# ON THE FLOW MAP FOR 2D EULER EQUATIONS WITH UNBOUNDED VORTICITY

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Abstract. In Part I, we construct a class of examples of initial velocities for which the unique solution to the Euler equations in the plane has an associated flow map that lies in no Hölder space of positive exponent for any positive time. In Part II, we explore inverse problems that arise in attempting to construct an example of an initial velocity producing an arbitrarily poor modulus of continuity of the flow map.

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## Overview

In [22], V. I. Yudovich described what is still the weakest class of initial velocities for which solutions to the 2D Euler equations are known to be well-posed. (This extended his bounded vorticity result of [21].) Any solution with initial velocity in this class, which we will call $\mathbb{Y}$, has a unique velocity field and flow map that are continuous in space and time with explicit upper bounds on their spatial moduli of continuity (MOC). The velocity field is both Osgood- and Dini-continuous ((1.4, 11.1)).

In this paper we initiate the investigation of the fundamental question of how "bad" the MOC of the flow can be:

> **Fundamental question**: Given any strictly increasing concave MOC, $f$, and positive time, $t$, does there exist an initial velocity in the class $\mathbb{Y}$ for which any MOC of the flow map at time $t$ is at least as large as $f$ on some nonempty open interval, $(0, a)$?

Past studies of properties of the flow map tend to ask the opposite question: "How smooth is the flow map?" (A notable exception is an example by Bahouri and Chemin in [2], upon which we build in Part I.) In particular, [17] is an important recent study of the smoothness of flow maps for initial velocities in $\mathbb{Y}$.

If the answer to our fundamental question is "yes," it would support the idea that the class $\mathbb{Y}$ is near the edge of uniqueness for solutions to the Euler equations. What is meant by this unavoidably imprecise statement is that, although uniqueness of a solution to the Euler equations does not necessarily require the uniqueness (or even existence) of a (classical) flow (for instance, see [6]), the two ideas seem to be closely entwined. This is evident in the observations of Yudovich in [22] as well as in the approach of Vishik in [20], which relies on properties of the flow.

On the other hand, if the answer to this question is "no," then it means that there is some hope of extending $\mathbb{Y}$ to obtain a larger class of initial velocities for which both existence and uniqueness in 2D can be proven. And if the answer is "no" there still remains the question of characterizing those MOC that *can* be achieved.

To answer our fundamental question "yes" we have little choice but to specially construct an initial velocity for which we can obtain a lower bound on the MOC of its flow that is greater than a given $f$. To answer our fundamental question "no" we have little choice but to show that the upper bound on a MOC of the flow that results from the classical theory cannot be arbitrarily large because of some underlying property of the Euler equations.

These two opposing approaches involve very different kinds of techniques, the former more closely tied to classical fluid mechanics the latter more closely tied to the theory of functional equations. In Part I we explore the first approach while in Part II we explore the second approach, in each case obtaining partial answers, but leaving the final answer unresolved.

This paper is organized as follows:

**Part I**: Yudovich showed in [21] that for bounded initial vorticity the flow map lies in the Hölder space of exponent $e^{-Ct}$ for all positive time $t$. Bahouri and Chemin in [2] showed that this regularity of the flow was in a sense optimal by constructing an example for which the flow lies in no Hölder space of exponent higher than $e^{-t}$. We extend the example of Bahouri and Chemin in [2] to a class of initial vorticities in $\mathbb{Y}$ having a point singularity. We show that for some such initial vorticities the flow lies in no Hölder space of positive exponent for any positive time (Corollary 5.1). In Section 6 we indicate a possible approach to extending this result to obtain still poorer MOC, a subject of future work.

**Part II**: A MOC of the flow map can be derived in terms of a MOC of the vector field, as long as the vector field's MOC satisfies an Osgood condition (see (1.4)). We examine the inverse problem: given a MOC of the flow, obtain a MOC of a vector field from which it can be derived. We do this first for a general flow and vector field, in a manner that has application beyond solutions to the Euler equations, then specialize to solutions to the Euler equations with Yudovich velocity, where there are further restrictions on both MOC.

We will show that if the MOC, $\Gamma(t, x)$, of the flow map is concave for all $t > 0$ then one can find a necessarily concave MOC, $\mu$, of the velocity field that yields an upper bound on the MOC of the flow map at least as large as $\Gamma(t_0, x)$ at any fixed time $t_0 > 0$ (Theorem 9.12, Remark (9.13)). We identify additional constraints ((10.10)) required on the MOC, $\mu$, however, and it is left as an open problem whether such constraints can be satisfied. We show that $\mu$ is Dini-continuous in Section 11 and explore a useful implication of this property. Given the constraints on $\mu$ identified in (10.10), we show in Section 12 how the $L^p$-norms of the Yudovich vorticity field can be recovered from $\mu$ ((12.2) and Theorem 12.2).

## Part I: Yudovich velocities

### 1. Introduction

The Euler equations describe the flow of an incompressible, constant-density, zero-viscosity fluid in a stationary frame, with the effects of temperature and other physical factors ignored. Letting $v$ be the velocity field and $p$ the pressure field for the fluid, we can write these equations in dimensionless form as

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p &= f \quad \text{on } [0, T] \times \Omega, \\ \operatorname{div} v &= 0 \quad \text{on } [0, T] \times \Omega, \\ v &= v^0 \quad \text{on } \{0\} \times \Omega. \end{cases}$$

Here, $f$ is the external force, $v^0$ the initial velocity, $T > 0$, and $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, the domain in which the fluid lies. When $\Omega$ is not all of $\mathbb{R}^d$, we impose the *no-penetration* boundary conditions, $v \cdot \mathbf{n} = 0$ on $\partial\Omega$, where $\mathbf{n}$ is the outward unit normal to the boundary.

In this paper neither the external force nor the effect of the boundary will play an important role, though the dimension will. Thus, we will assume that $\Omega = \mathbb{R}^2$, which means that we can also write the Euler equations in their vorticity formulation and can allow $T$ to be arbitrarily large:

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega &= 0 \qquad \text{on } [0, T] \times \Omega, \\ v &= K * \omega \quad \text{on } [0, T] \times \Omega, \\ \omega &= \omega^0 \qquad \text{on } \{0\} \times \Omega. \end{cases}$$

Here, $\omega = \omega(v) = \partial_1 v^2 - \partial_2 v^1$ is the *vorticity* (scalar curl) of the velocity with $\omega^0 = \omega(v^0)$ and $K$ is the Biot-Savart kernel,

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \tag{1.1}$$

where $x^\perp = (-x_2, x_1)$. (We are following the common convention of giving $\omega$ a dual meaning both as a function and as a variable.)

Suppose that $\omega$ is a scalar field lying in $L^p$ for all $p$ in $[p_0, \infty)$ for some $p_0$ in $[1, 2)$. Let

$$\theta(p) = \|\omega\|_{L^p}, \quad \alpha(\epsilon) = \epsilon^{-1} \theta(\epsilon^{-1}), \tag{1.2}$$

and define

$$\mu(x) = \inf \left\{ x^{1-2\epsilon} \alpha(\epsilon) \colon \epsilon \text{ in } (0, 1/2] \right\}. \tag{1.3}$$

A classical result of measure theory is that $p \log \theta(p)$ is convex; this fact will play an important role in Part II.

**Definition 1.1.** We say that $\omega$ is a *Yudovich vorticity* if it is compactly supported (this is not essential, but will simplify our presentation) that satisfies the Osgood condition,

$$\int_0^1 \frac{dx}{\mu(x)} = \infty. \tag{1.4}$$

Examples of Yudovich vorticities are

$$\theta_0(p) = 1, \theta_1(p) = \log p, \ldots, \theta_m(p) = \log p \cdot \log^2 p \cdots \log^m p, \tag{1.5}$$

where $\log^m$ is log composed with itself $m$ times. These examples are described in [22] (see also [8].) Roughly speaking, the $L^p$–norm of a Yudovich vorticity can grow in $p$ only slightly faster than $\log p$. Such growth in the $L^p$–norms arises, for example, from a point singularity of the type $\log \log(1/|x|)$.

We define the class, $\mathbb{Y}$, of Yudovich velocities to be

$$\mathbb{Y} = \{K * \omega \colon \omega \text{ is a Yudovich vorticity}\}.$$

A Yudovich velocity will always lie in a space, $E_m$, as defined in [4]: Let $\sigma$ be a *stationary vector field*, meaning that $\sigma$ is of the form

$$\sigma = \left( -\frac{x_2}{r^2} \int_0^r \rho g(\rho)\, d\rho,\ \frac{x_1}{r^2} \int_0^r \rho g(\rho)\, d\rho \right) \tag{1.6}$$

for some $g$ in $C_C^\infty(\mathbb{R})$ with $\int_{\mathbb{R}^2} g = 1$. For any real number $m$, a vector $v$ belongs to $E_m$ if it is divergence-free and can be written in the form $v = m\sigma + v'$, where $v'$ is in $L^2(\mathbb{R}^2)$. $E_m$ is an affine space; having fixed the origin, $m\sigma$, in $E_m$, we can define a norm by $\|m\sigma + v'\|_{E_m} = \|v'\|_{L^2(\Omega)}$. Convergence in $E_m$ is equivalent to convergence in the $L^2$–norm to a vector in $E_m$.

Given $\theta$ as above, we define the function space,

$$\mathbb{Y}_\theta = \left\{ v \in E_m : \|\omega(v)\|_{L^p} \le C\theta(p) \text{ for all } p \text{ in } [p_0, \infty) \right\}, \tag{1.7}$$

for some constant $C$. We define the norm on $\mathbb{Y}_\theta$ to be

$$\|v\|_{\mathbb{Y}_\theta} = \|v\|_{E_m} + \sup_{p \in [p_0, \infty)} \|\omega(v)\|_{L^p} / \theta(p). \tag{1.8}$$

**Definition 1.2.** We say that a continuous function $f\colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$ is a modulus of continuity (MOC). When we say that a MOC, $f$, is $C^k$, $k \ge 0$, we mean that it is continuous on $[0, \infty)$ and $C^k$ on $(0, \infty)$.

A real-valued function or vector field, $v$, on a normed linear space, $X$, admits $f$ as a MOC if $|v(x) - v(y)| \le f(\|x - y\|_X)$ for all $x, y$ in $X$.

In Definition 1.2 we do not require $f$ to be concave: the MOC we will work with in Part I will each have that property (see Theorem 10.3), but we will not need this until Part II.

The final thing we must do before stating Yudovich's theorem is to define what we mean by a weak solution to the Euler equations.

**Definition 1.3** (Weak Euler Solutions)**.** Given an initial velocity $v^0$ in $\mathbb{Y}_\theta$, $v$ in $L^\infty([0, T]; \mathbb{Y}_\theta)$ is a weak solution to the Euler equations (without forcing) if $v(0) = v^0$ and

$$(\mathbf{E}) \qquad \frac{d}{dt} \int_\Omega v \cdot \varphi + \int_\Omega (v \cdot \nabla v) \cdot \varphi = 0$$

for all divergence-free $\varphi$ in $(H^1(\mathbb{R}^2))^2$.

Our form of the statement of Yudovich's theorem is a generalization of the statement of Theorem 5.1.1 of [4] from bounded to unbounded vorticity.

**Theorem 1.4** (Yudovich's Theorem for Unbounded Vorticity)**.** ***First part:*** *For any $v^0$ in $\mathbb{Y}$ there exists a unique weak solution $v$ of (E). Moreover, $v$ is in $C(\mathbb{R}; E_m) \cap L^\infty_{loc}(\mathbb{R}; L^\infty(\mathbb{R}^2))$ and*

$$\|\omega(t)\|_{L^p(\mathbb{R}^2)} = \|\omega^0\|_{L^p} \text{ for all } p_0 \le p < \infty. \tag{1.9}$$

***Second part****: The vector field has a unique continuous flow. More precisely, if $v^0$ is in $\mathbb{Y}_\theta$ then there exists a unique mapping $\psi$, continuous from $\mathbb{R} \times \mathbb{R}^2$ to $\mathbb{R}^2$, such that*

$$\psi(t, x) = x + \int_0^t v(s, \psi(s, x))\, ds.$$

*Let $\Gamma_t \colon [0, \infty) \to [0, \infty)$ be defined by $\Gamma_t(0) = 0$ and for $s > 0$ by*

$$\int_s^{\Gamma_t(s)} \frac{dr}{\mu(r)} = t. \tag{1.10}$$

*Then $\delta \mapsto \Gamma_t(\delta)$ is an upper bound on the MOC of the flow at time $t > 0$; that is, for all $x$ and $y$ in $\mathbb{R}^2$*

$$|\psi(t, x) - \psi(t, y)| \le \Gamma_t(|x - y|). \tag{1.11}$$

*Also, for all $x$ and $y$ in $\mathbb{R}^2$*

$$|v(t, x) - v(t, y)| \le \mu(|x - y|). \tag{1.12}$$

**Remark 1.5.** As we shall see in Theorem 10.3, $\mu$ is concave, giving it sublinear growth; hence, $\int_1^\infty \mu(r)^{-1}\, dr = \infty$. This makes $\Gamma_t(s)$ well-defined by (1.10) for all $s > 0$. Because $\mu$ is Osgood, $\Gamma_t(s)$ decreases to 0 as $s \to 0^+$.

Existence in the first part of Yudovich's theorem can be established, for instance, by modifying the approach on p. 311-319 of [14], which establishes existence under the assumption of bounded vorticity; the uniqueness argument is given by Yudovich in [22]. The second part is Theorem 2 of [22], the bound on the MOC of the flow following from working out the details of Yudovich's proof (see Sections 5.2 through 5.4 of [9]).

**Remark 1.6.** More properly, (1.10, 1.12) should be

$$\int_{2s}^{\Gamma_t(2s)} \frac{dr}{\mu(r)} = \frac{Ct}{2} \text{ and } |v(t, x) - v(t, y)| \le C\mu(|x - y|/2),$$

where $C$ is an absolute constant. We use the simpler forms in (1.10, 1.12), however, because they only result in changes in insignificant constants.

## 2. Square-symmetric vorticity

Ignoring for the moment the Euler equations, we will assume that the vorticity has certain symmetries, and from these symmetries deduce some useful properties of the divergence-free velocity having the given vorticity. In Section 3, we will then consider a solution to the Euler equations whose initial vorticity possesses these symmetries.

For convenience, we number the quadrants in the plane $Q_1$ through $Q_4$, starting with

$$Q_1 = \{(x_1, x_2) : x_1 \ge 0, x_2 \ge 0\}$$

and moving counterclockwise through the quadrants.

**Definition 2.1.** We say that a Yudovich vorticity (vorticity as in Definition 1.1) is *symmetric by quadrant*, or SBQ, if $\omega$ is compactly supported and $\omega(x) = \omega(x_1, x_2)$ is odd in $x_1$ and $x_2$; that is, $\omega(-x_1, x_2) = -\omega(x_1, x_2)$ for $x_1 \neq 0$ and $\omega(x_1, -x_2) = -\omega(x_1, x_2)$ for $x_2 = 0$—so also $\omega(-x) = \omega(x)$ when $x$ lies on neither axis.

**Lemma 2.2.** *Let $\omega$ be SBQ. Then there exists a unique vector field $v$ in $E_0 \cap \mathbb{Y}$ with $\omega(v) = \omega$, and $v$ satisfies the following:*

(1) $v_2(x_1, 0) = 0$ *for all $x_1$ in $\mathbb{R}$;*
(2) $v_1(0, x_2) = 0$ *for all $x_2$ in $\mathbb{R}$;*
(3) $v(0, 0) = 0$.

*If, in addition, $\omega \geq 0$ in $Q_1$, then*

(4) $v_1(x_1, 0) \geq 0$ *for all $x_1 \geq 0$.*

*Proof.* Let $p$ be in $[1, 2)$ and let $q > 2p/(2 - p)$. By Proposition 3.1.1 p. 44 of [4], for any vorticity $\omega$ in $L^p$ there exists a unique divergence-free vector field $v$ in $L^p + L^q$ whose curl is $\omega$, with $v$ being given by the Biot-Savart law,

$$v = K * \omega. \tag{2.1}$$

Here, $K$ is the Biot-Savart kernel of (1.1), which decays like $C/|x|$ with a singularity of order $C/|x|$ at the origin.

Because $\omega$ is compactly supported and lies in $L^2(\mathbb{R}^2)$, $\omega$ is in $L^p(\mathbb{R}^2)$, and (2.1) gives our velocity $v$, unique in all the spaces $L^p + L^q$. Also, because $\int_{\mathbb{R}^2} \omega = 0$, $v$ is in $(L^2)^2 = E_0$ (see the comment following Definition 1.3.3 of [4], for instance).

Then

$$\begin{aligned} v_1(x_1, 0) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y_2}{|x - y|^2} \omega(y)\, dy = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \omega(y)\, dy \\ &= \frac{1}{2\pi} \sum_{j=1}^{4} \int_{Q_j} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \omega(y)\, dy. \end{aligned}$$

Making the changes of variables, $u = (-y_1, y_2)$, $u = -y$, and $u = (y_1, -y_2)$ on $Q_2$, $Q_3$, and $Q_4$, respectively, in all cases the determinant of the Jacobian is $\pm 1$, and we obtain

$$\begin{aligned} v_1(x_1, 0) = \frac{1}{2\pi} \Bigg[ &\int_{Q_1} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \omega(y)\, dy - \int_{Q_1} \frac{u_2}{(x_1 + u_1)^2 + u_2^2} \omega(u)\, du \\ &+ \int_{Q_1} \frac{u_2}{(x_1 + u_1)^2 + u_2^2} \omega(u)\, du - \int_{Q_1} \frac{u_2}{(x_1 - u_1)^2 + u_2^2} \omega(u)\, du \Bigg] \end{aligned}$$

or

$$v_1(x_1, 0) = \frac{1}{\pi} \int_{Q_1} (f_1(x_1, y) - f_2(x_1, y))\, \omega(y)\, dy, \tag{2.2}$$

where

$$f_1(x_1, y) = \frac{y_2}{(x_1 - y_1)^2 + y_2^2}, \quad f_2(x_1, y) = \frac{y_2}{(x_1 + y_1)^2 + y_2^2}. \tag{2.3}$$

It follows from $(x_1 - y_1)^2 + y_2^2 \le (x_1 + y_1)^2 + y_2^2$ on $Q_1$ that $f_1(x_1, y) > f_2(x_1, y)$ for all $x_1, y_1 > 0$. Conclusion (4) then follows from (2.2).

By the Biot-Savart law of (2.1),

$$\begin{aligned} v_2(x_1, -x_2) &= (K_2 * \omega)(x_1, -x_2) \\ &= \int_{\mathbb{R}^2} K_2(x_1 - y_1, -x_2 - y_2)\omega(y_1, y_2)\, dy \\ &= \int_{\mathbb{R}^2} K_2(x_1 - y_1, x_2 + y_2)\omega(y_1, -y_2)\, dy \\ &= \int_{\mathbb{R}^2} K_2(x_1 - y_1, x_2 - (-y_2))\omega(y_1, -y_2)\, dy \\ &= -v_2(x_1, x_2). \end{aligned}$$

Here we used $K_2(x_1, -x_2) = -K_2(x_1, x_2)$ and the symmetry of $\omega$. A similar calculation shows that $v_1(-x_1, x_2) = -v_1(x_1, x_2)$. Thus, the velocity along the $x$-axis is directed along the $x$-axis and the velocity along the $y$-axis is directed along the $y$-axis, so the axes are preserved by the flow. In particular, the origin is fixed. This gives conclusions (1)-(3). □

Lemma 2.3 is Proposition 2.1 of [2] (see also Proposition 5.3.1 of [4]).

**Lemma 2.3.** *Let $\omega$ be SBQ with*

$$\omega = 2\pi \mathbf{1}_{[0,1]\times[0,1]} \tag{2.4}$$

*in $Q_1$. Then there exists a constant $C > 0$ such that*

$$v_1(x_1, 0) \ge 2x_1 \log(1/x_1) \tag{2.5}$$

*for all $x_1$ in $(0, C]$.*

The following lemma is a slight generalization of Lemma 2.3 that will give us our key inequality.

**Lemma 2.4.** *Let $\omega$ be SBQ with*

$$\omega = 2\pi \mathbf{1}_{[0,r]\times[0,r]} \tag{2.6}$$

*on $Q_1$ for some $r$ in $(0, 1)$. Then for any $\lambda$ in $(0, 1)$ there exists a right neighborhood of the origin, $\mathcal{N}$, such that*

$$v_1(x_1, 0) \ge 2(1 - \lambda)x_1 \log(1/x_1) \tag{2.7}$$

*for all $x_1$ in $(0, r^{1/\lambda}] \cap \mathcal{N}$.*

**Remark 2.5.** The neighborhood $\mathcal{N}$ depends only upon $\lambda$; in particular, it is independent of $r$.

*Proof.* The result follows from scaling the result in Lemma 2.3. Indeed, if we write $\omega^r(x)$ for the function $\omega$ defined by (2.6) then $\omega^1$ is the function defined by (2.4) and $\omega^r(\cdot) = \omega^1(\cdot/r)$. Letting $v^r = K * \omega^r$ we see that $v^r(x) = rv^1(x/r)$, since then $\omega(v^r(x)) = r(1/r)\omega(v^1)(x/r) = \omega^1(x/r) = \omega^r(x)$ and $v^r(x)$ is divergence-free. It follows from Lemma 2.3 that for all $x_1$ such that $x_1/r$ lies in $[0, C]$,

$$\begin{aligned} v_1^r(x_1, 0) &= rv_1^1(x_1/r, 0) > r2(x_1/r)\log(1/(x_1/r)) \\ &= 2x_1\left[\log(1/x_1) + \log r\right]. \end{aligned}$$

Thus, if $x_1^\lambda \le r$ then $\log r \ge \lambda \log x_1 = -\lambda \log(1/x_1)$ so

$$v_1^r(x_1, 0) > 2x_1(1 - \lambda)\log(1/x_1).$$

Thus, (2.7) holds for all $x_1$ in $[0, r^{1/\lambda}] \cap [0, rC]$. But $r^{1/\lambda} \le rC$ if and only if $r \le C^{\lambda/(1-\lambda)}$, which gives us the right neighborhood, $\mathcal{N} = (0, C^{1/(1-\lambda)})$. □

**Definition 2.6.** We say that $\omega$ is square-symmetric if $\omega$ is SBQ and $\omega(x_1, x_2) = \omega(\max\{x_1, x_2\}, 0)$ on $Q_1$.

Being square-symmetric means that a vorticity is SBQ and is constant in absolute value along the boundary of any square centered at the origin.

**Lemma 2.7.** *Assume that $\omega$ is square-symmetric, finite except possibly at the origin, and $\omega(x_1, 0)$ is non-increasing for $x_1 > 0$. Then for any $\lambda$ in $(0, 1)$*

$$v_1(x_1, 0) \ge C_\lambda \omega(x_1^\lambda, 0) x_1 \log(1/x_1) \tag{2.8}$$

*for all $x_1$ in the neighborhood $\mathcal{N}$ of Lemma 2.4, where $C_\lambda = (1 - \lambda)/\pi$.*

*Proof.* We can write $\omega$ on $Q_1$ as

$$\omega(x) = 2\pi \int_0^1 \alpha(s) \mathbf{1}_{[0,s]\times[0,s]}(x)\, ds, \tag{2.9}$$

for some measurable, nonnegative function $\alpha\colon (0, 1) \to [0, \infty)$. This means that

$$\omega(x_1, 0) = 2\pi \int_{x_1}^1 \alpha(s)\, ds. \tag{2.10}$$

Let $V(s)$ be the value of $v_1(x_1, 0)$ that results from assuming that $\omega$ is given by (2.6). By Lemma 2.2, $V(s) > 0$. Then because the Biot-Savart law of (2.1) is linear, and using Lemma 2.4, for all $x_1$ in the neighborhood $\mathcal{N}$,

$$\begin{aligned}
v_1(x_1, 0) &= \int_0^1 \alpha(s) V(s)\, ds \\
&= \int_0^{x_1^\lambda} \alpha(s) V(s)\, ds + \int_{x_1^\lambda}^1 \alpha(s) V(s)\, ds \\
&\ge \int_{x_1^\lambda}^1 \alpha(s) V(s)\, ds \\
&\ge 2\pi \left( \int_{x_1^\lambda}^1 \alpha(s)\, ds \right) \frac{2(1-\lambda)}{2\pi} x_1 \log(1/x_1) \\
&= \frac{1-\lambda}{\pi} \omega(x_1^\lambda, 0) x_1 \log(1/x_1).
\end{aligned}$$

In the final inequality, $V(s)$ is bounded as in Lemma 2.4 because $x_1^\lambda \le s$ in the integrand. □

**Remark 2.8.** Properly speaking, we must allow the function $\alpha$ of (2.9) to be a distribution since, for instance, to obtain $\omega$ of Lemma 2.4, we would need $\alpha = \delta_r$. We could avoid this complication, however, by assuming that $\omega$ is strictly decreasing and that $\omega(x_1, 0)$ is sufficiently smooth as a function of $x_1 > 0$.

## 3. Evolution of square-symmetric vorticity

We now assume that our initial vorticity is square-symmetric, and consider what happens to the solution to $(E)$ over time.

**Theorem 3.1.** *Assume that $\omega^0$ is square-symmetric, finite except possibly at the origin, and $\omega^0(x_1, 0)$ is nonnegative and non-increasing for $x_1 > 0$. Then for any $\lambda$ in $(0, 1)$,*

$$v_1(t, x_1, 0) \ge C_\alpha \omega^0(\Gamma_t(2^{\lambda/2} x_1^\lambda), 0) x_1 \log(1/x_1) := L(t, x_1) \tag{3.1}$$

*for all $x_1$ in the neighborhood $\mathcal{N}$ of Lemma 2.4 and all time $t \ge 0$, where $\Gamma_t$ is defined as in Theorem 1.4. The constant $C_\lambda = (1-\lambda)/\pi$.*

*Further, let $x_1(t)$ be the minimal solution to*

$$\frac{dx_1(t)}{dt} = L(t, x_1), \quad x_1(0) = a \tag{3.2}$$

*with $a > 0$ in $\mathcal{N}$, and $(0, t_a)$ being the time of existence. Then $\psi^1(t, a, 0) \ge x_1(t)$ for all $t$ in $[0, t_a)$.*

**Remark 3.2.** In our applications of Theorem 3.1 in the next two sections, $L$ will be Osgood continuous in space, so that a unique (and explicit) solution to (3.2) exists for all time. Hence, there will be no need to determine a minimal solution and we will have $t_a = \infty$.

*Proof of Theorem 3.1.* Since $\omega^0(x_1, x_2) = -\omega^0(x_1, -x_2)$, if $\omega(t, x_1, x_2)$ is a solution to $(E)$ then $-\omega(t, x_1, -x_2)$ is also a solution. But the solution to $(E)$ is unique by Theorem 1.4, so we conclude that $\omega(t, x_1, x_2) = -\omega(t, x_1, -x_2)$. Similarly, $\omega(t, x_1, x_2) = -\omega(t, -x_1, x_2)$, and we see that $\omega$ is SBQ. By Lemma 2.2, then, it follows that the flow transports vorticity in $Q_k$, $k = 1, \dots, 4$, only within $Q_k$, because of the direction of $v$ along the axes for all $t \geq 0$. Therefore, $\omega(t)$ is also nonnegative in $Q_1$ for all time.

Our approach then will be to produce a point-by-point lower bound $\overline{\omega}(t)$ on $\omega(t)$ that satisfies all the requirements of Lemma 2.7. In particular, it is SBQ, so $\omega(t) - \overline{\omega}(t)$ is SBQ and nonnegative in $Q_1$. It follows from Lemma 2.2 that $v_1(t, x_1, 0) - \overline{v}_1(t, x_1, 0) \geq 0$ for all $t \geq 0$, where $\omega(\overline{v}(t)) = \overline{\omega}(t)$. Thus, the lower bound on $\overline{v}_1(t, x_1, 0)$ coming from Lemma 2.7 will also be a lower bound on $v_1(t, x_1, 0)$. We now determine $\overline{\omega}(t)$.

Because conclusion (3) of Lemma 2.2 holds for all time, $\omega$ being SBQ for all time, the origin is fixed by the flow; that is $\psi(t, 0) = \psi^{-1}(t, 0) = 0$ for all $t$. Also, the Euler equations are time reversible, and the function $\Gamma_t$ of (1.10) depends only upon the Lebesgue norms of the vorticity, which are preserved by the flow; therefore, $\Gamma_t$ is a bound on the modulus of continuity of $\psi^{-1}(t, \cdot)$ as well. Thus,

$$\left|\psi^{-1}(t, x)\right| = \left|\psi^{-1}(t, x) - \psi^{-1}(t, 0)\right| \leq \Gamma_t(|x|).$$

In $Q_1$, the value of $\omega(t, x)$, then, is bounded below by using the minimum value of $\omega^0$ on the circle of radius $\Gamma_t(|x|)$ centered at the origin, since this is the furthest away from the origin that $\psi^{-1}(t, x)$ can lie, and $\omega^0$ decreases with the distance from the origin. That is,

$$\omega(t, x) = \omega^0(\psi^{-1}(t, x)) \geq \omega^0(\Gamma_t(|x|), 0)$$

because $\omega^0$ is square-symmetric.

Since $\sqrt{2}\max\{x_1, x_2\} \geq |x|$, $\Gamma_t$ is nondecreasing, and $\omega_0$ is nonincreasing on $Q_1$, $\omega^0(\Gamma_t(\sqrt{2}\max\{x_1, x_2\}), 0) \leq \omega^0(\Gamma_t(|x|), 0)$ on $Q_1$. Letting

$$\overline{\omega}(t, x_1, x_2) = \omega^0(\Gamma_t(\sqrt{2}\max\{x_1, x_2\}), 0)$$

we see that $\overline{\omega}$ is square-symmetric, and on $Q_1$, $\overline{\omega}(t, x) \leq \omega(t, x)$, so $\overline{\omega}$ is our desired lower bound on $\omega$.

Then from (2.8),

$$\begin{aligned} v^1(x_1, 0) &\geq C_\lambda \overline{\omega}(t, x_1^\lambda, 0) x_1 \log(1/x_1) \\ &= C_\lambda \omega^0(\Gamma_t((\sqrt{2}\max\{x_1, 0\})^\lambda), 0) x_1 \log(1/x_1) \\ &= C_\lambda \omega^0(\Gamma_t(2^{\lambda/2} x_1^\lambda), 0) x_1 \log(1/x_1). \end{aligned}$$

That the minimal solution exists to (3.2) on $[0, t_a)$ for some $t_a > 0$ and the inequality, $\psi^1(t, a, 0) \geq x_1(t)$ for all $t$ in $[0, t_a)$, are classical results; see, for instance, Theorems 2.1 and 4.2 Chapter III of [7]. □

## 4. Bounded vorticity

We now apply Theorem 3.1 to the first in the sequence of Yudovich's vorticity bounds in (1.5) in which we have bounded vorticity. We assume that $\omega$ is square-symmetric with $\omega^0 = \mathbf{1}_{[0,1/2]\times[0,1/2]}$ in $Q_1$ so that $\|\omega^0\|_{L^1 \cap L^\infty} = 1$. We have,

$$\mu(r) = \inf\left\{r^{1-2\epsilon}/\epsilon : \epsilon \text{ in } (0,1/2]\right\} = \inf\left\{g(\epsilon) : \epsilon \text{ in } (0,1/2]\right\},$$

where $g(\epsilon) = r^{1-2\epsilon}/\epsilon$. Then

$$g'(\epsilon) = C\frac{r^{1-2\epsilon}(2\epsilon\log(1/r) - 1)}{\epsilon^2},$$

which is zero when $\epsilon = \epsilon_0 := 1/(2\log(1/r))$ if $r < 1$ and $\epsilon_0 < 1/2$, and never zero otherwise. But

$$\begin{aligned} \epsilon_0 < 1/2 \iff \frac{1}{2\log(1/r)} < 1/2 \iff \log(1/r) > 1 \\ \iff \frac{1}{r} > e \iff r < e^{-1}, \end{aligned}$$

so the condition $r < 1$ is redundant.

Assume that $r < e^{-1}$. Then $g(\epsilon)$ approaches infinity as $\epsilon$ approaches either zero or infinity; hence, $\epsilon_0$ minimizes $g$. Thus,

$$\begin{aligned} \mu(r) &= r^{1-2\epsilon_0}/\epsilon_0 = 2r(1/r)^{2\epsilon_0}\log(1/r) \\ &= 2re^{2\log(1/r)\epsilon_0}\log(1/r) = -2er\log(r). \end{aligned}$$

Then from (1.10),

$$\begin{aligned} \int_{x_1}^{\Gamma_t(x_1)} \frac{dr}{\mu(r)} &= -(2e)^{-1}\left[\log(-\log r)\right]_{x_1}^{\Gamma_t(x_1)} = t \\ &\implies \log(-\log(x_1) - \log(-\log(\Gamma_t(x_1))) = 2et \\ &\implies \Gamma_t(x_1) = x_1 e^{-2et} \end{aligned}$$

as long as $\Gamma_t(x_1) < e^{-1}$.

Thus, Theorem 3.1 gives

$$\begin{aligned} v^1(t, x_1, 0) &\geq C_\lambda \omega^0(2^{\lambda/2}x_1^{\lambda e^{-2et}}, 0)x_1\log(1/x_1) \\ &\geq C_\lambda x_1 \log(1/x_1) \end{aligned}$$

as long as $2^{\lambda/2}x_1^{\lambda e^{-2et}} < 1/2$.

Solving $dx_1(t)/dt = C_\lambda x_1 \log(1/x_1)$ with $x_1(0) = a$ gives

$$\psi^1(t, a, 0) \geq x_1(t) = a^{\exp(-C_\lambda t)},$$

which applies for sufficiently small $a$.

Since $\psi(t, 0, 0) = 0$,

$$\frac{|\psi(t, a, 0) - \psi(t, 0, 0)|}{a^\alpha} = \frac{\left|\psi^1(t, a, 0)\right|}{a^\alpha} \geq a^{\exp(-C_\lambda t) - \alpha},$$

which is infinite for any $\alpha > e^{-C_\lambda t}$. This shows that the flow can be in no Hölder space $C^\alpha_\lambda$ for $\alpha > e^{-C_\lambda t}$, reproducing, up to a constant, the result of [2] (or see Theorem 5.3.1 of [4].)

## 5. Yudovich's higher examples

Assume that $m \geq 2$ and let $\omega^0$ have the symmetry described in Theorem 3.1 with

$$\omega^0(x_1, 0) = \log^2(1/x_1) \cdots \log^m(1/x_1) = \theta_m(1/x_1)/\log(1/x_1), \tag{5.1}$$

for $x_1$ in $(0, 1/\exp^m(0))$, and $\omega^0$ equal to zero elsewhere in the first quadrant. Then by Lemma 5.2,

$$\theta(p) = \|\omega^0\|_{L^p} \leq C \log p \cdots \log^{m-1} p = \theta_{m-1}(p)$$

for all $p$ larger than some $p^*$, with $\theta_{m-1}$ given by (1.5).

Adapting an observation of Yudovich's in [22], if $\mu$ is the function of (1.3) associated with the admissible function $\theta$, then letting $\epsilon_0 = 1/2\log(1/r)$ for $r < e^{-p^*}$,

$$\begin{aligned}\mu(r) &\leq (r^{1-2\epsilon_0}/\epsilon_0)\theta(1/\epsilon_0) = -Crr^{1/\log r} \log r \theta(\log(1/r)) \\ &= Cr \log(1/r) \log^2(1/r) \cdots \log^m(1/r) \\ &= Cr\theta_m(1/r).\end{aligned}$$

Then, if we define the upper bound on the modulus of continuity of the flow by (1.10), we have

$$\begin{aligned}-C\left[\log^{m+1}(1/r)\right]_s^{\Gamma_t(s)} &= \int_s^{\Gamma_t(s)} \frac{dr}{Cr\theta_m(1/r)} \\ &\leq \int_s^{\Gamma_t(s)} \frac{dr}{\mu(r)} = t \\ &\implies -C\log^{m+1}(1/\Gamma_t(s)) \leq -C\log^{m+1}(1/s) + t \\ &\implies \log^{m+1}(1/\Gamma_t(s)) \geq \log^{m+1}(1/s) - Ct \\ &\implies \log^m(1/\Gamma_t(s)) \geq e^{-Ct}\log^m(1/s).\end{aligned}$$

Using this bound, we have, from (3.1),

$$\begin{aligned}v^1(t, &x_1, 0) \geq C_\lambda \omega^0(\Gamma_t(2^{\lambda/2}x_1^\lambda), 0)x_1 \log(1/x_1) \\ &= C_\lambda \log^2(1/\Gamma_t(2^{\lambda/2}x_1^\lambda)) \cdots \log^m(1/\Gamma_t(2^{\lambda/2}x_1^\lambda))x_1 \log(1/x_1) \\ &\geq C_\lambda e^{-Ct} \log^2(1/\Gamma_t(2^{\lambda/2}x_1^\lambda)) \cdots \log^m(1/2^{\lambda/2}x_1^\lambda)x_1 \log(1/x_1) \\ &\geq C'_\lambda e^{-Ct} \log^2(1/\Gamma_t(2^{\lambda/2}x_1^\lambda)) \cdots \log^m(1/x_1)x_1 \log(1/x_1),\end{aligned} \tag{5.2}$$

as long as $x_1 > 0$ is sufficiently small, where $C'_\alpha$ depends on $\lambda$. (When $m > 2$, the argument $1/\Gamma_t(2^{\lambda/2}x_1^\lambda)$ appears in each of the $\log^2, \ldots, \log^{m-1}$ factors above, but not in the log factor. When $m = 2$ it appears in none of the factors.)

Specializing to the case $m = 2$ and combining the previous two inequalities, the explicit dependence of the bound in (5.2) on $\Gamma_t$ disappears, and we have

$$v^1(t, x_1, 0) \geq C'_\lambda e^{-Ct} \log^2(1/x_1) x_1 \log(1/x_1) = C'_\lambda e^{-Ct} x_1 \theta_2(1/x_1).$$

Solving for

$$\frac{dx_1(t)}{dt} = C'_\lambda e^{-Ct} x_1 \theta_2(1/x_1) \tag{5.3}$$

with $x_1(0) = a$, we get

$$\log^3(1/x_1(t)) = \log^3(1/a) + C'_\lambda \left(e^{-Ct} - 1\right),$$

so

$$\begin{aligned} \psi^1(t, a, 0) \geq x_1(t) &= \exp\left(-(-\log a)^{\exp\left(C'_\lambda(e^{-Ct}-1)\right)}\right) \\ &= e^{-(-\log a)^\gamma}, \end{aligned}$$

where $\gamma = \exp\left(C'_\lambda(e^{-Ct} - 1)\right)$.

Observe that $\gamma < 1$ for all $t > 0$. Thus, for any $\alpha$ in $(0, 1)$ and all $t > 0$,

$$\begin{aligned} \|\psi\|_{C^\alpha} &\geq \lim_{a \to 0^+} \frac{\psi^1(t, a, 0) - \psi^1(t, 0, 0)}{a^\alpha} \geq \lim_{a \to 0^+} \frac{x_1(t)}{a^\alpha} \\ &= \lim_{a \to 0^+} \frac{e^{-(-\log a)^\gamma}}{e^{-(-\log a)\alpha}} = \lim_{u \to \infty} \frac{e^{-u^\gamma}}{e^{-\alpha u}} = \lim_{u \to \infty} e^{\alpha u - u^\gamma} = \infty. \end{aligned} \tag{5.4}$$

We conclude that the flow lies in no Hölder space of positive exponent for all positive time, a result that we state explicitly as a corollary of Theorem 3.1.

**Corollary 5.1.** *There exists initial velocities satisfying the conditions of Theorem 1.4 for which the unique solution to (E) has an associated flow lying, for all positive time, in no Hölder space of positive exponent.*

We used Lemma 5.2 above, and Lemma 5.3 is used in its proof.

**Lemma 5.2.** *Let $m \geq 2$ and let $\omega^0$ have the symmetry described in Theorem 3.1 with*

$$\omega^0(x_1, 0) = \log^2(1/x_1) \cdots \log^m(1/x_1) = \theta_m(1/x_1)/\log(1/x_1),$$

*for $x_1$ in $(0, 1/\exp^m(0))$, and $\omega^0$ equal to zero elsewhere in the first quadrant. Then*

$$\|\omega^0\|_{L^p} \sim \log p \cdots \log^{m-1} p = \theta_{m-1}(p)$$

*for large $p$.*

*Proof.* Because of the symmetry of $\omega^0$,

$$\begin{aligned} \|\omega^0\|_{L^p}^p &= 4 \int_0^{1/\exp^m(0)} 2 \int_0^{x_1} (\omega^0(x_1, 0))^p \, dx_2 \, dx_1 \\ &= 8 \int_0^{1/\exp^m(0)} x_1 \left[\log^2(1/x_1) \cdots \log^m(1/x_1)\right]^p \, dx_1. \end{aligned} \tag{5.5}$$

Making the change of variables, $u = \log(1/x_1) = -\log x_1$, it follows that $x_1 = e^{-u}$ and $du = -(1/x_1)\,dx_1$ so $dx_1 = -e^{-u}\,du$. Thus,

$$\begin{aligned}\|\omega^0\|_{L^p}^p &= 8\int_\infty^{\exp^{m-1}(0)} e^{-u}\left[\log u\cdots\log^{m-1}u\right]^p(-e^{-u})\,du\\ &= 8\int_{\exp^{m-1}(0)}^\infty e^{-2u}\left[\log u\cdots\log^{m-1}u\right]^p\,du\\ &\geq 8\int_p^\infty e^{-2u}\left[\log p\cdots\log^{m-1}p\right]^p\,du\\ &= 4e^{-2p}\left[\log p\cdots\log^{m-1}p\right]^p,\end{aligned}$$

the inequality holding for all sufficiently large $p$. Asymptotically, then, $\|\omega^0\|_{L^p} \geq e^{-2}\log p\cdots\log^{m-1}p$.

For the upper upper bound on $\|\omega^0\|_{L^p}$, we use Lemma 5.3 to obtain, for all sufficiently large $p$,

$$\begin{aligned}\|\omega^0\|_{L^p}^p &\leq 8\left(\int_0^p + \int_p^\infty\right)e^{-2u}\left[\log u\cdots\log^{m-1}u\right]^p\,du\\ &\leq 8\int_0^p e^{-2u}\left[\log p\cdots\log^{m-1}p\right]^p\,du\\ &\qquad\qquad + 8\int_p^\infty e^{-2u}\left[\log p\cdots\log^{m-1}p\,e^{\frac{u}{p}-1}\right]^p\,du\\ &\leq 8\left(\log p\cdots\log^{m-1}p\right)^p\left[\int_0^p e^{-2u}\,du + e^{-p}\int_p^\infty e^{-u}\,du\right]\\ &= 8\left(\frac{1+e^{-2p}}{2}\right)\left(\log p\cdots\log^{m-1}p\right)^p.\end{aligned}$$

Thus, asymptotically, $\|\omega^0\|_{L^p} \leq \log p\cdots\log^{m-1}p$, completing the proof. □

**Lemma 5.3.** *Let $m$ be a positive integer. Then for sufficiently large $p$,*

$$\log(xp)\cdots\log^{m-1}(xp) \leq (\log p\cdots\log^{m-1}p)e^{x-1} \tag{5.6}$$

*for all $x \geq 1$.*

*Proof.* We prove this for $m = 3$, the proof for other values of $m$ being entirely analogous. First, by taking the logarithm of both sides of (5.6), that equation holds if and only if

$$\begin{aligned}f(x) &:= \log\log(xp) + \log\log\log(xp)\\ &\leq g(x) := \log(\log p\log\log p) + x - 1.\end{aligned}$$

Because equality holds for $x = 1$, our result will follow if we can show that $f' \leq g'$ for all $x \geq 1$ and sufficiently large $p$. This is, in fact, true, since

$$f' = \frac{1}{x\log(xp)} + \frac{1}{x\log(xp)\log\log(xp)} \leq 1 = g'$$

for all $x \geq 1$ and $p \geq e^e$. □

## 6. Remarks

It is natural to try to extend the analysis of Section 5 to Yudovich initial vorticities for $m > 2$. But this is, in fact, quite difficult, because when $m > 2$ the explicit dependence of the bound in (5.2) on $\Gamma_t$ remains, so we must also bound $\log^k(1/\Gamma_t(s))$ for $k = 2, \dots, m-1$. Doing so makes the analog of (5.3) no longer exactly integrable. It is clear that one obtains a worse bound on the modulus of continuity (MOC) than for $m = 2$, but it is not at all clear what happens as we take $m$ to infinity.

We chose to give the initial vorticity $SBQ$ symmetry because such symmetry works well with the symmetry of the Biot-Savart law to produce a lower bound on the velocity along the $x$- or $y$-axes. Having made this choice, the rest of our choices concerning the vorticity were inevitable, up to interchanging the roles of $x$ and $y$ or changing the sign of the vorticity. Because the initial vorticity is $SBQ$, the function $f = f_1 - f_2$, where $f_1$ and $f_2$ are defined in (2.3), controls the bound on the velocity, and $f$ is continuous except for a singularity at $y = (x_1, 0)$, where it goes to positive infinity (for $x_1 > 0$). Thus, whatever lower bound we derive on $v_1(x_1, 0)$, it will increase the fastest at a singularity of $|\omega(t)|$ that lies along the $x$-axis and this effect is most pronounced when $\omega$ is of a single sign in $Q1$ (this follows from (2.2)). The lower bound on the MOC of the flow then follows from allowing a point $a = (a_1, 0)$ to approach the singularity and looking at how large the appropriate difference quotient becomes, as in (5.4). But to do this, we need control on the position of the singularity of $|\omega(t)|$, and, when assuming $SBQ$, the origin is the one point at which we have the most control—the singularity doesn't move at all.

Thus, the assumption of $SBQ$ naturally leads us to assume a point singularity at the origin. Then, because it appears that we can only bound from below the effect on $v_1(x_1, 0)$ of the vorticity outside of the square on which a point lies (actually, an even larger square because of the exponent $\lambda$ in Lemma 2.7), we are naturally led to the assumption that $|\omega^0|$ decreases with the distance from the origin, which leads to Lemma 2.7.

A possible way to around these difficulties is to maintain symmetry by quadrant of the initial vorticity, but to drop the constraint of square symmetry, pinching or cutting out the singularity as in Figure 1. We also require that $|\omega^0|$ be a decreasing function of $|x_1|$ alone.

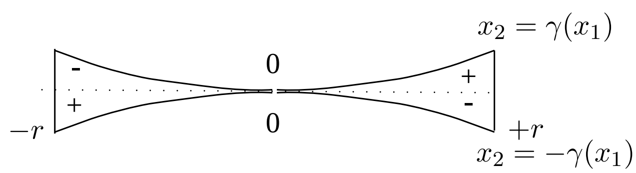


*Figure 1*

The support of the initial vorticity is now $\Omega_0$, the region lying between the curves $x_2 = \pm\gamma(x_1)$ and the vertical lines $x_1 = \pm r$. We require that $\gamma$

be even and that $\gamma'(0) = 0$ (else we gain little over our existing example). One can show that (2.8, 3.1) continue to hold, though now the neighborhood on which they hold shrinks rapidly as we make $\gamma$ pinch more tightly (that is, vanish more quickly as the origin is approached). But by pinching the initial vorticity this way we can sharpen the singularity of $\omega^0$ at the origin while leaving its $L^p$-norms and so $\Gamma_t$ unchanged. This then would increase the lower bound in (3.1)—if, that is, we can insure that the geometry of the support of the vorticity is not changed too radically over a short time. This is the subject of a future work.

## Part II: Inverse problems

### 7. Introduction

The various mappings in Part I can be described schematically as (see Section 1 for definitions)

$$v^0 \mapsto v(t,x) \mapsto \psi(t,x), \tag{7.1}$$

$$v^0 \mapsto \omega^0 \mapsto \theta(p) \mapsto \mu(x) \mapsto \Gamma_t(x). \tag{7.2}$$

That is, the initial velocity gives the velocity at all time which gives the flow at all time, and the initial velocity also gives the initial vorticity, whose $L^p$-norms, $\theta(p)$, give the MOC of the the velocity field which gives the MOC of the flow. The Osgood condition on $\mu$ insures that both mappings in (7.1) and the last mapping in (7.2) are well-defined.

The mappings in (7.1) are trivial to invert: $v(t,x) = \partial_t\psi(t,\psi^{-1}(t,x))$ and $v^0 = v(0,\cdot)$. The first mapping in (7.2) is easily inverted as well using the Biot-Savart law, (2.1). The remaining three mappings in (7.2), which are the topic of this part of the paper, are another matter.

Our interest in inverting $\theta(p) \mapsto \mu(x)$ and $\mu(x) \mapsto \Gamma_t(x)$ stems from an attempt to answer the question, "Can the upper bound on the MOC of the flow map given by (1.10) be arbitrarily poor?" More precisely, we have the following two questions:

> **Question 1**: Given a fixed time $t_0 > 0$ and any (concave) MOC, $f$, does there exist a (concave) MOC, $\mu$, such that the associated $\Gamma_{t_0} \geq f$, at least near the origin?
>
> **Question 2**: If we obtain $\mu$ from $f$ as in Question 1, can we find a function $\theta$ that inverts the map, $\theta(p) \mapsto \mu(x)$?

If we place no restrictions on the MOC, $\mu$, other than the almost minimal ones that $\mu$ is $C^1$ and strictly increasing then we can answer Question 1 affirmatively fairly easily (see Theorem 9.4). In this form, Question 1 is equivalent to some results in functional equations from the early 1960s due to Kordylewski and Kuczma ([11, 12]) and Choczewski ([5]).

We will find, however, in Section 10 that $\mu$ must be concave (and have further properties as well). Adding only the fundamental constraint that $\mu$

be concave (a common constraint for flows associated with transport equations) we give an affirmative answer to Question 1 in Remark (9.13), but only after a fairly lengthy digression into results, due primarily to Zdun [23] and Targoński and Zdun [19], on iteration (sub)groups. In Section 8, we give some of these results along with proofs in a form more directly applicable to our purposes. In Section 9 we consider Question 1, extending in a small way the results of Zdun and using them to answer Question 1.

After characterizing the required properties of $\mu$ in Section 10, and describing some useful implications of these properties in Section 11, we give a complete answer to Question 2 in Theorem 12.2.

Remaining open is whether Question 1 can be answered affirmatively when the additional properties of $\mu$ given in the three equivalent conditions of (10.10) are required to hold.

Finally, in Section 13, we give one approach to inverting (approximately) the map, $\omega^0 \mapsto \theta(p)$. This approach yields an additional constraint on the third derivative of $\mu$; it is unclear, however, whether this constraint is an artifact of the method of inversion or whether it, or some similar constraint, is an essential requirement.

In what follows, we will have need to distinguish among the following three degrees of concavity (or, similarly, convexity):

**Definition 7.1.** Assume that $f$ is a twice differentiable function on some open interval of $\mathbb{R}$. Then we say that

(1) $f$ is *concave* if $f'' \le 0$;
(2) $f$ is *strictly concave* if $f'$ is strictly decreasing;
(3) $f$ is *strongly strictly concave* if $f'' < 0$.

Observe that (3) $\implies$ (2) $\implies$ (1).

## 8. Properties of the MOC of the flow map

The relation between the (time-independent) MOC of a vector field and of its flow is given in the following, classical lemma (which was used in the proof of Theorem 1.4):

**Lemma 8.1.** *Let $\Omega$ be a domain in $\mathbb{R}^n$ and $v$ be a vector field on $\Omega$ that admits a MOC, $\mu$, that satisfies the Osgood condition, (1.4). Then $v$ has a unique associated flow, $\psi$, continuous from $\mathbb{R} \times \Omega$ to $\mathbb{R}^n$, such that*

$$\psi(t,x) = x + \int_0^t v(\psi(s,x))\,ds \tag{8.1}$$

*for all $x$ in $\Omega$. For $t \ge 0$ define $\Gamma_t \colon [0,\infty) \to [0,\infty)$ by $\Gamma_t(0) = 0$ and for $x > 0$ by*

$$I_t(x) := \int_x^{\Gamma_t(x)} \frac{dr}{\mu(r)} = t \tag{8.2}$$

*for all $t \geq 0$. Then $\Gamma_t$ is a modulus of continuity for the flow, $\psi$, in the sense of (1.11).*

*Proof.* The proof is classical. See, for instance, Theorem 5.2.1 of [4]. (Chemin's theorem is stated for log-Lipschitz vector fields, but the proof applies for any vector field having a MOC satisfying Osgood's condition.) □

We will alternately write $\Gamma(t, x)$ and $\Gamma_t(x)$.

**Theorem 8.2.** *Assume that $\mu$ is a MOC with $\mu > 0$ on $(0, \infty)$ and that it satisfies the Osgood condition, (1.4). Then (8.2) uniquely defines $\Gamma\colon [0, \infty)^2 \to [0, \infty)$ with $\Gamma_0$ = identity map and, for all $t > 0$, $\Gamma_t$ strictly increasing, $\Gamma_t(0) = 0$, and $\Gamma_t(x) > x$ for all $x > 0$. Also, $\Gamma$ is continuously differentiable.*

*If $\mu$ is strictly increasing on $(0, a)$ for some $0 < a \leq \infty$ then $\Gamma_t' > 1$ on $(0, \Gamma_t^{-1}(a))$ for all $t > 0$.*

*Moreover, viewing $\mu$ as a 1-vector (velocity) field on $[0, \infty)$, $\Gamma$ satisfies the transport equation,*

$$\begin{cases} \partial_t \Gamma(t, x) - \mu(x) \partial_x \Gamma(t, x) = 0, & (t, x) \text{ in } [0, \infty) \times [0, \infty), \\ \Gamma(0, x) = x, & x \text{ in } [0, \infty). \end{cases} \tag{8.3}$$

*Also viewing each $\Gamma_t$, $t \geq 0$, as 1-vector fields on $[0, \infty)$, $\Gamma$ is the flow associated with the velocity field, $\mu$. That is, $\Gamma$ is its own flow:*

$$\begin{cases} \partial_t \Gamma(t, x) = \mu(\Gamma(t, x)), & (t, x) \text{ in } [0, \infty) \times [0, \infty), \\ \Gamma(0, x) = x, & x \text{ in } [0, \infty). \end{cases} \tag{8.4}$$

*Finally, the following identity holds for all $(t, x)$ in $[0, \infty) \times [0, \infty)$:*

$$\mu(\Gamma_t(x)) = \Gamma_t'(x) \mu(x). \tag{8.5}$$

*Proof.* The conclusions in the first paragraph of this lemma, with the exception of the last sentence, follow immediately from (8.2).

Taking the derivative with respect to $x$ of (8.2),

$$I_t'(x) = \frac{\Gamma_t'(x)}{\mu(\Gamma_t(x))} - \frac{1}{\mu(x)} = 0$$

for all $t, x > 0$. This gives (8.5) and shows that $\Gamma$ is continuously differentiable in space and that if $\mu$ is strictly increasing on $(0, a)$ then $\Gamma_t' > 1$ on $(0, \Gamma_t^{-1}(a))$ for all $t > 0$.

Taking the derivative of $I_t(x)$ with respect to $t$, we have

$$\partial_t I_t(x) = \frac{\partial_t \Gamma_t(x)}{\mu(\Gamma_t(x))} = 1$$

so $\mu(\Gamma_t(x)) = \partial_t \Gamma_t(x)$ for all $t, x > 0$, which is (8.4). It follows, then, that $\Gamma$ is continuously differentiable in time and that $\partial_t \Gamma_t(x) = \Gamma_t'(x) \mu(x)$, or,

$$\mu(x) = \frac{\partial_t \Gamma_t(x)}{\Gamma_t'(x)}, \tag{8.6}$$

an equality that hold for all $x$ and is independent of $t \geq 0$. This is (8.3). □

Moreover, we have the following simple but important lemma:

**Lemma 8.3.** *Let $\mu$ be as in Theorem 8.2. The following are equivalent:*

(1) *The MOC, $\mu = \partial_t \Gamma_t|_{t=0}$, is differentiable and concave on $(0, a)$.*
(2) *$\Gamma_t$ is twice differentiable on $(0, \infty)$ and concave on $(0, \Gamma_t^{-1}(a))$ for all $t > 0$.*
(3) *$\Gamma_t$ is twice differentiable on $(0, \infty)$ and concave on $(0, \Gamma_t^{-1}(a))$ for all $t$ in $(0, \delta)$ for some $\delta > 0$.*

*Here, $0 < a \le \infty$. Furthermore, strong strict concavity of $\Gamma_t$ on $(0, \Gamma_t^{-1}(a))$ for all $t > 0$ implies strict concavity of $\mu$ on $(0, a)$.*

*Proof.* That $\mu = \partial_t \Gamma_t|_{t=0}$ follows from (8.6), since $\Gamma_0(x) = x$. Taking the derivative of (8.5) with respect to $x$ gives

$$\begin{aligned}\mu'(\Gamma_t(x)) &= \frac{(\mu(\Gamma_t(x)))'}{\Gamma_t'(x)} = \frac{\Gamma_t'(x)\mu'(x) + \Gamma_t''(x)\mu(x)}{\Gamma_t'(x)} \\ &= \mu'(x) + \Gamma_t''(x)\frac{\mu(x)}{\Gamma_t'(x)},\end{aligned}$$

or,

$$\Gamma_t''(h) = \left(\mu'(\Gamma_t(h)) - \mu'(h)\right)\frac{\Gamma_t'(h)}{\mu(h)}. \tag{8.7}$$

Since $\Gamma_t(h) > h$ for all $t, h > 0$, $\Gamma_t''(h)$ and $\mu'(\Gamma_t(h)) - \mu'(h)$ have the same sign for all $t, h > 0$.

Suppose that (3) holds and let $h$ lie in $(0, a)$. Then for all sufficiently small $t > 0$, $h$ lies in $(0, \Gamma_t^{-1}(a))$ which shows by (8.7) that $\mu'(\Gamma_t(h)) \le \mu'(h)$. But $\Gamma_t(h)$ decreases to $h$ as $t \to 0^+$, so $\mu'$ is increasing at $h$. Hence, $\mu'$ is increasing for all $h$ in $(0, a)$ meaning that $\mu$ is concave on $(0, a)$. That is, (3) $\implies$ (1).

Now assume that (1) holds. This shows directly from (8.7) that $\Gamma_t''(h)$ is concave; that is, (1) $\implies$ (2). That (2) $\implies$ (3) is immediate.

The statement involving strong strict concavity is a small modification of the argument above. □

**Remark 8.4.** Other than the implication involving strict concavity, one need only assume in Lemma 8.3 that $\mu$ is continuous and that $\Gamma_t$ is continuously differentiable, as is shown in the results of Zdun [23] that we discuss in the next section.

It follows from (8.4) that $\partial_t \Gamma_t(x) > 0$, so $\Gamma_t(x)$ is an increasing function of $t$ for fixed $x > 0$; that is, the MOC of the flow gets worse with time. This observation leads to the following:

**Lemma 8.5.** *If $\mu$ is (strictly) increasing then the map, $\Gamma(\cdot, x)$ is a (strictly) increasing convex function for all $x > 0$ and $\partial_x \Gamma(\cdot, x)$ is (strictly) increasing.*

*Proof.* From (8.4), $\partial_t \Gamma_t(x) = \mu(\Gamma_t(x))$, and as observed above, $\Gamma_t(x)$ increases with $t$. Since $\mu$ also increases by assumption, it follows that $\partial_t \Gamma_t(x)$

increases with $t$; that is, $t \mapsto \Gamma_t(x)$ is convex. From (8.3), $\partial_t \Gamma_t(x) = \mu(x)\Gamma_t'(x)$, and we conclude that $t \mapsto \Gamma_t'(x)$ is strictly increasing. □

In the context of iteration (semi)groups, all of the results in this section, with the possible exception of Lemma 8.5, are known, in broader generality, and are due primarily to Zdun [23] and Targoński and Zdun [19] (also see Section 3.3 of [18], which summarizes many of the key results in [23]). We make use of these connections, along with some of the deeper results in [23, 19], in the next section.

## 9. Inverting the MOC of the flow map

In Section 8 we characterized the properties of the MOC, $\Gamma$, of the flow map, in particular as regards concavity properties of $\Gamma_t$, and found that it is easy to obtain the corresponding MOC, $\mu$, of the vector field from (8.6). But this formula requires that we know $\Gamma_t$ for all $t$ in a neighborhood of the origin. In this section we attempt to answer Question 1 of Section 7, in which we only know $\Gamma_{t_0}$ at one time, $t_0 > 0$. In this case, we do not expect to obtain a unique $\mu$ and so do not expect to obtain a unique $\Gamma$. We are especially interested in determining whether we can find a $\mu$ that is concave.

Our starting point will be Theorem 9.4, in which we construct a $\mu$ that is strictly increasing, but only on some interval $[0, a)$. This is adequate for our uses, but complicates all subsequent arguments because we have to keep track of the interval on which various functions are guaranteed to be strictly increasing or concave: this is the purpose of introducing Definitions 9.1 and 9.2. The essential meaning of the theorems are easier to grasp, however, if one ignores any statements involving $\iota$, $V$, or $J$, and just imagines that $\mu$ is strictly increasing on all of $[0, \infty)$. In any case, these definitions are required along the way, but not in the statement Theorem 9.12, the main result of this section.

**Definition 9.1.** For any function, $f$, on $[0, \infty)$ we define

$$\iota(f) = \sup\{a \in [0, \infty] \colon f \text{ is strictly increasing on } [0, a)\},$$
$$V(f) = \sup\{a \in [0, \infty] \colon f \text{ is concave on } (0, a)\}.$$

**Definition 9.2.** Let $f$ be a $C^1$ MOC and let $a = \iota(f(x) - x)$. We say that a MOC, $f$, such that $f(x) > x$ for all $x$ in $(0, \infty)$ is *acceptable* or *globally acceptable* if $a = \infty$ and is *locally acceptable* if $a > 0$. We define $J(f) = a$.

**Remark 9.3.** It follows from Definition 9.2 that $f' > 1$ on $(0, J(f))$. Also, $f$ concave is compatible with $f$ being acceptable.

We now answer Question 1 of Section 7 affirmatively.

**Theorem 9.4.** *Fix $t_0 > 0$. Given any $f$ that is a globally acceptable MOC there exists a continuous MOC, $\mu$, satisfying the Osgood condition, (1.4),* *with* $\mu > 0$ *on* $(0, \infty)$*, such that*

$$I(x) := \int_x^{f(x)} \frac{dr}{\mu(r)} = t_0 \tag{9.1}$$

*for all* $x > 0$*. If* $\lim_{x \to \infty} x/f(x) = 0$ *then* $\lim_{x \to \infty} \mu(x)/x = 0$*. If* $f$ *is* $C^k$*,* $k \geq 1$*, then* $\mu$ *can be chosen to be* $C^{k-1}$*.*

*Proof.* Choose $a > 0$ arbitrarily, then choose a smooth $\mu$, strictly increasing on the interval $[a, f(a)]$ such that the following two conditions are satisfied:

$$\int_a^{f(a)} \frac{ds}{\mu(s)} = t_0, \tag{9.2}$$

$$\mu(f(a)) = f'(a)\mu(a). \tag{9.3}$$

It is easy to see that we can find a function $\mu$ on $[a, f(a)]$ that satisfies (9.2, 9.3) if and only if we choose $\mu(a)$ so that

$$\frac{f(a) - a}{f'(a)t_0} < \mu(a) < \frac{f(a) - a}{t_0}. \tag{9.4}$$

Since $f'(a) > 1$ this is always possible.

Next define $\mu(x)$ for $x$ in the interval $[f^{-1}(a), a]$ by

$$\mu(x) = \frac{\mu(f(x))}{f'(x)} \tag{9.5}$$

and note that if $f$ is concave then $\mu$ is strictly increasing on $[f^{-1}(a), a]$ because $\mu(f(x))$ is strictly increasing and $f'$ is decreasing on that interval. Also, $\mu$ is continuous, in particular at $a$ by (9.3).

Suppose that $f$ lies in $C^2((0, \infty))$. Then taking the derivative of (9.5),

$$\mu'(x) = \mu'(f(x)) - \mu(f(x)) \frac{f''(x)}{f'(x)}^2.$$

Hence, $\mu$ is in $C^1((f^{-1}(a), a)$. To insure that $\mu'$ is continuous at $a$, we simply require that $\mu$ be chosen on the interval $[a, f(a)]$ such that

$$\mu'(a) = \mu'(f(a)) - \mu(f(a)) \frac{f''(a)}{f'(a)}^2, \tag{9.6}$$

$\mu'(a)$ being a right-sided derivative and $\mu'(f(a))$ a left-sided derivative. Equality in (9.6) can be assured by changing the definition of $\mu$ on $[a, f(a)]$ an arbitrarily small amount near the endpoints. Hence, we can make (9.6) hold under the same conditions (9.4) while retaining the other properties of $\mu$ already established. But once (9.6) holds, the continuity of $f$, $f'$, and $f''$ makes $\mu'$ continuous on $[f^{-1}(a), f(a)]$. A straightforward extension of this argument shows that if $f$ lies in $C^k((0, \infty))$ then $\mu$ can be chosen to lie in $C^{k-1}([f^{-1}(a), f(a)])$.

Extending this definition of $\mu$ inductively to $[f^{-n}(a), f^{-n+1}(a)]$ we unambiguously define $\mu$ on all of $(0, f(a)]$ and the resulting $\mu$ is positive, continuous, satisfies (9.5) for all $x$ in $(0, a]$, is strictly increasing if $f$ is concave, and if $f$ lies in $C^k((0, \infty))$ then $\mu$ lies in $C^{k-1}((0, f(a)])$.

Next, extend $\mu$ to the interval $[f(a), f(f(a))]$ using $\mu(f(x)) = f'(x)\mu(x)$, the complement of (9.5), and inductively extend $\mu$ to all of $(0, \infty)$. Then $\mu$ satisfies (9.5) for $x > a$ as well and if $f$ lies in $C^k((0, \infty))$ then $\mu$ lies in $C^{k-1}((0, \infty))$. (Even if $f$ is concave, extending $\mu$ in this way, does not insure that it is an increasing function for $x > f(a)$, for though $\mu$ is increasing on the interval $[a, f(a)]$, $f'$ would be decreasing.)

Defining $I\colon (0, \infty) \to [0, \infty)$ as in (9.1), it follows from (9.5) that

$$I'(x) = \frac{f'(x)}{\mu(f(x))} - \frac{1}{\mu(x)} = 0$$

for all $x > 0$. Therefore, $I$ is a constant function. But by construction, $I(a) = t_0$ so $I(x) = t_0$ for all $x > 0$. That is, $\mu$ satisfies (9.1) for all $x > 0$.

It follows by the absolute continuity of the integral and the fact that $f$ is *strictly* increasing that

$$\int_0^a \frac{ds}{\mu(s)} = \sum_{n=0}^{\infty} \int_{f^{-(n+1)}(a)}^{f^{-n}(a)} \frac{ds}{\mu(s)} = \sum_{n=0}^{\infty} t_0 = \infty, \tag{9.7}$$

so that $\mu$ satisfies the Osgood condition, (1.4), and also that we must have

$$\lim_{s \to 0^+} \mu(s) = 0, \tag{9.8}$$

so that we can extend $\mu$ continuously to $[0, \infty)$ by setting $\mu(0) = 0$. □

**Remark 9.5.** After expressing the relation in (9.1) in the form $\mu(f(x)) = f'(x)\mu(x)$, we can view our construction of the function $\mu$ as an application of Theorem 2.1 of [13], which is due to Kordylewski and Kuczma ([11, 12]). That $\mu$ is $C^{k-1}$ when $f$ is $C^k$ can be seen as an application of Theorem 4.1 of [13], which is due to Choczewski ([5]). Also see Theorem 6.2 of [23] (quoted in Proposition 3.3.45 of [18]).

Since $\mu(f(x)) = f'(x)\mu(x)$, we have

$$\mu'(f(x)) = \mu'(x) + \mu(x)\frac{f''(x)}{f'(x)}. \tag{9.9}$$

If $f$ is concave then $\mu'(f(x)) \le \mu'(x)$. It does not, however, follow that $\mu'(x)$ is a decreasing function of $x$—for this, we need more information, as in Lemma 8.3.

We now show how the problem of inverting the relation in (9.1) to obtain $\mu$ from $f$ is related to iteration theory. We start with the following definition, adapted to our setting from [23] (see also [24] and Section 3.3 of [18]):

**Definition 9.6.** A *continuous iteration group of MOC* (CIG) is a family, $G = (f^t)_{t \in \mathbb{R}}$, of MOC such that

(1) For all $t > 0$, $f^t(x) > x$.
(2) For all $s, t$ in $\mathbb{R}$, $f^s \circ f^t = f^{s+t}$.
(3) $f^0$ is the identity.
(4) As a map from $\mathbb{R}$ to $[0, \infty)$, $t \mapsto f^t(x)$ is continuous for all $x$ in $[0, \infty)$.

Furthermore (refer to Definitions 9.1 and 9.2 for definitions of $J$ and $V$),

- If, for all $t > 0$, $f^t$ is locally acceptable with $J(f^t) \ge f^{-t}(a)$ for some $0 < a < \infty$ then we say that $G$ is *locally acceptable* and define $J(G)$ to be the supremum of all such $a$. If $a = \infty$ then we say that $G$ is *acceptable* or *globally acceptable*.
- If, for all $t > 0$, $V(f^t) \ge f^{-t}(a)$ for some $0 < a < \infty$ then we say that $G$ is *locally concave* and define $V(G)$ to be the supremum of all such $a$. If $a = \infty$ then we say that $G$ is *concave* or *globally concave*.
- We say that $G$ is $C^k$, $k \ge 0$, if $f^t$ is $C^k$ for all $t$ in $\mathbb{R}$.
- We say that$f$ is *embedded* in $G$ if $f^1 = f$.

Let $\mu$ be a MOC and let $\Gamma$ be the corresponding MOC of the flow given by Theorem 8.2. If we let $f = \Gamma_1$ then because the flows $(\Gamma_t)_{t \in \mathbb{R}}$ form a group under composition, letting $f^t = \Gamma_t$, $f$ is embedded in the CIG, $(f^t)_{t \in \mathbb{R}}$.

Now suppose, starting with only with an acceptable MOC, $f$, that we can find a $C^1$ CIG, $G = (f^t)_{t \in \mathbb{R}}$, for $f$ with each $f^t$ also strictly increasing. Then

$$\begin{aligned}
\partial_t f^t(x) &= \lim_{h \to 0} \frac{f^{t+h}(x) - f^t(x)}{h} = \lim_{h \to 0} \frac{f^t(f^h(x)) - f^t(x)}{h} \\
&= \lim_{h \to 0} \frac{f^t(f^h(x)) - f^t(x)}{f^h(x) - x} \lim_{h \to 0} \frac{f^h(x) - f^0(x)}{h} \\
&= (f^t)'(x) \partial_s f^s(x)|_{s=0}.
\end{aligned}$$

Hence the function,

$$\mu(x) := \frac{\partial_t f^t(x)}{(f^t)'(x)} = \partial_s f^s(x)|_{s=0}$$

is well-defined, with the first expression independent of $t$ in $\mathbb{R}$.

Also, arguing as in the proof of Lemma 5.2 part II of [23],

$$\mu(f^t(x)) = \lim_{h \to 0} \frac{f^h(f^t(x)) - f^t(x)}{h} = \lim_{h \to 0} \frac{f^{t+h}(x) - f^t(x)}{h} = \partial_t f^t(x)$$

and, applying the chain rule,

$$\begin{aligned}
\mu(f^t(x)) &= \frac{\partial f^s(f^t(x))}{\partial s}\Big|_{s=0} = \frac{\partial f^t(f^s(x))}{\partial s}\Big|_{s=0} = (f^t)'(f^0(x)) \frac{\partial f^t(x)}{\partial s}\Big|_{s=0} \\
&= (f^t)'(x) \mu(x).
\end{aligned}$$

Letting $I(t, x) = \int_x^{f^t(x)} \frac{dr}{\mu(r)}$, we conclude from these two relations for $\mu(f^t(x)$ that $\partial_t I(t, x) = 1$ and $\partial_x I(t, x) = 0$, and from this it follows that for all $x > 0$,

$$\int_x^{f^t(x)} \frac{dr}{\mu(r)} = t.$$

This includes (9.1) in the special case, $t_0 = 1$. (In other words, (9.1) is satisfied, where it is now convenient to set $t_0 = 1$, with no loss of generality.) In fact, by Lemma 8.3, we can invert (9.1) to obtain $\mu$ from $f$ if and only if there *is* some such CIG, $G = (f^t)_{t\in\mathbb{R}}$, in which case $\Gamma_t = f^t$. Finally, observe that $\iota(\mu) = J(G)$.

Combining the results of Section 8, Theorem 9.4, and [23] with the observations above, we have:

**Theorem 9.7.** *Suppose that $f$ is a $C^k$, $k \geq 1$, globally concave globally acceptable MOC. Then there exists a (in fact, an infinite number of) $C^k$ locally acceptable CIG, $G$, embedding $f$ with $J(G)$ arbitrarily large. Let $G = (f^t)_{t\in\mathbb{R}}$ be any such CIG embedding $f$ with $a = J(G)$. Then $\mu := \partial_t f^t|_{t=0}$ is $C^{k-1}$, satisfies the Osgood condition, and $\iota(\mu) = a$. Moreover, the following are equivalent:*

1. *$\mu$ is concave (on $(0, b)$);*
2. *$G$ is locally concave with $V(G) \geq b$;*
3. *for some $\delta > 0$, $f^t$ is concave (on $(0, f^{-t}(b))$) for all $t$ in $(0, \delta)$ .*

**Remark 9.8.** That $G$ is $C^k$ in Theorem 9.7 is proved below following (9.13).

Left open in Theorem 9.7 is the question of whether any concave acceptable MOC is embeddable in a concave CIG. We cannot prove this, and indeed it may not be true, but for our purposes, the weaker result in Theorem 9.12 will suffice (see Remark (9.13)). Before proceeding to the proof of Theorem 9.12, however, let us look at some illustrative examples.

Let $a_m = 1/\exp^{m+1}(1)$ for for any $m = 0, 1, \ldots$ define $\mu_m \colon [0, a_m) \to [0, \infty)$ with $\mu_m(0) = 0$ and

$$\mu_m(x) = x\theta_m(1/x) \text{ for } 0 < x < a_m. \tag{9.10}$$

where $\theta_m$ is defined in (1.5). It is straightforward to verify that $\mu_m$ is continuous, $C^\infty$ on $(0, a_m)$, concave, and increasing (strictly increasing for $m \geq 1$). Extend $\mu$ arbitrarily to $[0, \infty)$ in such a way as to maintain these properties.

Now define $f_m^t$ by

$$\int_x^{f_m^t(x)} \frac{ds}{\mu_m(s)} = t.$$

We can exactly integrate this to give, for $0 < x < a_m$,

$$\log^{m+1}(1/f_m^t(x)) - \log^{m+1}(1/x) = -t,$$

whose solution is

$$f_m^t(x) = 1/\exp^m(e^{-t}\log^m(1/x)) = F_m(e^{-t}F_m^{-1}(x)),$$

where $F_m(x) = 1/\exp^m(x)$. Or, we can write,

$$f_m^t(x) = h_m(t + h_m^{-1}(x)), \tag{9.11}$$

where

$$h_m(x) = \frac{1}{\exp^{m+1}(-x)},$$

which we note is strictly increasing.

It is easy to verify directly from (9.11) that $G_m = (f_m^t)_{t\in\mathbb{R}}$ is a CIG embedding $f = f_m^1$ and it follows from Theorem 9.7 that $G_m$ is a concave CIG. In fact, in our setting, any $C^1$ CIG, $G = (f^t)_{t\in\mathbb{R}}$, must be of the form

$$f^t(x) = h(t + h^{-1}(x)) \tag{9.12}$$

for some $h\colon (-\infty,\infty) \to (0,\infty)$. (The function $h$ is called the *generating function* of $G$.) This follows from Theorem 7.1 Chapter I of [23] (quoted in Theorem 3.3.29 of [18]), where it is also proven that $h$ must be strictly increasing with $h(-\infty) = 0$, $h(\infty) = \infty$ and that $h$ is nearly unique in the sense that if $f^t(x) = h_j(t + h_j^{-1}(x))$, $j = 1, 2$, then there exists some $a$ in $\mathbb{R}$ such that $h_1(\cdot) = h_2(a + \cdot)$. (In the terminology of [23], $f$ satisfies property $P3^\circ$ with $\overline{a} = 0$, $\overline{b} = +\infty$.)

By Theorem 1.2 Chapter II of [23], $h$ is differentiable on $(-\infty,\infty)$ and $h'$ never vanishes so, in fact, $h$ is *strictly* increasing. Also, by Theorem 9.7,

$$\begin{aligned}\mu(x) &= \partial_t f^t(x)|_{t=0} = h'(t + h^{-1}(x))|_{t=0}\\ &= h'(h^{-1}(x)) = \frac{1}{(h^{-1}(x))'},\end{aligned} \tag{9.13}$$

which is Lemma 4.2 of [23]. The last expression for $\mu$ shows that if $\mu$ is $C^{k-1}$ then $h$ is $C^k$. Since, by Theorem 9.4, $\mu$ can be chosen to be $C^{k-1}$ if $f$ is $C^k$, this completes the proof of Theorem 9.7 promised in Remark (9.8).

(The last expression for $\mu$ in (9.13) also leads to a direct expression for $h^{-1}$ in terms of $\mu$; namely, $h^{-1}(x) = h^{-1}(a) + \int_x^a (\mu(s))^{-1}\, ds$ for any fixed choice of $a > 0$ and assigned value of $h^{-1}(a)$. This in turn leads to the relation in (9.12) and to the statement regarding the uniqueness of $h$.)

For $f_m$, we have

$$h_m^{-1}(x) = -\log^{m+1}(1/x)$$

and

$$h_m'(x) = -\frac{\exp^{m+1}(-x)\cdots\exp(-x)(-1)}{\exp^{m+1}(-x)^2} = \frac{\exp^m(-x)\cdots\exp(-x)}{\exp^{m+1}(-x)}$$

so

$$\begin{aligned}\mu_m(x) &= h_m'(h_m^{-1}(x)) = \frac{\log x\cdots\log^m(x)}{1/x} = x\log x\cdots\log^m(x)\\ &= x\theta_m(1/x),\end{aligned}$$

in agreement with (9.10).

In more generality, we have Theorem 9.9. (See Definition 7.1 for our distinctions among degrees of concavity.)

**Theorem 9.9.** *Suppose that $f$ is a $C^k$, $k \geq 3$, concave acceptable MOC embedded in a $C^k$ locally acceptable CIG, $G = (f^t)_{t \in \mathbb{R}}$, given by Theorem 9.7 with $a = J(G)$. There exists a $C^k$ generating function, $h\colon (-\infty, \infty) \to (0, \infty)$, for $f^t$ as in (9.12), with $h(-\infty) = 0$ and $h(\infty) = \infty$. Any such $h$ must be strictly increasing on $\mathbb{R}$ and strongly strictly convex on $(-\infty, b)$, where $b = h^{-1}(a)$. If $f^t(x) = h_j(t + h_j^{-1}(x))$, $j = 1, 2$, then there exists some $c$ in $\mathbb{R}$ such that $h_1(\cdot) = h_2(c + \cdot)$. For any $t, x > 0$,*

$$\int_x^{f^t(x)} \frac{ds}{\mu(s)} = t,$$

*where $\mu(x) = h'(h^{-1}(x)) = 1/(h^{-1}(x))'$ is $C^{k-1}$ and strictly increasing on $(0, a)$.*

*Let $0 < \overline{a} \leq a$ and let $\overline{b} = h^{-1}(\overline{a})$. $G$ (and hence $\mu$) is locally concave with $V(G) = \overline{a}$ if and only if $\log h'$ is concave on $(-\infty, \overline{b})$, and $\mu$ is strongly strictly concave on $(0, \overline{a})$ if and only if $\log h'$ is concave on $(-\infty, \overline{b})$. Finally, if $\mu$ is (strongly strictly) concave on $(0, \overline{a})$ then on $(-\infty, \overline{b})$, $\log h$ is strictly increasing and (strongly strictly) concave with $(\log h')' \leq (\log h)'$, strict inequality holding when $\mu$ is strongly strictly concave.*

*Proof.* The existence of a generating function $h$ satisfying (9.12) and possessing the stated properties follows from Theorem 7.1 Chapter I of [23]. By Theorem 9.7, $\mu$ is $C^{k-1}$, and by the comment following (9.13), $h$ is $C^k$.

By (9.12), $f^t(h(x - t)) = h(x)$, so

$$(f^t)'(h(x - t))h'(x - t) = h'(x).$$

But $(f^t)' > 1$ on $f^{-t}(a)$ by Theorem 9.7 so we conclude that $h'$ is increasing on $(-\infty, b)$; that is, the condition that $h$ be (non-strictly) convex is already required simply for the CIG to be any strictly increasing CIG embedding $f$, as given by Theorem 9.7. More important, we conclude that $f^t$ is as differentiable as $h$.

Writing (9.13) as

$$\mu(h(x)) = h'(x) \tag{9.14}$$

(which shows that $h$ is strictly increasing on $\mathbb{R}$) we have

$$\mu'(h(x))h'(x) = h''(x). \tag{9.15}$$

Since $h$ is strictly increasing, $\mu$ will be strictly increasing on $(0, \overline{a})$ if and only if $h$ is strictly convex on $(-\infty, \overline{b})$. Taking another derivative gives

$$\mu'(h(x))h''(x) + \mu''(h(x))(h'(x))^2 = h'''(x)$$

so

$$\mu''(h(x))(h'(x))^2 = h'''(x) - \mu'(h(x))h''(x) = h'''(x) - \frac{(h''(x))^2}{h'(x)},$$

or,

$$\mu''(h(x))h'(x) = \frac{h'''(x)h'(x) - (h''(x))^2}{h'(x)^2} = \left(\frac{h''(x)}{h'(x)}\right)' \\ = (\log h')''(x). \tag{9.16}$$

Thus, $\mu$ is strictly increasing and (strongly strictly) concave on $(0, \overline{a})$ if and only if $h$ is strictly convex while $\log h'$ is (strongly strictly) concave on $(-\infty, \overline{b})$.

Since $\mu$ is concave on $(0, \overline{a})$, we have $\mu'(x) \le \mu(x)/x$ (with strict inequality if $\mu$ is strongly strictly convex) on $(0, \overline{a})$. From (9.14, 9.15),

$$\frac{\mu(h(x))}{h(x)} = \frac{h'(x)}{h(x)} = (\log h)'(x), \\ \mu'(h(x)) = \frac{h''(x)}{h'(x)} = (\log h')'(x), \tag{9.17}$$

so we conclude that $0 < (\log h')' \le (\log h)'$ on $(-\infty, \overline{b})$ with strict inequality when $\mu$ is strongly strictly convex on $(0, \overline{a})$. Differentiating $(9.17)_1$ then substituting $(9.17)_2$ gives

$$(\log h)''(x) = \frac{h(x)\mu'(h(x))h'(x) - \mu(h(x))h'(x)}{(h(x))^2} \\ = \frac{h'(x)}{(h(x)^2} \left[\mu'(h(x))h(x) - \mu(h(x))\right] \le 0, \tag{9.18}$$

again using $\mu'(x) \le \mu(x)/x$ . Thus, if $\mu$ is (strongly strictly) concave on $(0, \overline{a})$ then $\log h$ must be (strongly strictly) concave on $(-\infty, \overline{b})$. □

**Remark 9.10.** Much of Theorem 9.9 can be obtained assuming only that $(f^t)_{t\in\mathbb{R}}$ is $C^1$, as in Theorem 3.22 of [19].

Let $h$, $a$, and $b$ be as in Theorem 9.9. The conclusion in Theorem 9.9 that $h(-\infty) = 0$ is equivalent to $\mu$ satisfying the Osgood condition, for a change of variables gives

$$\int_0^1 \frac{ds}{\mu(s)} = \int_0^1 \frac{ds}{h'(h^{-1}(s))} = \int_{h^{-1}(0)}^{h^{-1}(1)} \frac{h'(u)}{h'(u)}\, du = h^{-1}(1) - h^{-1}(0),$$

which is infinite if and only if $h^{-1}(0) = -\infty$.

Since $h' > 0$, we can write $h'$ uniquely in the form

$$h' = e^g \tag{9.19}$$

for some function $g\colon \mathbb{R} \to \mathbb{R}$. Then $g = \log h'$ and $h'' = g'e^g$. But $h'' > 0$ on $(-\infty, b)$ by Theorem 9.9, meaning that $g' > 0$ on $(-\infty, b)$ and hence $g$ is a strictly increasing function on $(-\infty, b)$. Again by Theorem 9.9, $G = (f^t)_{t\in\mathbb{R}}$ is locally concave with $0 < V(G) = \overline{a} \le a$ if and only if $g$ is concave on $(-\infty, h^{-1}(\overline{a}))$. Also, $0 = \mu(0) = \mu(h(-\infty)) = h'(-\infty)$ so $g(-\infty) = -\infty$. Thus, we have the following immediate corollary of Theorem 9.9:

**Corollary 9.11.** *Let $f$, $G = (f^t)_{t\in\mathbb{R}}$, and $h$ be as in Theorem 9.9 with $a = J(G)$. Then $h' = e^g$ for some $C^{k-1}$ function, $g\colon \mathbb{R} \to \mathbb{R}$, strictly increasing on $(-\infty, h^{-1}(a))$ with $g(-\infty) = -\infty$. Furthermore, $G$ is locally concave with $0 < V(G) = \overline{a} \le a$ if and only if $g$ is concave on $(-\infty, \overline{b})$, where $\overline{b} = h^{-1}(\overline{a})$, and $\mu$ is strongly strictly concave on $(0, \overline{a})$ if and only if $g$ is strongly strictly concave on $(-\infty, \overline{b})$.*

We are now in a position to prove that given a concave acceptable MOC there always exists a larger concave acceptable MOC that is embeddable in a concave CIG: this is Theorem 9.12.

**Theorem 9.12.** *Let $f$ be any $C^k$ globally concave globally acceptable MOC, $k \ge 3$. Then for any $a > 0$ there exists a $C^{k+1}$ globally concave globally acceptable MOC, $\overline{f}$, embedded in a $C^{k+1}$ globally concave globally acceptable CIG with $\overline{f} > f$ on $(0, a)$. The associated function $\mu$ is concave and $C^k$ and the generating function $h$ is $C^{k+1}$. Furthermore, if $f$ is strongly strictly concave then $\mu$ is strongly strictly concave on $(0, a)$.*

*Proof.* Let $G = (f_t)_{t\in\mathbb{R}}$ be any $C^k$ locally acceptable CIG embedding $f$ as given by Theorem 9.7 and let $h$ be the corresponding $C^k$ generating function given by Theorem 9.9. Let $a' = J(G)$ and let $a = h(h^{-1}(a') - 1)$, which we note can be made arbitrarily large.

By (9.12), $f(h(x-1)) = f^1(h(x-1)) = h(x)$, so

$$f'(h(x-1))h'(x-1) = h'(x).$$

Taking the logarithm of both sides gives

$$\log f'(h(x-1)) + g(x-1) = g(x) \tag{9.20}$$

since $g = \log h'$. But $f' > 1$ so

$$g(x) > g(x-1) \text{ for all } x \text{ in } \mathbb{R}. \tag{9.21}$$

Taking the derivative of (9.20) gives

$$\frac{f''(h(x-1))h'(x-1)}{f'(h(x-1)} + g'(x-1) = g'(x).$$

But $f'' \le 0$, $f' > 0$, and $h' > 0$ on all of $\mathbb{R}$ so we conclude that

$$g'(x) \le g'(x-1) \text{ for all } x \text{ in } \mathbb{R} \tag{9.22}$$

and we note that strict inequality holds if $f$ is strongly strictly concave.

We emphasize that (9.21, 9.22) hold globally for all $x$ in $\mathbb{R}$.

For any $x$ in $\mathbb{R}$ let

$$\overline{g}(x) = \int_x^{x+1} g(s)\,ds. \tag{9.23}$$

Since $\overline{g}$ is the mean value of $g$ on $(x, x+1)$ and $g$ is strictly increasing on $(-\infty, h^{-1}(a'))$,

$$\overline{g}(x) > g(x) > \overline{g}(x-1) \text{ for all } x \text{ in } I := (-\infty, h^{-1}(a)). \tag{9.24}$$

By (9.21), $\overline{g}'(x) = g(x+1) - g(x) > 0$ on $\mathbb{R}$ so $\overline{g}$ is strictly increasing on $\mathbb{R}$. By (9.22), $\overline{g}''(x) = g'(x+1) - g'(x) \le 0$ on $\mathbb{R}$, so $\overline{g}$ is concave on all of $\mathbb{R}$. If $f$ is strongly strictly concave then strict inequality holds in (9.22) so $\overline{g}$ is strongly strictly concave. In either case, we also have $\overline{g}(-\infty) = -\infty$.

Now let

$$\overline{h}(x) = \int_{-\infty}^{x} e^{\overline{g}(s)}\, ds.$$

It follows from (9.24) that

$$\overline{h}(x) > h(x) > \overline{h}(x-1) \text{ for all } x \text{ in } I \tag{9.25}$$

so that also

$$h^{-1}(x) < \overline{h}^{-1}(x) + 1 \text{ for all } x \text{ in } (0, a) = h(I).$$

Letting

$$\overline{f}^t(x) := \overline{h}(t + \overline{h}^{-1}(x))$$

it follows from Theorem 9.9 and Corollary 9.11 that $(\overline{f}^t)_{t\in\mathbb{R}}$ is globally concave and globally acceptable and the corresponding function $\overline{\mu}$ is also strictly increasing and concave with

$$\int_x^{\overline{f}^t t(x)} \frac{dr}{\overline{\mu}(r)} = t$$

for all $t, x > 0$. Because $\overline{h} > h$ on $I$ and $\overline{h}^{-1}(x) > h^{-1}(x) - 1$ on $(0, a)$, we have

$$\overline{f}^2(x) = \overline{h}(2 + \overline{h}^{-1}(x)) > h(2 + \overline{h}^{-1}(x)) > h(1 + h^{-1}(x)) = f(x) \tag{9.26}$$

as long as $2 + \overline{h}^{-1}(x) < h^{-1}(a)$ and $x < a$. But by (9.25), $\overline{h}^{-1} < h^{-1}$ on $(0, a)$ so $2 + \overline{h}^{-1}(x) < h^{-1}(a)$ will hold if $2 + h^{-1}(x) < h^{-1}(a)$, which in turn holds if $x < h(2 + h^{-1}(a))$. But $h$ is strictly increasing on all of $\mathbb{R}$ by Theorem 9.9 so this is a weaker condition than $x < a$, so (9.26) holds on $(0, a)$.

If $f$ is strongly strictly concave then so is $\overline{g}$ as observed above and hence $\overline{\mu}$ is as well by Corollary 9.11.

Now let $\mu = 2\overline{\mu}$ and $j^t = \overline{f}^{2t}$. Then

$$\int_x^{j^t(x)} \frac{dr}{\mu(r)} = \int_x^{\overline{f}^{2t}(x)} \frac{dr}{2\overline{\mu}(r)} = \frac{2t}{2} = t$$

so by Theorem 9.7, $(j^t)_{t\in\mathbb{R}}$ is a globally acceptable globally concave CIG embedding $j = j^1$ with $j = \overline{f}^2 > f$. The smoothness of $\overline{f}$, $(j^t)_{t\in\mathbb{R}}$, and $\mu$ follow from the extra level of differentiability given to $\overline{g}$ and hence to $\overline{h}$ by (9.23). □

**Remark 9.13.** Letting $\Gamma_t = \overline{f}^t$ in Theorem 9.12 gives an affirmative answer to Question 1 of Section 7.

**Remark 9.14.** If we change the limits of integration in (9.23) to go from $x-1$ to $x$ then we obtain a concave CIG embedding a concave function that is less than $f$. This leads to the obvious question of whether it is possible to iterate the procedure in the proof of Theorem 9.12, alternately producing over or underestimates of the previous step, to obtain a concave $CIG$ embedding $f$ itself.

In Section 10 we look in detail at the properties of the MOC, $\mu$, that arise when it is a bound on the modulus of continuity of the vector field associated with a solution to the Euler equations. We will find not only that $\mu$ must be concave but also that it must satisfy the additional constraint in (10.10). Whether this additional constraint can be accommodated in Theorem 9.12 for all $f$ is an open question.

## 10. MOC of the Eulerian velocity

We now return to the topic of Section 1, where $\mu$ is the MOC of the solution (the velocity) of the Euler equations that is derived from $\theta(p)$, the $L^p$-norms of the solution's vorticity.

To avoid trivialities, we assume throughout that $\theta$ is never zero.

We will write (1.3) in the form,

$$\mu(x) = \inf_{\epsilon \in \mathcal{A}} \left\{ x^{1-2\epsilon} \alpha(\epsilon) \right\}, \quad \mathcal{A} = (0, 1/2], \tag{10.1}$$

where $\alpha(\epsilon) = \epsilon^{-1}\theta(\epsilon^{-1})$, as in (1.2).

Since $\theta(p) = \|\omega^0\|_{L^p}$ for some vorticity, $\omega^0$, it inherits some important properties from basic results of measure theory, as in Lemma 10.1.

**Lemma 10.1.** *Suppose that $\theta(p)$ is the $L^p$-norm of some function, $f$, lying in $L^p$ for all $p$ in $[p_0, \infty)$ and with $\|f\|_{L^\infty}$ possibly finite but nonzero. Then $\varphi(p) := p \log \theta(p)$ is convex and $C^\infty$ on $(p_0, \infty)$, and, unless $\theta$ is in $L^\infty$, $\varphi(p)$ is strictly increasing for sufficiently large $p$. Also, $\log \theta(\epsilon^{-1})$ is convex and $\log \alpha$ is strictly convex.*

*Proof.* That $\varphi(p)$ is $C^\infty$ and convex is classical (see, for instance, Exercise 4(b) Chapter 3 of [15]), and it follows from this that it is eventually strictly increasing, unless $f$ is in $L^\infty$.

Then

$$\log \theta(\epsilon^{-1}) = \epsilon \left( \epsilon^{-1} \log \theta(\epsilon^{-1}) \right) = \epsilon \log \varphi(\epsilon^{-1})$$

so

$$\begin{aligned}
(\log \theta(\epsilon^{-1}))'' &= \left( \log \varphi(\epsilon^{-1}) - \frac{1}{\epsilon} (\log \varphi)'(\epsilon^{-1}) \right)' \\
&= -\frac{1}{\epsilon^2} (\log \varphi)'(\epsilon^{-1}) + \frac{1}{\epsilon^3} (\log \varphi)''(\epsilon^{-1}) + \frac{1}{\epsilon^2} (\log \varphi)'(\epsilon^{-1}) \\
&= \frac{1}{\epsilon^3} (\log \varphi)''(\epsilon^{-1}).
\end{aligned} \tag{10.2}$$

Since $\log\varphi$ is convex it follows that $\log\theta(\epsilon^{-1})$ is convex. Then since $\log\alpha(\epsilon) = -\log\epsilon + \log\theta(\epsilon^{-1})$ and $-\log\epsilon$ is strictly convex, $\log\alpha$ is strictly convex. $\square$

The function, $\lambda\colon \mathbb{R}\to\mathbb{R}$ defined by

$$\lambda(r) := r + \log(\mu(e^{-r})) \tag{10.3}$$

will play a large role in what follows as will the function

$$A(x) := \frac{x\mu'(x)}{\mu(x)} = x(\log\mu(x))'. \tag{10.4}$$

We first establish, in Proposition 10.2, some properties of these functions that follow simply from $\mu$ being strictly increasing and concave (properties that we show hold for $\mu$ in Theorem 10.3).

**Proposition 10.2.** *Assume that $\mu$ is a strictly increasing (strictly) concave twice continuously differentiable MOC. Then $\log\mu$ is strictly increasing and strongly strictly concave on $(0,\infty)$, $\lambda$ is (strictly) increasing on $\mathbb{R}$ with $\lambda' < 1$, $\mu(x)/x$ is (strictly) decreasing on $(0,\infty)$, and $\log\mu(x)/x$ is strictly increasing and strictly concave on $(0,a)$ for some $a > 0$. Also,*

$$0 < A(x) \le 1 \text{ for all } x > 0 \tag{10.5}$$

*with strict inequality if $\mu$ is strictly concave. Moreover, if $\mu$ satisfies the Osgood condition, (1.4), then*

$$\limsup_{x\to 0} A(x) = 1. \tag{10.6}$$

*Proof.* That $\log\mu$ is strictly increasing and strongly strictly concave follows directly from the assumed properties of $\mu$.

Letting $r = -\log x$, we can write $\lambda(r)$ variously as

$$\lambda(r) = \log\left(\frac{\mu(x(r))}{x(r)}\right) = \log(e^r\mu(e^{-r})) = r + \log\mu(e^{-r}). \tag{10.7}$$

Then

$$\begin{aligned} \lambda'(r) &= \frac{d}{dx}\log\left(\frac{\mu(x)}{x}\right)\frac{dx}{dr} = \frac{d}{dx}\log\left(\frac{\mu(x)}{x}\right)(-e^{-r}) \\ &= \frac{x}{\mu(x)}\frac{x\mu'(x)-\mu(x)}{x^2}(-x) = \frac{1}{\mu(x)}\left(\mu(x) - x\mu'(x)\right) \\ &= 1 - \frac{x\mu'(x)}{\mu(x)} = 1 - A(x). \end{aligned} \tag{10.8}$$

Because $\mu$ is strictly increasing, $A(x) > 0$ and $\lambda' < 1$. Because $\mu$ is (strictly) concave, $\mu'$ is (strictly) decreasing so by the mean value theorem,

$$\mu'(x) \le \frac{\mu(x)-\mu(0)}{x} = \frac{\mu(x)}{x}$$

so $A \le 1$ and hence, and equivalently, $\lambda' \ge 0$ with strict inequalities when $\mu$ is strictly concave. This gives (10.5) and shows that $\lambda$ is (strictly) increasing.

Now,

$$\left(\frac{\mu(x)}{x}\right)' = \frac{x\mu'(x) - \mu(x)}{x^2} \le 0$$

by (10.5) with strict inequality when $\mu$ is strictly concave. Thus, $\mu(x)/x$ is (strictly) decreasing.

Finally, let $g(x) = \log\mu(x)/x$. Then $\log\mu(x) = xg(x)$ and $(\log\mu(x))' = g(x) + xg'(x)$ so $xg'(x) = (\log\mu(x))' - g(x)$. But $\mu(0) = 0$ so $g$ is negative on $(0, a)$ for sufficiently small $a > 0$. Hence, $g' > 0$ on $(0, a)$ since $\log\mu$ is increasing as we showed above. Then, $(\log\mu(x))'' = 2g'(x) + xg''(x)$ so

$$xg''(x) = (\log\mu(x))'' - 2g'(x)$$

and we conclude that $g$ is strictly concave on $(0, a)$.

To prove (10.6), suppose that

$$0 \le \alpha := \limsup_{x\to 0} A(x) < 1.$$

Then there exists $\epsilon > 0$ such that $A \le \alpha$ on $(0, \epsilon)$. Thus, $(\log\mu)'(x) \le \alpha/x$ on $(0, \epsilon)$, and integrating from $x < \epsilon$ to $\epsilon$ gives

$$\log\mu(\epsilon) - \log\mu(x) \le \alpha\,(\log\epsilon - \log x)$$

so

$$\mu(x) \ge \frac{\mu(\epsilon)}{\epsilon^\alpha} x^\alpha.$$

But this means that $\mu$ does not satisfy (1.4). Hence, (10.6) holds. □

We show in Theorem 10.3 that when $\mu$ is given by (10.1), we can say more about the functions $\lambda$ and $A$, as well as about $\mu$ itself. The proof of this theorem relies on determining the value of $\epsilon$ that minimizes the expression in (10.1). This is natural, for as we will see in the next section, $\mu$ can be defined in terms of a Legendre transformation (see (12.2)), and finding the minimizing $\epsilon$ is the usual way to calculate the Legendre transformation for strictly concave functions.

**Theorem 10.3.** *Assume that* $\log\alpha$ *is strictly convex and twice continuously differentiable. The function* $\mu$ *given by (10.1) is continuous on* $[0, \infty)$ *with* $\mu(0) = 0$*,* $\mu$ *is strictly increasing and concave, and* $\mu$ *is twice continuously differentiable and positive on* $(0, \infty)$*;* $\log\mu(x)$ *and* $\lambda(r) := r + \log(\mu(e^{-r}))$ *are each strictly increasing and strictly concave on* $(0, \infty)$ *and* $\mathbb{R}$*, respectively;* $\mu(x)/x$ *is strictly decreasing with* $\lim_{x\to\infty}\mu(x)/x = 0$*; and* $A$ *is strictly decreasing with*

$$A(0) := \lim_{x\to 0^+} A(x) = 1. \tag{10.9}$$

*Furthermore, if* $p\log\theta(p)$ *is convex and twice continuously differentiable then* $\log\alpha$ *is strictly convex and twice continuously differentiable,* $\lambda$ *is strongly* *strictly concave, and each of the following equivalent conditions hold:*

$$\begin{aligned} \lambda''(r) + (\lambda'(r))^2 &\ge 0 \quad \textit{for all } r \textit{ in } \mathbb{R}, \\ x^2\mu''(x) - x\mu'(x) + \mu(x) &\ge 0 \quad \textit{for all } x > 0, \\ \mu''(x) &\ge \left(\frac{\mu(x)}{x}\right)' \quad \textit{for all } x > 0. \end{aligned} \tag{10.10}$$

*Proof.* The expression for $\mu$ in (10.1) shows that it is continuous on $[0, \infty)$, positive on $(0, \infty)$, and strictly increasing. The function $x \mapsto x^{1-2\epsilon}$ is concave for all $\epsilon$ in $[0, 1/2]$, so for any $\gamma$ in $[0, 1]$ and $x, y$ in $(0, \infty)$,

$$\begin{aligned} \mu(\gamma x + (1-\gamma)y) &= C \inf_{\epsilon \in \mathcal{A}} \left\{ (\gamma x + (1-\gamma)y)^{1-2\epsilon} \alpha(\epsilon) \right\} \\ &\ge C \inf_{\epsilon \in \mathcal{A}} \left\{ (\gamma x^{1-2\epsilon} + (1-\gamma) y^{1-2\epsilon}) \alpha(\epsilon) \right\} \\ &\ge C\gamma \inf_{\epsilon \in \mathcal{A}} \left\{ x^{1-2\epsilon} \alpha(\epsilon) \right\} + C(1-\gamma) \inf_{\epsilon \in \mathcal{A}} \left\{ y^{1-2\epsilon} \alpha(\epsilon) \right\} \\ &= \gamma \mu(x) + (1-\gamma)\mu(y), \end{aligned}$$

where $\mathcal{A} = (0, 1/p_0]$. It follows that $\mu$ and hence $\log \mu$ is concave. Because $\mu(x)/x = \inf_{\epsilon \in \mathcal{A}} \left\{ x^{-2\epsilon} \alpha(\epsilon) \right\}$ it strictly decreases to 0. (Some of these facts also follow from Proposition 10.2, given the properties of $\mu$.)

We have, as in (10.7), and using (10.1),

$$\lambda(r) = \log\left(\frac{\mu(x)}{x}\right) = \inf_{\epsilon \in \mathcal{A}} \left\{ -2\epsilon \log x + \log \alpha(\epsilon) \right\} = \inf_{\epsilon \in \mathcal{A}} \left\{ g(r, \epsilon) \right\}, \tag{10.11}$$

where

$$g(r, \epsilon) = 2\epsilon r + \log \alpha(\epsilon).$$

Since $\log \alpha$ is strictly convex so is $g(r, \cdot)$. Thus, $g(r, \cdot)$ always achieves it minimum at a unique $\epsilon = \epsilon(r)$ with

$$\partial_\epsilon g(r, \epsilon)|_{\epsilon = \epsilon(r)} = 2r + \alpha'(\epsilon(r))/\alpha(\epsilon(r)) = 0. \tag{10.12}$$

Moreover, the function $g$ is twice continuously differentiable in both variables, so $\epsilon(r)$ is continuously differentiable by the implicit function theorem.

Writing (10.12) as

$$(\log \alpha)'(\epsilon(r)) = -2r, \tag{10.13}$$

since $\log \alpha$ is strictly convex, $(\log \alpha)'$ strictly increases, so $(\log \alpha)'(\epsilon)$ strictly decreases as $\epsilon$ decreases. But as $r$ increases, $-2r$ strictly decreases; hence, $\epsilon(r)$ is a strictly decreasing function of $r$, and hence also invertible. Moreover, because $(\log \alpha)'$ strictly increases, we must have $\epsilon(r) \to 0$ as $r \to \infty$.

Also from (10.12),

$$\alpha'(\epsilon(r)) = -2r\alpha(\epsilon(r)). \tag{10.14}$$

But,

$$\lambda(r) = g(r, \epsilon(r))$$

so

$$\lambda'(r) = \frac{d}{dr} g(r, \epsilon(r)) = \partial_r g(r, \epsilon(r)) + \partial_\epsilon g(r, \epsilon(r))\epsilon'(r) \\ = \partial_r g(r, \epsilon(r)) = 2\epsilon(r) \tag{10.15}$$

by (10.12) so $\lambda$ is strictly increasing. Since $\epsilon$ is strictly decreasing it follows that $\lambda$ is strictly concave. And because $\epsilon$ is continuously differentiable, $\lambda$, and hence $\mu$, are twice continuously differentiable.

But (10.8, 10.15) give

$$\epsilon(r) = \frac{1}{2}\left(1 - A(x(r))\right),$$

so $A(x(r)) \to 1$ as $r \to \infty$ or $A(x) \to 1$ as $x \to 0^+$, giving (10.9) (this also follows from (10.6)). This also shows that $A$ is strictly decreasing.

Now assume that $p \log \theta(p)$ is convex. From the proof of Lemma 10.1, we see that $\log \alpha$ is strictly convex and so all of the conclusions above, in particular, that $\lambda$ is strictly concave and $\epsilon$ is invertible, hold. We also have, as in the proof of Lemma 10.1, that

$$\phi(\epsilon) := \log \theta(1/\epsilon) = \log \epsilon + \log \alpha(\epsilon) \tag{10.16}$$

is convex.

Letting $\eta \colon (0, 1/p_0] \to (0, \infty)$ be the inverse of the map, $r \mapsto \epsilon(r)$, we have, using (10.14),

$$\phi''(\epsilon) = -\frac{1}{\epsilon^2} + \left(\frac{\alpha'(\epsilon)}{\alpha(\epsilon)}\right)' = -\frac{1}{\epsilon^2} - 2(\eta(\epsilon))' \\ = -\frac{1}{\epsilon^2} - \frac{2}{\epsilon'(\eta(\epsilon))} = -\frac{1}{\epsilon^2} - \frac{4}{\lambda''(\eta(\epsilon))}. \tag{10.17}$$

Or, expressed in the variable $r$ and using $\epsilon(r) = (1/2)\lambda'(r)$,

$$\phi''(\epsilon(r)) = -\frac{4}{(\lambda'(r))^2} - \frac{4}{\lambda''(r)}.$$

By Lemma 10.1, $\phi$ is convex and we conclude that $1/\lambda''(r) + 1/(\lambda'(r))^2 \le 0$ so that $\lambda''(r) < 0$ and

$$\frac{\lambda''(r) + (\lambda'(r))^2}{\lambda''(r)\lambda'(r)} \le 0.$$

Then since $\lambda''(r)\lambda'(r) < 0$, $(10.10)_1$ holds.

It remains to show the equivalence of the three conditions in (10.10). Let $r = -\log x$ as in the proof of Proposition 10.2. Starting with (10.8) one can show that

$$\lambda''(r) = x^2 \frac{\mu''(x)}{\mu(x)} - x^2 \left(\frac{\mu'(x)}{\mu(x)}\right)^2 + x\frac{\mu'(x)}{\mu(x)}. \tag{10.18}$$

Then from (10.8, 10.18),

$$\begin{aligned}\lambda''(r) + (\lambda'(r))^2 &= x^2\frac{\mu''(x)}{\mu(x)} - x^2\left(\frac{\mu'(x)}{\mu(x)}\right)^2 + x\frac{\mu'(x)}{\mu(x)} + \left(1 - \frac{\mu'(x)}{\mu(x)}x\right)^2 \\ &= 1 - \frac{x\mu'(x)}{\mu(x)} + \frac{x^2\mu''(x)}{\mu(x)} = \frac{x^2\mu''(x) - x\mu'(x) + \mu(x)}{\mu(x)}.\end{aligned}$$

This gives the equivalence of $(10.10)_1$ and $(10.10)_2$, and a simple calculation shows that $(10.10)_3$ is a re-expression of $(10.10)_2$. (Note that integrating $(10.10)_3$ does not contradict (10.5), because the concavity of $\mu$ means that neither $\mu(x)/x$ nor $\mu'(x)$ converges to 0 as $x \to 0$.) □

## 11. Yudovich velocity fields are Dini-continuous

We say that $\mu$ is a Dini MOC if the MOC $S_\mu \colon [0, \infty) \to [0, \infty)$ defined by

$$S_\mu(x) = \int_0^x \frac{\mu(s)}{s}\,ds \tag{11.1}$$

exists (that is, if the integral is finite for any $x > 0$ and hence for all $x > 0$ and $S_\mu$ is as in Definition 1.2). If a function has a Dini MOC we say that the function is Dini-continuous.

The function $S_\mu$ can be used to re-express $(10.10)_3$ as $S''_\mu(x) \le \mu''(x)$, meaning that $S_\mu$ is more strictly convex than $\mu$.

It is shown in [17] that the sequence of example Yudovich velocity fields derived from (1.5) are Dini-continuous. In fact, it follows from Proposition 11.1 that all Yudovich velocity fields are Dini-continuous and the MOC $\mu$ and $S_\mu$ are essentially the same, as we show in Proposition 11.1. These are perhaps the most significant properties of Yudovich velocity fields. (Also see Remark (11.2).)

It follows trivially from (10.5) that $\mu \le S_\mu$ and $\mu' \le S'_\mu$ for any concave MOC, $\mu$. (Allowing that $S_\mu$ may be infinite and in that case defining $S'_\mu$ to be infinite.) In addition, for Yudovich velocity fields, $\mu'' \ge S''_\mu$, a consequence of Lemma 10.1, Theorem 10.3, and Proposition 11.1.

**Proposition 11.1.** *Assume that $\mu$ is a strictly increasing (strongly) strictly concave Osgood MOC and that (10.9) holds. Then $\mu$ is Dini-continuous while $S_\mu$ is strictly increasing and (strongly) strictly concave, $S_\mu$ lies in the same germ as $\mu$, and $S'_\mu$ lies in the same germ as $\mu'$ at the origin. Moreover, the equivalent of (10.10) holds for $S_\mu$ if it holds for $\mu$.*

*Proof.* Assume that (10.9) holds. It follows that for some $x > 0$ we have $s\mu'(s) \ge (1/2)\mu(s)$ for all $s < x$. Hence,

$$S_\mu(x) = \int_0^x \frac{\mu(s)}{s}\,ds \le 2\int_0^x \mu'(s)ds = 2\mu(x) < \infty.$$

That is, $\mu$ must satisfy not only the Osgood condition, (1.4), but the Dini condition. Then since $S_\mu(0)$ must be zero we can apply L'Hospital's rule to conclude that

$$\lim_{x\to 0}\frac{S_\mu(x)}{\mu(x)}=\lim_{x\to 0}\frac{S_\mu'(x)}{\mu'(x)}=\lim_{x\to 0}\frac{\mu(x)}{x\mu'(x)}=\lim_{x\to 0}\frac{1}{A(x)}=1.$$

Also, $S_\mu'(x)=\mu(x)/x$ is strictly decreasing by Proposition 10.2 so $S_\mu$ is strictly concave. (And if $\mu$ is strongly strictly concave then so too is $S_\mu$.)

Now assume that (10.10) holds. Then the the equivalent of $(10.10)_3$ holds for $S_\mu$ if and only if $S''_{S_\mu}-S''_\mu\le 0$, as was observed above for $\mu$. Then

$$\begin{aligned} S''_{S_\mu}-S''_\mu\le 0 &\iff \left(\int_0^x \frac{S_\mu(s)}{s}\,ds\right)''-\left(\int_0^x \frac{\mu(s)}{s}\,ds\right)''\le 0\\ &\iff \left(\frac{S_\mu(x)}{x}\right)'-\left(\frac{\mu(x)}{x}\right)'\le 0\\ &\iff xS_\mu'(x)-S_\mu(x)-(x\mu'(x)-\mu(x))\le 0\\ &\iff \mu(x)-S_\mu(x)-x\mu'(x)+\mu(x)\le 0\\ &\iff j(x):=S_\mu(x)+x\mu'(x)-2\mu(x)\ge 0. \end{aligned}$$

By (10.9), $j(0)=0$, and we have

$$\begin{aligned} j'(x)&=\frac{\mu(x)}{x}+\mu'(x)+x\mu''(x)-2\mu'(x)\\ &=\frac{x^2\mu''(x)-x\mu'(x)+\mu(x)}{x}. \end{aligned}$$

Then $j'(x)\ge 0$ for all $x>0$ if $(10.10)_2$ holds, and it follows that $j(x)\ge 0$ for all $x>0$. □

**Remark 11.2.** It is shown in [3] by Charles Burch that if $v$ is a velocity field with a concave MOC, $\mu$, then $Rv$ has a MOC, $\nu$, given by

$$\nu(x)=c\left(S_\mu(x)+x\int_x^\infty \frac{\mu(s)}{s^2}\,ds\right).$$

Here, $R$ can be a Riesz transform. (This result appears in an earlier form as Lemma 1 of [16] by Victor Shapiro. Also see [10].)

For a Yudovich velocity, $v$ lies in $L^\infty([0,\infty)\times\mathbb{R}^2)$, so we can choose $\mu$ to be bounded, making $\nu(x)$ finite for all $x>0$. Since $\nu'(x)=c\int_x^\infty \mu(s)s^{-2}\,ds$, $\nu$ is strictly increasing. That $\nu(0)=0$ then follows directly if $\nu'(0)<\infty$ and by applying L'Hospital's rule, otherwise. Noting that $\nu''(x)=-c\mu'(x)/x^2<0$, we see that $\nu$ is strictly concave. It is, in general, neither Osgood nor Dini, as we can see by looking at $\mu_2$ of (9.10). (For bounded vorticity, however, which corresponds to $\mu_1$, $\nu$ is both Osgood and Dini.)

## 12. Inverting the defining relation for the MOC of the velocity

Our intent in this section is start with a MOC, $\mu$, having all of the properties stated in Theorem 10.3 and invert the relation in (10.1) to obtain a function

$\alpha$ and hence $\theta$ that satisfies all of the properties of Lemma 10.1. Such a $\theta$ can then give the $L^p$-norms of a Yudovich vorticity as in Definition 1.1.

That $\mu$ is concave follows directly from (10.1) and requires no special properties of $\theta$. When $\theta$ comes from the $L^p$-norms of a vorticity field, however, $p \log \theta(p)$ must be convex and $\log \alpha$ must be strictly convex (see Lemma 10.1). It is $\log \alpha$ being strictly convex that is critical to obtaining an inverse, as we can see by re-expressing (10.1) in terms of a Legendre transformation.

**Definition 12.1.** Let $f \colon I \to \mathbb{R}$ be a strictly convex function on an interval, $I$. We define its *Legendre transformation*, $f^*$, by

$$f^*(x) = \sup_{\epsilon \in I} \{x\epsilon - f(\epsilon)\}.$$

The domain of $f^*$ consists of all $x$ in $\mathbb{R}$ for which the supremum is finite.

We can write (10.1) in terms of a Legendre transformation. From (10.11),

$$\begin{aligned} \log\left(\frac{\mu(x)}{x}\right) &= \inf_{\epsilon \in \mathcal{A}} \{-2\epsilon \log x + \log \alpha(\epsilon)\} \\ &= -\sup_{\epsilon \in \mathcal{A}} \{2\epsilon \log x - \log \alpha(\epsilon)\} = -(\log \alpha)^*(2 \log x). \end{aligned} \tag{12.1}$$

Thus,

$$\mu(x) = x e^{-(\log \alpha)^*(2 \log x)}.$$

Because we have restricted the Legendre transformation to strictly convex functions, $f^*$ is also strictly convex, and the Legendre transformation is an involution ($(f^*)^* = f$). See, for instance, Section 14 of [1]. Hence, letting $u = 2 \log x$, (12.1) becomes $(\log \alpha)^*(u) = -\lambda(-u/2)$. Letting $\overline{\lambda}(s) = -\lambda(-s/2)$, we have

$$\begin{aligned} \log \alpha(x) &= (\overline{\lambda})^*(x) = \sup_{\epsilon \in \mathbb{R}} \left\{x\epsilon - \overline{\lambda}(\epsilon)\right\} = \sup_{\epsilon \in \mathbb{R}} \{(-x)(-\epsilon) - (-\lambda(-\epsilon/2))\} \\ &= \sup_{\epsilon \in \mathbb{R}} \{(-2x)\epsilon - (-\lambda(\epsilon))\} = (-\lambda)^*(-2x). \end{aligned}$$

Thus,

$$\alpha(x) = e^{(-\lambda)^*(-2x)}. \tag{12.2}$$

As long as $\lambda$ is strictly concave, so that $-\lambda$ is strictly convex, we can perform the inversion.

There are three limitations of using (12.2) alone. First, $\lambda$ may be strictly concave only near the origin. Second, it is not clear what the domain of $\alpha$ is. In particular, we need the domain to include 0: as we will see, $\mu$ satisfying the Osgood condition is required to insure this. Third, it is not clear from (12.2) that (10.10) is enough to insure that $p \log \theta(p)$ is convex. For this reason, we give an explicit method for inverting (10.1) in Theorem 12.2.

**Theorem 12.2.** *Let $\mu$ be a strictly increasing $C^2$ MOC satisfying the Osgood condition, (1.4), with $\lambda\colon \mathbb{R} \to \mathbb{R}$ and $\epsilon\colon \mathbb{R} \to \mathbb{R}$ given by*

$$\lambda(r) = r + \log \mu(e^{-r}), \quad \epsilon(r) = \frac{1}{2}\lambda'(r). \tag{12.3}$$

*Assume that there is a neighborhood $\mathcal{N}$ of $r = \infty$ on which $\lambda(r)$ is strictly concave (or, equivalently, a neighborhood of the origin on which $A$, given by (10.4), is strictly decreasing). Then $\epsilon$ is invertible on $\mathcal{N}$, and calling that inverse, $\eta$, and letting*

$$\alpha(\epsilon) = C_0 \exp\left\{\lambda(\eta(\epsilon)) - 2\eta(\epsilon)\epsilon\right\}, \tag{12.4}$$

*we have $\theta(p) = p^{-1}\alpha(p^{-1})$ in a neighborhood of $p = \infty$ for some $C_0 > 0$. The function $\log \alpha$ is strictly convex in a neighborhood of the origin. If (10.10) holds then $\log \theta(1/\epsilon)$ is convex in a neighborhood of the origin, while $p \log \theta(p)$ is convex in a neighborhood of $\infty$.*

*Proof.* First observe that $\lambda$ strictly concave in a neighborhood of infinity is equivalent to $A$ being strictly decreasing in a neighborhood of the origin by virtue of (10.8). Combined with (10.6) this gives that (10.9) holds.

Examining the proof of Theorem 10.3, the starting point now being the function $\mu$ rather than the function $\alpha$, we see that the first equality in (10.11) along with (10.15) define $\epsilon$ as a function of $r = -\log x$, where $\epsilon$ gives the location of the infimum in the defining relation, (10.1), between $\alpha$ and $\mu$. The invertibility of $\epsilon = \epsilon(r)$ required $\lambda$ to be strictly concave (this also gave $\epsilon' < 0$) and the condition on $\mu$ in (10.9) insures that $\epsilon(\infty) = 0$; together, these two facts give the invertibility of $\epsilon$ on $\mathcal{N}$.

Using (10.13), we have

$$\frac{d}{dr}(\log \alpha(\epsilon(r))) = (\log \alpha)'(\epsilon(r))\epsilon'(r) = -2r\epsilon'(r).$$

Hence,

$$\begin{aligned}\log \alpha(\epsilon(r)) &= -2\int r\epsilon'(r)\,dr = -2\left[r\epsilon(r) - \int \epsilon(r)\,dr\right]\\ &= 2\left[-r\epsilon(r) + \int \frac{1}{2}\lambda'(r)\,dr\right] = \lambda(r) - r\lambda'(r) + C\end{aligned}$$

so

$$\alpha(\epsilon(r)) = C\exp\left[\lambda(r) - r\lambda'(r)\right] = C\exp\left[\lambda(r) - 2r\epsilon(r)\right], \tag{12.5}$$

and (12.4) is just a re-expression of (12.5).

Because $\alpha(x) = x^{-1}\theta(x^{-1})$, we have

$$\log \alpha(x) = -\log x + \log \theta\left(\frac{1}{x}\right) = -\log x + x\varphi\left(\frac{1}{x}\right),$$

where $\varphi(p) = p \log \theta(p)$ as in Lemma 10.1, so

$$(\log \alpha)'(x) = -\frac{1}{x} + \varphi\left(\frac{1}{x}\right) - \frac{1}{x}\varphi'\left(\frac{1}{x}\right),$$

$$(\log \alpha)''(x) = \frac{1}{x^2} + \frac{1}{x^3}\varphi''\left(\frac{1}{x}\right).$$

Hence, taking the derivative of (10.13) gives

$$-2 = (\log \alpha)''(\epsilon(r))\epsilon'(r) = \epsilon'(r)\left[\frac{1}{\epsilon(r)^2} + \frac{1}{\epsilon(r)^3}\varphi''\left(\frac{1}{\epsilon(r)}\right)\right].$$

Since $\epsilon(r)$ is strictly decreasing on $\mathcal{N}$, $\log \alpha$ is strictly convex on $\mathcal{N}$. Also,

$$\begin{aligned}\varphi''\left(\frac{1}{\epsilon(r)}\right) &= -\epsilon(r)\left[2\frac{\epsilon(r)^2}{\epsilon'(r)} + 1\right] = -\frac{\lambda'(r)}{2}\left[\frac{(\lambda'(r))^2}{\lambda''(r)} + 1\right] \\ &= -\frac{\lambda'(r)}{2\lambda''(r)}\left[(\lambda'(r))^2 + \lambda''(r)\right].\end{aligned} \tag{12.6}$$

This shows that $\varphi$ is convex if (10.10) holds, as long as $\lambda$ is strictly concave. The convexity of $\log \theta(1/\epsilon)$ then follow as in the proof of Lemma 10.1. □

As an example, let us apply Theorem 12.2 to the first Yudovich vorticity, where $\theta(p) = C$, which corresponds to $\mu_1$ of Section 9. As we know from Section 4, $\mu(x) = -x \log x$ for sufficiently small $x$, where here and in what follows we ignore immaterial constants.

Then from (12.3),

$$\lambda(r) = \log\left(e^r(-e^{-r}\log(e^{-r})\right) = \log r$$

and so $\epsilon(r) = \frac{1}{2}\lambda'(r) = 1/(2r)$. Thus, $\epsilon$ is invertible for *all* $r$ with inverse $\eta(\epsilon) = 1/(2\epsilon)$. Thus, $\eta(\epsilon)\epsilon = \frac{1}{2}$ and (12.4) gives

$$\alpha(\epsilon) = C_0 \exp\left\{\log(2 \cdot 1/(2\epsilon)) - 2\frac{1}{2}\right\} = \frac{C}{\epsilon}$$

and thus $\theta(p) = p^{-1}(Cp) = C$, recovering $\theta$ to within a multiplicative constant.

The higher Yudovich examples cannot be inverted exactly using Theorem 12.2 because $\epsilon(r)$ becomes a transcendental function of $r$ that cannot be inverted in closed form. The following proposition is of some use in this regard, however.

**Proposition 12.3.** *If $\epsilon$ is overestimated then the expression in (12.4) underestimates $\alpha$. That is, assume that $\overline{\epsilon}\colon \mathbb{R} \to (0, 1/2)$ is $C^1$ and strictly decreasing with $\overline{\epsilon}(\infty) = 0$ and $\overline{\epsilon} \geq \epsilon$,and let*

$$\overline{\alpha}(\overline{\epsilon}(r)) = C \exp\left[\lambda(r) - 2r\overline{\epsilon}(r)\right]. \tag{12.7}$$

*Then $\overline{\alpha}(x) \leq \alpha(x)$ for all sufficiently large $x$.*

*Proof.* From (12.7), $\overline{\alpha}(\overline{\epsilon}(r))$ is an increasing function of $r$, so

$$\begin{aligned}\overline{\alpha}(\epsilon(r)) &\le \overline{\alpha}(\overline{\epsilon}(r)) = C\exp\left[\lambda(r) - 2r\overline{\epsilon}(r)\right] \le C\exp\left[\lambda(r) - 2r\epsilon(r)\right] \\ &= \alpha(\epsilon(r)).\end{aligned}$$

□

For example, suppose that for $\mu = \mu_m$, $m \ge 2$, we approximate $\epsilon(r)$ by $1/(2r)$, which is the exact $\epsilon(r)$ for the first Yudovich example. As can be easily verified, this is an overestimate of the true $\epsilon(r)$, and we will obtain $\overline{\alpha}(\epsilon) = \epsilon^{-1}\theta_m(\epsilon^{-1})$ as an underestimate of $\alpha(\epsilon)$. This is a kind of dual to the overestimate in [22] of what we call $\mu_m$ from $\theta_m$ using this same estimate for $\epsilon$.

## 13. Recovering $\omega^0$ from its $L^p$-norms

It remains to invert the second map in (7.2); namely, $\omega^0 \mapsto \theta(p)$. We should expect to invert this map neither uniquely nor exactly. Lack of uniqueness arises because any rearrangement or sign change of $\omega^0$ yields the same $\theta$. Since, however, we are interested in square-symmetric vorticities, as in Definition 2.6, the lack of uniqueness is not a problem.

The inability to invert exactly is a more complex issue. To see what is involved, let $\lambda$ be the distribution function for $\omega^0$; that is, $\lambda(x)$ = measure of $\{t \colon |\omega^0(t)| > x\}$. It is classical that

$$\theta(p)^p = \|\omega^0\|_{L^p}^p = p\int_0^\infty x^{p-1}\lambda(x)\,dx = p\mathcal{M}\lambda(p), \tag{13.1}$$

where $\mathcal{M}$ is the Mellin transform. If $\omega^0$ lies in $L^{p_0} \cap L^p$ for all $p \ge p_0$ then $\lambda(p)$ decays faster than any polynomial in $p$ and it is easy to see from (13.1) (or directly from the definition of the $L^p$-norm) that $\varphi(p) := p\log\theta(p)$ is complex-analytic in the right-half plane, $\operatorname{Re} p > p_0$. Of necessity, then, $\varphi$ must at least be real-analytic (and real-valued) on $(p_0, \infty)$ to perform the inversion exactly, and we should not expect this to be the case.

Instead, we must look for a way to make an approximate inversion. Toward this end, we will take an approach using (13.1) that is, in a sense, a generalization of one proof of Stirling's approximation.

To motivate this approximation, we first show how to obtain an approximation for $\varphi$ from $\lambda$. Assume that a smooth $\lambda$ is given and let

$$I_p = \int_0^\infty x^{p-1}p\lambda(x)\,dx = \int_0^\infty e^{(p-1)\log x - \rho(x)}\,dx, \tag{13.2}$$

where $\rho = -\log\lambda$. Being a distribution function, $\lambda$ is decreasing, hence $\rho$ is increasing. Suppose also that $\rho$ is convex. Then $f(x) = x^{p-1}\lambda(x)$ has a unique maximum at $x = x_p$, where

$$\rho'(x_p) = (p-1)/x_p. \tag{13.3}$$

Moreover, $x_p$ must increase to $\infty$ as $p \to \infty$.

For $x$ near $x_p$, we thus have

$$\begin{aligned}(p-1)\log x-\rho(x)&\approx(p-1)\left[\log x_p+\frac{1}{x_p}(x-x_p)-\frac{1}{2x_p^2}(x-x_p)^2\right]\\&\quad-\left[\rho(x_p)+\rho'(x_p)(x-x_p)+\frac{\rho''(x_p)}{2}(x-x_p)^2\right]\\&=(p-1)\log x_p-\rho(x_p)-\frac{1}{2}\left[\frac{p-1}{x_p^2}+\rho''(x_p)\right](x-x_p)^2.\end{aligned}\tag{13.4}$$

This approximation should be a good one for all sufficiently large $p$ if

$$\lim_{x\to\infty}\frac{\rho'''(x)}{\rho''(x)}=0,\tag{13.5}$$

an assumption we now add.

Differentiating (13.3) with respect to $p$ and using the chain rule gives

$$\rho''(x_p)=\frac{1}{x_p\frac{dx_p}{dp}}-\frac{p-1}{x_p^2},$$

so

$$(p-1)\log x-\rho(x)\approx(p-1)\log x_p-\rho(x_p)-\frac{(x-x_p)^2}{2x_p\frac{dx_p}{dp}}.$$

Thus,

$$I_p\approx\int_0^\infty e^{((p-1)\log x_p-\rho(x_p))}\exp\left(-\frac{1}{2x_p\frac{dx_p}{dp}}(x-x_p)^2\right)\,dx.$$

Assuming the Gaussian in the integrand is sufficiently sharp, we have

$$\begin{aligned}I_p&\approx x_p^{p-1}e^{-\rho(x_p)}\int_{-\infty}^\infty\exp\left(-\frac{1}{2x_p\frac{dx_p}{dp}}(x-x_p)^2\right)\,dx\\&=x_p^{p-1}e^{-\rho(x_p)}\left(\frac{1}{2x_p\frac{dx_p}{dp}}\right)^{-1/2}\sqrt{\pi}=\sqrt{2\pi}x_p^{p-\frac{1}{2}}e^{-\rho(x_p)}\left(\frac{dx_p}{dp}\right)^{\frac{1}{2}}.\end{aligned}$$

(Even if the Gaussian is not sharp, this approximation is at most a factor of two overestimate.)

Since $\theta(p)^p=pI_p$, we have

$$\begin{aligned}\theta(p)&\approx p^{\frac{1}{p}}\left[\sqrt{2\pi}x_p^{p-\frac{1}{2}}e^{-\rho(x_p)}\left(\frac{dx_p}{dp}\right)^{1/2}\right]^{\frac{1}{p}}\\&=(2\pi)^{\frac{1}{2p}}x_p^{1-\frac{1}{2p}}e^{-\rho(x_p)/p}\left[\frac{dx_p}{dp}\right]^{\frac{1}{2p}}.\end{aligned}\tag{13.6}$$

Also,

$$\frac{d\rho(x_p)}{dp} = \rho'(x_p)\frac{dx_p}{dp} = \frac{p-1}{x_p}\frac{dx_p}{dp} = (p-1)\frac{d}{dp}\log x_p.$$

Integrating by parts gives

$$\rho(x_p) = \int (p-1)\frac{d}{dp}\log x_p\,dp = C + (p-1)\log x_p - \int \log x_p\,dp.$$

Substituting this into (13.6) gives

$$\begin{aligned}\theta(p) &\approx (2\pi)^{\frac{1}{2p}} x_p^{1-\frac{1}{2p}} \exp\left(-\frac{C}{p} - \frac{p-1}{p}\log x_p + \frac{1}{p}\int \log x_p\,dp\right)\left[\frac{dx_p}{dp}\right]^{\frac{1}{2p}} \\ &= (2\pi)^{\frac{1}{2p}} x_p^{-\frac{1}{p}} e^{-\frac{C}{p}} \exp\left(\frac{1}{p}\int \log x_p\,dp\right)\left[\frac{dx_p}{dp}\right]^{\frac{1}{2p}}\end{aligned}$$

and hence,

$$\varphi(p) = p\log\theta(p) \approx -C - \log x_p + \int \log x_p\,dp + \frac{1}{2}\log\left[\frac{dx_p}{dp}\right].$$

This approximation will hold if its derivative,

$$-\frac{d\log x_p}{dp} + \log x_p + \frac{1}{2}\frac{\frac{d^2x_p}{dp^2}}{\frac{dx_p}{dp}} \approx (p\log\theta(p))' = \varphi'(p), \tag{13.7}$$

approximately holds.

From (13.3), $\log x_p = \log(p-1) - \log\rho'(x_p)$, and differentiating gives

$$\frac{d\log x_p}{dp} = \frac{1}{p-1} - \frac{\rho''(x_p)}{\rho'(x_p)}\frac{dx_p}{dp}.$$

But $x_p$ is increasing and hence so is $\log x_p$, and $\rho$ is increasing and convex, so all derivatives above are nonnegative. We conclude that $|\frac{d\log x_p}{dp}| < \frac{1}{p-1}$ and hence vanishes as $p \to \infty$. We also add the assumption that

$$\frac{\frac{d^2x_p}{dp^2}}{\frac{dx_p}{dp}} \to 0 \text{ as } p \to \infty. \tag{13.8}$$

Therefore, for sufficiently large $p$, we have

$$x_p \approx e^{\varphi'(p)} =: \beta(p).$$

Then (13.3) becomes $\rho'(\beta(p)) \approx p/\beta(p)$ so that (estimating $p-1$ by $p$)

$$\frac{d}{dp}\rho(\beta(p)) = \rho'(\beta(p))\beta'(p) \approx p(\log\beta)'(p) = p\varphi''(p). \tag{13.9}$$

The function $\rho$ increases, so the requirement that $\varphi$ be convex enters here.

Integrating from a sufficiently large $q$ to $p > q$ gives

$$\begin{aligned}\rho(\beta(p)) &\approx \rho(\beta(q)) + \int_q^p s\varphi''(s)\,ds \\ &= \rho(\beta(q)) + p\varphi'(p) - q\varphi'(q) - \int_q^p \varphi'(p)\,dp \\ &= C_q + p\varphi'(p) - \varphi(p).\end{aligned} \tag{13.10}$$

Now assume that (13.10) holds exactly, and hence so does (13.9); differentiating it implicitly gives

$$\begin{aligned}\rho''(\beta(p))[\beta'(p)]^2 &= p\varphi'''(p) + \varphi''(p) - \rho'(\beta(p))\beta''(p) \\ &= p\varphi'''(p) + \varphi''(p) - \rho'(\beta(p))\left[[\varphi''(p)]^2 + \varphi'''(p)\right]e^{\varphi'(p)} \\ &= \left[p - \rho'(\beta(p))e^{\varphi'(p)}\right]\varphi'''(p) + \varphi''(p) - \rho'(\beta(p))[\varphi''(p)]^2 e^{\varphi'(p)}.\end{aligned}$$

But,

$$\rho'(\beta(p))e^{\varphi'(p)} = \frac{p\varphi''(p)}{\beta'(p)}e^{\varphi'(p)} = \frac{p\varphi''(p)}{\varphi''(p)e^{\varphi'(p)}}e^{\varphi'(p)} = p,$$

so

$$\rho''(\beta(p))[\beta'(p)]^2 = \varphi''(p) - p[\varphi''(p)]^2. \tag{13.11}$$

Thus, to insure that $\rho$ is convex (which we assumed to obtain a unique solution to (13.3)) we must add the condition that $\varphi''(x) \le \frac{1}{x}$ for all sufficiently large $x$. (Then also $|\varphi''(p)| = |\frac{d \log x_p}{dp}| < \frac{1}{p}$, as above.) Hence, with this condition, (13.3) continues to hold (exactly).

Differentiating (13.11) logarithmically gives

$$\begin{aligned}\frac{\rho'''(\beta(p))}{\rho''(\beta(p))}\beta'(p) &= \frac{d}{dp}\log\left[\varphi''(p) - p[\varphi''(p)]^2\right] - 2\frac{d}{dp}\left[\log\beta'(p)\right] \\ &= \frac{\varphi'''(p) - 2p\varphi'''(p)\varphi''(p) - [\varphi''(p)]^2}{\varphi''(p) - p[\varphi''(p)]^2} - 2\frac{\beta''(p)}{\beta'(p)}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\rho'''(\beta(p))}{\rho''(\beta(p))} &= \frac{\frac{\varphi'''(p)}{\varphi''(p)} - 2p\varphi'''(p) - \varphi''(p)}{\varphi''(p) - p[\varphi''(p)]^2}e^{-\varphi'(p)} \\ &\qquad - 2\frac{[\varphi''(p)]^2 + \varphi'''(p)\varphi''(p)}{[\varphi''(p)]^2}e^{-\varphi'(p)} \\ &= \frac{\frac{\varphi'''(p)}{\varphi''(p)} - 2p\varphi'''(p) - \varphi''(p)}{\varphi''(p) - p[\varphi''(p)]^2}e^{-\varphi'(p)} - 2\left[1 + \frac{\varphi'''(p)}{\varphi''(p)}\right]e^{-\varphi'(p)}.\end{aligned} \tag{13.12}$$

This places a condition on $\varphi$ that insures that the condition (13.5) on $\rho$ holds. Or we could place the following conditions on $\varphi$, the first of which strengthens the condition imposed earlier that $\varphi''(x) \le \frac{1}{x}$ for all sufficiently large $x$:

(1) $\frac{1}{x} - \varphi''(x)$ is bounded away from zero for all sufficiently large $x$;
(2) $\frac{x\varphi'''(x)}{\varphi''(x)} e^{-\varphi'(x)}$, $\frac{\varphi'''(x)}{[\varphi''(x)]^2} e^{-\varphi'(x)} \to 0$ as $x \to \infty$.

To ensure that the assumption in (13.8) holds, we calculate,

$$\frac{\frac{d^2 x_p}{dp^2}}{\frac{dx_p}{dp}} = \frac{\beta''(p)}{\beta'(p)} = \frac{[\varphi''(p)]^2 + \varphi'''(p)}{\varphi''(p)} = \varphi''(p) + \frac{\varphi'''(p)}{\varphi''(p)}.$$

Condition (1) directly gives $\varphi''(p) \to 0$, and integrating Condition (1) gives $e^{-\varphi'(p)} > Cp^{-1}$. It then follows from Condition (2) that $\varphi'''(p)/\varphi''(p) \to 0$.

What we have done is to give a rough derivation of the following:

> If $\varphi$ is convex and satisfies the two conditions above then (13.10) can be used to approximately determine $\rho$, and hence a square-symmetric $\omega^0$, from $\varphi$.

We note that each of the examples of Yudovich in (1.5) satisfy both of these conditions, and (13.10) can be used to determine $\omega^0$ approximately. For instance, when $m = 1$, $\varphi(p) = p \log\log p$, $\varphi'(p) = \log\log p + \frac{1}{\log p}$, $p\varphi'(p) - \varphi(p) = \frac{p}{\log p}$, and $\beta(p) = \log p \, e^{1/\log p}$. For large $p$, then, we have $\beta(p) \approx \log p$ so $\beta^{-1}(x) \approx e^x$. Then from (13.10),

$$\rho(x) \approx C + \beta^{-1}(x)\varphi'(\beta^{-1}(x)) - \varphi(\beta^{-1}(x)) \approx C + \frac{\beta^{-1}(x)}{\log \beta^{-1}(x)} = C + \frac{e^x}{x}.$$

Thus, $\lambda(x) \approx e^{-e^x/x}$, and since $e^{x/2} < e^x/x < e^x$ for all $x > 1$, this $\lambda$ corresponds to $\omega^0$ square-symmetric with

$$\omega^0(x) = f(x_1)\log(2\log(1/x_1))\mathbf{1}_{(0,r)} = f(x_1)\left(\log 2 + \log\log(1/x_1)\right)\mathbf{1}_{(0,r)}$$

in the first quadrant for some $0 < r < e^{-1}$, where $\frac{1}{2} < f(x) < 1$. This is in agreement with Lemma 5.2.

**Remark 13.1.** Condition (1) is fairly natural, as the need for $\varphi$ to be convex derives, ultimately, from Hölder's inequality. Condition (2) arose from trying to insure that the approximation in (13.4) is accurate. Assuming Condition (1) holds, $1/\varphi''(x) > x$, so

$$\frac{x\left|\varphi'''(x)\right|}{\left|\varphi''(x)\right|} > x^2\left|\varphi'''(x)\right|, \quad \frac{\varphi'''(x)}{[\varphi''(x)]^2} > x^2\left|\varphi'''(x)\right|.$$

Integrating Condition (1) gives $e^{-\varphi'(x)} > Cx^{-1}$, but for the Yudovich examples in (1.5), $e^{-\varphi'(x)}$ decreases much more slowly ($(\log x)^{-1}$ for $m = 1$). Thus, Condition (2) can be roughly viewed as saying that $|\varphi'''(x)|$ strays not too far from $x^{-2}$. Finally, both conditions can be expressed in terms of $\lambda$, and hence in terms of $\mu$, by using (12.6), leading to a condition on the third derivative of $\mu$. The resulting forms of the conditions are not, however, immediately enlightening.

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