On uniqueness and properties of the flow map for weak solutions to the 2D Euler equations

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Outline

Classical flow

- Let $v\colon I \times \Omega \to \mathbb{R}^d$ be a time-varying velocity field, where $I = [0, T)$ is a time interval, and the domain, Ω, lies in $\mathbb{R}^d$, $d \geq 1$. An associated (classical) *flow* or *flow map* for v is a function ψ in $C(I \times \Omega; \mathbb{R}^d)$ such that

$$\psi(t, x) = x + \int_0^t v(s, \psi(s, x))\, ds$$

  for all (t, x) in $I \times \Omega$.
- Continuity of v is (more than) enough to ensure the existence of a classical flow (Peano's existence theorem). Uniqueness requires more knowledge of the velocity field.

Modulus of continuity (MOC)

Definition (Modulus of continuity)

We say that a continuous function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ is a modulus of continuity (MOC). When we say that a MOC, f, is C^k, $k \ge 0$, we mean that it is continuous on $[0,\infty)$ and C^k on $(0,\infty)$.

- A real-valued function or vector field, f, on Ω admits μ as a MOC if

$$|f(x) - f(y)| \le \mu(|x-y|) \text{ for all } x, y \text{ in } \Omega.$$

- MOC will often be strictly increasing and concave, but we do not make that part of the definition.
- A function having a MOC does not have a unique MOC. In particular, if μ is a MOC for f then any $\nu \ge \mu$ is a MOC for f.

Osgood

- A MOC, μ, is Osgood if

$$\int_0^1 \frac{dx}{\mu(x)} = \infty.$$

- If f has an Osgood MOC then f is *Osgood continuous*.
- Lipschitz and log-Lipschitz functions are Osgood continuous.

Osgood gives uniqueness of flow

Lemma (Classical)

Suppose that the velocity field, $v(t, \cdot)$, admits an Osgood MOC, μ, independent of t. Then v has a unique associated flow, ψ, continuous from $I \times \Omega$ to $\mathbb{R}^d$; that is, for all x in Ω,

$$\psi(t, x) = x + \int_0^t v(\psi(s, x))\, ds.$$

For t in I, define $\Gamma_t \colon [0, \infty) \to [0, \infty)$ by $\Gamma_t(0) = 0$ and for $x > 0$ by

$$\int_x^{\Gamma_t(x)} \frac{dr}{\mu(r)} = t.$$

Then Γ_t is a modulus of continuity for the flow, $\psi(t, \cdot)$.

We will alternately write $\psi_t(x)$ and $\psi(t, x)$, $\Gamma_t(x)$ and $\Gamma(t, x)$.

Uniqueness of 2D Euler

- The largest known class of velocities for which uniqueness of solutions to the Euler equations is known (without adding a restriction on the sign of the vorticity) is due to Misha Vishik 1999. It uses borderline Besov spaces, which he introduced.
- For Π in $C([1,\infty))$, define the space

$$B_\Gamma = \{f \in \mathcal{S}'(\mathbb{R}^2) : \sum_{j=-1}^{N} \|\Delta_j f\|_{L^\infty} = O(\Gamma(N))\},$$

$$\|f\|_\Gamma = \sup_{N \geq 1} \frac{1}{\Gamma(N)} \sum_{j=-1}^{N} \|\Delta_j f\|_{L^\infty}.$$

- For a function in B_Γ, the $B^0_{\infty,1}$-norm diverges, but in a controlled way.
- The precise definition of B_Π does not concern us here, only the function Π. The *faster* Π increases the larger the space, B_Π.

Stated a little roughly Vishik's result is:

Theorem (Vishik 1999)

Let Π in $C([1, \infty))$ be such that $\nu(x) = x^2 \Pi(x^{-1})$ is an Osgood MOC. If ω_1, ω_2 are the vorticities for two solutions to the Euler equations with

$$\omega_1, \omega_2 \text{ in } L^\infty([0, T]; L^{p_0} \cap B_\Pi)$$

for some $p_0 < 2$ then the two solutions are identical.

$\Pi(x) = x \log x$ is an example of a suitable Π.

Vishik's class in 2D

- Vishik's is a uniqueness class: existence in this class for ω^0 in $L^{p_0} \cap B_\Pi$ is not known at all in 3D and is only known for a subclass in 2D.
- Suppose that Γ in $C([1, \infty))$ is such that $\Pi(x) = x\Gamma(x)$ satisfies the condition in Vishik's uniqueness theorem. Then in 2D, we *almost* have that for ω^0 in $L^{p_0} \cap L^{p_1} \cap B_\Gamma$ there exists a unique solution in $L^\infty([0, T]; L^{p_0} \cap L^{p_1} \cap B_\Gamma)$ when $p_0 < 2 < p_1$:
    - Vishik 1999 showed this for $\Gamma(x) = \log(x)^\kappa$, globally in time for $0 < \kappa \leq \frac{1}{2}$ and locally in time for $\frac{1}{2} < \kappa \leq 1$.
    - Cozzi and Kelliher 2007 showed this for $p_0 = p_1 = 2$, globally in time for $\kappa < 1$, locally in time for $\kappa = 1$.
- For $\kappa \leq \frac{1}{2}$, Vishik 1999 obtained an Osgood bound on the MOC of the velocity globally in time, and hence a unique classical flow.
- For $\frac{1}{2} < \kappa < 1$ and $p_0 = p_1 = 2$, even existence of a classical flow is not known, since continuity of the vector field is not known.

Yudovich's uniqueness class

There is a large and easier to work with subclass of Vishik's uniqueness class due to Yudovich 1995:

- Let ω lie in L^p for all p in $[p_0, \infty)$ for some p_0 in $[1, 2)$. Let

$$\theta(p) = \|\omega\|_{L^p}, \quad \alpha(\epsilon) = \epsilon^{-1}\theta(\epsilon^{-1}),$$
$$\mu(x) = \inf\left\{x^{1-2\epsilon}\alpha(\epsilon) \colon \epsilon \text{ in } (0, 1/2]\right\}.$$

- We say that ω is a *Yudovich vorticity* if μ is Osgood. The set of all associated L^2-velocities is called, $\mathbb{Y}$.
- Examples of Yudovich vorticities are

$$\theta_0(p) = 1, \theta_1(p) = \log p, \ldots, \theta_m(p) = \log p \cdot \log^2 p \cdots \log^m p.$$

- A classical result of measure theory is that $p \log \theta(p)$ is convex; this fact will play an important role.

Uniqueness of velocity and flow for Yudovich solutions

- Yudovich 1995 showed that solutions to the Euler equations lying in $L^\infty([0, T]; \mathbb{Y})$ are unique. This applies to dimensions 2 and higher.
- In 2D, v^0 in $\mathbb{Y}$ gives v in $L^\infty([0, T]; \mathbb{Y})$ so one has both existence and uniqueness in Yudovich's class.
- A surprising coincidence is that the function μ is both central to Yudovich's uniqueness argument for the Eulerian velocities (after a change of variables) and to establishing the uniqueness of the classical flow, μ being an Osgood MOC for the velocity field.
- Is this more than a coincidence?

MOC of the flow as a proxy for uniqueness

- The more rapidly a MOC increases near the origin the more irregular the function.
- Given a class, $\mathcal{C}$, of initial velocities we will use a measure of how rapidly a MOC of a corresponding flow can increase as a proxy for how near to the edge of uniqueness the class brings the Euler equations.
- This is a very imperfect proxy. It is lent some credence by Yudovich's coincidence. That a subclass of the initial velocities of Vishik can have unique solutions without any apparent control on the MOC of the velocity (and perhaps not even a continuous vector field) and hence of the flow argues against it. But Yudovich's subclass is more physically meaningful than Vishik's.
- In any case, it leads to the following question, which is of interest in its own right:

  How bad can the MOC of the flow be for v^0 in $\mathcal{C}$, and specifically for v^0 in $\mathbb{Y}$?

How bad can the MOC of the flow be?

Specifically, working in $\mathbb{Y}$, we ask the question:

> **Fundamental question** *(FQ): Given any strictly increasing concave MOC, f, and $t > 0$, does there exist an initial velocity in the class $\mathbb{Y}$ for which any MOC of the flow map at time t is at least as large as f on some nonempty open interval, $(0, a)$?*

We require f to be strictly increasing and concave because the definition of μ gives that both μ and $\Gamma(t, \cdot)$ have these properties (as we show later). Also, we only care about MOC near the origin.

- Forward approach: For any strictly increasing concave MOC, f, show how to construct a v^0 whose flow has only MOC poorer than f. Answers FQ, "**Yes**."
- Inverse approach: Show that there exists a MOC, f, that can be achieved by no v^0. Answers FQ, "**No**."

Forward approach

We will speak mostly of the inverse approach, but as regards the forward approach:

- Yudovich 1963 showed that for bounded initial vorticity the flow map lies in the Hölder space of exponent e^{-Ct} for all positive time, t. That is, $\Gamma_t(x) = Cx^{e^{-Ct}}$ is a MOC for the flow.
- Bahouri and Chemin 1995 showed that this regularity of the flow was optimal by constructing an example for which the flow lies in no Hölder space of exponent higher than e^{-t}. That is, $\Gamma_t(x) = Cx^{e^{-t}}$ cannot be a MOC for any C.
- In my thesis 2005, I extended the example of Bahouri and Chemin to specific initial vorticities in $\mathbb{Y}$ having a point singularity like $\log |x|$, showing the flow lies in no Hölder space of positive exponent for any positive time. That is, $\Gamma_t(x) = Cx^{\alpha}$ cannot be a MOC for the flow at time $t > 0$ for any C and any $\alpha > 0$.

Inverse approach

- Since μ is the MOC of $v(t, \cdot)$, Γ defined by

$$\int_x^{\Gamma_t(x)} \frac{dr}{\mu(r)} = t$$

  is a modulus of continuity for the flow, $\psi(t, \cdot)$.

- So we could try answering FQ by showing either that:
  - There is a strictly increasing concave MOC, f, for which no strictly increasing Osgood MOC, μ, gives $\Gamma_1 \geq f$. Answers FQ, "**No**."
  - For any strictly increasing concave MOC, f, there is a strictly increasing Osgood MOC, μ, giving $\Gamma_1 \geq f$. Supports the answer, "**Yes**," to FQ.

Mappings to invert in inverse approach

1. $v^0 \mapsto v(t,x) \mapsto \psi(t,x)$:

   the initial velocity gives the velocity at all time which gives the flow at all time.

2. $v^0 \mapsto \omega^0 \mapsto \theta(p) \mapsto \mu(x) \mapsto \Gamma_t(x)$:

   the initial velocity gives the initial vorticity, whose L^p-norms give the MOC of the the velocity field, which gives the MOC of the flow.

- The Osgood condition on μ insures that both mappings in *(1)* and the last mapping in *(2)* are well-defined.
- The mappings in *(1)* are trivial to invert: $v(t,x) = \partial_t \psi(t, \psi^{-1}(t,x))$ and $v^0 = v(0,\cdot)$. The first mapping in *(2)* is easily inverted as well using the Biot-Savart law. The remaining three mappings in *(2)* are another matter.
- We need to understand the forward maps better before we can invert them.

Range of $\theta(p) \mapsto \mu(x)$

Recall that

$$\theta(p) = \|\omega\|_{L^p}, \quad \alpha(\epsilon) = \epsilon^{-1}\theta(\epsilon^{-1}),$$
$$\mu(x) = \inf\left\{x^{1-2\epsilon}\alpha(\epsilon) \colon \epsilon \text{ in } (0, 1/2]\right\}.$$

Because $p \log \theta(p)$ is convex, $\log \alpha$ is strictly convex.

Define $\lambda \colon \mathbb{R} \to \mathbb{R}$ and $A \colon (0, \infty) \to (0, \infty)$ by

$$\lambda(r) := r + \log(\mu(e^{-r})),$$
$$A(x) := \frac{x\mu'(x)}{\mu(x)} = x(\log \mu(x))'.$$

One can show that:

- μ is continuous on $[0, \infty)$, positive on $(0, \infty)$, and strictly increasing.
- $\mu(x)$ is concave, since $x \mapsto x^{1-2\epsilon}$ is concave for all ϵ in $[0, 1/2]$.
- λ is strictly increasing and, because $\log \alpha$ is strictly convex, λ is strictly concave.
- A is strictly decreasing with $A(0) := \lim_{x \to 0^+} A(x) = 1$.

$\theta(p) \mapsto \mu(x)$ and the Legendre transformation

Using $r = -\log x$, $\lambda(r) = r + \log(\mu(e^{-r}))$ becomes

$$\lambda(-\log x) = -\log x + \log(\mu(e^{-(-\log x)})) = \log\left(\frac{\mu(x)}{x}\right)$$
$$= \log\left(\frac{\inf_{\epsilon\in(0,1/2]}\left\{x^{1-2\epsilon}\alpha(\epsilon)\right\}}{x}\right) = \inf_{\epsilon\in(0,1/2]}\{-2\epsilon\log x + \log\alpha(\epsilon)\}$$
$$= -\sup_{\epsilon\in(0,1/2]}\{2\epsilon\log x - \log\alpha(\epsilon)\} = -(\log\alpha)^*(2\log x).$$

Definition (Legendre transformation)

Let $f\colon I \to \mathbb{R}$ be a strictly convex function on an interval, I. We define its *Legendre transformation*, f^*, by

$$f^*(x) = \sup_{\epsilon\in I}\left\{x\epsilon - f(\epsilon)\right\}.$$

The domain of f^* consists of all x in $\mathbb{R}$ for which the supremum is finite.

Inverting $\theta(p) \mapsto \mu(x)$

Because we have restricted the Legendre transformation to strictly convex functions, f^* is also strictly convex, and the Legendre transformation is an involution ($(f^*)^* = f$). Hence, letting $u = 2\log x$,

$$\lambda(-\log x) = -(\log\alpha)^*(2\log x)$$

becomes $(\log\alpha)^*(u) = -\lambda(-u/2)$. Letting $\overline{\lambda}(s) = -\lambda(-s/2)$, we have

$$\begin{aligned}\log\alpha(x) &= (\overline{\lambda})^*(x) = \sup_{\epsilon\in\mathbb{R}}\left\{x\epsilon - \overline{\lambda}(\epsilon)\right\} = \sup_{\epsilon\in\mathbb{R}}\left\{(-x)(-\epsilon) - (-\lambda(-\epsilon/2))\right\} \\ &= \sup_{\epsilon\in\mathbb{R}}\left\{(-2x)\epsilon - (-\lambda(\epsilon))\right\} = (-\lambda)^*(-2x).\end{aligned}$$

Thus,

$$\alpha(x) = e^{(-\lambda)^*(-2x)}.$$

Inverting $\theta(p) \mapsto \mu(x)$

- Because θ came from the L^p-norms of a function, λ was strictly concave, and this was needed to apply the (inverse) Legendre transform to $-\lambda$. Thus, this measure-theoretic origin of θ was needed to be able to invert the relationship between θ and μ.
- A more explicit form of the inversion can be given in terms of λ and λ'. This overcomes three limitations of simply using the Legendre transformation:
  1. We may only have λ strictly concave near the origin because our other inversions will not be perfect.
  2. It is not clear what the domain of α is. In particular, we need the domain to include 0. It turns out that μ satisfying the Osgood condition is required to insure this.
  3. $p \log \theta(p)$ convex does not easily follow from the Legendre transformation inversion, and, indeed, further restrictions on λ are required to insure this. This is easier to see in the explicit form of the inversion.

Inverting $\omega^0 \mapsto \theta(p)$

- This inversion amounts to asking the question of whether we can find a measurable function in the plane whose L^p-norms match a given function, θ. We only care that they match asymptotically with large p, because that is what determines the MOC near the origin.
- We should expect to invert this map neither uniquely nor exactly. Lack of uniqueness arises because any rearrangement or sign change of ω^0 yields the same θ. Thus, at best we should expect to obtain the distribution function, γ, for ω^0; that is,

$$\gamma(x) = \text{ measure of } \left\{ t \colon |\omega^0(t)| > x \right\}.$$

- The inability to invert exactly is a more complex issue.

Range of $\omega^0 \mapsto \theta(p)$

To see what is involved, start with the classical relation,

$$\theta(p)^p = \|\omega^0\|_{L^p}^p = p \int_0^\infty x^{p-1} \gamma(x)\, dx = p \mathcal{M} \gamma(p),$$

where $\mathcal{M}$ is the Mellin transform. If ω^0 lies in $L^{p_0} \cap L^p$ for all $p \geq p_0$ then $\gamma(p)$ decays faster than any polynomial in p and it is easy to see that

$$\varphi(p) := p \log \theta(p)$$

is complex-analytic in the right-half plane, $\operatorname{Re} p > p_0$. Of necessity, then, φ must at least be real-analytic (and real-valued) on (p_0, ∞) to perform the inversion exactly, and we should not expect this to be the case.

But we only need an inversion that applies asymptotically in large p, as that is what determines the MOC, μ, of the velocity field near zero.

Inverting $\omega^0 \mapsto \theta(p)$ as in Stirling's approximation

To perform an approximate inversion, one can take an approach using the Mellin transform that is, in a sense, a generalization of one proof of Stirling's approximation. The result (which still needs some refinement to obtain a full proof) is:

Assume that

1. $\frac{1}{x} - \varphi''(x)$ is bounded away from zero for all sufficiently large x;
2. $\frac{x\varphi'''(x)}{\varphi''(x)} e^{-\varphi'(x)}, \ \frac{\varphi'''(x)}{[\varphi''(x)]^2} e^{-\varphi'(x)} \to 0$ as $x \to \infty$.

Then letting $\beta(p) := e^{\varphi'(p)}$,

$$\gamma(\beta(p)) \approx \frac{C}{\beta(p)^p} e^{\varphi(p)}.$$

(1) is probably an essential assumption; (2) may just be an artifact of the proof.

The three critical inversions

- The need to invert $\theta(p) \mapsto \mu(x)$ is very specific to Yudovich velocities.
- The same is true of inverting $\omega^0 \mapsto \theta(p)$, though it has somewhat more general application.
- The remaining inversion, however, has, in principle, application to any system of ODEs.

Inverting $\mu(x) \mapsto \Gamma_t(x)$

The defining relation between μ and Γ is

$$I_t(x) := \int_x^{\Gamma_t(x)} \frac{dr}{\mu(r)} = t.$$

It follows that $\partial_t \Gamma_t(x) > 0$ and, because μ is strictly increasing, $\Gamma_t'(x) > 0$.

Taking derivatives gives

$$I_t'(x) = \frac{\Gamma_t'(x)}{\mu(\Gamma_t(x))} - \frac{1}{\mu(x)} = 0, \quad \partial_t I_t(x) = \frac{\partial_t \Gamma_t(x)}{\mu(\Gamma_t(x))} = 1$$

so

$$\mu(\Gamma_t(x)) = \Gamma_t'(x)\mu(x) = \partial_t \Gamma_t(x).$$

Evaluating the second equality at $t = 0$ and using $\Gamma_0(x) = x$ gives

$$\mu(x) = \frac{\partial_t \Gamma_t(x)}{\Gamma_t'(x)}\bigg|_{t=0} = \partial_t \Gamma_t(x)\bigg|_{t=0}.$$

Thus, it is easy to find μ given Γ. But that is not what we want.

Inverting $\mu(x) \mapsto \Gamma_1(x)$

Given a MOC, f, does there exists a MOC, μ, such that $\forall x > 0$,

$$\int_x^{f(x)} \frac{ds}{\mu(s)} = 1?$$

Defining Γ by

$$\int_x^{\Gamma_t(x)} \frac{ds}{\mu(s)} = t$$

would then give a Γ with $f = \Gamma_1$.

What if we require μ have other properties, such as being concave?

If μ is concave then f must be concave. Whether, given f concave one can invert this relation to obtain a concave μ is equivalent to an open problem in the theory of iteration semigroups.

The flow map as a group

Let $f^t = \Gamma_t$ for all t in $\mathbb{R}$. Then $f^0 = id$, $f^{-t} = (f^t)^{-1}$, and

$$f^{s+t} = f^s \circ f^t$$

so $G = (f^t)_{t \in \mathbb{R}}$ is a group under composition. Since,

$$\int_x^{f^t(x)} \frac{ds}{\mu(s)} = t,$$

if μ is *strictly increasing* then $f^t(x) - x$ is *strictly increasing*.

Definition

We say that a MOC, f, is *acceptable* if f is C^1 and $f(x) - x$ is *strictly increasing*.

If f is an acceptable MOC it is strictly increasing with $f' > 1$.

$\mu(\Gamma_t(x)) = \Gamma_t'(x)\mu(x)$ becomes $\mu(f^t(x)) = (f^t)'(x)\mu(x)$.

Continuous iteration group

We specialize the definition of a continuous iteration group in Zdun 1979 to our acceptable MOC.

Definition

A *continuous iteration group of MOC* (CIG) is a family, $(f^t)_{t \in \mathbb{R}}$, such that:

1. Each f^t is an acceptable MOC for all $t > 0$.
2. For all s, t in $\mathbb{R}$, $f^s \circ f^t = f^{s+t}$.
3. f^0 is the identity.
4. For any fixed x in $[0, \infty)$ the map $t \mapsto f^t(x)$ is continuous.

We say that the CIG is concave or C^k if each f^t, $t > 0$, is concave or C^k. We say that the MOC, f, is *embedded* in the CIG if $f^1 = f$.

So, given a concave acceptable f, we want to find a CIG in which it is embedded.

Basic inversion theorem

Theorem (Basic inversion theorem, K 2010)

Given any f that is an acceptable MOC for a flow there exists a continuous MOC, μ, satisfying the Osgood condition with $\mu > 0$ on $(0, \infty)$, such that

$$\int_x^{f(x)} \frac{dr}{\mu(r)} = 1 \tag{1}$$

for all $x > 0$. If f is concave then for any $x_0 > 0$ we can make μ strictly increasing on $[0, x_0]$. If f is C^k, $k \geq 1$, then μ can be made C^{k-1}.

This can be proved by expressing the relation in (1) as the functional equation, $\mu(f(x)) = f'(x)\mu(x)$, and using a construction due to Kordylewski and Kuczma 1960, 1962. The regularity of μ follows from an argument of Choczewski 1963.

Some ideas in the proof of basic inversion theorem

- The construction involves choosing μ appropriately (in particular, making it *strictly increasing*) on the interval $[f^{-1}(x_0), x_0]$ then extending it to the left and right by insisting that $\mu(f(x)) = f'(x)\mu(x)$ for all x.
- The relation $\mu(x) = \mu(f(x))/f'(x)$ and the concavity of f then insure that μ is *strictly increasing* on $[0, x_0]$. This is not a limitation for our applications, since MOC are only important near the origin.

For simplicity of presentation, we suppress the complications that arise from μ being *strictly increasing* only in a neighborhood of the origin, stating our arguments and results as though they were for all x. An exception, however, is our main result which is *stated* in a way that accounts for this complication.

Concave inversion

Left open is the question of whether all concave acceptable MOCs are embeddable in a concave CIG; in fact, this is an open problem in the theory of functional equations. (With different assumptions on the function f, both existence and uniqueness of such a concave CIG is known.)

But for our purposes the weaker result below will suffice.

Theorem (Concave inversion theorem, K 2010)

Let f be any C^k concave acceptable MOC, $k \geq 3$. For any $a > 0$ there exists a C^{k+1} concave acceptable MOC, $\overline{f}$, embedded in a concave C^{k+1} CIG and such that $\overline{f} > f$ on (0, a). The associated function μ is concave and C^k.

Zdun 1979

Once we have μ, defining $G = (f^t)_{t \in \mathbb{R}}$ by

$$\int_x^{f^t(x)} \frac{dr}{\mu(r)} = t, \qquad (2)$$

G is a CIG embedding $f = f^1$. We have the following combination of results from Zdun 1979 and our previous theorem.

Theorem (Primarily Zdun 1979)

Suppose that f is a C^k, $k \geq 1$, concave acceptable MOC. Then there exists a (in fact, an infinite number of) strictly increasing C^k CIG, $(f^t)_{t \in \mathbb{R}}$, embedding f. Let $(f^t)_{t \in \mathbb{R}}$ be any such CIG embedding f. Then $\mu := \partial_t f^t|_{t=0}$ is C^{k-1} and Osgood, and satisfies (2). Moreover, the following are equivalent:

1. *μ is (strictly) concave;*
2. *$(f^t)_{t \in \mathbb{R}}$ is (strictly) concave;*
3. *f^t is (strictly) concave for all t in $(0, \delta)$ for some $\delta > 0$.*

Since $\mu(f^t(x)) = (f^t)'(x)\mu(x)$, we have

$$\mu'(f^t(x)) = \mu'(x) + \mu(x)\frac{(f^t)''(x)}{(f^t)'(x)},$$

or,

$$(f^t)''(x) = \left(\mu'(f^t(x)) - \mu'(x)\right)\frac{(f^t)'(x)}{\mu(x)}.$$

Since f^t is acceptable for all $t > 0$, $f^t(x) > x$ and $(f^t)'(x) > 1$. Hence, $(f^t)''(x)$ and $\mu'(f^t(x)) - \mu'(x)$ have the same sign for all $t, x > 0$.

Proof for *strictly* concave:

- *(1)* $\implies$ *(2)*: μ' is strictly increasing so $\mu'(f^t(x)) > \mu'(x)$; hence, $(f^t)''(x) > 0$.
- *(2)* $\implies$ *(3)*: immediate.
- *(3)* $\implies$ *(1)*: For all x and all $0 < h < f^\delta(x) - x$, $\mu'(x+h) - \mu'(x) > 0$. Hence, μ is strictly concave.

The generating function

To give an idea of the proof of the concave inversion theorem, we need to introduce the concept of a *generating function* for the CIG, $G = (f^t)_{t \in \mathbb{R}}$. This is a function, $h\colon (-\infty, \infty) \to (0, \infty)$ such that

$$f^t(x) = h(t + h^{-1}(x))$$

for all t and x. It turns out that such a generating function always exists, is unique (to within a translation), and is always *strictly increasing*. We have,

$$\mu(x) = h'(h^{-1}(x) = 1/(h^{-1}(x))'.$$

Since $\mu(h(x)) = h'(x)$,

$$h''(x) = \mu'(h(x))h'(x).$$

Hence, μ is *strictly increasing* if and only if h is strictly convex.

Since $h' > 0$, $h' = e^g$ for some $g\colon \mathbb{R} \to \mathbb{R}$. Then

$$h'' = e^g g'.$$

Combining the above, if $G = (F^t)_{t \in \mathbb{R}}$ is a CIG with generating function, h, and corresponding function, μ, then TFAE:

- μ is strictly increasing,
- h is strictly convex,
- $g = \log h'$ is strictly increasing.

Moreover, a simple calculation gives

$$\mu''(h(x))h'(x) = \frac{h'''(x)h'(x) - (h''(x))^2}{h'(x)^2} = \left(\frac{h''(x)}{h'(x)}\right)' = (\log h')''(x).$$

Thus, $(f^t)_{t \in \mathbb{R}}$ (and hence μ) is concave if and only if g is concave.

Key idea in proof of concave inversion theorem

A long calculation shows that if $f = f^1$ is concave then

$$g(x) > g(x-1),\ g'(x) \le g'(x-1)$$

for all $x > 0$. Define $\overline{g}$ by

$$\overline{g}(x) = \int_x^{x+1} g(s)\,ds.$$

g strictly increasing gives $\overline{g}'(x) = g(x+1) - g(x) > 0$, so $\overline{g}$ is *strictly increasing*. And $\overline{g}''(x) = g'(x+1) - g'(x) \le 0$, so $\overline{g}$ is concave. Hence, the resulting $G = (\overline{f}^t)_{t \in \mathbb{R}}$ is concave.

Since $\overline{g}$ is the mean value of g on $(x, x+1)$ and g is *strictly increasing*,

$$\overline{g}(x) > g(x) > \overline{g}(x-1)$$

We can use this to bound $\overline{f}^1$ in terms of $\overline{f}$.

Two questions and one remark

1. $p \log \|\omega^0\|_{L^p}$ convex adds the constraint that λ be concave, or, equivalently, that $\mu''(x) \geq (\mu(x)/x)'$. Can we invert the map $\mu \mapsto f = \Gamma_1$ and satisfy this constraint? Or does this require additional constraints on f and if so what are they?
2. What other reasonable constraints other than μ being *strictly increasing* and concave can we accommodate and still invert the map?
3. A Yudovich velocity field is not only Osgood continuous, but Dini continuous, meaning that

$$\int_0^x \frac{\mu(s)}{s}\, ds < \infty$$

for all $x > 0$.

Obrigado!