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VANISHING VISCOSITY AND THE ACCUMULATION OF VORTICITY ON THE BOUNDARY †

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Abstract. We say that the vanishing viscosity limit holds in the classical sense if the velocity for a solution to the Navier-Stokes equations converges in the energy norm uniformly in time to the velocity for a solution to the Euler equations. We prove, for a bounded domain in dimension 2 or higher, that the vanishing viscosity limit holds in the classical sense if and only if a vortex sheet forms on the boundary.

Key words. Vanishing viscosity, incompressible fluid mechanics

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1. Introduction

It is well known that for radially symmetric initial vorticity in a disk the velocity for a solution to the Navier-Stokes equations converges in the energy norm uniformly in time to the velocity for a solution to the Euler equations. It was shown recently in [8] that for such initial data in a disk, it also happens that a vortex sheet forms on the boundary as the viscosity vanishes. By a vortex sheet, we mean a velocity field whose vorticity, as a finite Borel signed measure (an element of the dual space of $C(\overline{\Omega})$), is supported along a curve—the boundary, in this case.

It turns out that this phenomenon is in a sense more universal: either both types of limits hold or neither holds for an arbitrary bounded domain in $\mathbb{R}^d$, $d \geq 2$, with C^2-boundary and with no particular assumption on the symmetry of the initial data. More precisely, the vanishing viscosity limit in the classical sense (condition (B) of Section 3) holds if and only if a vortex sheet of a particular type forms on the boundary (conditions (E) and (E_2) of Section 3). Now, however, the vortex sheet has vorticity belonging to the dual space of $H^1(\Omega)$ rather than $C(\overline{\Omega})$. We show this in Theorem 3.1 for no-slip boundary conditions and in Theorem 5.1 for characteristic boundary conditions on the velocity.

2. Background

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with C^2-boundary Γ, and let $\mathbf{n}$ be the outward unit normal vector to Γ. A classical solution $(\overline{u}, \overline{p})$ to the Euler equations satisfies

$$(EE) \begin{cases} \partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{p} = \overline{f} \text{ and } \operatorname{div} \overline{u} = 0 \text{ on } [0,T] \times \Omega, \\ \overline{u} \cdot \mathbf{n} = 0 \text{ on } [0,T] \times \Gamma \text{ and } \overline{u} = \overline{u}^0 \text{ on } \{0\} \times \Omega. \end{cases}$$

These equations describe the motion of an incompressible fluid of constant density and zero viscosity. The initial velocity $\overline{u}^0$ must at least lie in

$$H = \left\{ u \in (L^2(\Omega))^d : \operatorname{div} u = 0 \text{ in } \Omega, u \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\}$$

endowed with the L^2-norm, which, along with

$$V = \left\{ u \in (H^1(\Omega))^d : \operatorname{div} u = 0 \text{ in } \Omega, u = 0 \text{ on } \Gamma \right\}$$

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endowed with the H^1-norm, are the classical spaces of fluid mechanics.

We assume that $\overline{u}^0$ is in $C^{k+\epsilon}(\Omega)\cap H$, $\epsilon>0$, where $k=1$ for two dimensions and $k=2$ for 3 and higher dimensions, and that $\overline{f}$ is in $C^1([0,t]\times\Omega)$ for all $t>0$. Then as shown in [7] (Theorem 1 and the remarks on p. 508-509), there is some $T>0$ for which there exists a unique solution,

$$\overline{u} \text{ in } C^1([0,T];C^{k+\epsilon}(\Omega)), \tag{2.1}$$

to (EE). In two dimensions, T can be arbitrarily large, though it is only known that some nonzero T exists in three and higher dimensions.

The Navier-Stokes equations describe the motion of an incompressible fluid of constant density and positive viscosity ν. A classical solution to the Navier-Stokes equations with no-slip boundary conditions can be defined in analogy to (EE) by

$$(NS)\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + f \text{ and } \operatorname{div} u = 0 \text{ on } [0,T]\times\Omega, \\ u=0 \text{ on } [0,T]\times\Gamma \text{ and } u = u_\nu^0 \text{ on } \{0\}\times\Omega, \end{cases}$$

where u_ν^0 is in H and f is in $L^1([0,T];L^2(\Omega))$. We will work, however, with weak solutions to the Navier-Stokes equations. (See, for instance, Chapter III of [10].) Such weak solutions lie in $L^\infty([0,T];H)\cap L^2([0,T];V)$.

In Section 3 we prove various equivalent forms of the vanishing viscosity limit for no-slip boundary conditions, including the formation of a vortex sheet on the boundary, and remark briefly on their derivation in Section 4. In Section 5 we extend the results of Section 2 to characteristic boundary conditions. We discuss, in Section 6, our results in relation to those in [8] on vortex sheet formation for a disk. Finally, in Section 7, we include some technical lemmas employed in Section 3.

3. Equivalent forms of the vanishing viscosity limit

Let $\overline{u}$ be a classical solution to (EE) in Ω and $u=u^\nu$ be a weak solution to (NS) in Ω as in Section 2, and assume that $u_\nu^0\to\overline{u}^0$ in H and $f\to\overline{f}$ in $L^1([0,T];L^2(\Omega))$ as $\nu\to 0$.

Let $\gamma_{\mathbf{n}}$ be the boundary trace operator for the normal component of a vector field (see Lemma 7.1). Let $\mathcal{M}(\Omega)$ be the space of finite Borel signed measures on $\overline{\Omega}$—$\mathcal{M}(\Omega)$ is the dual space of $C(\overline{\Omega})$. Let μ in $\mathcal{M}(\overline{\Omega})$ be the measure supported on Γ for which $\mu|_\Gamma$ corresponds to Lebesgue measure on Γ (arc length for $d=2$, area for $d=3$, etc.). Then μ is also a member of $H^1(\Omega)'$.

We define the vorticity $\omega(u)$ to be the $d\times d$ antisymmetric matrix

$$\omega(u) = \frac{1}{2}\left[\nabla u - (\nabla u)^T\right]. \tag{3.1}$$

When working specifically in two dimensions, we can alternately define the vorticity as the scalar curl of u:

$$\omega(u) = \partial_1 u^2 - \partial_2 u^1. \tag{3.2}$$

Letting $\omega = \omega(u)$ and $\overline{\omega} = \omega(\overline{u})$, we define the following conditions:

$$
\begin{aligned}
&(A) && u \to \overline{u} \text{ weakly in } H \text{ uniformly on } [0,T],\\
&(A') && u \to \overline{u} \text{ weakly in } (L^2(\Omega))^d \text{ uniformly on } [0,T],\\
&(B) && u \to \overline{u} \text{ in } L^\infty([0,T];H),\\
&(C) && \nabla u \to \nabla \overline{u} - \langle \gamma_{\mathbf{n}} \cdot, \overline{u}\mu\rangle \text{ in } ((H^1(\Omega))^{d\times d})' \text{ uniformly on } [0,T],\\
&(D) && \nabla u \to \nabla \overline{u} \text{ in } (H^{-1}(\Omega))^{d\times d} \text{ uniformly on } [0,T],\\
&(E) && \omega \to \overline{\omega} - \frac{1}{2}\left\langle \gamma_{\mathbf{n}}(\cdot - \cdot^T), \overline{u}\mu\right\rangle \text{ in } ((H^1(\Omega))^{d\times d})' \text{ uniformly on } [0,T].
\end{aligned}
$$

In conditions (C), (D), and (E), the convergence is in the weak* topology of the given spaces. In (C) and (E), $((H^1(\Omega))^{d\times d})'$ is the dual space of $(H^1(\Omega))^{d\times d}$; in (D), $(H^{-1}(\Omega))^{d\times d}$ is the dual space of $(H_0^1(\Omega))^{d\times d}$. Thus, condition (C) means that

$$
(\nabla u(t), M) \to (\nabla \overline{u}(t), M) - \int_\Gamma (M \cdot \mathbf{n}) \cdot \overline{u}(t) \text{ in } L^\infty([0,T])
$$

for any M in $(H^1(\Omega))^{d\times d}$, condition (D) means that

$$
(\nabla u(t), M) \to (\nabla \overline{u}(t), M) \text{ in } L^\infty([0,T])
$$

for any M in $(H_0^1(\Omega))^{d\times d}$, and condition (E) means that

$$
(\omega(t), M) \to (\overline{\omega}(t), M) - \frac{1}{2}\int_\Gamma ((M - M^T) \cdot \mathbf{n}) \cdot \overline{u}(t) \text{ in } L^\infty([0,T])
$$

for any M in $(H^1(\Omega))^{d\times d}$.

In two dimensions, defining the vorticity as in Equation (3.2), we also define the following two conditions:

$$
\begin{aligned}
&(E_2) && \omega \to \overline{\omega} - (\overline{u} \cdot \boldsymbol{\tau})\mu \text{ in } (H^1(\Omega))' \text{ uniformly on } [0,T],\\
&(F_2) && \omega \to \overline{\omega} \text{ in } H^{-1}(\Omega) \text{ uniformly on } [0,T].
\end{aligned}
$$

Here, $\boldsymbol{\tau}$ is the unit tangent vector on Γ that is obtained by rotating the outward unit normal vector $\mathbf{n}$ counterclockwise by 90 degrees.

Condition (E_2) means that

$$
(\omega(t), f) \to (\overline{\omega}(t), f) - \int_\Gamma (\overline{u}(t) \cdot \boldsymbol{\tau}) f \text{ in } L^\infty([0,T])
$$

for any f in $H^1(\Omega)$, while condition (F_2) means that

$$
(\omega(t), f) \to (\overline{\omega}(t), f) \text{ in } L^\infty([0,T])
$$

for any f in $H_0^1(\Omega)$.

Theorem 3.1. *Conditions (A), (A'), (B), (C), (D), and (E) are equivalent. In two dimensions, conditions (E_2) and (F_2) are equivalent to the other conditions when Ω is simply connected.*

The last 5 words were not, but should have been, in the published version.

Proof. $(\mathbf{A}) \iff (\mathbf{A'})$: Let v be in $(L^2(\Omega))^d$. By Lemma 7.3, $v = w + \nabla p$, where w is in H and p is in $H^1(\Omega)$. Then assuming (A) holds,

$$
(u(t), v) = (u(t), w) \to (\overline{u}(t), w) = (\overline{u}(t), v)
$$

uniformly over t in $[0,T]$, so (A') holds. The converse is immediate.

(A) $\iff$ **(B)**: The forward implication is proved in Theorem 1 of [4]. The backward implication is immediate.

(A′) $\implies$ **(C)**: Assume that (A') holds and let M be in $(H^1(\Omega))^{d\times d}$. Then

$$(\nabla u(t), M) = -(u(t), \operatorname{div} M) \to -(\overline{u}(t), \operatorname{div} M) \text{ in } L^\infty([0,T]).$$

But,

$$-(\overline{u}(t), \operatorname{div} M) = (\nabla \overline{u}(t), M) - \int_\Gamma (M \cdot \mathbf{n}) \cdot \overline{u},$$

giving (C).

(C) $\implies$ **(D)**: This follows simply because $H_0^1(\Omega) \subseteq H^1(\Omega)$.

(D) $\implies$ **(A)**: Assume (D) holds, and let v be in H. Then $v = \operatorname{div} M$ for some M in $(H_0^1(\Omega))^{d\times d}$ by Corollary 7.5, so

$$\begin{aligned}(u(t), v) &= (u(t), \operatorname{div} M) = -(\nabla u(t), M) + \int_\Gamma (M \cdot \mathbf{n}) \cdot u(t) \\ &= -(\nabla u(t), M) \to -(\nabla \overline{u}(t), M),\end{aligned}$$

uniformly over $[0,T]$. But,

$$-(\nabla \overline{u}(t), M) = (\overline{u}(t), \operatorname{div} M) - \int_\Gamma (M \cdot \mathbf{n}) \cdot \overline{u}(t) = (\overline{u}(t), v),$$

from which (A) follows.

Now assume that $d = 2$.

(A′) $\implies$ **(E$_2$)**: Assume that (A') holds and let f be in $H^1(\Omega)$. Then

$$\begin{aligned}(\omega(t), f) &= -(\operatorname{div} u^\perp(t), f) = (u^\perp(t), \nabla f) = -(u(t), \nabla^\perp f) \\ &\to -(\overline{u}(t), \nabla^\perp f) \text{ in } L^\infty([0,T])\end{aligned}$$

where $u^\perp = -\langle u^2, u^1 \rangle$ and we used the identity $\omega(u) = -\operatorname{div} u^\perp$. But,

$$\begin{aligned}-(\overline{u}(t), \nabla^\perp f) &= (\overline{u}^\perp(t), \nabla f) = -(\operatorname{div} \overline{u}^\perp(t), f) + \int_\Gamma (\overline{u}(t)^\perp \cdot \mathbf{n}) f \\ &= -(\operatorname{div} \overline{u}^\perp(t), f) - \int_\Gamma (\overline{u}(t) \cdot \boldsymbol{\tau}) f = (\overline{\omega}(t), f) - \int_\Gamma (\overline{u}(t) \cdot \boldsymbol{\tau}) f,\end{aligned}$$

giving (E_2).

(E$_2$) $\implies$ **(F$_2$)**: Follows for the same reason that $(C) \implies (D)$.

(F$_2$) $\implies$ **(A)**: Assume (F_2) holds, and let v be in H. Then $v = \nabla^\perp f$ for some f in $H_0^1(\Omega)$ (f is called the stream function for v), and

$$\begin{aligned}(u(t), v) &= (u(t), \nabla^\perp f) = -(u^\perp(t), \nabla f) = (\operatorname{div} u^\perp(t), f) \\ &= -(\omega(t), f) \to -(\overline{\omega}(t), f) \text{ in } L^\infty([0,T]).\end{aligned}$$

But,

$$-(\overline{\omega}(t),f)=(\operatorname{div}\overline{u}^{\perp}(t),f)=-(\overline{u}^{\perp}(t),\nabla f)=(\overline{u}(t),\nabla^{\perp}f)$$
$$=(\overline{u}(t),v),$$

which shows that (A) holds.

What we have shown so far is that (A), (A'), (B), (C), and (D) are equivalent, as are (E_2) and (F_2) in two dimensions. It remains to show that (E) is equivalent to these conditions as well. We do this by establishing the implications $(C) \Longrightarrow (E) \Longrightarrow (A)$.

(C) $\Longrightarrow$ **(E)**: Follows directly from Equation (3.1).

(E) $\Longrightarrow$ **(A)**: Let v be in H and let x be the vector field in $(H^2(\Omega)\cap H_0^1(\Omega))^d$ solving $\Delta x = v$ on Ω (x exists and is unique by standard elliptic theory). Then, utilizing Lemma 7.6 twice (and suppressing the explicit dependence of u and $\overline{u}$ on t),

$$\begin{aligned}(u,v)&=(u,\Delta x)=-(\nabla u,\nabla x)+\int_\Gamma(\nabla x\cdot\mathbf{n})\cdot u=-(\nabla u,\nabla x)\\&=-2(\omega(u),\omega(x))-\int_\Gamma(\nabla u x)\cdot\mathbf{n}=-2(\omega(u),\omega(x))\\&\to -2(\omega(\overline{u}),\omega(x))+2\frac{1}{2}\int_\Gamma((\omega(x)-\omega(x)^T)\cdot\mathbf{n})\cdot\overline{u}\\&=-2(\omega(\overline{u}),\omega(x))+2\int_\Gamma(\omega(x)\cdot\mathbf{n})\cdot\overline{u}\\&=-(\nabla\overline{u},\nabla x)+\int_\Gamma(\nabla\overline{u}x)\cdot\mathbf{n}+2\int_\Gamma(\omega(x)\cdot\mathbf{n})\cdot\overline{u}\\&=-(\nabla\overline{u},\nabla x)+2\int_\Gamma(\omega(x)\cdot\mathbf{n})\cdot\overline{u}\\&=(\overline{u},\Delta x)-\int_\Gamma(\nabla x\cdot\mathbf{n})\cdot\overline{u}+2\int_\Gamma(\omega(x)\cdot\mathbf{n})\cdot\overline{u}\\&=(\overline{u},v)-\int_\Gamma((\nabla x)^T\cdot\mathbf{n})\cdot\overline{u}.\end{aligned}\tag{3.3}$$

Thus, $(E) \Longrightarrow (A)$ if and only if

$$\int_\Gamma((\nabla x)^T\cdot\mathbf{n})\cdot\overline{u}=0.\tag{3.4}$$

But, $(\operatorname{div}(\nabla x)^T)^j=\partial_j\partial_i x^j=\partial_i\operatorname{div}x$ or $\operatorname{div}(\nabla x)^T=\nabla\operatorname{div}x$. Similarly, $\operatorname{div}(\nabla\overline{u})^T=\nabla\operatorname{div}\overline{u}=0$. It follows that

$$\begin{aligned}\int_\Gamma((\nabla x)^T\cdot\mathbf{n})\cdot\overline{u}&=((\nabla x)^T,\nabla\overline{u})+(\nabla\operatorname{div}x,\overline{u})\\&=(\nabla x,(\nabla\overline{u})^T)-(\operatorname{div}x,\operatorname{div}\overline{u})+\int_\Gamma\overline{u}\cdot\mathbf{n}\operatorname{div}x=((\nabla\overline{u})^T,\nabla x)\\&=-(\operatorname{div}(\nabla\overline{u})^T,x)+\int_\Gamma((\nabla\overline{u})^T\cdot\mathbf{n})x=0.\end{aligned}$$

□

4. Remarks

The equivalent conditions of Theorem 3.1 complement those of [4], [11], [12], and [6].

It is only in the proof of $(A) \implies (B)$—in which we quote a result of Kato's in [4]—where the requirement that $\overline{u}$ be a classical solution to the Euler equations and that $f \to \overline{f}$ in $L^1([0,T];L^2)$ is used; in fact, it is the only place where the fact that u and $\overline{u}$ are solutions to the Navier-Stokes and Euler equations, respectively, appear in the proof at all. That is, assuming only that u is a vector field parameterized by ν that lies in $L^\infty([0,T];H) \cap L^2([0,T];V)$ and that $\overline{u}$ is a vector field lying in $L^\infty([0,T];H \cap H^1(\Omega))$, all of the implications in the proof of Theorem 3.1 remain valid except for $(A) \implies (B)$.

The proof of $(A) \implies (B)$ in [4] consists, using our terminology, of proving $(B) \implies (A) \implies (i) \implies (ii) \implies (B)$, where (i) and (ii) are the conditions,

$$(i) \qquad \nu \int_0^T \|\nabla u\|^2_{L^2(\Omega)} \to 0,$$

$$(ii) \qquad \nu \int_0^T \|\nabla u\|^2_{L^2(\Gamma_{c\nu})} \to 0,$$

with $\Gamma_{c\nu}$ a boundary layer of width proportional to ν.

If we weaken the regularity of Γ from C^2 to only locally Lipschitz, then the proof of $(E) \implies (A)$ fails because we would only have x in $(H^1_0(\Omega))^d$. Kato's proof that $(A) \implies (B)$ also requires a C^2 boundary.

In two dimensions, we need only have convergence of the vorticity away from the boundary—condition (F_2)—to insure that the vanishing viscosity limit holds. In particular, it follows that formation of a vortex sheet on the boundary of a type other than that given in (E_2) is inconsistent with u being a solution to (NS). In higher dimensions it is an open problem whether the analogous statement is true; that is, whether $(F) \implies (A)$, where (F) is the condition,

$$(F) \qquad \omega \to \overline{\omega} \text{ in } (H^{-1}(\Omega))^{d \times d} \text{ uniformly on } [0,T].$$

The remarks that follow attempt to give some insight into the nature of this problem.

One approach to proving that $(F) \implies (A)$ is to prove that $(F) \implies (D)$, since we have $(D) \implies (A)$. So suppose that (F) holds, and let M be in $(H^1_0(\Omega))^{d \times d}$. For any vector field v,

$$(\nabla v, M) = (\nabla v - (\nabla v)^T, M) + ((\nabla v)^T, M) = 2(\omega(v), M) + (\nabla v, M^T).$$

Thus,

$$(\nabla u, M - M^T) = 2(\omega(u), M) \to 2(\omega(\overline{u}), M) = (\nabla \overline{u}, M - M^T).$$

If M is antisymmetric then $M - M^T = 2M$ and we conclude that (D) holds for antisymmetric matrix fields in $(H^1_0(\Omega))^{d \times d}$. But if M in $(H^1_0(\Omega))^{d \times d}$ is symmetric,

$$2(\omega(u), M) = (\nabla u, M) - ((\nabla u)^T, M) = (\nabla u, M - M^T) = 0,$$

so $(\omega(u), M) = (\omega(\overline{u}), M) = 0$, and we can conclude nothing from this approach.

But some use can still be made of this observation. Let v be any element of H. Then from Corollary 7.5, for some M in $(H^1_0(\Omega))^{d \times d}$,

$$(u, v) = (u, \operatorname{div} M) = -(\nabla u, M).$$

Now, if we could insure that M can be chosen to be antisymmetric, then if (F) holds for M so does (D), as we just showed, and

$$-(\nabla u, M) \to -(\nabla \overline{u}, M) = (\overline{u}, v),$$

and (A) would follow.

In two dimensions, we can choose

$$M = \begin{pmatrix} 0 & -f \\ f & 0 \end{pmatrix}, \tag{4.1}$$

where f is the stream function for v as in the proof of $(F_2) \implies (A)$, which gives a slight variation on the proof of that same implication.

In three dimensions, however, it is not possible to find such an M. To see this, suppose that M in $(H_0^1(\Omega))^{d \times d}$ is antisymmetric. Then can write M as

$$M = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

with $a = b = c = 0$ on Γ. In this form the condition $\operatorname{div} u = \operatorname{div} \operatorname{div} M = 0$ is automatically satisfied, and letting $\tilde{\omega}$ be the vector $\langle c, -b, a \rangle$, we see that

$$u = \operatorname{div} M = \operatorname{curl} \tilde{\omega}, \tag{4.2}$$

where curl is the usual three-dimensional operator. But curl maps $H \cap C^\infty(\Omega)$ bijectively onto itself when Γ is C^∞ (see, for instance, [2]), so in general we only have $\tilde{\omega} \cdot \mathbf{n} = 0$ on Γ. That is, the condition that M be antisymmetric is not compatible with the condition that it vanish on Γ.

Finally, let

$$E(\Omega) = \left\{ u \in (L^2(\Omega))^d \colon \operatorname{div} u \in L^2(\Omega), u \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\},$$

with $\|u\|_{E(\Omega)} = \|u\|_{L^2(\Omega)} + \|\operatorname{div} u\|_{L^2(\Omega)}$. It is easy to see from the proofs of $(A') \implies (C)$ and $(D) \implies (A)$ that condition (D) can be weakened from convergence in the dual space of $(H_0^1(\Omega))^{d \times d}$ to convergence in the dual space of $(E(\Omega))^d$. (This is advantageous as a sufficient condition, though not as a necessary one.)

Returning to Equation (4.2), the condition $\tilde{\omega} \cdot \mathbf{n} = 0$ on Γ does not translate to $M \cdot \mathbf{n} = 0$ on Γ. Hence, M does not lie in $(E(\Omega))^d$ so we cannot use this weakening of condition (D) to conclude that (A) holds.

5. Characteristic boundary conditions on the velocity

We modify (NS) by allowing the velocity on the boundary to be equal to a nonzero time-varying vector field b, which is required, however, to be tangential to the boundary. This gives

$$(NS_b) \begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f \text{ and } \operatorname{div} u = 0 \text{ on } [0,T] \times \Omega, \\ u = b \text{ on } [0,T] \times \Gamma \text{ and } u = u_\nu^0 \text{ on } \{0\} \times \Omega, \end{cases}$$

where, as before, u_ν^0 is in H.

We require sufficient regularity on b so that (NS_b) is well-posed. For simplicity, we will assume that $b \cdot \mathbf{n} = 0$ on Γ with

$$b \in L^\infty([0,T]; H^{3/2}(\Gamma)), \partial_t b \in L^2([0,T]; H^{-1/2}(\Gamma)) \tag{5.1}$$

so that b lifts (extends) to a vector field, which we also call b, with

$$b \in L^\infty([0,T]; H \cap H^2(\Omega)),\ \partial_t b \in L^2([0,T]; L^2(\Omega)).$$

The assumption on b in Equation (5.1) is not the weakest possible, but the assumption on $\partial_t b$ can probably not be weakened.

We can then use the equation that corresponds to $u - b$ (which lies in $L^\infty([0,T];H) \cap L^2([0,T];V)$ for classical solutions) to define a weak solution to (NS_b). (This is essentially what is done in Section 4 of [12].)

With such solutions to (NS_b) in place of those for (NS)—but without changing the formulation of (EE)—we define the condition,

$$(C^b) \qquad \nabla u \to \nabla \overline{u} - \langle \gamma_{\mathbf{n}} \cdot, (\overline{u} - b)\mu \rangle \text{ in } ((H^1(\Omega))^{d \times d})' \text{ uniformly on } [0,T],$$

$$(E^b) \qquad \omega \to \overline{\omega} - \frac{1}{2} \left\langle \gamma_{\mathbf{n}} (\cdot - \cdot^T), (\overline{u} - b)\mu \right\rangle \text{ in } ((H^1(\Omega))^{d \times d})' \text{ uniformly on } [0,T],$$

and in two dimensions, the condition,

$$(E_2^b) \qquad \omega \to \overline{\omega} - ((\overline{u} - b) \cdot \boldsymbol{\tau})\mu \text{ in } (H^1(\Omega))' \text{ uniformly on } [0,T].$$

Theorem 3.1 then becomes:

Theorem 5.1. *Let u be a solution to (NS_b) and $\overline{u}$ be a solution to (EE). Conditions (A), (A'), (B), (C^b), (D), and (E^b) are equivalent. In two dimensions, conditions (E_2^b) and (F_2) are equivalent to the other conditions.*

Proof. The proof of $(A') \implies (C^b)$ is identical to the proof of $(A') \implies (C)$ except that a boundary term is included in the first step: this term leads to the "$-b$" in condition (C^b). A similar comment applies to the proof of $(A') \implies (E_2^b)$ and $(C) \implies (E^b)$. The proof of $(C) \implies (D)$ and $(E_2) \implies (F_2)$ are unaffected by the presence of the vector field b, as are all other implications except for $(A) \implies (B)$ and $(E^b) \implies (A)$.

(A) $\implies$ **(B)**: In [12] it is shown, using our terminology, that $(B) \implies (i) \implies (ii') \implies (B)$, where (ii') is the condition

$$(ii') \qquad \nu \int_0^T \|\nabla_{\boldsymbol{\tau}} u_{\boldsymbol{\tau}}\|^2_{L^2(\Gamma_{\delta(\nu)})} \to 0 \text{ or } \nu \int_0^T \|\nabla_{\boldsymbol{\tau}} u_{\mathbf{n}}\|^2_{L^2(\Gamma_{\delta(\nu)})} \to 0.$$

Here, $\nabla_{\boldsymbol{\tau}}$ is the gradient only in the direction tangential to the boundary, $u_{\boldsymbol{\tau}}$ and $u_{\mathbf{n}}$ are the components of the velocity tangential and normal to the boundary, respectively, and $\Gamma_{\delta(\nu)}$ is a boundary layer whose width $\delta(\nu)$ is of arbitrary order larger than ν.

But, $(B) \implies (A)$ is immediate, and $(A) \implies (i)$ follows from combining the argument on pages 232 through 233 of [12] with the proof of the implication $(A) \implies (i)$ on page 90 of [4] (in Kato's terminology, this is (ii) implies (iii)). This gives $(A) \implies (B)$.

($\mathbf{E^b}$) $\implies$ **(A)**: Adapting the argument of $(E) \implies (A)$, we see that the first boundary integral in Equation (3.3) does not vanish, since now $u = b$ on Γ. Also, $\overline{u}$ becomes $\overline{u} - b$ in the boundary integrals involving $\omega(x)$. This leads to

$$\begin{aligned}(u,v) &= (\overline{u}, \Delta x) + \int_\Gamma (\nabla x \cdot \mathbf{n}) \cdot b - \int_\Gamma (\nabla x \cdot \mathbf{n}) \cdot \overline{u} + 2 \int_\Gamma (\omega(x) \cdot \mathbf{n}) \cdot (\overline{u} - b) \\ &= (\overline{u}, v) - \int_\Gamma ((\nabla x)^T \cdot \mathbf{n}) \cdot (\overline{u} - b).\end{aligned}$$

The last boundary integral vanishes for the same reason that Equation (3.4) holds, $\overline{u}-b$ being in H, completing the proof. □

6. Radially symmetric initial vorticity in a disk

We assume, in this section only, that Ω is the unit disk D and that the initial vorticity $\overline{\omega}^0$ is radially symmetric. In this case, the solution to (EE) is stationary: $\overline{\omega}(t)=\overline{\omega}^0$ for all time t.

The vanishing viscosity limit in the classical sense of condition (B) holds in this setting under fairly general circumstances. Under the assumptions of Section 2 on the regularity of the initial velocity, and assuming that $b=0$, this convergence is implicit in [4] at least for zero forcing (see [5]), but was first explicitly proved by Matsui (see [9]). For nonzero b having the regularity assumed in Equation (5.1), the convergence is a simple consequence of the condition (ii') in the proof of Theorem 5.1 as observed by Wang in [12]. For substantially lower regularity on $\overline{u}^0$ and on b than we assume, the convergence is established in [8].

More precisely, the authors of [8] assume that $u^0=\overline{u}^0$ and $f=\overline{f}=0$ (which are not significant limitations, since one can handle $u^0\to\overline{u}^0$ in H by using the triangle inequality, and nonzero forcing presents no real difficulties), with

$$\begin{aligned}\overline{u}^0\in R^1(D)&=\left\{v\in(L^2(D))^2\colon v(x)=s(|x|)x^\perp \text{ for some } s,\omega(v)\in L^1(D)\right\}\\&=\left\{v\in H\colon \omega(v)\in L^1(D),\omega(v)\text{ radially symmetric}\right\}.\end{aligned}$$

They assume that $b(t,\cdot)=\alpha(t)$—that is, $b(t)$ is constant on the boundary—and that $\alpha\in\mathrm{BV}([0,T])$, the space of bounded variation functions. They prove (combining Propositions 9.6 and 9.7 of [8]) that

$$\omega\to\overline{\omega}-(B(2\pi)^{-1}-b\cdot\boldsymbol{\tau})\mu \text{ in } \mathcal{M}(\overline{D}) \text{ uniformly on } [0,T], \tag{6.1}$$

where

$$B=\int_D\overline{\omega}^0.$$

But, on Γ, $u^0\cdot\boldsymbol{\tau}$ is constant, so by Green's theorem,

$$B=\int_\Gamma\overline{u}^0\cdot\boldsymbol{\tau}=2\pi\overline{u}^0(x)\cdot\boldsymbol{\tau}(x)$$

for any point x on Γ, and we see that Equation (6.1) is the same as condition (E_2^b), except that the convergence is stronger.

That is, both conditions (B) and (E_2^b) hold for a disk, except that the convergence in (E_2^b) is in $\mathcal{M}(\overline{\Omega})$, which is stronger convergence than that of (E_2^b). What we have shown is that either both conditions (B) and (E_2^b) hold or neither condition holds for a given initial velocity in a general bounded domain in the plane—and in the analogous sense, in $\mathbb{R}^d$. It was the question of whether this was, in fact, the case that motivated this paper.

The regularity we assume in Equation (5.1) corresponds to α lying in $H^1([0,T])$, which is considerably stronger than the assumption in [8] that α lie in $\mathrm{BV}([0,T])$. And their assumption on the regularity of $\overline{u}^0$ is far lower than our assumption that $\overline{u}^0$ lies in $C^{1+\epsilon}(\Omega)\cap H$. Without the assumption of radial symmetry, however, it seems unlikely that one can weaken our assumptions in Equation (5.1) on b to any significant degree, since these assumptions go to the heart of establishing the existence of the corresponding weak solutions of (NS_b). Weakening the regularity assumptions on $\overline{u}^0$ would seem equally impossible, since the boundedness of $\nabla\overline{u}$ on $[0,T]\times\Omega$ is indispensable in Kato's argument showing that $(A) \Longrightarrow (B)$.

7. Some technical lemmas

In this section we assume only that Ω is bounded and that Γ is locally Lipschitzian, which of course includes the case that Γ is C^2.

The various integrations by parts that we make are justified by Lemma 7.1, which is Theorem 1.2 p. 7 of [10] for locally Lipschitz domains. (Temam states the theorem for C^2 boundaries but the proof for locally Lipschitz boundaries is the same, using a trace operator for Lipschitz boundaries in place of that for C^2 boundaries: see p. 117-119 of [3], in particular, Theorem 2.1 p. 119.)

LEMMA 7.1. *Let*

$$E(\Omega)=\left\{v\in(L^2(\Omega))^d\colon\ \operatorname{div}v\in L^2(\Omega)\right\}$$

with $\|v\|_{E(\Omega)}=\|v\|_{L^2(\Omega)}+\|\operatorname{div}v\|_{L^2(\Omega)}$. *There exists an extension of the trace operator* $\gamma_{\mathbf{n}}\colon (C_0^\infty(\overline{\Omega}))^d\to C^\infty(\Gamma)$ *defined by* $u\mapsto u\cdot\mathbf{n}$ *on* Γ *to a continuous linear operator from* $E(\Omega)$ *onto* $H^{-1/2}(\Gamma)$. *The kernel of* $\gamma_{\mathbf{n}}$ *is the space* $E_0(\Omega)$*—the completion of* $C_0^\infty(\Omega)$ *in the* $E(\Omega)$ *norm. For all* u *in* $E(\Omega)$ *and* f *in* $H^1(\Omega)$,

$$(u,\nabla f)+(\operatorname{div}u,f)=\int_\Gamma(u\cdot\mathbf{n})\overline{f}. \tag{7.1}$$

LEMMA 7.2. *Assume that* u *is in* $(\mathcal{D}'(\Omega))^d$ *with* $(u,v)=0$ *for all* v *in* $\mathcal{V}$. *Then* $u=\nabla p$ *for some* p *in* $\mathcal{D}'(\Omega)$. *If* u *is in* $(L^2(\Omega))^d$ *then* p *is in* $H^1(\Omega)$; *if* u *is in* H *then* p *is in* $H^1(\Omega)$ *and* $\Delta p=0$.

Proof. For u in $(\mathcal{D}'(\Omega))^d$ see Proposition 1.1 p. 10 of [10]. For u in $(L^2(\Omega))^d$ the result follows from a combination of Theorem 1.1 p. 107 and Remark 4.1 p. 55 of [3] (also see Remark 1.4 p. 11 of [10]). □

LEMMA 7.3. *For any* u *in* $(L^2(\Omega))^d$ *there exists a unique* v *in* H *and* p *in* $H^1(\Omega)$ *such that* $u=v+\nabla p$.

Proof. This follows, for instance, from Theorem 1.1 p. 107 of [3], which holds for an arbitrary domain, along with Lemma 7.2. □

LEMMA 7.4. *For any* f *in* $L^2(\Omega)$ *and* a *in* $(H^{1/2}(\Gamma))^d$ *satisfying the compatibility condition,*

$$\int_\Omega f=\int_\Gamma a\cdot\mathbf{n}$$

there exists a (non-unique) solution v *in* $(H^1(\Omega))^d$ *to* $\operatorname{div}v=f$ *in* Ω, $v=a$ *on* Γ.

Proof. This follows from Lemma 3.2 p. 126-127, Remark 3.3 p. 128-129, and Exercise 3.4 p. 131 of [3] (and see the comment on p. 67 of [1]). □

COROLLARY 7.5. *For any* v *in* H *there exists a matrix-valued function* M *in* $(H_0^1(\Omega))^{d\times d}$ *such that* $v=\operatorname{div}M$.

Proof. Let v be in H and observe that

$$\int_\Omega v^i=\int_\Omega v\cdot\nabla x_i=-\int_\Omega\operatorname{div}v\,x_i+\int_\Gamma(v\cdot\mathbf{n})x_i=0.$$

Thus, we can apply Lemma 7.4 to each component v^i using $a \equiv 0$ to obtain a vector w^i in $H_0^1(\Omega)$ satisfying $\operatorname{div} w^i = v^i$ on Ω, $w^i = 0$ on Γ. Forming a matrix-valued function whose rows are $w^1, w^2, \dots, w^d$ gives M. □

LEMMA 7.6. *Assume that u is in $(H^2(\Omega))^d$ or $(C^1(\Omega))^d$ with $\operatorname{div} u = 0$ and v is in $(H^1(\Omega))^d$. Then*

$$(\nabla u, \nabla v) = 2(\omega(u), \omega(v)) + \int_\Gamma (\nabla u v) \cdot \mathbf{n}.$$

Proof. Directly from Equation (3.1),

$$\begin{aligned} 2\omega(u) \cdot \omega(v) &= \frac{1}{2}(\nabla u - (\nabla u)^T) \cdot (\nabla v - (\nabla v)^T) \\ &= \nabla u \cdot \nabla v - (\nabla u)^T \cdot \nabla v. \end{aligned} \tag{7.2}$$

Since $\operatorname{div} u = 0$, we have $(\nabla u)^T \cdot \nabla v = \partial_j v^i \partial_i u^j = \partial_j (v^i \partial_i u^j) = \operatorname{div}(\nabla u v)$, so if u and v are both in $(C^\infty(\Omega))^d$ with $\operatorname{div} u = 0$ then

$$\begin{aligned} 2\int_\Omega \omega(u) \cdot \omega(v) &= \int_\Omega \nabla u \cdot \nabla v - (\nabla u)^T \cdot \nabla v \\ &= \int_\Omega \nabla u \cdot \nabla v - \int_\Omega \operatorname{div}(\nabla u v) = \int_\Omega \nabla u \cdot \nabla v - \int_\Gamma (\nabla u v) \cdot \mathbf{n}. \end{aligned}$$

The result then follows by the density of $C^\infty(\Omega)$ in $H^1(\Omega)$, $H^2(\Omega)$, and $C^1(\Omega)$. □

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