Boundary Layer Analysis of the Navier-Stokes Equations with Generalized Navier Boundary Conditions

SIAM Conference on Analysis of Partial Differential Equations
San Diego, CA

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14 November 2011

The central problem

How does the behavior of an incompressible fluid at low viscosity compare to the behavior of an inviscid fluid in the presence of a boundary?

- The central problem is when the fluid does not move on the boundary.
- The behavior of the fluid in a layer near the boundary is typically the focus.
- The production of vorticity on the boundary due to the tangential derivative of the pressure can lead to separation of the boundary layer, massively complicating any analysis.

We first briefly consider situations in which the central problem can be solved.

Special initial data special geometry

Ex1: A disk with radially symmetric initial vorticity. The solution to the Euler equations is stationary, and the nonlinear term for both the Navier-Stokes and Euler equations vanishes.

Ex2: Plane parallel channel flow, in which the velocity field is everywhere parallel to the 3D periodic channel's walls. Initial velocity $u_0 = (v_1(z), v_2(x, z), 0)$. The nonlinear terms have only tangential derivatives.

Ex3: Flow in a periodic circular pipe (a solid flat torus). Initial velocity, $u_0 = v_\theta(r)\mathbf{e}_\theta + v_z(r, \theta)\mathbf{e}_z + 0\mathbf{e}_r$. The nonlinear terms have only tangential derivatives.

In each case:

- The symmetry persists in time.
- The nonlinearity has only tangential derivatives of the velocity.
- The tangential derivative of the pressure is zero.

Why does the vanishing viscosity limit hold?

Point of view 1:

The weakened nonlinearity allows one to apply any of the equivalent conditions of Kato, for instance, the vanishing of

$$\nu \int_0^t \|\nabla_{\boldsymbol{\tau}} u_{\boldsymbol{\tau}}\|^2_{L^2(\Gamma_\delta(\nu))}$$

in a boundary layer of width $\delta(\nu) > C\nu$, a la Temam and Wang 1998 and Wang 2001.

Point of view 2:

The vanishing of the tangential derivative of the pressure prevents vorticity production on the boundary, a la Lighthill 1963, allowing convergence.

This is in the spirit of Temam and Wang 1998, who show that as long as $\|\nabla p \cdot \boldsymbol{\tau}\|_{L^2(\Gamma)}$ does not blow up too fast as $\nu \to 0$, the vanishing viscosity limit will hold (with a bound on the rate of convergence).

Point of view 3: Various direct arguments, especially for Ex1.

Focus changes

So in these three examples the focus shifts to how strong the convergence is, how fast the rate of convergence is, how weak the initial data can be, and how irregular the forcing or rotation of the boundary can be.

And, what is happening in the boundary layer, which, though weakened, is still of great interest.

I do not know the complete history of these problems, but:

Ex1: Radially symmetric initial vorticity in a disk: Matsui 1994, Bona and Wu 2002, Wang 2001, Nussenzveig Lopes, Lopes Filho, and Mazzucato 2008 and with the addition of Taylor 2008.

Ex2: Plane parallel channel flow: Wang 2001, Mazzucato and Taylor 2008, Mazzucato, Niu, and Wang 2011.

Ex3: Circular pipe flow: Wang 2001, Mazzucato and Taylor 2011, Han, Mazzucato, Niu, and Wang 2011 (preprint).

Lighthill's condition (1963)

The Navier-Stokes equations are

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u, \quad \operatorname{div} u = 0.$$

If we impose $u = 0$ on the boundary, then on a 2D boundary,

$$\nabla p \cdot \boldsymbol{\tau} = \nu \Delta u \cdot \boldsymbol{\tau} = \nu \nabla^{\perp} \omega(u) \cdot \boldsymbol{\tau} = -\nu \nabla \omega(u) \cdot \mathbf{n},$$

where $(\mathbf{n}, \boldsymbol{\tau})$ are the unit normal, tangent vectors, and $\omega(u)$ is the vorticity (scalar curl) of u. Thus,

$$\nabla \omega(u) \cdot \mathbf{n} = -\frac{1}{\nu} \nabla p \cdot \boldsymbol{\tau}.$$

This says that the production of vorticity is controlled by the tangential derivative of the pressure on the boundary.

A fourth example (2D)

A *confined eddy* is a compactly supported, radially symmetric vorticity having total mass zero.

Such eddies were studied in the whole plane by Nussenzveig Lopes, Lopes Filho, and Zheng 1999, where it is observed that:

- The vorticity and velocity vanish outside the eddy.
- They are stationary solutions to the Euler equations.
- This is true as well for a superposition of such eddies.

Hence, placed inside a bounded domain, a superposition is still a stationary solution to the Euler equations and one that vanishes on the boundary. This makes it is easy to establish the vanishing viscosity limit making the most basic of energy arguments.

The solution to the Navier-Stokes equations immediately breaks the symmetry of the initial data, a full nonlinearity is established, vorticity accumulates on the boundary, and a boundary layer forms. Yet the vanishing viscosity limit holds. This would be an interesting example to explore, perhaps starting by investigating the Kato-like conditions.

Different boundary conditions

Another change to the central problem is to use more amenable boundary conditions. I will speak of *generalized Navier* boundary conditions:

$$u \cdot \mathbf{n} = 0, \quad [\mathbf{S} u \, \mathbf{n}]_{\tan} + \mathcal{A} \, u = 0,$$

where $\mathbf{S} u = (\nabla u + (\nabla u)^T)/2$ is the symmetric gradient and $\mathcal{A}$ is a type $(1, 1)$ tensor on the boundary. $\mathcal{A}$ maps a tangent field on the boundary, such as u, to a tangent field on the boundary.

Special cases:

1. $\mathcal{A} = \alpha \mathbf{I}$ for $\alpha > 0$, reduces to *Navier* boundary conditions. Here, α is the friction parameter and the slip on the boundary is proportional to the tangential component of the stress.
2. When $\mathcal{A} =$ the shape operator (Weingarten map) we get

$$u \cdot \mathbf{n} = 0, \quad (\operatorname{curl} u) \times \mathbf{n} = 0.$$

Navier boundary conditions

Navier boundary conditions are variously called Navier friction, Navier slip, or simply Navier, or simply slip, boundary conditions (other names have been used as well). They were Navier's original boundary conditions.

There has been intermittent interest in Navier BCs over the years, but:

- Revival of active interest in the mathematical community working on the vanishing viscosity limit started with the paper of Clopeau, Mikelić, and Robert 1998, which gives a vanishing viscosity result in two dimensions.
- The work of J-M Coron 1995 on the controllability of the 2D Navier-Stokes equations with Navier boundary conditions, initiated interest in these boundary conditions in the PDE control theory community.

Three papers

By now there is a fairly substantial mathematical literature on the subject, but three papers are of particular concern to us here:

1. Iftimie and Planas 2006: Establishes the vanishing viscosity limit in 3D for the first time with a rate of convergence of order $\nu^{1/2}$.
2. Iftimie and Sueur 2010: Performs a Prandtl-like boundary layer analysis and establishes the optimal rate of convergence of order $\nu^{3/4}$.
3. Masmoudi and Rousset 2010 (preprint): Obtains bounds in conormal Sobolev spaces uniformly in viscosity and convergence uniform in time and space (without a bound on the rate of convergence).

Iftimie and Sueur 2010

Let:

- u^ν, $\nu > 0$, be a (strong) solution to the Navier-Stokes equations
- u^0 be the solution to the Euler equations
- fixed time of existence, $T > 0$, independent of ν

Iftimie and Sueur construct a corrector, which we will call θ, so that

$$\left\| u^\epsilon - u^0 - \theta \right\|_{L^\infty(0,T;L^2)} \leq C\nu, \quad \left\| u^\epsilon - u^0 - \theta \right\|_{L^2(0,T;H^1)} \leq C\nu^{\frac{1}{2}}.$$

Their corrector is of the form

$$\theta(t,x) = \nu^{\frac{1}{2}} v(t, x, \nu^{-\frac{1}{2}}\varphi(x)),$$

where $\varphi(x)$ is a smoothed distance from the boundary.

Prandtl-like equation for v

The boundary "profile" satisfies the equation:

$$\partial_t v - \partial_z^2 v + \frac{u^0 \cdot \mathbf{n}}{\varphi(x)} z \partial_z v + \left[u^0 \cdot \nabla v + v \cdot \nabla u^0 \right]_{tan} = 0$$

with boundary conditions (at $z = 0$)

$$\partial_z v(t, x, 0) - [\partial_z v(t, x, 0) \cdot \mathbf{n}] \, \mathbf{n} = -2 \left[\mathbf{S} u^0(t, x) \mathbf{n} + \alpha u^0(t, x) \right]_{tan},$$
$$v \cdot \mathbf{n} = 0.$$

Additional correctors are needed in analysis, but discarded at the end:

1. div v is order $\nu^{1/2}$; to reduce the impact of this, an additional, order-ν corrector, w, is added. But because of its smaller order, only the first-order corrector, v, is needed to obtain the convergence rates.
2. The normal component of $u^0 \cdot \nabla v + v \cdot \nabla u^0$ is not dealt with by v. To handle it, a pressure corrector of order ν is included in the analysis. Like w, it is not needed to obtain the convergence rates.

Our boundary layer corrector

In our approach we avoid these complications, following an approach that is more in the tradition of Temam and Wang, and which uses an explicit corrector. But we pay a price for simplicity: our corrected convergence rate in the vanishing viscosity limit is only $C\nu^{3/4}$, the same as for the *optimal* uncorrected rate.

Tuesday at 11:30 I will discuss our boundary layer corrector. But main points:

- It is divergence-free.
- It is explicit and decays exponentially away from the boundary.
- Because it is explicit, it is easy to obtain estimates on higher-order derivatives of our corrector; this is critical for obtaining uniform in time and space convergence rates.
- It is most easily expressed and interpreted in special coordinates I will talk about.
- Gung-Min will speak a bit on these same coordinates in his talk, which comes just before mine.

Masmoudi and Rousset

The philosophy in Masmoudi and Rousset 2010 (opinion is entering here) is that since in the Prandtl theory it is derivatives in the normal direction that are expected to be the largest, the appropriate space to work in is one that dampens down these derivatives.

The conormal space, H^m_{co}, is essentially the Sobolev space, H^m, with only normal derivatives weighted by a smoothed distance from the boundary.

Masmoudi and Rousset prove existence and uniqueness of solutions to the Navier-Stokes equations with Navier boundary conditions for initial velocity in H^m_{co}, $m > 6$, for a finite time, $T > 0$, depending on the initial velocity but independent of small viscosity. They produce estimates in H^m_{co} independent of small viscosity.

They also prove convergence of u^ν to u^0 uniform in time and space, though without a bound on the rate of convergence.

I am suppressing a number of technical details here.

Uniform convergence

To obtain uniform-in-time-and-space convergence of the uncorrected or corrected difference, we use the uniform-in-viscosity estimates in Masmoudi and Rousset 2010 in the conormal Sobolev space, H^m_{co}, and the following anistropic Agmons inequality:

Theorem (Gie, K)

Let Ω be a bounded domain in $\mathbb{R}^3$ with C^{m+1}-boundary, $m \geq 3$, and let Γ_a be the tubular neighborhood of fixed width $a > 0$ interior to Ω. Suppose that f and ∇f lie in the space $H^m_{co}(\Omega)$. Then

$$\|f\|_{L^\infty(\Gamma_a)} \ll_{m,a} \|f\|_{L^2(\Omega)}^{\frac{1}{2}-\frac{1}{2m}} \|f\|_{H^m_{co}(\Omega)}^{\frac{1}{2m}} \left[\|f\|_{L^2(\Omega)} + \|\nabla f\|_{H^m_{co}(\Omega)}\right]^{\frac{1}{2}},$$

$$\|f\|_{L^\infty(\Omega \setminus \Gamma_a)} \ll_{m,a} \|f\|_{L^2(\Omega)}^{1-\frac{3}{2m}} \|f\|_{H^m_{co}(\Omega)}^{\frac{3}{2m}}.$$

What we prove

Theorem (Gie, K)

Assume that $u_0 \in H \cap H^m(\Omega)$ *and* Γ *is* C^{m+2} *for* $m \geq 5$. *Then*

$$\left\| u^\nu - u^0 \right\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa \nu^{\frac{3}{4}},$$
$$\left\| u^\nu - u^0 \right\|_{L^2(0,T;H^1(\Omega))} \leq \kappa \nu^{\frac{1}{4}},$$

for a constant $\kappa = \kappa(T, \overline{\alpha}, u_0, f)$, $\overline{\alpha} = \|\mathcal{A}\|_{C^m(\Gamma)}$. *If* $m > 6$ *then*

$$\left\| u^\nu - u^0 \right\|_{L^\infty([0,T]\times\Gamma_a)} \leq \kappa \nu^{\frac{3}{8} - \frac{3}{8(m-1)}},$$
$$\left\| u^\nu - u^0 \right\|_{L^\infty([0,T]\times\Omega\setminus\Gamma_a)} \leq \kappa \nu^{\frac{3}{4} - \frac{9}{8m}},$$

where now $\kappa = \kappa(T, \overline{\alpha}, m, a, u_0, f)$ *and* Γ_a *is the interior tubular neighborhood of* Ω *with fixed width* $a > 0$.