

# VANISHING VISCOSITY IN THE PLANE FOR VORTICITY IN BORDERLINE SPACES OF BESOV TYPE

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ABSTRACT. The existence and uniqueness of solutions to the Euler equations for initial vorticity in  $B_\Gamma \cap L^{p_0} \cap L^{p_1}$  was proved by Misha Vishik, where  $B_\Gamma$  is a borderline Besov space parameterized by the function  $\Gamma$  and  $1 < p_0 < 2 < p_1$ . Vishik established short time existence and uniqueness when  $\Gamma(n) = O(\log n)$  and global existence and uniqueness when  $\Gamma(n) = O(\log^{\frac{1}{2}} n)$ . For initial vorticity in  $B_\Gamma \cap L^2$ , we establish the vanishing viscosity limit in  $L^2(\mathbb{R}^2)$  of solutions of the Navier-Stokes equations to a solution of the Euler equations in the plane, convergence being uniform over short time when  $\Gamma(n) = O(\log n)$  and uniform over any finite time when  $\Gamma(n) = O(\log^\kappa n)$ ,  $0 \leq \kappa < 1$ , and we give a bound on the rate of convergence. This allows us to extend the class of initial vorticities for which both global existence and uniqueness of solutions to the Euler equations can be established to include  $B_\Gamma \cap L^2$  when  $\Gamma(n) = O(\log^\kappa n)$  for  $0 < \kappa < 1$ .

## 1. INTRODUCTION

We consider an incompressible fluid of constant density and nonzero viscosity extending throughout the plane—described by the Navier-Stokes equations—and ask whether its velocity as a function of time and space converges in the energy norm to the velocity of an inviscid fluid—described by the Euler equations—having the same initial velocity. This is the so-called *vanishing viscosity limit*, which is of interest primarily in two settings: weak solutions in the whole space (or a periodic domain) and solutions of any kind in a domain with boundary, these being the two settings where knowledge of the limit is most wanting.

Here, we focus on a particular class of weak solutions in the plane. (Very little is known about the vanishing viscosity limit for weak solutions in higher dimensions.) This class of weak solutions arises in an issue closely related to the vanishing viscosity limit, namely, uniqueness of solutions to the Euler equations in a given class of weak solutions.

There are two results that reach the edge of what is known about uniqueness of solutions to the Euler equations in the plane. In [10], Yudovich established uniqueness (and existence) of solutions to the Euler equations

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with bounded initial vorticity, extending this result in [11] to a class of initial velocities with unbounded vorticities, which we will call  $\mathbb{Y}$ , with the restriction that the  $L^p$ -norms of the initial vorticity not grow much faster than  $\log p$ . (These results are for a bounded domain, but extend easily to the whole plane.) In [9], Vishik established the uniqueness of solutions for velocities whose vorticities lie in  $L^\infty([0, T]; B_\Gamma \cap L^{p_0})$ ,  $1 < p_0 < 2$  (or, in  $n$  dimensions,  $1 < p_0 < n$ ) and where  $B_\Gamma$  is defined in Section 2. This was under the assumption that  $\Gamma(n)$  not grow much faster than  $n \log n$ . Except for certain technical restrictions placed on  $\Gamma$ , this class of solutions includes those generated by initial vorticities in  $\mathbb{Y}$ . Vishik, however, was only able to establish existence of a solution in his uniqueness class for initial vorticities in  $B_\Gamma \cap L^{p_0} \cap L^{p_1}$ ,  $1 < p_0 < 2 < p_1 \leq \infty$  and  $\Gamma = O(\log n)$  (for a more detailed statement see Theorem 2.2).

We give a bound on the rate of convergence of the vanishing viscosity limit in the  $L^2$ -norm for initial vorticities in  $B_\Gamma \cap L^2$  with  $\Gamma = O(\log n)$ . We also extend the class of initial vorticities for which both existence and uniqueness can be established globally in time. (See Theorem 2.3, Corollary 2.4, and Corollary 2.5.)

Related results appear in [1], where convergence in the energy norm uniformly over finite time is shown for bounded initial vorticity, and in [4] where convergence in the same norm is shown for initial velocity in  $\mathbb{Y}$ . In [3], convergence for initial velocity in  $B_{\infty,1}^1$  is shown uniformly over finite time in the  $B_{\infty,1}^0$ -norm with a bound on the rate of convergence. The rates in these references are discussed following Corollary 2.5, below.

## 2. BACKGROUND AND STATEMENT OF MAIN RESULTS

The Navier-Stokes equations are given by

$$(NS) \quad \begin{cases} \partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} - \nu \Delta \tilde{u} = -\nabla \tilde{p} \\ \operatorname{div} \tilde{u} = 0 \\ \tilde{u}|_{t=0} = u^0 \end{cases}$$

and the Euler equations by

$$(E) \quad \begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p \\ \operatorname{div} u = 0 \\ u|_{t=0} = u^0. \end{cases}$$

Here,  $\tilde{u}$ ,  $\tilde{p}$ ,  $u$ , and  $p$  are tempered distributions. For the solutions we will be working with,  $\tilde{u}$  and  $u$  will lie  $L^\infty([0, T]; H^1)$ , allowing use to make sense of the nonlinear terms in (NS) and (E).

In the plane, the vorticity of a fluid is given by

$$\omega = \omega(u) = \partial_1 u_2 - \partial_2 u_1.$$

We now define the Littlewood-Paley operators. We begin with the following lemma:

**Lemma 2.1.** *There exist two radial functions  $\chi \in S$  and  $\varphi \in S$  such that*

$$\text{supp } \chi \subset \{\xi \in \mathbb{R}^2 : 0 \leq |\xi| \leq \frac{4}{3}\}, \quad \text{supp } \varphi \subset \{\xi \in \mathbb{R}^2 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\},$$

and

$$\text{supp } \chi(\xi) + \sum_{j=0}^{\infty} \varphi_j(\xi) = 1,$$

where  $\varphi_j(\xi) = \varphi(2^{-j}\xi)$  (so  $\check{\varphi}_j(x) = 2^{jn}\check{\varphi}(2^jx)$ ).

*Proof.* This is classical. See [7].  $\square$

Observe that, if  $|j - j'| \geq 2$ , then  $\text{supp } \varphi_j \cap \text{supp } \varphi_{j'} = \emptyset$ , and, if  $j \geq 1$ , then  $\text{supp } \varphi_j \cap \text{supp } \chi = \emptyset$ .

Let  $f \in S'$ . We define, for any integer  $j$ ,

$$\Delta_j f = \begin{cases} 0, & j < -1, \\ \chi(D)f = \check{\chi} * f, & j = -1, \\ \varphi(D)f = \check{\varphi}_j * f, & j > -1, \end{cases}$$

and

$$S_j f = \sum_{k=-\infty}^{j-1} \Delta_k f = \chi(2^{-j}D)f.$$

As in [9], let  $\Gamma : \mathbb{R} \rightarrow [1, \infty)$  be a locally Lipschitz continuous monotonically nondecreasing function that satisfies conditions (i)-(iii) p. 771 of [9]. Condition (i) is that  $\Gamma = 1$  on the interval  $(-\infty, -1]$  and  $\lim_{\beta \rightarrow \infty} \Gamma(\beta) = \infty$ . For the other (minor technical) conditions see [9].

Define the space

$$B_\Gamma = \{f \in S'(\mathbb{R}^2) : \sum_{j=-1}^N \|\Delta_j f\|_{L^\infty} = O(\Gamma(N))\}$$

with the norm

$$\|f\|_\Gamma = \sup_{N \geq -1} \frac{1}{\Gamma(N)} \sum_{j=-1}^N \|\Delta_j f\|_{L^\infty}.$$

The following fundamental result for initial vorticities in  $B_\Gamma$  is from Theorems 7.1 and 8.1 of [9]:

**Theorem 2.2** (Vishik). *Define  $\Gamma_1 : \mathbb{R} \rightarrow [1, \infty)$  by*

$$\Gamma_1(\beta) = \begin{cases} 1, & \beta < -1, \\ (\beta + 2)\Gamma(\beta), & \beta \geq -1 \end{cases}$$

and add the assumption (on  $\Gamma$ ) that  $\Gamma_1$  is convex. Finally, assume that  $\Gamma$  satisfies

$$(\beta + 2)\Gamma'(\beta) \leq C \tag{2.1}$$

for almost all  $\beta \in [-1, \infty)$ . Given initial vorticity  $\omega^0$  in  $B_\Gamma \cap L^{p_0} \cap L^{p_1}$  with  $1 < p_0 < 2 < p_1 \leq \infty$  there exists a short-time solution to (E) unique in the class of vorticities lying in  $L^\infty([0, T]; L^{p_0} \cap L^{p_1}) \cap C_{w^*}([0, T]; B_{\Gamma_1})$ . With the added assumption that

$$\Gamma'(\beta)\Gamma_1(\beta) \leq C \quad (2.2)$$

for almost all  $\beta \geq -1$ , there exists a solution to (E) unique in the class of vorticities lying in  $L_{loc}^\infty([0, \infty); L^{p_0} \cap L^{p_1}) \cap C_{w^*}([0, \infty); B_{\Gamma_1})$ . Here,  $C_{w^*}$  is the space of weak\*-continuous functions (see [9] for details).

Observe that the vorticity degrades immediately in that (as far as is known) it belongs to a larger space at all positive times than it does at time zero.

**Remark 2.1.** In Theorem 2.3, Corollary 2.4, and Corollary 2.5 below, for the case where  $\lim_{n \rightarrow \infty} \Gamma(n) = \infty$ , the symbol  $C$  represents an unspecified absolute constant (that is, independent of the initial data). For the case where  $\Gamma(n)$  is bounded in  $n$ , the constant  $C$  depends on both the  $L^2$ -norm and the  $B_{\infty,1}^0$ -norm of initial vorticity. This dependence arises in (4.1) below.

**Theorem 2.3.** Let  $\Gamma : \mathbb{R} \rightarrow [0, \infty)$  (without making any of the assumptions on  $\Gamma$  of [9]) and assume that  $u^0$  is in  $L^2$  with  $\omega^0 = \omega(u^0)$  in  $B_\Gamma \cap L^2$ . Then there exists a unique solution  $\tilde{u}$  to (NS) and a (not necessarily unique) solution  $u$  to (E), both lying in  $L^\infty([0, \infty); H^1(\mathbb{R}^2))$ . For any such  $u$ ,

$$\|\tilde{u} - u\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))} \leq C(\nu T)^{1/2} \|\omega^0\|_{L^2} \exp\left(e^{C\alpha T \Gamma(-\log(\nu T)/2)}\right) \quad (2.3)$$

for all  $T > 0$ , where  $\alpha = \|\omega^0\|_{B_\Gamma}$ .

*Proof.* The existence of a global-in-time solution to (E) with vorticity in  $L^\infty([0, \infty); L^p)$  for  $\omega^0$  in  $L^p(\mathbb{R}^2)$ ,  $p > 1$ , is due to Yudovich in [10] (see, for instance, Theorem 4.1 p. 126 of [5]). The existence and uniqueness of solutions to (NS) lying in  $L^\infty([0, T]; L^2) \cap L^2([0, T]; L^2)$  for  $u^0$  in  $L^2(\mathbb{R}^2)$  is classical (see, for instance, Theorems III.3.1 and III.3.2 of [6]). Because our solutions to (NS) are in the whole plane, all  $L^p$ -norms of the vorticity are non-increasing, so, in fact,  $\tilde{u}$  lies in  $L^\infty([0, \infty); H^1(\mathbb{R}^2))$ .

The proof of (2.3) is contained in the sections that follow.  $\square$

It is possible to loosen the finite energy requirement in Theorem 2.3 that  $u^0$  lie in  $L^2(\mathbb{R}^2)$ , allowing it to lie, for instance, in the space  $E_m$  of [2].

Without restrictions on  $\Gamma$  it is of course possible that the right-hand side of (2.3) will not go to zero with  $\nu$ . In order to establish the vanishing viscosity limit,  $\Gamma(n)$  cannot grow any faster than  $C \log n$ . We have the following immediate corollary of Theorem 2.3:

**Corollary 2.4.** *When  $\Gamma(n) = O(\log n)$ ,  $\tilde{u} \rightarrow u$  in  $L^\infty([0, T]; L^2(\mathbb{R}^2))$  for  $T < (C\alpha)^{-1}$ , with*

$$\begin{aligned} & \|\tilde{u} - u\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))} \\ & \leq C\|\omega^0\|_{L^2}(\nu T)^{1/2} \exp\left(\left(-\frac{1}{2}\log(\nu T)\right)^{C\alpha T}\right). \end{aligned} \quad (2.4)$$

In Corollary 2.5, we extend the class of solutions for which both existence and uniqueness of solutions to (E) can be demonstrated globally in time. Note that we obtain uniqueness in Corollary 2.5 in spite of lacking knowledge of whether the solution to (E) remains in the class  $L^\infty([0, T]; B_{\Gamma_1})$  for arbitrarily large  $T$ , this being (almost) the class for which Vishik demonstrates uniqueness in [9] (see the comment on p. 771 of [9]).

**Corollary 2.5.** *When  $\Gamma(n) = O(\log^\kappa n)$  with  $0 \leq \kappa < 1$ , the solution  $u$  to (E) is unique in  $L^\infty([0, \infty); H^1(\mathbb{R}^2))$ . Also,  $\tilde{u} \rightarrow u$  in  $L_{loc}^\infty([0, \infty); L^2(\mathbb{R}^2))$ , and for all  $T > 0$ ,*

$$\|\tilde{u} - u\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))} \leq C(\nu T)^{1/2}\|\omega^0\|_{L^2} \exp\left(e^{C\alpha T \log^\kappa(-\log(\nu T)/2)}\right). \quad (2.5)$$

*Proof.* The rate in (2.5) follows immediately from Theorem 2.3. By (2.5), any solution  $u$  to (E) lying in  $L^\infty([0, \infty); H^1(\mathbb{R}^2))$  is the strong limit in  $L_{loc}^\infty([0, \infty); L^2(\mathbb{R}^2))$  of the solutions  $\tilde{u}$  to (NS); since strong limits are unique, we conclude that the solution  $u$  is unique.  $\square$

In Corollary 2.5, one can show that a solution to (E) in  $L^\infty([0, \infty); H^1(\mathbb{R}^2))$  is unique without using the vanishing viscosity limit. Indeed, given a solution  $u$  to (E) with initial data  $u_0$ , we construct in the proof of Theorem 2.3 a sequence of  $C^\infty$  solutions  $u_n$  to (E) with initial data  $S_n u_0$ . We then show that  $\omega_0 \in B_\Gamma$  implies  $\|u_n - u\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))}$  goes to 0 as  $n$  approaches infinity (see (3.2) and (4.2) in the sections that follow), where  $\Gamma$  satisfies the conditions in Corollary 2.5. Since the sequence  $u_n$  is uniquely determined by the initial data  $u_0$ , two solutions to (E) with the same initial data and initial vorticity in  $B_\Gamma$  will have the same approximating sequence and will therefore be equal on  $[0, T]$ .

The restriction (2.1) on  $\Gamma$  ensures that  $\Gamma(N)$  grows no faster than  $C \log N$  for large  $N$ . Therefore, Corollaries 2.4 and 2.5 establish a rate of convergence for the entire short time existence and uniqueness class in [9]. Similarly, the assumption (2.2) on  $\Gamma$  ensures that  $\Gamma(N)$  grows no faster than  $C \log^{\frac{1}{2}} N$  for large  $N$ . Therefore, Corollary 2.5 establishes a rate of convergence for the entire global existence and uniqueness class in [9] as well.

For bounded initial vorticity, Chemin shows in [1] that

$$\begin{aligned} \|\tilde{u} - u\|_{L^\infty([0, T]; L^2(\mathbb{R}^2))} & \leq (4\nu T)^{\frac{1}{2}} \exp(-C\|\omega^0\|_{L^2 \cap L^\infty} T) \\ & \quad \times \|\omega^0\|_{L^2 \cap L^\infty} e^{1 - \exp(-C\|\omega^0\|_{L^2 \cap L^\infty} T)}. \end{aligned}$$

Since  $B_{\infty,1}^0 \subseteq L^\infty$ , we would expect this rate to be slower than that of (2.5) with  $\kappa = 0$ , which it is. As  $T$  approaches 0, though, this rate gets closer to being  $C(\nu T)^{1/2}$ , so it is not much worse than (2.5) with  $\kappa = 0$  for small times. Chemin's rate is substantially better than that of (2.4) and of (2.5) with  $0 < \kappa < 1$ ; however, the two spaces  $B_{O(\log^\kappa n)}$  and  $L^\infty$  are not comparable for  $0 < \kappa \leq 1$ , since the vorticity can be unbounded for the first space while  $\Gamma(n) = O(n)$  for the second.

Hmidi and Keraani show in [3] that for  $\omega^0$  in  $B_{\infty,1}^0$ ,

$$\|\tilde{u} - u\|_{L^\infty([0,T];B_{\infty,1}^0)} \leq c(\nu T)^{1/2}(1 + \nu T)^{1/2}e^{e^{\nu T}},$$

where the constants depend on the  $B_{\infty,1}^0$  norm of  $\omega^0$ . This is the same rate (in a different space) as that in (2.5) with  $\kappa = 0$ , up to the dependence of constants on time and on the initial data.

### 3. BASIC VANISHING VISCOSITY ARGUMENT

We now begin the proof of Theorem 2.3. Let

$$u_n = \text{the solution to (E) with initial velocity } u_n^0,$$

where  $u_n^0$ ,  $n = 1, 2, \dots$ , is a divergence-free initial velocity smoothed to lie in  $C^\infty$  and such that  $u_n^0 \rightarrow u^0$  in  $L^2(\mathbb{R}^2)$  as  $n \rightarrow \infty$ .

Letting

$$X = L^\infty([0, T]; L^2(\mathbb{R}^2))$$

we have, for any solution  $u$  to (E) in  $L^\infty([0, \infty); H^1(\mathbb{R}^2))$ ,

$$\|\tilde{u} - u\|_X \leq \|\tilde{u} - u_n\|_X + \|u - u_n\|_X.$$

A straightforward energy argument (see [4] for instance) shows that

$$\begin{aligned} \|\tilde{u}(t) - u_n(t)\|_{L^2}^2 &\leq C\nu t \|\omega^0\|_{L^2} \|\omega(u_n^0)\|_{L^2} + \|u^0 - u_n^0\|_{L^2}^2 \\ &\quad + 2 \int_0^t \int_{\mathbb{R}^2} |\tilde{u}(s, x) - u_n(s, x)|^2 |\nabla u_n(s, x)| \, dx \, ds. \end{aligned}$$

As long as we insure that the initial velocity is smoothed in such a way that

$$\|\omega(u_n^0)\|_{L^2} \leq C \|\omega^0\|_{L^2} \tag{3.1}$$

we can conclude from Gronwall's inequality that

$$\|\tilde{u}(t) - u_n(t)\|_{L^2}^2 \leq (C\nu t \|\omega^0\|_{L^2}^2 + \|u^0 - u_n^0\|_{L^2}^2) e^{2 \int_0^t \|\nabla u_n\|_{L^\infty}}$$

so

$$\|\tilde{u} - u_n\|_X \leq \left( (C\nu T)^{1/2} \|\omega^0\|_{L^2} + \|u^0 - u_n^0\|_{L^2} \right) e^{\int_0^T \|\nabla u_n\|_{L^\infty}},$$

using  $(A^2 + B^2)^{1/2} \leq A + B$  for  $A, B \geq 0$ .

The energy argument for bounding  $\|u - u_n\|_X$  is identical except that the term involving  $\nu$  is absent and of course we have  $u$  in place of  $\tilde{u}$ . (In this energy argument, although the norm of  $u(t)$  in  $H^1(\mathbb{R}^2)$  does not appear, the membership of  $u(t)$  in  $H^1(\mathbb{R}^2)$  for almost all  $t$  is required to insure the

vanishing of one of the two nonlinear terms, so we *are* using the membership of  $u$  in  $L^\infty([0, \infty); H^1(\mathbb{R}^2))$ .

We thus have

$$\|u - u_n\|_X \leq \|u^0 - u_n^0\|_{L^2} e^{\int_0^T \|\nabla u_n\|_{L^\infty}} \quad (3.2)$$

and so

$$\begin{aligned} \|\tilde{u} - u\|_X &\leq (C\nu T)^{1/2} \|\omega^0\|_{L^2} e^{\int_0^T \|\nabla u_n\|_{L^\infty}} \\ &\quad + 2\|u^0 - u_n^0\|_{L^2} e^{\int_0^T \|\nabla u_n\|_{L^\infty}}. \end{aligned} \quad (3.3)$$

Now suppose we can show that for some sequence  $(u_n^0)_{n=1}^\infty$  of approximations to  $u^0$  satisfying (3.1),

$$\|u^0 - u_n^0\|_{L^2} e^{\int_0^T \|\nabla u_n\|_{L^\infty}} \rightarrow 0 \text{ as } \nu \rightarrow 0. \quad (3.4)$$

Then letting  $n = f(\nu)$  with  $f(\nu) \rightarrow \infty$  as  $\nu \rightarrow 0$ , the second term in (3.3) will vanish with the viscosity. By choosing  $f$  to increase to infinity sufficiently slowly, we can always make the first term in (3.3) vanish with the viscosity as well. Thus, to establish the vanishing viscosity limit, we need only show that (3.4) holds; to determine a bound on the rate of convergence, however, we must choose the function  $f$  explicitly.

What we have done is in effect decouple the vanishing viscosity limit from the Navier-Stokes equations and from the viscosity itself. Also, we have yet to use the information we gain from  $\omega^0$  lying in  $B_\Gamma$ ; this information is encoded in the approximate solution  $u_n$  and will be exploited in the next section.

#### 4. CONVERGENCE IN (3.4)

To smooth the initial velocity let

$$u_n^0 = S_n u^0.$$

Then  $\omega_n^0 = S_n \omega^0$  and (3.1) is satisfied. Also,

$$\begin{aligned} \|u^0 - u_n^0\|_{L^2} &= \|(\text{Id} - S_n)u^0\|_{L^2} = \left\| \sum_{q=n+1}^{\infty} \Delta_q u^0 \right\|_{L^2} \leq \sum_{q=n+1}^{\infty} \|\Delta_q u^0\|_{L^2} \\ &\leq C \sum_{q=n+1}^{\infty} 2^{-q} \|\Delta_q \nabla u^0\|_{L^2} \\ &\leq C \left( \sum_{q=n+1}^{\infty} 2^{-2q} \right)^{1/2} \left( \sum_{q=n+1}^{\infty} \|\Delta_q \nabla u^0\|_{L^2}^2 \right)^{1/2} \\ &\leq C 2^{-n} \left( \sum_{q=n+1}^{\infty} \|\Delta_q \omega^0\|_{L^2}^2 \right)^{1/2} \leq C \|\omega^0\|_{L^2} 2^{-n}, \end{aligned}$$

where we used Minkowski's inequality, Bernstein's inequality, and the Cauchy-Schwarz inequality. From Lemma 4.1, below,

$$\begin{aligned} \|\nabla u_n(t)\|_{L^\infty} &\leq C \left( \|\omega_n^0\|_{L^2} + \|\omega_n^0\|_{B_{\infty,1}^0} \right) e^{Ct\|\omega_n^0\|_{B_{\infty,1}^0}} \\ &\leq C \left( \|\omega^0\|_{L^2} + \alpha\Gamma(n) \right) e^{Ct\alpha\Gamma(n)} \leq C\alpha\Gamma(n)e^{C\alpha t\Gamma(n)}, \end{aligned} \quad (4.1)$$

where  $\alpha = \|\omega^0\|_{B_\Gamma}$ . When  $\lim_{n \rightarrow \infty} \Gamma(n) = \infty$ , (4.1) holds for an absolute constant  $C$  for all sufficiently large  $n$ ; it holds for all  $n$  for a constant that depends upon the initial vorticity. (See Remark (2.1)). This applies as well to the inequalities that follow. Also, in (4.1) we used

$$\|\omega_n^0\|_{B_{\infty,1}^0} = \sum_{q \geq -1} \|\Delta_q \omega_n^0\|_{L^\infty} \leq \sum_{q=-1}^{n+1} \|\Delta_q \omega^0\|_{L^\infty} \leq \alpha\Gamma(n).$$

Thus,

$$\int_0^T \|\nabla u_n(t)\|_{L^\infty} \leq \frac{C\alpha\Gamma(n)}{C\alpha\Gamma(n)} \left( e^{C\alpha T\Gamma(n)} - 1 \right) \leq e^{C\alpha T\Gamma(n)}$$

and

$$\|u^0 - u_n^0\|_{L^2} e^{\int_0^T \|\nabla u_n\|_{L^\infty}} \leq C\|\omega^0\|_{L^2} 2^{-n} \exp\left(e^{C\alpha T\Gamma(n)}\right). \quad (4.2)$$

To bound the rate of convergence of  $\tilde{u}$  to  $u$ , we must decide how to choose  $n$  as a function of  $\nu$  in (3.3). Using (4.2), we have

$$\|\tilde{u} - u\|_X \leq C\|\omega^0\|_{L^2} \left( (\nu T)^{1/2} + 2^{-n} \right) \exp\left(e^{C\alpha T\Gamma(n)}\right).$$

Viewing this as a sum of two rates, when  $n = -(1/2)\log(\nu T)$  the two rates are equal. If  $n$  increases more rapidly as  $\nu \rightarrow 0$  then the first term decreases more slowly as  $\nu \rightarrow 0$ ; if  $n$  increases more slowly as  $\nu \rightarrow 0$  then the second term decreases more slowly as  $\nu \rightarrow 0$ . Since the slower decreasing of the two terms limits the convergence rate, we conclude that letting  $n = -(1/2)\log(\nu T)$  optimizes the convergence rate, giving the bound in Theorem 2.3 and completing its proof.

**Lemma 4.1.** *Let  $v$  be a  $C^\infty$ -solution to (E) with initial velocity  $v^0$ , where  $\omega^0$  is in  $L^{p_0} \cap B_{\infty,1}^0$ , with  $p_0$  in  $(1, \infty)$ . Then*

$$\|\nabla v(t)\|_{L^\infty} \leq C \left( \|\omega^0\|_{L^{p_0}} + \|\omega^0\|_{B_{\infty,1}^0} \right) e^{Ct\|\omega^0\|_{B_{\infty,1}^0}}.$$

*Proof.* We have,

$$\begin{aligned} \|\nabla v(t)\|_{L^\infty} &\leq \|\Delta_{-1}\nabla v(t)\|_{L^\infty} + \sum_{q \geq 0} \|\Delta_q \nabla v(t)\|_{L^\infty} \\ &\leq C\|\Delta_{-1}\omega(t)\|_{L^{p_0}} + C \sum_{q \geq 0} \|\Delta_q \omega(t)\|_{L^\infty} \\ &\leq C\|\omega^0\|_{L^{p_0}} + C\|\omega(t)\|_{B_{\infty,1}^0}. \end{aligned}$$



Here we used Bernstein's inequality with the Calderon-Zygmund inequality for the first term and Lemma 4.2 for the sum.

From Theorem 4.2 of [8],

$$\|\omega(t)\|_{B_{\infty,1}^0} \leq C(1 + \log(\|g(t)\|_{L^p} \|g^{-1}(t)\|_{L^p})) \|\omega^0\|_{B_{\infty,1}^0},$$

where  $g$  is the flow associated to  $v$ ; that is,

$$g(t, x) = x + \int_0^t v(s, g(s, x)) ds.$$

It follows from Gronwall's inequality that

$$\|g(t)\|_{L^p}, \|g^{-1}(t)\|_{L^p} \leq \exp \int_0^t \|\nabla v(s)\|_{L^\infty} ds.$$

Combining the three inequalities above gives

$$\|\nabla v(t)\|_{L^\infty} \leq C\|\omega^0\|_{L^{p_0}} + C \left(1 + 2 \int_0^t \|\nabla v(s)\|_{L^\infty} ds\right) \|\omega^0\|_{B_{\infty,1}^0},$$

and the proof is completed by another application of Gronwall's inequality.  $\square$

In the proof of Lemma 4.1 we used the existence of a flow associated with a smooth solution to (E), which allowed us to apply Theorem 4.2 of [8]. This is where our approach differs markedly from that of Vishik's in [9], where required properties of the flow are inferred from the membership of the vorticity in the spaces  $L^{p_0} \cap B_\Gamma$  and  $L^{p_0} \cap L^{p_1}$  and where the constraints on the values of  $p_0$  and  $p_1$  of Theorem 2.2 are required. Vishik also requires that  $p_0 < 2$  so that the velocity can be recovered uniquely from the vorticity using the Biot-Savart law, since he uses the vorticity formulation of a weak solution to (E) in [9]. By contrast, in Theorem 2.3 we in effect require that  $p_0 = p_1 = 2$ , so that we can make the basic energy argument in Section 3.

It is also possible to prove Lemma 4.1 using an argument like that in [3].

**Lemma 4.2.** *Let  $v$  be a divergence-free vector field in  $L_{loc}^2(\mathbb{R}^2)$  with vorticity  $\omega$ . Then there exists an absolute constant  $C$  such that for all  $q \geq 0$  (that is, avoiding the low frequencies),*

$$\|\Delta_q \nabla v\|_{L^\infty} \leq C \|\Delta_q \omega\|_{L^\infty}.$$

*Proof.* Since  $v$  is a divergence-free vector field in  $L_{loc}^2(\mathbb{R}^2)$  it possesses a (unique) stream function  $\psi$ ; that is,  $v = \nabla^\perp \psi = (-\partial_2 \psi, \partial_1 \psi)$ , and  $\omega = \Delta \psi$ . Therefore  $\Delta_q v = \nabla^\perp \Delta_q \psi$  and  $\Delta_q \psi = \Delta^{-1} \Delta_q \omega$ , so  $\nabla \Delta_q v = \nabla \nabla^\perp \Delta^{-1} \Delta_q \omega$ . It follows that

$$\|\Delta_q \nabla v\|_{L^\infty} \leq C \sup_{i,j} \|\Delta_q \partial_i \partial_j \Delta^{-1} \omega\|_{L^\infty}.$$

But,

$$\begin{aligned} \|\Delta_q \partial_i \partial_j \Delta^{-1} \omega\|_{L^\infty} &= \|\mathcal{F}^{-1}(\varphi_q(\xi) \frac{\xi_i \xi_j}{|\xi|^2} \widehat{\omega}(\xi))\|_{L^\infty} \\ &= \|\mathcal{F}^{-1}(\varphi_q(\xi) h_q(\xi) \widehat{\omega}(\xi))\|_{L^\infty} = \|\Delta_q(\check{h}_q(\xi) * \omega)\|_{L^\infty}, \end{aligned}$$

where

$$h_q(\xi) = \chi(2^{-3-q}\xi)(1 - \chi(2^{-q+1}\xi)) \frac{\xi_i \xi_j}{|\xi|^2},$$

$\chi$  and  $\varphi$  being defined in Lemma 2.1. Observe that  $h_q = 1$  on the support of  $\varphi_q$ . Because  $h_q(\xi) = h(2^{-q}\xi)$ , where

$$h(\xi) = \chi(2^{-3}\xi)(1 - \chi(2\xi)) \frac{\xi_i \xi_j}{|\xi|^2},$$

$\check{h}_q(x) = 2^{2q} \check{h}(2^q x)$ , and thus by a change of variables,  $\|\check{h}_q\|_{L^1} = \|\check{h}\|_{L^1} = C$ .

Then using Young's convolution inequality,

$$\begin{aligned} \|\Delta_q(\check{h}_q(\xi) * \omega)\|_{L^\infty} &= \|\check{h}_q(\xi) * \Delta_q \omega\|_{L^\infty} \leq \|\check{h}_q\|_{L^1} \|\Delta_q \omega\|_{L^\infty} \\ &\leq C \|\Delta_q \omega\|_{L^\infty}, \end{aligned}$$

which completes the proof.  $\square$

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