Boundary layer associated with the Darcy-Brinkman-Boussinesq model for convection in porous media

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Viscous equations

The *infinite Prandtl-Darcy number Darcy-Brinkman-Boussinesq system* (IPDDBB) in dimensionless form:

$$\begin{cases} -\epsilon \Delta \mathbf{v}^\epsilon + \mathbf{v}^\epsilon + \nabla p^\epsilon = \gamma T^\epsilon \mathbf{k} & \text{on } (0, t^*) \times \Omega, \\ \operatorname{div} \mathbf{v}^\epsilon = 0 & \text{on } (0, t^*) \times \Omega, \\ \partial_t T^\epsilon + \mathbf{v}^\epsilon \cdot \nabla T^\epsilon = \Delta T^\epsilon & \text{on } (0, t^*) \times \Omega, \\ \mathbf{v}^\epsilon = 0, \ T^\epsilon = f & \text{on } (0, t^*) \times \partial\Omega, \\ T^\epsilon(0) = T_0 & \text{on } \{0\} \times \Omega. \end{cases} \tag{1}$$

Ω	Periodic channel, $[0, 2\pi]^{d-2} \times (0, 1)$ in $\mathbb{R}^d$, d = 2, 3, containing a porous medium
$\mathbf{v}^\epsilon$, T^ϵ	velocity, temperature of a fluid in Earth's gravity
ϵ	Brinkman-Darcy number
γ	Rayleigh-Darcy number
f	smooth function on the boundary

Inviscid equations

Formally setting the Brinkman-Darcy number to zero gives the *infinite Prandtl-Darcy number Darcy-Boussinesq system* (IPDDB):

$$
\begin{cases}
\mathbf{v}^0 + \nabla p^0 = \gamma T^0 \mathbf{k} & \text{on } (0, t^*) \times \Omega, \\
\operatorname{div} \mathbf{v}^0 = 0 & \text{on } (0, t^*) \times \Omega, \\
\partial_t T^0 + \mathbf{v}^0 \cdot \nabla T^0 = \Delta T^0 & \text{on } (0, t^*) \times \Omega, \\
\mathbf{v}^0 \cdot \mathbf{n} = 0, \;\; T^0 = f & \text{on } (0, t^*) \times \partial\Omega, \\
T^0(0) = T_0 & \text{on } \{0\} \times \Omega.
\end{cases} \tag{2}
$$

- The well-posedness of weak solutions to (1) is standard. The well-posedness as well as further regularity of (2) was established by Fabrie 1986 and Ly and Titi for slightly different boundary conditions.
- Payne and Straughan 1998 establish the vanishing viscosity limit, showing that $\mathbf{v}^\epsilon \to \mathbf{v}^0$ in $L^\infty([0, t^*; L^2)$ as $\epsilon \to 0$, bounding the rate of convergence by $C\epsilon^{1/2}$.

Why a boundary layer expansion is possible

- Both the viscous and inviscid equations are fully nonlinear, but the nonlinearity occurs in the transport-diffusion term, which is of the form $\partial_t T + \mathbf{v} \cdot \nabla T = \Delta T$.
- The boundary conditions are the same for the transport-diffusion term, which results in no boundary layer (to first order) in the temperature.
- The transport-diffusion equation is coupled through the forcing term with a linear velocity equation: a damped and driven Stokes problem for the viscous equations or a simple Helmholtz decomposition for the inviscid equation.
- Because the different boundary conditions appear only in the linear equation (in the velocity), the boundary layer is linear.
- The nonlinearity in the transport-diffusion equation complicates things considerably, but in the end the explicit form of the boundary layer corrector allows us to control everything—at least to first order.

2D and 3D result

We will construct a boundary layer corrector, $\boldsymbol{\theta}^\epsilon$, such that:

Theorem (KTW)

For constants, C, depend only on T_0, f, and t,

$$\begin{aligned}
\left\|\mathbf{v}^\epsilon - \mathbf{v}^0 - \boldsymbol{\theta}^\epsilon\right\|_{L^\infty(0,t;L^2)} &\le C\epsilon^{1/2}, \\
\left\|\mathbf{v}^\epsilon - \mathbf{v}^0 - \boldsymbol{\theta}^\epsilon\right\|_{L^\infty(0,t;H^1)} &\le C, \\
\left\|T^\epsilon - T^0\right\|_{L^\infty(0,t;L^2)} &\le C\epsilon^{1/2}, \\
\left\|T^\epsilon - T^0\right\|_{L^2(0,t;H^1)} &\le C\epsilon^{1/2}, \\
\left\|\nabla p^\epsilon - \nabla p^0\right\|_{L^\infty(0,t;L^2)} &\le C\epsilon^{1/2}.
\end{aligned}$$

Assumptions on initial temperature are described two slides from now.

Theorem (KTW)

When $d = 2$,

$$\left\| T^\epsilon - T^0 \right\|_{L^\infty(0,t;H^1)}, \ \left\| \partial_t (T^\epsilon - T^0) \right\|_{L^2(0,t;L^2)} \le C\epsilon^{1/4},$$

and

$$\begin{aligned} \left\| \mathbf{v}^\epsilon - \mathbf{v}^0 - \boldsymbol{\theta}^\epsilon \right\|_{L^\infty((0,t)\times\Omega)} &\le C\epsilon^{1/8}, \\ \left\| T^\epsilon - T^0 \right\|_{L^\infty((0,t)\times\Omega)} &\le C\epsilon^{3/8}. \end{aligned}$$

Each of the constants, C, depends only on T_0, f, and t.

The proof relies on an extension of the "anisotropic embedding lemma" of Temam and Wang 1996.

Compatible initial data

- To construct a boundary layer corrector, we need smooth solutions to the inviscid equations down to $t = 0$.
- To ensure this, we assume that the initial temperature is compatible with the boundary data, possibly because it has been "properly prepared" by being the solution to (2) for some positive time.
- Specifically, we need that for some $k \geq 6$ (when $d = 3$)

$$\mathbf{v}^0, T^0 \in C^k([0, t^*] \times \overline{\Omega}).$$

- The compatibility conditions are way too lengthy to write down explicitly, but derive from a straightforward application of Temam 1982.

Coordinates

- $\Omega = [0, 2\pi]^{d-1} \times (0, 1)$, periodic in the horizontal direction(s).
- In 3D we will use coordinates (x, y, z) and in 2D (x, z) so that z is always the vertical coordinate.
- We alternately use (x_1, x_2, x_3) or (x, y, z) as convenient.

Heuristic derivation of boundary layer corrector

- $T^\epsilon = T^0 = f$ on the boundary, so we do not expect a boundary layer for the temperature field.
- The corrector, $\boldsymbol{\theta}^\epsilon = \mathbf{v}^\epsilon - \mathbf{v}^0$ with $q^\epsilon = p^\epsilon - p^0$, satisfies:

$$-\epsilon \Delta \boldsymbol{\theta}^\epsilon + \boldsymbol{\theta}^\epsilon + \nabla q^\epsilon = \epsilon \Delta \mathbf{v}^0, \quad \operatorname{div} \boldsymbol{\theta}^\epsilon = 0,$$

$$\boldsymbol{\theta}^\epsilon \big|_{z=0,1} = -\mathbf{v}^0 \big|_{z=0,1}.$$

- At $z = 0$, we use the stretched coordinate, $Z = z\epsilon^{-\alpha}$, and assume that

$$\boldsymbol{\theta}^\epsilon(x, y, z; t) = \boldsymbol{\theta}(x, y, Z; t), q^\epsilon(x, y, z; t) = q(x, y, Z; t).$$

- Neglecting terms of order ϵ gives, for $j = 1, d-1$,

$$-\epsilon^{1-2\alpha} \frac{\partial^2 \theta_j}{\partial Z^2} + \theta_j + \frac{\partial q}{\partial x_j}.$$

- Since the viscous term must be effective in the boundary layer, we surmise that $\alpha = 1/2$.

- Thus,

$$-\frac{\partial^2 \theta_j}{\partial Z^2} + \theta_j + \frac{\partial q}{\partial x_j} = 0, \quad j = 1, d-1.$$

- Then θ_3 is of order $\epsilon^{1/2}$ since

$$\operatorname{div} \boldsymbol{\theta}^\epsilon = \partial_x \theta_1 + \partial_y \theta_2 + \epsilon^{-1/2} \partial_Z \theta_3 = 0.$$

- The resulting Prandtl-type equations can be solved exactly:

$$\begin{aligned} q_0 &\equiv 0, \\ \theta_j &= -v_j^0(x, y, 0; t) e^{-Z}, j = 1, 2, \\ \theta_3 &= \sqrt{\epsilon}\left(\frac{\partial v_1^0(x, y, 0; t)}{\partial x} + \frac{\partial v_2^0(x, y, 0; t)}{\partial y}\right)(1 - e^{-Z}). \end{aligned}$$

Heuristic derivation of boundary layer corrector (cont.)

- The vertical corrector is of order $\sqrt{\epsilon}$ in the interior since

$$\theta_3 = \quad \sqrt{\epsilon}\left(\frac{\partial v_1^0(x, y, 0; t)}{\partial x} + \frac{\partial v_2^0(x, y, 0; t)}{\partial y}\right)(1 - e^{-Z}).$$

  This is as for the Stokes problem (Temam and Wang 1996).
- We construct a similar corrector at the opposite boundary, $z = 1$.
- Since the corrector at one boundary is nonzero (though small) at the opposite boundary, we multiply each by cutoff functions supported near each boundary and add them together.
- The resulting corrector we will continue to call $\boldsymbol{\theta}^{\epsilon}$.

The boundary layer corrector *exactly* solves:

$$\begin{cases} -\epsilon\Delta\boldsymbol{\theta}^\epsilon + \boldsymbol{\theta}^\epsilon = \mathbf{f}^\epsilon & \text{on } (0, t^*) \times \Omega, \\ \operatorname{div} \boldsymbol{\theta}^\epsilon = 0 & \text{on } (0, t^*) \times \Omega, \\ \boldsymbol{\theta}^\epsilon = -\mathbf{v}^0 & \text{on } (0, t^*) \times \partial\Omega. \end{cases}$$

There is a long explicit expression for $\mathbf{f}^\epsilon$, but all we need is

$$\begin{aligned} \|\mathbf{f}^\epsilon\| &\le C\epsilon^{\frac{1}{2}}, \\ \|\partial_t^m \partial_x^n \mathbf{f}^\epsilon\|_{L^\infty(0,t^*;L^2)} &\le C\epsilon^{1/2} \quad (d = 2), \\ \left\|\partial_t^m \partial_x^n \partial_y^l \mathbf{f}^\epsilon\right\|_{L^\infty(0,t^*;L^2)} &\le C\epsilon^{1/2} \quad (d = 3). \end{aligned}$$

I will now give some idea of how we prove

$$\left\|\mathbf{v}^\epsilon - \mathbf{v}^0 - \boldsymbol{\theta}^\epsilon\right\|_{L^\infty((0,t)\times\Omega)} \leq C\epsilon^{1/8}.$$

Let

$$\mathbf{w}_v^\epsilon = \mathbf{v}^\epsilon - \mathbf{v}^0 - \boldsymbol{\theta}^\epsilon, \quad w_T^\epsilon = T^\epsilon - T^0.$$

1. Start by proving the first theorem, which gives convergence of the velocity, temperature, and pressure in L^2 or H^1 spatial norms using a fairly standard energy argument.
2. In proving convergence of the pressure one obtains, using estimates on the Stokes problem,

$$\left\|\mathbf{w}_v^\epsilon\right\|_{L^\infty(0,t^*;H^2)} \leq C\epsilon^{-1/2}.$$

3. Using the two-dimensional Agmon's inequality gives

$$\|\mathbf{w}_v^\epsilon\|_{L^\infty((0,t^*)\times\Omega)} \le C \|\mathbf{w}_v^\epsilon\|_{L^\infty(0,t^*;L^2)}^{1/2} \|\mathbf{w}_v^\epsilon\|_{L^\infty(0,t^*;H^2)}^{1/2} \le C\left(\epsilon^{1/2}\right)^{1/2}\left(\epsilon^{-1/2}\right)^{1/2} \le C.$$

4. Since $\mathbf{v}^0$ and θ^ϵ are uniformly bounded in $L^\infty((0,t^*)\times\Omega)$, it follows that

$$\|\mathbf{v}^\epsilon\|_{L^\infty((0,t^*)\times\Omega)} \le C.$$

5. Use this uniform bound on the viscous velocity to improve the energy argument in the first theorem to give

$$\|w_T^\epsilon\|_{L^\infty(0,t;H^1)}, \ \|\partial_t w_T^\epsilon\|_{L^2(0,t;L^2)} \le C\epsilon^{1/4}.$$

6. Use the improved temperature convergence to improve the energy argument for velocity convergence to give

$$\|\partial_x \mathbf{w}_v^\epsilon\|_{L^\infty(0,t^*;L^2)} \le C\epsilon^{1/4},$$
$$\|\partial_x \mathbf{w}_v^\epsilon\|_{L^\infty(0,t^*;H^1)} \le C\epsilon^{-1/4}.$$

7. Extend the anistropic embedding lemma of Temam and Wang 1996:

Theorem (Slight extension of Temam and Wang 1996)

For all $u \in H^1_{0,per}(\Omega)$

$$\|u\|_{L^\infty(\Omega)} \le C(\|u\|_{L^2}^{\frac{1}{2}} \|\partial_z u\|_{L^2}^{\frac{1}{2}} + \|\partial_z u\|_{L^2}^{1/2} \|\partial_x u\|_{L^2}^{1/2} + \|u\|_{L^2}^{1/2} \|\partial_x \partial_z u\|_{L^2}^{1/2}),$$

where one or both sides of the inequality could be infinite.

8. Apply the anisotropic embedding lemma to give

$$\begin{aligned}
\|\mathbf{w}_v^\epsilon\|_{L^\infty((0,t^*)\times\Omega)} &\le C\Big[\|\mathbf{w}_v^\epsilon\|_{L^\infty(0,t^*;L^2)}^{1/2}\|\mathbf{w}_v^\epsilon\|_{L^\infty(0,t^*;H^1)}^{1/2} \\
&\quad + \|\mathbf{w}_v^\epsilon\|_{L^\infty(0,t^*;H^1)}^{1/2}\|\partial_x\mathbf{w}_v^\epsilon\|_{L^\infty(0,t^*;L^2)}^{1/2} \\
&\quad + \|\mathbf{w}_v^\epsilon\|_{L^\infty(0,t^*;L^2)}^{1/2}\|\nabla\partial_x\mathbf{w}_v^\epsilon\|_{L^\infty(0,t^*;L^2)}^{1/2}\Big] \\
&\le C\left(\left(\epsilon^{\frac{1}{2}}\right)^{\frac{1}{2}} + \left(\epsilon^{1/4}\right)^{1/2} + \left(\epsilon^{1/2}\epsilon^{-1/4}\right)^{1/2}\right) \\
&\le C\epsilon^{1/8}.
\end{aligned}$$

Further problems

1. Obtain higher order correctors for $\mathbf{v}^\epsilon$, T^ϵ, and p^ϵ and determine the resulting convergence rates.
2. Extend the 2D uniform in space convergence to apply to a bounded domain in $\mathbb{R}^2$ with C^2 boundary. The main issue is the anisotropic embedding lemma, not the corrector.
3. Try to improve the $1/8$ exponent in

$$\left\|\mathbf{v}^\epsilon - \mathbf{v}^0 - \boldsymbol{\theta}^\epsilon\right\|_{L^\infty((0,t)\times\Omega)} \le C\epsilon^{1/8}$$

   to $1/2$, the same scaling as in the energy norm.
4. Try to improve convergence in 3D, though uniform in space and time convergence might not be possible.