$$\frac{d}{dt} \int_\Omega u \cdot v + \int_\Omega (u \cdot \nabla u) \cdot v + \nu \int_\Omega \nabla u \cdot \nabla v - \nu \int_\Gamma (\kappa - \alpha) u \cdot v = 0 \tag{NS}$$

for all v in V. (We make sense of the initial condition $u(0) = u^0$ as in [22].)

This formulation of a weak solution is equivalent to that in (2.11) and (2.12) of [5]. This follows from the identity

$$2 \int_\Omega D(u) \cdot D(v) = \int_\Omega \nabla u \cdot \nabla v - \int_\Gamma \kappa u \cdot v,$$

which holds for all u and v in V. To establish this identity, let u and v be in $V \cap H^2(\Omega)$ and observe that $2D(u) \cdot D(v) = \nabla u \cdot \nabla v + \nabla u \cdot (\nabla v)^T$. Also, because u is divergence-free, $\nabla u \cdot (\nabla v)^T = \partial_i u^j \partial_j v^i = \partial_j(\partial_i u^j v^i) = \operatorname{div}(v \cdot \nabla u)$. Then, using (4.1), the identity follows from

$$\int_\Omega \nabla u \cdot (\nabla v)^T = \int_\Omega \operatorname{div}(v \cdot \nabla u) = \int_\Gamma (v \cdot \nabla u) \cdot \mathbf{n} = -\int_\Gamma \kappa u \cdot v$$

and the density of $H^2(\Omega) \cap V$ in V.

Our second formulation of a weak solution will be identical to that of Problem 3.2 on page 191 of [22], except that the operator A of [22] will also include the boundary integral of (5.1). Accordingly, we define the operators A and B by

$$(Au, v)_{V,V'} = \int_\Omega \nabla u \cdot \nabla v - \int_\Gamma (\kappa - \alpha) u \cdot v,$$
$$(Bu, v)_{V,V'} = \int_\Omega (u \cdot \nabla u) \cdot v$$

for all u and v in V.

By (2.5),

$$|(Au, v)_{V,V'}| \leq \|u\|_V \|v\|_V + C \|u\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} \leq C \|u\|_V \|v\|_V . \tag{5.2}$$

Thus, $A : L^2([0,T]; V) \to L^2([0,T]; V')$, and, as it does for the classical version of the space V (for which vectors are zero on Γ), $B : L^2([0,T]; V) \to L^1([0,T]; V')$. Thus, if u is a solution as in Definition 5.1, then $-\nu Au - Bu$ lies in $L^1([0,T]; V')$ and

$$\frac{d}{dt} \langle u, v \rangle = (-\nu Au - Bu, v)_{V,V'}$$

for all v in V. It follows from Lemma 1.1 on page 169 of [22] that u is in $C([0,T]; H)$. This not only makes sense of the initial condition $u(0) = u^0$ but also shows that the following formulation of a weak solution is equivalent to that of Definition 5.1.

Definition 5.2. *Given viscosity $\nu > 0$ and initial velocity u^0 in H, u in $L^2([0,T]; V)$ is a weak solution to the Navier–Stokes equations if $u(0) = u^0$ and*

$$\begin{cases} u' \in L^1([0,T]; V'), \\ u' + \nu Au + Bu = 0 \text{ on } (0,T), \\ u(0) = u^0, \end{cases}$$

where $u' := \partial_t u$.

From here on we will refer to either of the formulations in Definitions 5.1 and 5.2 as (NS).

6. Existence and uniqueness. We can obtain existence and uniqueness of a solution to (NS) assuming only that the initial velocity is in H.

Theorem 6.1. *Assume that Γ is C^2 and α is in $L^\infty(\Gamma)$. Let u^0 be in H and let $T > 0$. Then there exists a solution u to (NS). Moreover, u is in $L^2([0,T]; V) \cap C([0,T]; H)$, u' is in $L^2([0,T]; V')$, and we have the energy inequality*

$$\|u(t)\|_{L^2(\Omega)} \leq e^{C(\alpha)\nu t} \|u^0\|_{L^2(\Omega)}. \tag{6.1}$$

The constant $C(\alpha)$ is zero if α is nonnegative on Γ.

Sketch of proof. Existence of a solution to (NS) proceeds as in the first proof of existence in [15, pp. 75–77], though using the analogue of the energy inequality on page 130 of [16]. Using (4.5), we have, formally,

$$\frac{1}{2} \frac{d}{dt} \|u\|^2_{L^2(\Omega)} + \nu \|\nabla u\|^2_{L^2(\Omega)} = \nu \int_\Gamma (\mathbf{n} \cdot \nabla u) \cdot u \leq C\nu \|u\|^2_{L^2(\Gamma)} , \tag{6.2}$$

Typo fixed from published version: integral over domain should have been integral over the boundary.

where $C = \sup_\Gamma|\kappa - \alpha|$. Arguing exactly as in [16], it follows that

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \nu\|\nabla u\|^2_{L^2(\Omega)} \le C\nu\|u\|^2_{L^2(\Omega)}.$$

Integrating over time gives

$$\begin{aligned}\|u(t)\|^2_{L^2(\Omega)} &+ \nu\int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)}\,ds \\ &\le \|u^0\|^2_{L^2(\Omega)} + C\nu\int_0^t \|u(s)\|^2_{L^2(\Omega)}\,ds.\end{aligned} \tag{6.3}$$

The energy bound, (6.1), then follows from Gronwall's lemma. (If α is nonnegative, then, in fact, energy is nonincreasing—in the absence of forcing—so $C(\alpha) = 0$. This follows from the equation preceding (2.16) of [5].)

The proofs of regularity in time and space and of uniqueness proceed exactly as in the proof of Theorem 3.2 on page 199 of [22], though in the proof of uniqueness we must account for the presence of the boundary integral in (NS). ∎

7. Additional regularity. If we assume extra regularity on the initial velocity, that regularity will be maintained for all time. This is crucial for establishing the vanishing viscosity limit in section 8, where we must impose stronger regularity on the initial velocity to obtain existence of a solution to the Euler equations.

THEOREM 7.1. *Assume that Γ is $C^{2,1/2+\epsilon}$ and that α is in $H^{1/2+\epsilon}(\Gamma)+C^{1/2+\epsilon}(\Gamma)$ for some $\epsilon > 0$. Let u^0 be in $\mathcal{W}$ with initial vorticity ω^0, and let u be the unique solution to (NS) given by Theorem 6.1 with corresponding vorticity ω. Let $T > 0$. Then*

$$u' \in L^2([0,T];V) \cap C([0,T];H).$$

If, in addition, ω^0 is in $L^\infty(\Omega)$ (so u^0 is compatible), then

$$u \in C([0,T];H^2(\Omega)),\ \omega \in C([0,T];H^1(\Omega)) \cap L^\infty([0,T]\times\Omega).$$

Proof. Regularity of u'. We prove that u' lies in $L^2([0,T];V) \cap L^\infty([0,T];H)$ in three steps as in the proof of Theorem 3.5 on pages 202–204 of [22]. In this proof, Temam uses a Galerkin approximation sequence (u_m) to the solution u. We employ the same sequence, though using the basis of Corollary A.3 rather than that of [22]; this is the only change to step (i).

No change to step (ii) of Temam's proof is required, because the bound on $\|u_m'(0)\|^2_{L^2}$ in (3.88) of [22], which does not involve boundary integrals, still holds.

In step (iii), an additional term of

$$\nu\int_\Gamma(\kappa-\alpha)|u_m'|^2$$

appears on the right side of (3.94) of Temam's proof, which we bound by

$$C\nu\|u_m'\|_{L^2(\Omega)}\|\nabla u_m'\|_{L^2(\Omega)} \le \frac{\nu}{2}\|\nabla u_m'\|^2_{L^2(\Omega)} + C\nu\|u_m'\|^2_{L^2(\Omega)}.$$

Then (3.95) of Temam's proof becomes

$$\frac{d}{dt}\|u_m'(t)\|^2_{L^2(\Omega)} \le \phi_m(t)\|u_m'(t)\|^2_{L^2(\Omega)},$$

where

$$\phi_m(t) = \left(\frac{2}{\nu} + C\nu\right) \|u_m(t)\|^2_{L^2(\Omega)},$$

and the proof that u' lies in $L^2([0,T];V) \cap L^\infty([0,T];H)$ is completed as in [22].

Regularity of u and ω. To establish the regularity of u and ω we follow the bootstrap argument in the second half of the proof of Theorem 2.3 in [5]. For completeness, we give a full account of this argument here, adapting it to multiply connected domains.

Because u' is in $L^2([0,T];V)$ we can argue as in the paragraph preceding Definition 5.2, with u' playing the role of u, that u' is in $C([0,T];H)$. The membership of u' in $L^2([0,T];V)$ also gives u in $H^1([0,T];V)$; by one-dimensional Sobolev embedding, u is then in $C^{1/2}([0,T];V)$ and hence in $C([0,T];V)$. It follows that $u \cdot \nabla u$ is in $C([0,T];L^q(\Omega))$ for all q in $[1,2)$ (see, for instance, Theorem 1.4.4.2 on page 28 of [9]).

Now let $\beta > 0$ and let $\Phi := -u \cdot \nabla u - u' + \beta u$. Then Φ is in $C([0,T];L^q(\Omega))$ for all q in $[1,2)$ by our observations above. Because of the additional regularity we have imposed on Γ and on α over that assumed in Theorem 6.1, $g := (2\kappa - \alpha)u \cdot \boldsymbol{\tau}$ lies in $C([0,T];H^{1/2}(\Gamma))$ (see, for instance, Theorem 1.4.1.1 on page 21 and Theorem 1.4.4.2 on page 28 of [9]).

Let $a : V \times V \to \mathbb{R}$ be defined by $a(u,v) = (Au + \beta u, v)_{V,V'}$ and require that $\beta > 0$ be sufficiently large that $a(v,v) \ge \|v\|_V^2$ for all v in V. This is possible because α is in $L^\infty(\Gamma) \subseteq H^{1/2+\epsilon}(\Gamma) + C^{1/2+\epsilon}(\Gamma)$ by one-dimensional Sobolev embedding. Also by (5.2) we see that $|a(u,v)| \le C \|u\|_V \|v\|_V$. Applying the Lax–Millgram lemma, we find that there exists a unique w in V such that $a(w,v) = \Phi$ for all v in V. By Definition 5.2, $w = u(t)$ for all t in $[0,T]$. By Definition 5.1, $u(t)$ is also the unique variational solution at time t in $[0,T]$ to Stokes's problem,

$$\begin{cases} -\nu\Delta u + \nabla p + \beta u = \Phi & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ \omega(u) = g & \text{on } \Gamma. \end{cases}$$

Formally, the vorticity formulation of the above system is

$$\begin{cases} -\nu\Delta \omega + \beta \omega = \omega(\Phi) & \text{in } \Omega, \\ \omega = g & \text{on } \Gamma. \end{cases}$$

Because $\omega(\Phi)$ is in $C([0,T];H^{-1,q}(\Omega))$, standard elliptic theory gives a unique solution ω in $C([0,T];H^{1,q}(\Omega))$. Because u is in $C([0,T];H)$, there exists an associated stream function φ in $C([0,T] \times \Omega)$ that is constant on each component of Γ at time t in $[0,T]$; this follows directly from (3.1). Letting ψ be the unique solution to

$$\begin{cases} \Delta \psi = \omega & \text{in } \Omega, \\ \psi = \varphi & \text{on } \Gamma, \end{cases}$$

it follows that $u = \nabla^\perp \psi$, because u and $\nabla^\perp \psi$ have the same vorticity ($\operatorname{curl} \nabla^\perp = \Delta$) and their stream functions (namely, φ and ψ) share the same value on Γ.

By Theorem 2.5.1.1 on page 128 of [9], ψ is in $C([0,T];H^{3,q}(\Omega))$; u, then, is in $C([0,T];H^{2,q}(\Omega))$ and hence in $C([0,T] \times \Omega)$ by Sobolev embedding. But u is in $C([0,T];V)$, so $u \cdot \nabla u$ and also Φ are in $C([0,T];L^2(\Omega))$. Passing through the same argument again, this time with $q = 2$, gives u in $C([0,T];(H^2(\Omega))^2)$. □

With Theorem 7.1, we have a replacement for Theorem 2.3 of [5] that applies regardless of the sign of α. Since the nonnegativity of α is used nowhere else in [5] and [17], all the results of both of those papers apply to simply connected domains as well regardless of the sign of α—with the regularity we have assumed on Γ and α.

To remove the restriction on the domain being simply connected, it remains only to show that Lemmas 2 and 3 of [17] remain valid for multiply connected domains. We show this for Lemma 2 of [17] in Theorem A.2. Because, however, for multiply connected domains there is no longer a unique vector field in $\mathcal{W}$ with a given vorticity, we must define a *vector field* to be compatible, rather than its *vorticity*, as was done in [17].

Definition 7.2. *A vector field v in $\mathcal{W}$ is called compatible if $\omega(v)$ is in $L^\infty(\Omega)$.*

As for Lemma 3 of [17], we need only use Corollary 3.2 to replace the term $\|\omega(\cdot,t)\|_{L^p(\Omega)}^{1-\theta}$ with $(\|\omega(\cdot,t)\|_{L^p(\Omega)} + \|u(\cdot,t)\|_{L^2(\Omega)})^{1-\theta}$ in the proof of Lemma 3 in [17]. Lemma 3 of [17] then follows with no other changes in the proof—only the value of the constant C changes.

We thus have the following theorem, which is only a slight modification of Proposition 1 of [17], and which applies to multiply connected domains and unsigned α.

Theorem 7.3. *Assume that Γ and α are as in Theorem 7.1. Let q be in $(2,\infty]$, and assume that u^0 is in V with initial vorticity ω^0 in $L^p(\Omega)$ for some p in $[q,\infty]$. Let $T>0$. Then there exists a unique solution u to (NS) with corresponding vorticity ω, and for all p in $[q,\infty]$,*

$$\|\omega(t)\|_{L^p} \le \|\omega^0\|_{L^p} + C_0 \tag{7.1}$$

a.e. in $[0,T]$. The constant C_0, which is independent of p, is given by

$$C_0 = C(T,\alpha,\kappa,q)e^{C(\alpha)\nu T}\max\{|\Omega|^{1/2},1\}\left(\|u^0\|_{L^2(\Omega)} + \|\omega^0\|_{L^q(\Omega)}\right).$$

Also, u is in $L^\infty([0,T];C(\overline{\Omega}))\cap L^\infty([0,T];V)$, the norm of u in this space being bounded over any finite range of viscosity ν.

Proof. Approximate u^0 by a sequence of compatible vector fields via Theorem A.2, and let u_n be the corresponding solutions to (NS) given by Theorem 7.1. The argument in the proof of Lemma 3 of [17] can be used to bound $\Lambda = \|(2\kappa-\alpha)u_n\cdot\boldsymbol{\tau}\|_{L^\infty(\Omega)}$ in terms of $\|\omega^0\|_{L^q(\Omega)}$, and this in turn gives the bound $\|\omega_n(t)\|_{L^p} \le \|\omega^0\|_{L^p} + C_0$. This bound holds for the solution u in the limit, as in the proof of Proposition 1 in [17]. (The constant $C(T,\alpha,\kappa,q)$ approaches infinity as q approaches 2, so it is not possible to extend this result to $p=2$.)

Finally, using Sobolev interpolation, (2.3), and Corollary 3.2,

$$\begin{aligned}\|u(t)\|_{C(\overline{\Omega})} &\le C\,\|u(t)\|_{L^2(\Omega)}^{\theta}\,\|u(t)\|_{H^{1,q}(\Omega)}^{1-\theta}\\ &\le C\,\|u(t)\|_{L^2(\Omega)}^{\theta}\,(\|\omega(t)\|_{L^q(\Omega)} + \|u(t)\|_{L^2(\Omega)})^{1-\theta},\end{aligned} \tag{7.2}$$

where $\theta = (q-2)/(2q-2)$. This norm is finite by (6.1), so u is also in $L^\infty([0,T];C(\overline{\Omega}))$ and its norm is uniformly bounded over any finite range of viscosity, as is its norm in $L^\infty([0,T];V)$. Explicitly,

$$\begin{aligned}\|u\|_{L^\infty([0,T];V)} = \|\nabla u\|_{L^\infty([0,T];L^2(\Omega))} &\le C\,\|\nabla u\|_{L^\infty([0,T];L^q(\Omega))}\\ &\le C(\|\omega\|_{L^\infty([0,T];L^q(\Omega))} + \|u\|_{L^\infty([0,T];L^2(\Omega))})\\ &\le C(T,\alpha,\kappa)e^{C(\alpha)\nu T},\end{aligned} \tag{7.3}$$

a bound we will use in section 8. In the second inequality above we used Corollary 3.2. ∎

8. Vanishing viscosity. In this section we bound the rate of convergence in $L^\infty([0,T];L^2(\mathbb{R}^2))$ of solutions to (NS) to the unique solution to the Euler equations for the class of (bounded or unbounded) *Yudovich vorticities*. To describe Yudovich vorticity, we need the following definition.

DEFINITION 8.1. *Let $\theta : [p_0,\infty) \to \mathbb{R}$ for some $p_0 > 1$. We say that θ is admissible if the function $\beta : (0,\infty) \to [0,\infty)$, defined for some $M > 0$ by*[2]

$$\beta(x) := \beta_M(x) := x \inf\left\{(M^\epsilon x^{-\epsilon}/\epsilon)\theta(1/\epsilon) : \epsilon \text{ in } (0, 1/p_0]\right\}, \tag{8.1}$$

satisfies

$$\int_0^1 \frac{dx}{\beta(x)} = \infty. \tag{8.2}$$

Because $\beta_M(x) = M\beta_1(x/M)$, this definition is independent of the value of M, though the presence of M in the definition will turn out to be convenient. Also, β is a monotonically increasing continuous function, with $\lim_{x\to 0^+} \beta(x) = 0$.

Yudovich proves in [26] that for a bounded domain in $\mathbb{R}^n$, if $\|\omega^0\|_{L^p} \le \theta(p)$ for some admissible function θ, then at most one solution to the Euler equations exists. Because of this, we call the class of all such vorticities *Yudovich vorticity*.

DEFINITION 8.2. *We say that a vector field v has Yudovich vorticity if $p \mapsto \|\omega(v)\|_{L^p(\Omega)}$ is an admissible function.*

Examples of admissible bounds on vorticity are

$$\theta_0(p) = 1, \theta_1(p) = \log p, \dots, \theta_m(p) = \log p \cdot \log\log p \cdots \log^m p, \tag{8.3}$$

where $\log^m$ is log composed with itself m times. These admissible bounds are described in [26] (see also [14].) Roughly speaking, the L^p-norm of a Yudovich vorticity can grow in p only slightly faster than $\log p$ and still be admissible. Such growth in the L^p-norm arises, for example, from a point singularity of the type $\log\log(1/|x|)$.

DEFINITION 8.3. *Given an initial velocity u^0 in V, $\overline{u}$ in $L^2([0,T];V)$ is a weak solution to the Euler equations if $\overline{u}(0) = u^0$ and*

$$\frac{d}{dt}\int_\Omega \overline{u}\cdot v + \int_\Omega (\overline{u}\cdot\nabla\overline{u})\cdot v = 0$$

for all v in V.

The existence of a weak solution to the Euler equations under the assumption that the initial vorticity ω^0 is in $L^p(\Omega)$ for some $p > 1$ (a weaker assumption than that of Definition 8.3 when $1 < p < 2$) was proved in [25]. By the result in [26] mentioned above, the solutions are unique in the class of all such solutions $\overline{u}$ for which $\omega(\overline{u})$ and $\overline{u}'$ lie in $L^\infty_{loc}(\mathbb{R};L^p(\Omega))$ for all p in an interval $[p_0,\infty)$.

THEOREM 8.4. *Assume that Γ and α are as in Theorem 7.1. Fix $T > 0$, let u^0 be in V, and assume that ω^0 is in $L^p(\mathbb{R}^2)$ for all p in $[2,\infty)$, with $\|\omega^0\|_{L^p} \le \theta(p)$ for some admissible function θ. Let u be the solution to (NS) for $\nu > 0$ given by Theorem 7.3 and $\overline{u}$ be the unique weak solution to the Euler equations for which $\omega(\overline{u})$ and $\overline{u}'$ are in $L^\infty_{loc}(\mathbb{R};L^p(\Omega))$ for all p in $[2,\infty)$, $\overline{u}$ and u both having initial velocity u^0. Then*

$$u(t) \to \overline{u}(t) \text{ in } L^\infty([0,T];L^2(\Omega)\cap L^2(\Gamma)) \text{ as } \nu \to 0.$$

[2]The definition of β in (8.1) differs from that in [14] in that it directly incorporates the factor of p that appears in the Calderón–Zygmund inequality; in [14] this factor is included in the equivalent of (8.2).

Also, there exists a constant $R = C(T, \alpha, \kappa)$, such that if we define the function $f : [0, \infty) \to [0, \infty)$ by

$$\int_{R\nu}^{f(\nu)} \frac{dr}{\beta(r)} = CT,$$

where β is defined as in (8.1), then

$$\begin{aligned} &\|u - \overline{u}\|_{L^\infty([0,T];L^2(\Omega))} \le f(\nu)^{1/2} \text{ and} \\ &\|u - \overline{u}\|_{L^\infty([0,T];L^2(\Gamma))} \le C'(T, \alpha, \kappa) f(\nu)^{1/4} \end{aligned} \tag{8.4}$$

for all ν in $(0, 1]$.

Proof. We let $w = u - \overline{u}$. It is possible to show that integral identity in Definition 5.1 extends to any v in $L^2([0, T]; V)$ in the form

$$\int_\Omega \partial_t u \cdot v + \int_\Omega (u \cdot \nabla u) \cdot v + \nu \int_\Omega \nabla u \cdot \nabla v - \nu \int_\Gamma (\kappa - \alpha) u \cdot v = 0$$

with a similar extension for the identity in Definition 8.3. Applying these identities with $v = w$ and subtracting give

$$\begin{aligned} &\int_\Omega w \cdot \partial_t w + \int_\Omega w \cdot (u \cdot \nabla w) + \int_\Omega w \cdot (w \cdot \nabla \overline{u}) \\ &\quad = \nu \int_\Gamma (\kappa - \alpha) u \cdot w - \nu \int_\Omega \nabla u \cdot \nabla w. \end{aligned} \tag{8.5}$$

Both $\partial_t u$ and $\partial_t \overline{u}$ are in $L^2([0, T]; V')$, so (see, for instance, Lemma 1.2 on page 176 of [22])

$$\int_\Omega w \cdot \partial_t w = \frac{1}{2} \frac{d}{dt} \|w\|^2_{L^2(\Omega)}.$$

Applying Lemma 2.1,

$$\begin{aligned} &\int_\Omega w \cdot (u \cdot \nabla w) \\ &\quad = \int_\Omega w^i u^j \partial_j w^i = \frac{1}{2} \int_\Omega u^j \partial_j \sum_i (w^i)^2 = \frac{1}{2} \int_\Omega u \cdot \nabla |w|^2 \\ &\quad = \frac{1}{2} \int_\Gamma (u \cdot \mathbf{n}) |w|^2 - \frac{1}{2} \int_\Omega (\operatorname{div} u) |w|^2 = 0, \end{aligned}$$

since $u \cdot \mathbf{n} = 0$ on Γ and $\operatorname{div} u = 0$ in Ω. Thus, integrating (8.5) over time,

$$\|w(t)\|^2_{L^2(\Omega)} \le K + 2 \int_0^t \int_\Omega |w|^2 |\nabla \overline{u}|, \tag{8.6}$$

where

$$\begin{aligned} K &= 2\nu \int_0^t \left[\int_\Gamma (\kappa - \alpha) u \cdot w - \int_\Omega \nabla u \cdot \nabla w \right] \\ &\le 2\nu \int_0^t \left[\int_\Gamma (\kappa - \alpha) u \cdot w + \int_\Omega \nabla u \cdot \nabla \overline{u} \right]. \end{aligned}$$

Applying (2.5) and then using (7.3) and its equivalent for solutions to the Euler equations (where the constant does not increase with time), we have

$$\left|\int_\Gamma (\kappa - \alpha) u \cdot w\right| \le \|\kappa - \alpha\|_{L^\infty(\Gamma)} \|u\|_{L^2(\Gamma)} \|w\|_{L^2(\Gamma)} \le C \|u\|_V \|w\|_V \le C(T, \alpha, \kappa) e^{C(\alpha)\nu T}. \tag{8.7}$$

By (7.3) we also have

$$\left|\int_\Omega \nabla u \cdot \nabla \overline{u}\right| \le \|\nabla u\|_{L^2(\Omega)} \|\nabla \overline{u}\|_{L^2(\Omega)} \le C(T, \alpha, \kappa) e^{C(\alpha)\nu T}, \tag{8.8}$$

so

$$K \le C(T, \alpha, \kappa) e^{C(\alpha)\nu T} \nu \le R\nu \tag{8.9}$$

for all ν in $(0, 1]$ for some constant R.

By (7.2), $\|u\|_{L^\infty([0,T]\times\Omega)} \le C$ for all ν in $(0, 1]$. It is also true that $\overline{u}$ is in $L^\infty([0, T] \times \Omega)$ (arguing, for instance, exactly as in (7.2)). Thus,

$$M = \sup_{\nu \in (0,1]} \||w|^2\|_{L^\infty([0,T]\times\Omega)}$$

is finite.

Also, because the L^p-norms of vorticity are conserved for $\overline{u}$, we have, by Corollary 3.2,

$$2 \|\nabla \overline{u}(t)\|_{L^p(\Omega)} \le Cp\|\omega^0\|_{L^p(\Omega)} + C\|u^0\|_{L^2(\Omega)} \le Cp(\theta(p) + 1/p) \tag{8.10}$$

for all $p \ge 2$. Because θ is admissible, so is $p \mapsto C[\theta(p) + 1/p]$, and its associated β function—call it $\overline{\beta}$—is bounded by a constant multiple of that associated to θ; that is, $\overline{\beta} \le C\beta$.

We now proceed as in [14]. Let s be in $[0, T]$, and let

$$A = |w(s, x)|^2, \quad B = |\nabla \overline{u}(s, x)|, \quad L(s) = \|w(s)\|_{L^2}^2.$$

Then

$$\begin{aligned}\int_{\mathbb{R}^2} |w(s, x)|^2 |\nabla \overline{u}(s, x)|\, dx &= \int_{\mathbb{R}^2} AB = \int_{\mathbb{R}^2} A^\epsilon A^{1-\epsilon} B \le M^\epsilon \int_{\mathbb{R}^2} A^{1-\epsilon} B \\ &\le M^\epsilon \|A^{1-\epsilon}\|_{L^{1/(1-\epsilon)}} \|B\|_{L^{1/\epsilon}} = M^\epsilon \|A\|_{L^1}^{1-\epsilon} \|B\|_{L^{1/\epsilon}} \\ &= M^\epsilon L(s)^{1-\epsilon} \|\nabla \overline{u}(s)\|_{L^{1/\epsilon}} \le CM^\epsilon L(s)^{1-\epsilon} \frac{1}{\epsilon}(\theta(1/\epsilon) + \epsilon).\end{aligned}$$

Since this is true for all ϵ in $[1/p_0, \infty)$, it follows that

$$2 \int_{\mathbb{R}^2} |\nabla \overline{u}(s, x)||w(s, x)|^2\, dx \le C\overline{\beta}(L(s)) \le C\beta(L(s)).$$

From (8.6) and (8.9), then, we have

$$L(t) \le R\nu + C \int_0^t \beta(L(r))\, dr.$$

By Lemma 1.1,

$$\int_{R\nu}^{L(t)} \frac{ds}{C\beta(s)} = \left(-\int_{L(t)}^{1} + \int_{R\nu}^{1} \right) \frac{ds}{C\beta(s)} \le \int_0^t ds = t. \tag{8.11}$$

It follows that for all t in $(0, T]$,

$$\int_{R\nu}^{1} \frac{ds}{\beta(s)} \le CT + \int_{L(t)}^{1} \frac{ds}{\beta(s)}. \tag{8.12}$$

As $\nu \to 0^+$, the left side of (8.12) becomes infinite because of (8.2); hence, so must the right side. But this implies that $L(t) \to 0$ as $\nu \to 0^+$ and that the convergence is uniform over $[0, T]$. It also follows from (8.11) that

$$\int_{R\nu}^{L(t)} \frac{dr}{\beta(r)} \le Ct \tag{8.13}$$

and that, as $\nu \to 0$, $L(t) \to 0$ uniformly over any finite time interval. The rate of convergence given in $L^\infty([0, T]; L^2(\Omega))$ in (8.4) can be derived from (8.13) precisely as in [14].

By (2.5),

$$\begin{aligned} \|u - \overline{u}\|_{L^2(\Gamma)} = \|w\|_{L^2(\Gamma)} &\le C \,\|\nabla w\|_{L^2(\Omega)}^{1/2} \|w\|_{L^2(\Omega)}^{1/2} \\ &\le C(T, \alpha, \kappa) e^{C(\alpha)\nu T} L(t)^{1/4}, \end{aligned}$$

from which the convergence rate for $L^\infty([0, T]; L^2(\Gamma))$ in (8.4) follows. ◻

The convergence rate in $L^\infty([0, T]; L^2(\Omega))$ established in Theorem 8.4 is the same as that established for the entire plane in [14], except for the presence of the constant C and the value of the constant R, which now increases with time (linearly, when α is nonnegative).

In the important special cased of bounded initial vorticity, one obtains the bound

$$\|u - \overline{u}\|_{L^\infty([0,T]; L^2(\mathbb{R}^2))} \le M^{1/2} \left(\frac{R\nu}{M} \right)^{\frac{1}{2} e^{-\epsilon T}} \tag{8.14}$$

for all t in $[0, T]$ for which $\nu < (M/R)e^{-2}$. Here, the R and M are defined as in the proof of Theorem 8.4, and $\epsilon = C\|\omega^0\|_{L^2 \cap L^\infty}$. When α is nonnegative, R is proportional to t, and (8.14) is essentially the same bound obtained by Chemin in [3] working in all of $\mathbb{R}^2$.

One can also calculate explicit bounds for the sequence of admissible vorticities in (8.3), obtaining bounds similar to that of (8.14), but with iterated exponentials. In general, it is not possible to obtain an explicit bound. The important point, however, is that, as described in section 5 of [14], it is possible to obtain an arbitrarily poor bound on the convergence rate for properly chosen initial vorticity. This is because the function f, which was defined implicitly in terms of β, can, conversely, be used to define β, and we can choose f so that it approaches zero arbitrarily slowly. (It is an open and difficult question whether initial vorticities actually exist that achieve arbitrarily slow convergence.)

In Theorem 8.4, we held α constant in (1.2) and let $\nu \to 0$, which is equivalent to letting $a \to 0$ linearly with ν in (1.1). One could modify the proof of Theorem 8.4 in an attempt to obtain the vanishing viscosity limit with slower than linear convergence of a to 0 by being explicit about the value of the constant C_0 in Theorem 7.3. This constant controls the bounds on both K and M in the proof of Theorem 8.4, which, along with the L^p-norms of the initial vorticity, ultimately determine the convergence rate. But C_0 increases to infinity with $\|\alpha\|_{L^\infty(\Gamma)}$, and the bounds on K and M each increase to infinity with C_0. The conclusion is that $\|\alpha\|_{L^\infty(\Gamma)}$ must be bounded over sufficiently small values of ν for the approach in the proof of Theorem 8.4 to remain valid. Thus, using our approach, we cannot significantly improve over the assumption that α remains fixed as $\nu \to 0$ in the vanishing viscosity limit.

9. No-slip boundary conditions. As long as α is nonvanishing, we can let $\gamma = 1/\alpha$ and re-express the Navier boundary conditions in (1.2) as

$$v \cdot \mathbf{n} = 0 \text{ and } 2\gamma(\mathbf{n} \cdot D(v)) \cdot \boldsymbol{\tau} + v \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma. \tag{9.1}$$

When γ is identically zero, we have the usual no-slip boundary conditions. An obvious question to ask is whether it is possible to arrange for γ to approach zero in such a manner that the corresponding solutions to the Navier–Stokes equations with Navier boundary conditions approach the solution to the Navier–Stokes equations with the usual no-slip boundary conditions in $L^\infty([0,T]; L^2(\Omega))$.

Let u^0 be an initial velocity in V, and assume that $\gamma > 0$ lies in $L^\infty(\Gamma)$. Fix a $\nu > 0$ and let

$$\begin{aligned} u &= \text{the unique solution to the Navier–Stokes equations} \\ &\quad\ \text{with Navier boundary conditions for } \alpha = 1/\gamma \text{ and} \\ \widetilde{u} &= \text{the unique solution to the Navier–Stokes equations} \\ &\quad\ \text{with no-slip boundary conditions,} \end{aligned}$$

in each case with the same initial velocity u^0.

If we let γ approach 0 uniformly on the boundary, we automatically have some control over u on the boundary.

LEMMA 9.1. *For sufficiently small* $\|\gamma\|_{L^\infty(\Gamma)}$,

$$\|u\|_{L^2([0,T];L^2(\Gamma))} \le \frac{\|u^0\|_{L^2(\Omega)}}{\sqrt{\nu}} \|\gamma\|^{1/2}_{L^\infty(\Gamma)}. \tag{9.2}$$

Proof. Assume that $\|\gamma\|_{L^\infty(\Gamma)}$ is sufficiently small that $\alpha > \kappa$ on Γ. Then, as in the proof of Theorem 6.1, we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \nu\|\nabla u(t)\|^2_{L^2(\Omega)} = \nu\int_\Gamma (\kappa - \alpha) u \cdot u,$$

so

$$\|u(t)\|^2_{L^2(\Omega)} \le \|u^0\|^2_{L^2(\Omega)} + 2\nu\int_0^t\int_\Gamma (\kappa - \alpha) u \cdot u.$$

But

$$\int_\Gamma (\kappa - \alpha) u \cdot u \le -\inf_\Gamma\{\alpha - \kappa\}\|u(t)\|^2_{L^2(\Gamma)},$$

so

$$\|u(t)\|^2_{L^2(\Omega)} \le \|u^0\|^2_{L^2(\Omega)} - 2\nu\inf_\Gamma\{\alpha - \kappa\}\|u\|^2_{L^2([0,t];L^2(\Gamma))}$$

and

$$\|u\|^2_{L^2([0,t];L^2(\Gamma))} \le \|u^0\|^2_{L^2(\Omega)}/(2\nu \inf_\Gamma \{\alpha - \kappa\}).$$

Then (9.2) follows because $\|\gamma\|_{L^\infty(\Gamma)} \inf_\Gamma \{\alpha - \kappa\} \to 1$ as $\|\gamma\|_{L^\infty(\Gamma)} \to 0$. ∎

If we assume enough smoothness of the initial data and of Γ, we can use (9.2) to establish convergence of u to $\widetilde{u}$ as $\|\gamma\|_{L^\infty(\Gamma)} \to 0$.

THEOREM 9.2. *Fix $T > 0$, assume that u^0 is in $V \cap H^3(\Omega)$ with $u^0 = 0$ on Γ, and assume that Γ is C^3. Then for any fixed $\nu > 0$,*

$$u \to \widetilde{u} \text{ in } L^\infty([0,T];L^2(\Omega)) \cap L^2([0,T];\dot{H}^1(\Omega)) \cap L^2([0,T];L^2(\Gamma)) \tag{9.3}$$

as $\gamma \to 0$ in $L^\infty(\Gamma)$. Here, $\dot{H}^1(\Omega)$ is the homogeneous Sobolev space.

Proof. First, u exists and is unique by Theorem 6.1; the existence and uniqueness of $\widetilde{u}$ are classical results. Because u^0 is in $H^3(\Omega)$ and Γ is C^3, $\widetilde{u}$ is in $L^\infty([0,T];H^3(\Omega))$ by the argument on page 205 of [22] following the proof of Theorem 3.6 of [22]. Hence, $\nabla\widetilde{u}$ is in $L^\infty([0,T];H^2(\Omega))$ and so in $L^\infty([0,T];C(\overline{\Omega}))$.

Arguing as in the proof of Theorem 8.4 with $w = u - \widetilde{u}$, we have

$$\begin{aligned}\int_\Omega \partial_t w \cdot w + \int_\Omega w \cdot (u \cdot \nabla w) + \int_\Omega w \cdot (w \cdot \nabla \widetilde{u}) + \nu \int_\Omega \nabla w \cdot \nabla w \\ - \nu \int_\Gamma (\kappa - \alpha) u \cdot w + \nu \int_\Gamma (\mathbf{n} \cdot \nabla \widetilde{u}) \cdot w = 0.\end{aligned}$$

(Even though w is not a valid test function for $\widetilde{u}$, we are working with sufficiently smooth solutions that this integration is still valid. Since w is divergence-free and tangential to the boundary, the pressure term for each equation integrates to zero.) But $\widetilde{u} = 0$ on Γ, so $w = u$ on Γ, and

$$\begin{aligned}\int_\Omega \partial_t w \cdot w + \int_\Omega w \cdot (w \cdot \nabla \widetilde{u}) + \nu \int_\Omega |\nabla w|^2 + \nu \int_\Gamma (\alpha - \kappa)|u|^2 \\ + \nu \int_\Gamma (\mathbf{n} \cdot \nabla \widetilde{u}) \cdot u = 0.\end{aligned}$$

For $\|\gamma\|_{L^\infty(\Gamma)}$ sufficiently small that $\alpha = 1/\gamma > \kappa$ on Γ, integrating over time gives

$$\|w(t)\|^2_{L^2(\Omega)} + \nu \int_0^t \|\nabla w\|^2_{L^2(\Omega)} \le K + 2 \int_0^t \int_\Omega |w|^2 |\nabla \widetilde{u}|, \tag{9.4}$$

where

$$\begin{aligned}K &= -2\nu \int_0^t \int_\Gamma (\mathbf{n} \cdot \nabla \widetilde{u}) \cdot u \le 2\nu \int_0^t \|\nabla \widetilde{u}\|_{L^2(\Gamma)} \|u\|_{L^2(\Gamma)} \\ &\le C\nu \int_0^t \|\widetilde{u}\|_{H^2(\Omega)} \|u\|_{L^2(\Gamma)} \le C\nu \|\widetilde{u}\|_{L^2([0,T];H^2(\Omega))} \|u\|_{L^2([0,T];L^2(\Gamma))}.\end{aligned}$$

By Theorem 3.10 on page 213 of [22], $\|\widetilde{u}\|_{L^2([0,T];H^2(\Omega))}$ is finite (though the bound on it in [22] increases to infinity as ν goes to 0), so by Lemma 9.1,

$$K \le C_1(\nu) \|\gamma\|^{1/2}_{L^\infty(\Gamma)}. \tag{9.5}$$

Because $\nabla \widetilde{u}$ is in $L^\infty([0, T]; C(\overline{\Omega}))$,

$$\int_0^t \int_\Omega |w|^2 |\nabla \widetilde{u}| \le C_2(\nu) \int_0^t \|w(s)\|_{L^2(\Omega)}^2 \, ds,$$

where $C_2(\nu) = \|\nabla \widetilde{u}\|_{L^\infty([0,T] \times \Omega)}$, and (9.4) becomes

$$\|w(t)\|_{L^2(\Omega)}^2 + \nu \int_0^t \|\nabla w\|_{L^2(\Omega)}^2 \le C_1(\nu) \|\gamma\|_{L^\infty(\Gamma)}^{1/2} + C_2(\nu) \int_0^t \|w(s)\|_{L^2(\Omega)}^2 \, ds.$$

By Gronwall's lemma,

$$\|w(t)\|_{L^2(\Omega)}^2 \le C_1(\nu) \|\gamma\|_{L^\infty(\Gamma)}^{1/2} e^{C_2(\nu) t},$$

and the convergence in $L^\infty([0, T]; L^2(\Omega))$ and thus also in $L^2([0, T]; \dot{H}^1(\Omega))$ follow immediately. Convergence in $L^2([0, T]; L^2(\Gamma))$ then follows directly from Lemma 9.1, since $\widetilde{u} = 0$ on Γ. ◻

10. The boundary layer. In [13], Kato investigates the vanishing viscosity limit of solutions of the Navier–Stokes equations with no-slip boundary conditions to a solution of the Euler equations for a bounded domain in $\mathbb{R}^d$, $d \ge 2$. What Kato shows is the following: Let u be the weak solution to the Navier–Stokes equations with no-slip boundary conditions and with $u(0)$ in H, and let $\overline{u}$ be the solution to the Euler equations, where sufficient smoothness is assumed for $\overline{u}(0)$ that $\nabla \overline{u}$ is bounded on $[0, T] \times \Omega$. Assume also that $u(0) \to \overline{u}(0)$ in H as $\nu \to 0$. Then the following are equivalent:

(i) $u(t) \to \overline{u}(t)$ in $L^2(\Omega)$ as $\nu \to 0$ uniformly over t in $[0, T]$;
(ii) $u(t) \to \overline{u}(t)$ in $L^2(\Omega)$ as $\nu \to 0$ weakly for all t in $[0, T]$;
(iii) $\nu \int_0^T \|\nabla u\|_{L^2(\Omega)}^2 \, dt \to 0$ as $\nu \to 0$;
(iii′) $\nu \int_0^T \|\nabla u\|_{L^2(\Gamma_{c\nu})}^2 \, dt \to 0$ as $\nu \to 0$.

Here, $\Gamma_{c\nu}$ is the boundary strip of width $c\nu$ with $c > 0$ fixed but arbitrary.

Let us return to the setting of Theorem 8.4 and compare the situation to that of [13]. We now have zero forcing and the same initial conditions for both (NS) and (E), which simplifies the analysis in [13] slightly, but we have different boundary conditions on (NS) and we have insufficient smoothness of u^0 for Kato's conditions to apply. However, we have already proven that condition (i) holds and hence also (ii), and since conditions (iii) and (iii') follow immediately from (7.1), no further work is required to show the equivalence of Kato's four conditions.

(It is also possible to directly adapt Kato's argument to our setting, thereby establishing the vanishing viscosity limit in the spirit of Kato. This requires, however, all of the results developed to prove the vanishing viscosity limit in Theorem 8.4 and considerably more effort besides.)

We can bound the rate at which the convergence in condition (iii′) occurs, giving us some idea of what is happening in the boundary layer.

THEOREM 10.1. *With the assumptions in Theorem 8.4,*

$$\nu \int_0^T \|\nabla u\|_{L^2(\Gamma_{c\nu})}^2 \, dt \le C(p, T, \alpha, \kappa, u^0) T (2c)^{1-2/p} \nu^{2-2/p}$$

for all p in $(2, \infty)$ and t in $[0, T]$.

Proof. We have, using (7.1) and Corollary 3.2,

$$\begin{aligned}\|\nabla u\|_{L^2(\Gamma_{\delta/2})} &\le \|z\nabla u\|_{L^2(\Gamma_\delta)} \le \|z\|_{L^{p'}(\Gamma_\delta)} \|\nabla u\|_{L^p(\Omega)} \\ &\le \|z\|_{L^{p'}(\Gamma_\delta)} \left(Cp \|\omega\|_{L^p(\Omega)} + C' \|u\|_{L^2(\Omega)}\right) \\ &\le \|z\|_{L^{p'}(\Gamma_\delta)} \left(Cp \left(\|\omega^0\|_{L^p} + C_0\right) + C' \|u\|_{L^2(\Omega)}\right),\end{aligned}$$

where $1/p' + 1/p = 1/2$. But

$$\|z\|_{L^{p'}} \le C\delta^{1/p'} = C\delta^{1/2-1/p}.$$

Substituting this into the earlier inequality, squaring the result, setting $\delta = 2c\nu$, and integrating over time conclude the proof. □

The proof of Theorem 10.1 shows that the square of the gradient of the velocity for a solution to (NS) with Navier boundary conditions vanishes in the L^2-norm nearly linearly with the width of the boundary layer. We could obtain linear convergence for appropriate smoother initial velocities if we could show that $\|\nabla u\|_{L^\infty(\Omega)}$ is bounded uniformly over small ν. It is not at all clear, however, whether such a result is obtainable. In any case, the behavior of the boundary layer for Navier boundary conditions is principally derived from the boundary conditions themselves and is not highly dependent upon the smoothness of the initial velocity.

This is in contrast to no-slip boundary conditions, where for smooth data probably the strongest general statement that can be made was made by Kato in [13] with his equivalent conditions for the vanishing viscosity limit. (See also the incremental improvement in [23] and [27].) For the less regular initial velocities that we assume in Theorem 8.4, it is quite possible that a condition stronger than Kato's condition (iii′) is required to imply convergence in the vanishing viscosity limit. This is because there is no known bound on $\|u\|_{L^2([0,T];L^\infty(\Omega))}$ uniform over small ν, which is required to achieve the vanishing viscosity limit using Osgood's lemma as in section 8. In fact, obtaining such a bound would almost certainly require obtaining a uniform bound on the L^p-norm of the vorticity for some $p > 2$, which is tantamount to establishing the vanishing viscosity limit to begin with, at least for smooth initial data.

Appendix. Compatible sequences. For p in $(1, \infty)$, define the spaces

$$X_0^p = H_0 \cap H^{1,p}(\Omega) \text{ and } X^p = H \cap H^{1,p}(\Omega) = X_0^p \oplus H_c, \tag{A.1}$$

each with the $H^{1,p}(\Omega)$-norm.

LEMMA A.1. *Let p be in $(1, \infty]$. For $p < 2$ let $\widehat{p} = p/(2-p)$, for $p > 2$ let $\widehat{p} = \infty$, and for $p = 2$ let $\widehat{p}$ be any value in $[2, \infty]$. Then for any v in X_0^p,*

$$\|v\|_{L^{\widehat{p}}(\Gamma)} \le C(p) \|\omega(v)\|_{L^p(\Omega)}.$$

Proof. For $p < 2$ and any v in X_0^p, we have

$$\begin{aligned}\|v\|_{L^{\widehat{p}}(\Gamma)} &\le C(p) \|v\|_{L^p(\Omega)}^{1-\lambda} \|\nabla v\|_{L^p(\Omega)}^{\lambda} \le C(p) \|\nabla v\|_{L^p(\Omega)} \\ &\le C(p) \|\omega(v)\|_{L^p(\Omega)},\end{aligned}$$

where $\lambda = 2(\widehat{p} - p)/(p(\widehat{p} - 1)) = 1$ if $p < 2$ and $\lambda = 2/p$ if $p \ge 2$. The first inequality follows from Theorem 3.1 on page 43 of [7], the second follows from (2.3), and the third from Lemma 3.1. □

Given a vorticity ω in $L^p(\Omega)$ with p in $(1,\infty)$, the Biot–Savart law gives a vector field v in H whose vorticity is ω. (That v is in $L^2(\Omega)$ follows as in the proof of Lemma A.1, Ω being bounded.) Let $v = u + h$, where u is in H_0 and h is in H_c. Then $\omega(u) = \omega$ as well, so we can define a function $K_\Omega\colon L^p(\Omega) \to H_0$ by $\omega \mapsto u$ having the property that $\omega(K_\Omega[\omega]) = \omega$. By (2.3) and Lemma 3.1, u is also in $H^{1,p}(\Omega)$, so, in fact, $K_\Omega\colon L^p(\Omega) \to X_0^p$ and is the inverse of the function ω. It is continuous by the same two lemmas. We can write the inequality in Lemma A.1, then, as $\|K_\Omega[\omega]\|_{L^{\widehat{p}}(\Gamma)} \le C(p)\,\|\omega\|_{L^p(\Omega)}$.

THEOREM A.2. *Assume that Γ is C^2 and α is in $L^\infty(\Gamma)$. Let v be in X^p for some p in $(1,\infty)$ and have vorticity ω. Then there exists a sequence (v_n) of compatible vector fields (Definition 7.2) whose vorticities converge strongly to ω in $L^p(\Omega)$. The vector fields (v_n) converge strongly to v in X^p and, if $p \ge 2$, also in V.*

Proof. Our proof is a minor adaptation of that of Lemma 2 of [17], which we first summarize. Let N_n be a tubular neighborhood of Γ of width $2/n$ (for n sufficiently large) and let $U_n = N_n \cap \Omega$. Define $d\colon U_n \to \mathbb{R}^+$ by $d(x) = \operatorname{dist}(x,\Gamma)$ and $r\colon U_n \to \Gamma$ by letting $r(x)$ be the nearest point to x on Γ. Define a cutoff function ζ_n in $C^\infty(\Omega)$ taking values in $[0,1]$ so that $\zeta_n \equiv 0$ on U_{n+1} and $\zeta_n \equiv 1$ on $\Omega \setminus U_n$, and let the sequence (η_k) be an approximation of the identity.

It is shown in [17] that β is a continuous extension operator from $L^{\widehat{p}}(\Gamma)$ into $L^p(\Omega)$, where

$$\beta(G)(x) := \zeta_n(x)(\eta_n * \omega)(x) + (1-\zeta_n(x))e^{-nd(x)}G(r(x))$$

and where $\widehat{p}$ is defined as in Lemma A.1. In calculating $\eta_n * \omega$, we extend ω by zero to all of $\mathbb{R}^2$. Defining $\Psi\colon L^{\widehat{p}}(\Gamma) \to L^{\widehat{p}}(\Gamma)$ by

$$\Psi(G) = (2\kappa - \alpha)K_\Omega[\beta_n(G)] \cdot \boldsymbol{\tau},$$

it is shown that Ψ is a contraction mapping for sufficiently large n and so has a unique fixed point, G^n. Finally, defining $\omega_n = \beta(G^n)$, the authors show that ω_n converges to ω in $L^p(\Omega)$ (this argument uses Lemma A.1). Key to this last step is demonstrating that $\|G^n\|_{L^{\widehat{p}}(\Gamma)}$ is bounded over n.

Since the authors of [17] are working in a simply connected domain, they can deal exclusively with vorticity. To adapt their proof to multiply connected domains, where we must recover the velocity with the proper harmonic component, requires only one change to their construction. We suppose that $v = u + h$ with $u \in X_0^p$ and h in H_c and define

$$\Psi(G) = (2\kappa - \alpha)(K_\Omega[\beta(G)] + h) \cdot \boldsymbol{\tau}.$$

In forming the difference $\Psi(G_1) - \Psi(G_2)$ the term $(2\kappa - \alpha)h \cdot \boldsymbol{\tau}$ cancels, and the existence of a unique fixed point G^n follows precisely as in [17].

We can now define

$$\omega_n = \beta(G_n),\ v_n = K_\Omega[\omega_n] + h,$$

and observe that *on* Γ,

$$\begin{aligned}\omega(v_n) &= \omega_n = \beta(G^n) = G^n = \Psi(G^n) = (2\kappa - \alpha)(K_\Omega[\beta(G^n)] + h) \cdot \boldsymbol{\tau} \\ &= (2\kappa - \alpha)(K_\Omega[\omega_n] + h) \cdot \boldsymbol{\tau} = (2\kappa - \alpha)v_n \cdot \boldsymbol{\tau},\end{aligned}$$

so v_n satisfies the Navier boundary conditions. (Note that we had to include the harmonic component h of the velocity in the definition of Ψ; we could not simply apply Lemma 2 of [17] to u and add h to the resulting vector field, because such a vector field would not, in general, satisfy the Navier boundary conditions.)

The convergence of ω_n to ω in $L^p(\Omega)$ is argued as in [17], except that now, to show that $\|G^n\|_{L^{\widehat{p}}(\Gamma)}$ is bounded over n, we have

$$\begin{aligned}\|G^n\|_{L^{\widehat{p}}(\Gamma)} &\le \|2\kappa - \alpha\|_{L^\infty} \|K_\Omega[\omega_n] + h\|_{L^{\widehat{p}}(\Gamma)} \\ &\le C(\|K_\Omega[\omega_n]\|_{L^{\widehat{p}}(\Gamma)} + \|h\|_{L^{\widehat{p}}(\Gamma)}) \\ &\le C\left(\|\omega\|_{L^p(\Omega)} + \frac{1}{2}\|G^n\|_{L^{\widehat{p}}(\Gamma)} + \|\nabla h\|_{L^p(\Omega)}\right)\end{aligned}$$

for n sufficiently large. Here, the bound on $\|K_\Omega[\omega_n]\|_{L^{\widehat{p}}(\Gamma)}$ is as in [17] and the bound on $\|h\|_{L^{\widehat{p}}(\Gamma)}$ follows from Theorem 3.1 on page 43 of [7] and (2.3) as in the proof of Lemma A.1. It follows that $\|G^n\|_{L^{\widehat{p}}(\Gamma)} \le C\|v\|_{X_p}$ for sufficiently small n, which is what is required to complete the proof of the convergence of ω_n to ω in $L^p(\Omega)$ as in [17].

To prove the convergence of v_n to v in X^p, we observe that

$$\begin{aligned}\|\nabla v - \nabla v_n\|_{L^p(\Omega)} &= \|\nabla u + \nabla h - (\nabla K_\Omega[\omega_n] + \nabla h)\|_{L^p(\Omega)} \\ &= \|\nabla(u - K_\Omega[\omega_n])\|_{L^p(\Omega)} \le Cp\,\|\omega(u - K_\Omega[\omega_n])\|_{L^p(\Omega)} \\ &= Cp\,\|\omega - \omega_n\|_{L^p(\Omega)},\end{aligned}$$

where we used Lemma 3.1. Then by (2.3), v_n converges strongly to v in X^p as well. Convergence in V for $p \ge 2$ follows since Ω is bounded. ∎

Our only use of Theorem A.2 is in the proofs of Theorem 7.3 and Corollary A.3. In both of these instances we need only the case $p \ge 2$. We include all the cases, however, for the same reason as in [17]: we hope that if the vorticity bound in Lemma 3 of [17] can be extended to p in $(1, 2)$, then the convergence in Proposition 1 of [17] can also be extended (for multiply connected Ω).

Corollary A.3. *Assume that Γ is C^2, and α is in $L^\infty(\Gamma)$. Then there exists a basis for V lying in $\mathcal{W}$ that is also a basis for H.*

Proof. The space $V = (V \cap H_0) \oplus H_c$ is separable because $V \cap H_0$ is the image under the continuous function K_Ω of the separable space $L^2(\Omega)$ and H_c is finite-dimensional. Let $\{v_i\}_{i=1}^\infty$ be a dense subset of V. Applying Theorem A.2 to each v_i and unioning all the sequences, we obtain a countable subset $\{u_i\}_{i=1}^\infty$ of $\mathcal{W}$ that is dense in V. Selecting a maximal independent set gives us a basis for V and for H as well, since V is dense in H. ∎

Acknowledgments. The author wishes to thank Josef Málek for suggesting that he explore the vanishing viscosity limit with Navier boundary conditions, and Misha Vishik for many useful discussions.

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