On the Uniqueness of Weak Solutions to the 2D Euler Equations

(properties of the flow)

2009 SIAM Conference on Analysis of PDEs

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9 December 2009

Bounded vorticity and the Euler equations

For initial vorticity in $L^1 \cap L^\infty(\mathbb{R}^2)$ there exists a unique weak solution, (v, p), to the Euler equations,

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p &= 0, \\ \operatorname{div} v &= 0. \end{cases}$$

Vorticity ($\omega = \omega(v) = \partial_1 v^2 - \partial_2 v^1$) is transported by the flow, and there exists a unique mapping ψ (the classical *flow*), continuous from $\mathbb{R} \times \mathbb{R}^2$ to $\mathbb{R}^2$, such that

$$\psi(t, x) = x + \int_0^t v(s, \psi(s, x))\, ds.$$

This is due to Yudovich 1963.

Moreover, both v and ψ have an explicit modulus of continuity (MOC) in the space variables:

$$|v(t,x) - v(t,y)| \le \mu(|x-y|),$$
$$|\psi(t,x) - \psi(t,y)| \le \Gamma_t(|x-y|),$$

for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}^2$, with

$$\begin{aligned} \mu(r) &= -Cr \log r, \quad r < 1/2, \\ \Gamma_t(r) &= Cr^{e^{-Ct}}, \qquad r \text{ suff. small.} \end{aligned}$$

The function μ, which is independent of time, can be derived from potential theory using the conservation of the L^p-norms of the vorticity; Γ_t comes from the proof of the existence of a classical flow for a log-Lipschitz velocity field.

The flow map: Hölder continuity

It follows from

$$|\psi(t,x) - \psi(t,y)| \le \Gamma_t(|x-y|) = C\,|x-y|^{e^{-Ct}}$$

that for fixed $t > 0$, the map $\psi(t,\cdot)$ lies in the Hölder space,

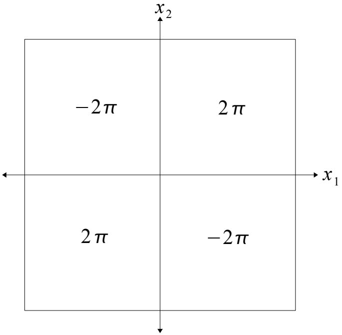
$$C^\alpha, \quad \alpha = e^{-Ct}.$$

(Sorry for the dual use of "C.")

This leads to the question of whether this upper bound on the Hölder continuity of the flow map is ever achieved.

Theorem (Bahouri & Chemin 1994)

For the initial vorticity, $\omega^0 = 2\pi \mathbb{1}_{[0,1]\times[0,1]}$, the flow map $\psi(t, \cdot)$ lies in no Hölder space, C^α, for $\alpha > e^{-t}$.



Definition

Let $\theta : [p_0, \infty) \to \mathbb{R}^+$ for some p_0 in $[1, 2)$. We say that θ is *admissible* if the function $\beta : (0, \infty) \to [0, \infty)$ defined by

$$\beta(x) := \inf \left\{ (x^{1-\epsilon}/\epsilon)\theta(1/\epsilon) : \epsilon \text{ in } (0, 1/p_0] \right\},$$

where C is a fixed absolute constant, satisfies

$$\int_0^1 \frac{dx}{\beta(x)} = \infty.$$

Examples of admissible bounds on vorticity are

$$\theta_0(p) = 1, \theta_1(p) = \log p, \ldots, \theta_m(p) = \log p \cdot \log^2 p \cdots \log^m p,$$

for which

$$\beta_m(x) \leq Cx\theta_{m+1}(1/x).$$

Definition

We say that a vector field v has *Yudovich vorticity* if for some admissible function $\theta : [p_0, \infty) \to \mathbb{R}^+$ with p_0 in $[1, 2)$, $\|\omega(v)\|_{L^p} \le \theta(p)$ for all p in $[p_0, \infty)$.

Roughly speaking, the L^p–norm of a Yudovich vorticity can grow in p only slightly faster than $\log p$ and still be admissible.

Such growth in the L^p–norms arises, for example, from a point singularity of the type $\log \log(1/|x|)$.

Theorem (Yudovich 1995)

First part: *For any initial velocity having Yudovich vorticity there exists a unique weak solution to the Euler equations.*
Second part: *The vector field has a unique continuous flow, along which the vorticity is transported. Let*

$$\mu(r) = \beta(r^2)/r,$$

and let $\Gamma_t : [0, \infty) \to [0, \infty)$ *with* $\Gamma_t(0) = 0$ *and, for* $s > 0$,

$$\int_s^{\Gamma_t(s)} \frac{dr}{\mu(r)} = t \text{ or equivalently } \int_{s^2}^{\Gamma_t(s)^2} \frac{dr}{\beta(r)} = t.$$

Then for all (t, x, y) *in* $\mathbb{R} \times \mathbb{R}^2$,

$$|v(t,x) - v(t,y)| \le \mu(|x - y|),$$
$$|\psi(t,x) - \psi(t,y)| \le \Gamma_t(|x - y|).$$

Bounded vorticity

The bounds on the modulus of continuity of the velocity field and the flow when specialized to bounded vorticity go as follows:

$$\theta(p) = 1$$

and, for r sufficiently small,

$$\begin{aligned} \beta(r) &= -r \log r, \\ \mu(r) &= \beta(r^2)/r = -r^2 \log(r^2)/r = \beta(r), \\ \Gamma_t(r) &= r^{e^{-Ct}}, \\ \psi(t, \cdot) &\in C^\alpha, \ \alpha = e^{-Ct}. \end{aligned}$$

(We have ignored unimportant constants here and elsewhere.)

This reproduces the upper bound on α for bounded vorticity.

Hölder space continuity for unbounded vorticity

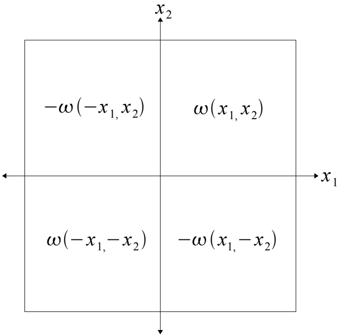
The obvious question is what happens for the higher Yudovich vorticities, especially whether the upper bound on the modulus of continuity of the flow given by Yudovich is achieved by a specific example.

We will find that for the next Yudovich example, $\theta(p) = \log p$, there exists initial vorticities for which the flow map, $\psi(t, \cdot)$, lies in no Hölder space of positive exponent for any $t > 0$.

The ultimate goal of this kind of analysis is to construct initial Yudovich vorticities for which the flow map has an arbitrarily poor modulus of continuity. Were this demonstrated, it would suggest that some aspect of uniqueness breaks down when going beyond Yudovich vorticities. *Maybe.*

Definition

We say that a compactly supported Yudovich vorticity ω is *symmetric by quadrant*, or SBQ, if $\omega(x) = \omega(x_1, x_2)$ is odd in x_1 and x_2; i.e, $\omega(-x_1, x_2) = -\omega(x_1, x_2)$ and $\omega(x_1, -x_2) = -\omega(x_1, x_2)$. (So also $\omega(-x) = \omega(x)$.)



The divergence-free velocity field, v, can be recovered from its vorticity, ω, by the Biot-Savart law:

$$v = K * \omega, \quad K(x) = \frac{x^\perp}{2\pi |x|^2}.$$

Lemma

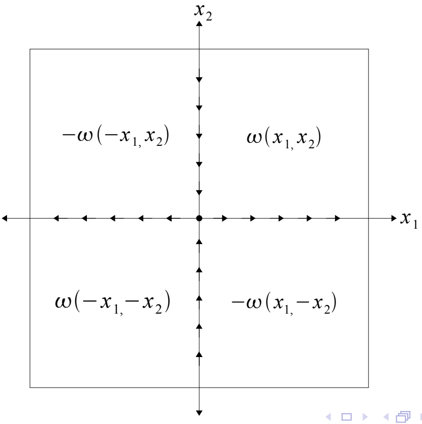
Let ω be SBQ. Then there exists a unique divergence-free vector field, v, in $(L^2(\mathbb{R}^2))^2$ with $\omega(v) = \omega$, and v satisfies the following:

1. $v_2(x_1, 0) = 0$ *for all* x_1 *in* $\mathbb{R}$;
2. $v_1(0, x_2) = 0$ *for all* x_2 *in* $\mathbb{R}$;
3. $v(0, 0) = 0$;
4. *If* $\omega \geq 0$ *in* Q_1 *then* $v_1(x_1, 0) \geq 0$ *for all* $x_1 \geq 0$.

SBQ vector field

SBQ is preserved by the flow. Therefore:

1. The origin remains fixed.
2. The quadrants do not mix.
3. $\omega \geq 0$ in Q_1 is preserved by the flow.



Our approach

1. Let ω be SBQ with $\omega = 2\pi \mathbb{1}_{[0,1]\times[0,1]}$. Then Bahouri and Chemin showed that $v_1(x_1, 0) \geq C x_1 \log(1/x_1) \; \forall \; x_1 \geq C$.
2. Scale Bahouri and Chemin's result to show that for $\omega = 2\pi \mathbb{1}_{[0,r]\times[0,r]}$, $r < 1$, $\lambda < 1$, one has $v_1(x_1, 0) \geq C_\lambda x_1 \log(1/x_1) \; \forall \; x_1^\lambda \leq r$.
3. Construct an SBQ initial vorticity, ω^0, that decreases in $Q1$ from a singularity at the origin by summing an infinite number of vorticities as in Step 2. Such a vorticity has "square symmetry," and we obtain a lower bound on $v_1^0(x_1, 0)$.
4. Use the upper bound on the modulus of continuity of the flow to bound $\omega(t)$ below by a square symmetric vorticity, $\overline{\omega}(t)$. The bound in step 3 then gives a lower bound on $v_1(t, x_1, 0)$.
5. Use the lower bound on $v_1(t, x_1, 0) = v_1(t, x_1, 0) - v_1(t, 0, 0)$ to obtain a lower bound on $|\psi(t, x_1, 0)| = |\psi(t, x_1, 0) - \psi(t, 0, 0)|$.

Lemma

Let ω be SBQ with $\omega = 2\pi \mathbb{1}_{[0,r]\times[0,r]}$ on Q_1 for some r in $(0,1)$. Then for any λ in $(0,1)$ there is a right neighborhood $\mathcal{N} = \mathcal{N}_\lambda$ of 0 for which

$$V(r, x_1) := v_1(x_1, 0) \geq 0,$$

where

$$V(r, x_1) \geq C x_1 \log(1/x_1)$$

for all x_1 in $\mathcal{N}$ for which $x_1^\lambda \leq r$. (The neighborhood, $\mathcal{N}$, depends only upon λ; in particular, it is independent of r.)

Observe that the bound on $V(r, x_1)$ does not depend upon r, but the domain on which it is bounded by the lemma shrinks as r shrinks. This lemma can be got from Bahouri and Chemin's bound for $r = 1$ by scaling.

Definition

We say that ω is square-symmetric if ω is SBQ and $\omega(x_1, x_2) = \omega(\max\{x_1, x_2\}, 0)$ on Q_1.

That is, the vorticity is SBQ and is constant in absolute value along the boundary of any square centered at the origin.

Lemma

Assume that ω is square-symmetric, finite except possibly at the origin, and $\omega(x_1, 0)$ is nonnegative and non-increasing for $x_1 > 0$. Then for any λ in $(0, 1)$ and any λ' in $(0, \lambda)$

$$v_1(x_1, 0) \geq C(1 - \lambda')\omega(x_1^\lambda, 0)x_1 \log(1/x_1)$$

for all x_1 in the neighborhood $\mathcal{N} = \mathcal{N}_{\lambda,\lambda'}$, where $C = 2/\pi$.

Proof.

Write ω on Q_1 as

$$\omega(x) = 2\pi \int_0^1 \alpha(r) \mathbb{1}_{[0,r]\times[0,r]}(x)\, dr,$$

for some $\alpha\colon (0,1) \to [0,\infty)$. In particular,

$$\omega(x_1, 0) = 2\pi \int_{x_1}^1 \alpha(r)\, dr.$$

Because the Biot-Savart law is linear,

$$v_1(x_1, 0) = \int_0^1 \alpha(r) V(r, x_1)\, dr.$$

□

Proof continued.

So,

$$\begin{aligned} v_1(x_1, 0) &= \int_0^1 \alpha(r) V(r, x_1)\, dr \\ &= \int_0^{x_1^\lambda} \alpha(r) V(r, x_1)\, dr + \int_{x_1^\lambda}^1 \alpha(r) V(r, x_1)\, dr \\ &\geq \int_{x_1^\lambda}^1 \alpha(r) V(r, x_1)\, dr \\ &\geq 2\pi \left(\int_{x_1^\lambda}^1 \alpha(r)\, dr \right) \frac{4}{2\pi} (1 - \lambda') x_1 \log(1/x_1) \\ &= \frac{2}{\pi} (1 - \lambda') \omega(x_1^\lambda, 0) x_1 \log(1/x_1). \end{aligned}$$

In the final inequality, $V(r, x_1)$ is bounded as in the previous lemma because $x_1^\lambda \leq r$ in the integrand.

□

Theorem

Assume that ω^0 is square-symmetric, finite except possibly at the origin, and $\omega^0(x_1, 0)$ is nonnegative and non-increasing for $x_1 > 0$. Then for any λ in $(0, 1)$ and any λ' in $(0, \lambda)$,

$$v_1(t, x_1, 0) \geq C(1 - \lambda')\omega^0(\Gamma_t(2^{\lambda/2} x_1^\lambda), 0) x_1 \log(1/x_1)$$

for all x_1 in the neighborhood $\mathcal{N} = \mathcal{N}_{\lambda,\lambda'}$ and all time $t \geq 0$. Further, let $L(t, x_1)$ be any continuous lower bound on $v_1(t, x_1, 0)$, for instance, the one above. Then if $x_1(t)$ is the solution to

$$\frac{dx_1(t)}{dt} = L(t, x_1)$$

with $x_1(0) = a > 0$ in $\mathcal{N}$, then $\psi^1(t, a, 0) \geq x_1(t)$ for all $t \geq 0$.

Idea of the proof.

The vorticity, $\omega(t)$, remains SBQ at $t > 0$, but is no longer square-symmetric. But $|\omega(t)|$ can be bounded below by $|\overline{\omega}(t)|$, with $\overline{\omega}(t)$ square-symmetric, by using the bound, Γ_t, on the modulus of the continuity of the flow to control how much the square-symmetric initial vorticity can be distorted in time t. The differential equation for x_1 follows from the definition of the flow, using $\psi(t, 0, 0) = 0$ for all t (the origin is fixed by the flow). □

For Bahouri and Chemin's example, $\omega^0 = 2\pi \mathbb{1}_{[0,1]\times[0,1]}$, the proof of this part is quite simple: the flow maps an open ball containing the origin homeomorphically into an open set and the image of the origin must lie in its interior.

Returning to the $\omega = 2\pi \mathbb{1}_{[0,1]\times[0,1]}$ example,

$$\begin{aligned} v^1(t, x_1, 0) &\geq C(1-\lambda')\omega^0(2(2^{\lambda/2} x_1^\lambda/2)^{e^{-Ct}}, 0) x_1 \log(1/x_1) \\ &\geq C(1-\lambda') x_1 \log(1/x_1). \end{aligned}$$

Solving $dx_1(t)/dt = C(1-\lambda')x_1 \log(1/x_1)$ with $x_1(0) = a$ gives

$$\psi^1(t, a, 0) \geq x_1(t) = a^{\exp(-C(1-\lambda')t)},$$

which applies for sufficiently small a. Since $\psi(t, 0, 0) = 0$,

$$\frac{|\psi(t, a, 0) - \psi(t, 0, 0)|}{a^\alpha} = \frac{\left|\psi^1(t, a, 0)\right|}{a^\alpha} \geq a^{\exp(-C(1-\lambda')t) - \alpha},$$

which is infinite for any $\alpha > \exp(-C(1-\lambda')t)$. This shows that the flow can be in no Hölder space C^α for $\alpha > \exp(-C(1-\lambda')t)$. But this is true for any λ' in $(0, 1)$, so the flow can be in no Hölder space C^α for $\alpha > e^{-Ct}$, reproducing, up to a constant, the result of Bahouri and Chemin.

Yudovich's next example

Letting ω^0 be square-symmetric with $\omega^0(x_1, 0) = \log\log(1/x_1)$ for $0 < x_1 < 1/e$ and zero for $x_1 \geq 1/e$ gives

$$\begin{aligned} \theta(p) &= C \log p, \\ \beta(r) &\leq Cr \log r \log\log(1/r). \end{aligned}$$

Rather than solving for Γ_t, one shows that

$$\log\log(1/\Gamma_t(s)) \geq \frac{1}{2} e^{-Ct} \log\log(1/s).$$

Then, for $x_1 > 0$ sufficiently small,

$$\begin{aligned} v^1(t, x_1, 0) &\geq C(1 - \lambda')\omega^0(\Gamma_t(2^{\lambda/2} x_1^\lambda), 0) x_1 \log(1/x_1) \\ &\geq C(1 - \lambda') \log\log(1/\Gamma_t(2^{\lambda/2} x_1^\lambda)) x_1 \log(1/x_1) \\ &\geq Ce^{-Ct} \log\log(1/2^{\lambda/2} x_1^\lambda) x_1 \log(1/x_1) \\ &\geq Ce^{-Ct} \log\log(1/x_1) x_1 \log(1/x_1). \end{aligned}$$

Solving for

$$\frac{dx_1(t)}{dt} = Ce^{-Ct} \log\log(1/x_1) x_1 \log(1/x_1) \qquad (1)$$

with $x_1(0) = a$, we get

$$\log^3(1/x_1(t)) = \log^3(1/a) + C\left(e^{-Ct} - 1\right),$$

so

$$\begin{aligned}\psi^1(t, a, 0) \geq x_1(t) &= \exp\left(-(-\log a)^{\exp\left(C(e^{-Ct}-1)\right)}\right) \\ &= e^{-(-\log a)^{\gamma}},\end{aligned}$$

where $\gamma = \exp\left(C(e^{-Ct} - 1)\right)$.

We had,

$$\psi^1(t, a, 0) \geq e^{-(-\log a)^\gamma},$$

where $\gamma = \exp\left(C(e^{-Ct} - 1)\right) < 1$ for all $t > 0$. Thus, for any α in $(0, 1)$ and all $t > 0$,

$$\begin{aligned} \|\psi\|_{C^\alpha} &\geq \lim_{a \to 0^+} \frac{\psi^1(t, a, 0) - \psi^1(t, 0, 0)}{a^\alpha} = \lim_{a \to 0^+} \frac{\psi^1(t, a, 0)}{a^\alpha} \\ &\geq \lim_{a \to 0^+} \frac{e^{-(-\log a)^\gamma}}{e^{(-\log a)\alpha}} = \lim_{u \to \infty} \frac{e^{-u^\gamma}}{e^{-\alpha u}} = \lim_{u \to \infty} e^{\alpha u - u^\gamma} = \infty. \end{aligned} \tag{2}$$

We conclude that the flow lies in no Hölder space of positive exponent for all positive time.

Problem with higher Yudovich vorticities

For the higher Yudovich vorticities,

$$\theta_m(p) = \log p \cdot \log^2 p \cdots \log^m p, \quad m < 2,$$

the initial vorticity is distorted too much at time $t > 0$ to obtain a square-symmetric lower bound for the vorticity at time t that blows up fast enough at the origin. Either more refined estimates, or something clever, are needed.