Some recent results for two extended Navier-Stokes systems

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Extended Navier-Stokes equations

Navier-Stokes equations ($\mathcal{P}$ is the Leray projector):

$$\begin{cases} \partial_t u + \mathcal{P}(u \cdot \nabla u - f) = \nu \mathcal{P} \Delta u & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

In the *extended*, or unconstrained, Navier-Stokes equations, we drop the divergence-free condition on u, but add a term that controls the divergence:

$$\begin{cases} \partial_t u + \mathcal{P}(u \cdot \nabla u - f) = \nu (\mathcal{P} \Delta u + \nabla \operatorname{div} u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \\ u(0) = u_0 & \text{in } \Omega. \end{cases}$$

Control of divergence

Taking the divergence of

$$\partial_t u + \mathcal{P}(u \cdot \nabla u - f) = \nu(\mathcal{P}\Delta u + \nabla \operatorname{div} u)$$

gives

$$\begin{cases} \partial_t g = \nu \Delta g & \text{in } \Omega, \\ \nabla g \cdot \mathbf{n} = 0 & \text{on } \Gamma, \\ g(0) = \operatorname{div} u_0 & \text{in } \Omega, \end{cases}$$

where

$$g = \operatorname{div} u.$$

So g satisfies the heat equation with Neumann boundary conditions.

If $\operatorname{div} u_0 = 0$ then $\operatorname{div} u = 0$ for all time, and extended Navier-Stokes reduces to Navier-Stokes.

Where do these equations come from?

In an equivalent form, the extended Navier-Stokes equations go back to Grubb and Solonnikov [1989, 1991].

Liu, Liu, and Pego [2005, 2007] studied these equations, motivated by the analysis of the stability of the following numeric scheme of Johnston and Liu [2004]:

$$\langle \nabla p^n, \nabla \phi \rangle = \langle f^n - u^n \cdot \nabla u^n + \nu \Delta u^n - \nu \nabla \operatorname{div} u^n, \phi \rangle \quad \forall \phi \in H^1(\Omega)$$

then

$$\frac{u^{n+1} - u^n}{\Delta t} - \nu \Delta u^{n+1} = f^n - u^n \cdot \nabla u^n - \nabla p^n,$$
$$u|_{\partial\Omega} = 0.$$

This is a weak-form pressure-Poisson equation for p^n whose solution is then used in the elliptic boundary value problem for u^{n+1}, and the process is iterated.

Stokes and Euler pressures

Liu, Liu, Pego [2005, 2007] analyze the extended Navier-Stokes equations in an equivalent form:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p_E + \nu \nabla p_S = \nu \Delta u + f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where

$$\nabla p_E = \mathcal{P}(u \cdot \nabla u - f) - u \cdot \nabla u + f,$$
$$\nabla p_S = \Delta u - \mathcal{P} \Delta u - \nabla \operatorname{div} u = [\Delta, \mathcal{P}] u,$$

are the Stokes and Euler pressures, respectively.

Strong versus weak solutions

- For a weak solution (details later) equality holds with respect to duality pairing with appropriate test functions and

$$u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1).$$

  Regularity of the pressure, if any, comes from recovering it from u.

  Equality will also hold as distributions.

- By a strong solution we mean that

$$u \in L^2(0, T; H^2) \cap H^1(0, T; L^2), \quad \nabla p \in L^2(0, T; L^2)$$

  so that the equations make sense as classical derivatives.

Well-posedness of strong solutions

The emphasis in Liu, Liu, Pego [2005, 2007] is on the stability of their numeric scheme and related schemes. They prove local well-posedness of strong solutions, however. Key to everything is a commutator estimate on $[\Delta, \mathcal{P}]u$:

Theorem (Liu, Liu, Pego 2005, 2007)

Let Ω be a connected, bounded domain with C^3 boundary. For any $\delta > 0$ there exists $C_\delta \geq 0$ such that for all u in $H^2 \cap H^1_0$,

$$\left\|[\Delta, \mathcal{P}]u\right\|^2 = \left\|\nabla p_S\right\|^2 \leq \left(\frac{1}{2} + \delta\right) \left\|\Delta u\right\|^2 + C_\delta \|\nabla u\|^2.$$

$\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ here and throughout.

- It is important that the constant in front of $\|\Delta u\|^2$ is < 1.
- The boundary must be fairly smooth.

Lack of coercivity

Iyer, Pego, and Zarnescu [2012] continue to focus on strong solutions. They obtain small data global existence results in 3D and in 2D. For u_0 in H^1_0, in 3D the smallness condition is on the the H^1-norm of u_0 and in 2D it is on the divergence of u_0.

Defining the *extended Stokes operator*,

$$Au := -\nu(\mathcal{P}\Delta u + \nabla \operatorname{div} u),$$

so that

$$\partial_t u + \mathcal{P}(u \cdot \nabla u - f) = -Au,$$

they show that A is neither positive on $H^2 \cap H^1_0$ nor is it coercive in the sense that an inequality of the form,

$$\langle u, Au \rangle \geq \epsilon \|\nabla u\|^2 - C\|\operatorname{div} u\|^2,$$

fails to hold for any $\epsilon > 0$.

Coercivity in another space

Define Qu to be the unique H^1-function with mean zero for which

$$\nabla Q(u) = (I - P)u.$$

Iyer, Pego, and Zarnescu show that for all sufficiently small ϵ there exists a $C = C(\epsilon)$ such that A is positive and coercive in the H^1-equivalent inner product,

$$(u, v)_\epsilon = (u, v) + \epsilon(\nabla u, \nabla v) + C(Q(u), Q(v)).$$

- The coercivity *relies upon the commutator estimate* on $[\Delta, P]u$ from Liu, Liu, and Pego.
- The coercivity is the basis of their small data global existence result.
- They also show that for some initial data, the L^2-norm of the velocity initially increases without forcing, before eventually decreasing.

Global-in-time solutions

To obtain global-in-time solutions, we need to go beyond the commutator estimate and coercivity approach, and look for weak solutions. These will ultimately lead to strong solutions and to higher regularity solutions, global-in-time for 2D.

The key new idea is very simple: decompose the velocity field, u, as

$$u = v + z,$$

where v and z lie in H_0^1 with

$$v \in V := \left\{ w \in (H_0^1(\Omega))^2 \colon \operatorname{div} w = 0 \right\}.$$

To do this, we need an orthogonal decomposition,

$$(H_0^1(\Omega))^d = V \oplus V^\perp.$$

Orthogonal decomposition of H_0^1

Let $g = \operatorname{div} u$ for a given $u \in H_0^1(\Omega)$, and let z be the solution to the stationary Stokes problem,

$$\begin{cases} -\Delta z + \nabla q = 0 & \text{in } \Omega, \\ \operatorname{div} z = g & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega \end{cases}$$

so that $v = u - z$ satsifies

$$\begin{cases} -\Delta v - \nabla q = -\Delta u & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $z \in H_0^1$ (even if $\partial\Omega$ is only *Lipschitz*) and so v lies in V. And,

$$(\nabla v, \nabla z) = -(v, \Delta z) = -(v, \nabla q) = 0,$$

showing orthogonality in H_0^1.

Divergence lifting

Since $g = \operatorname{div} u$ solves the heat equation with Neumann boundary conditions, g is determined solely by the divergence at time zero. Then z is determined uniquely from the stationary Stokes problem.

We view this as a lifting from the divergence to a vector field in $(H_0^1)^d$ having that divergence.

The vector field, z, being known for all time, we rewrite the extended Navier-Stokes equations with v being the vector field to solve for:

$$\begin{aligned}\partial_t v - \nu \mathcal{P} \Delta v + \mathcal{P}(v \cdot \nabla v) \\ = \widetilde{f} - \mathcal{P}(v \cdot \nabla z) - \mathcal{P}(z \cdot \nabla v) + \nu \nabla \operatorname{div} u,\end{aligned}$$

where $\widetilde{f} = \mathcal{P} f - \mathcal{P}(z \cdot \nabla z) - \partial_t z$.

Now, classical approaches to existence and uniqueness of the usual Navier-Stokes equations can be applied.

Weak solutions

Definition

We define a *weak solution* of the extended Navier-Stokes equations to be a function u such that

$$u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1_0), \qquad \operatorname{div} u \in C(0, T; L^2) \cap L^2(0, T; H^1),$$

and

$$\partial_t(u, \varphi) + ((u \cdot \nabla)u, \varphi) = -\nu(\nabla u, \nabla\varphi) + \langle f, \varphi\rangle, \tag{1}$$

$$\partial_t(\operatorname{div} u, h) = -\nu(\nabla \operatorname{div} u, \nabla h) \tag{2}$$

for almost all $t \in (0, T)$, and all test functions $\varphi \in V$, $h \in H^1$.

It follows from (2) that (1) also holds when $\varphi = \nabla q$ for any q in H^1. This is the key to showing that $\partial_t u + \mathcal{P}(u \cdot \nabla u - f) = \nu(\mathcal{P}\Delta u + \nabla \operatorname{div} u)$ holds as distributions (and almost everywhere for strong solutions).

Existence and uniqueness of weak solutions

Theorem (Ignatova, Iyer, K,Pego, Zarnescu)

Let $d = 2$ or 3, $T > 0$ be arbitrary, and assume that

$$u_0 \in L^2(\Omega) \quad \textit{and} \quad f \in L^2(0, T; V').$$

If either

$$\operatorname{div} u_0 \in L^2(\Omega) \textit{ and } \partial\Omega \textit{ is } C^2$$

or

$$\operatorname{div} u_0 \in H^2(\Omega) \textit{ and } \partial\Omega \textit{ is Lipschitz}$$

then there exists a weak solution, u, with initial data u_0 such that

$$u \in L^\infty(0, T; L^2) \cap L^2(0, T; H_0^1) \quad \textit{and} \quad \operatorname{div} u \in C^\infty(\Omega \times (0, T)).$$

If $\partial\Omega \subseteq \mathbb{R}^2$ is Lipschitz then weak solutions are unique.

Galerkin approximation

With the decomposition, $u = v + z$, the weak formulation gives

$$\partial_t(v, \varphi) + \nu(\nabla v, \nabla\varphi) + (v \cdot \nabla v, \varphi) = \langle \widetilde{f}, \varphi \rangle - (v \cdot \nabla z, \varphi) - (z \cdot \nabla v, \varphi)$$

with, as before,

$$\widetilde{f} = \mathcal{P}f - \mathcal{P}(z \cdot \nabla z) - \partial_t z.$$

This is the Navier-Stokes equations with different forcing and with two additional terms, both linear in v.

The existence of solutions—and higher regularity—are established using a Galerkin approximation of v.

The main new technical issue is obtaining sufficient regularity of z as it is lifted from $\operatorname{div} u$.

Regularity of divergence lifting

The lowest regularity term is $\partial_t z$, which comes from lifting $\partial_t g = \partial_t \operatorname{div} u$.

$\operatorname{div} u_0 \in H^2$ *and* $\partial\Omega$ *is Lipschitz:*

$\partial_t g$ satisfies the heat equation with an initial value of $\nu \Delta \operatorname{div} u_0$, which lies in $L^2(\Omega)$. Hence $\partial_t g \in C^0(0, T; L^2)$ and classical "lifting" via solving the Stokes problem gives

$$\partial_t z \in L^\infty(0, T; H^1).$$

$\operatorname{div} u_0 \in L^2(\Omega)$ *and* $\partial\Omega$ *is* C^2*:*

A duality argument gives

$$\|\partial_t z\| \leq C \|\partial_t g\|_{\tilde{H}^{-1}},$$

where $\tilde{H}^{-1}(\Omega)$ is the dual of $H^1(\Omega)$. Just enough regularity.

Strong solutions

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 domain and suppose

$$u_0 \in H_0^1 \cap H^2, \quad \operatorname{div} u_0 \in H^2, \quad \nabla \operatorname{div} u_0 \cdot \mathbf{n} = 0$$

and

$$f \in L^2(0, T; H^{-1}), \quad \partial_t f \in L^2(0, T; H^{-1}), \qquad f(0) \in L^2.$$

If u is the (unique) weak solution with initial data u_0, then

$$\partial_t u \in L^\infty(0, T; L^2) \cap L^2(0, T; H_0^1).$$

If, further, $f \in L^\infty(0, T; L^2)$, then $u \in L^\infty(0, T; H^2)$.

The Shirokoff-Rosales system

Shirokoff and Rosales [2011] introduced a pressure-Poisson system to provide a high-order, efficient time discrete scheme for the incompressible Navier-Stokes equations in irregular domains. The continuous time version of their scheme is

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f & \text{in } \Omega, \\ u \times \mathbf{n} = 0 & \text{on } \partial\Omega, \\ \operatorname{div} u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \Delta p = -\operatorname{div}((u \cdot \nabla)u) + \operatorname{div} f & \text{in } \Omega, \\ \nabla p \cdot \mathbf{n} = (\nu \Delta u - (u \cdot \nabla)u + \lambda u + f) \cdot \mathbf{n} - \mathcal{C} & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\lambda > 0$ and $\mathcal{C} = \mathcal{C}(t)$ is chosen to satisfy the compatibility condition necessary for solving for the pressure.

The heat equation appears again

Taking the divergence of the first equation, we see that

$$\begin{cases} \partial_t \operatorname{div} u - \nu \Delta \operatorname{div} u = 0 & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus if $\operatorname{div} u_0 = 0$, then for all $t \geq 0$, we must have $\operatorname{div} u = 0$ identically in Ω, and not just on $\partial\Omega$.

Damping of $u \cdot \mathbf{n}$

Explicitly,

$$\mathcal{C} = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} (\nu \Delta u + \lambda u) \cdot \mathbf{n} = \frac{1}{|\partial\Omega|} \int_{\Omega} (\nu \Delta \operatorname{div} u + \lambda \operatorname{div} u).$$

Using the boundary condition for p and the equation for u gives

$$\partial_t (u \cdot \mathbf{n}) + \lambda u \cdot \mathbf{n} = \mathcal{C} \quad \text{on } \partial\Omega.$$

If $\operatorname{div} u_0 = 0$ then $\operatorname{div} u = \mathcal{C} = 0$ for all time. If also $u_0 \cdot \mathbf{n} = 0$ then $u \cdot \mathbf{n} = 0$ for all time, and these equations reduce to the Navier-Stokes equations.

Well-posedness

Given $u \in H^1$ satisfying $u \times \mathbf{n} = 0$ on $\partial\Omega$, we define v, z_1, and z_2 as solutions of the Stokes problems,

$$\begin{cases} -\Delta z_1 + \nabla q_1 = 0 & \text{in } \Omega \\ \operatorname{div} z_1 = 0 & \text{in } \Omega \\ z_1 = \operatorname{Proj}_{\mathbf{n}} u & \text{on } \partial\Omega \end{cases} \qquad \begin{matrix} -\Delta z_2 + \nabla q_2 = 0 & \text{in } \Omega, \\ \operatorname{div} z_2 = \operatorname{div} u & \text{in } \Omega, \\ z_2 = 0 & \text{on } \partial\Omega, \end{matrix}$$

and

$$\begin{cases} -\Delta v + \nabla q = -\Delta u & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, z_1 carries the boundary condition, z_2 carries the divergence, and v lies in V.

Bounding z_1 and z_2

- Letting $z = z_1 + z_2$, we treat z and v much as for the extended Navier-Stokes systems.

- Estimates on z_2 are like those on z for the extended Navier-Stokes equations, the estimates coming from lifting the divergence, which satiisfies the heat equation with Dirichlet boundary conditions:

$$\|z_2(t)\|_{H^1} \le C\|\operatorname{div} u\|_{L^2} \le C\|\operatorname{div} u_0\|_{L^2}.$$

- For z_1, estimates on the stationary Stokes problem gives

$$\|z_1(t)\|_{H^1(\Omega)} \le C\|u \cdot \mathbf{n}(t)\|_{H^{1/2}(\partial\Omega)}.$$

Bounding $u \cdot \mathbf{n}$ and so z_1

From

$$\partial_t(u \cdot \mathbf{n}) + \lambda u \cdot \mathbf{n} = \mathcal{C} \quad \text{on } \partial\Omega,$$

applying Duhamel's principle gives

$$u \cdot \mathbf{n}(t, x) = e^{-\lambda t} u^0 \cdot \mathbf{n}(x) + \int_0^t e^{-\lambda(t-s)} \mathcal{C}(s) \, ds.$$

Using the expression for $\mathcal{C}$ and that $\operatorname{div} u$ satisfies the heat equation,

$$\begin{aligned}
\int_0^t e^{-\lambda(t-s)} \mathcal{C}(s) \, ds &= \frac{1}{|\partial\Omega|} \int_0^t e^{-\lambda(t-s)} \int_\Omega \big(\nu \Delta \operatorname{div} u + \lambda \operatorname{div} u\big)(s, x) \, dx \, ds \\
&= \frac{1}{|\partial\Omega|} \int_0^t e^{-\lambda(t-s)} \int_\Omega \big(\partial_s \operatorname{div} u + \lambda \operatorname{div} u\big)(s, x) \, dx \, ds.
\end{aligned}$$

Bounding $u \cdot \mathbf{n}$ and so z_1

But, $e^{\lambda s}(\partial_s \operatorname{div} u + \lambda \operatorname{div} u)(s, x) = \partial_s(e^{\lambda s} \operatorname{div} u(s, x))$, so integrating in time yields,

$$\int_0^t e^{-\lambda(t-s)} \mathcal{C}(s)\, ds = \frac{1}{|\partial\Omega|} \left[\operatorname{div} u(t) - e^{-\lambda t} \operatorname{div} u^0 \right].$$

Thus, bottom line,

$$\|z_1(t)\|_{H^1} \le C \|u \cdot \mathbf{n}\|_{H^{1/2}(\partial\Omega)} \le C \left(e^{-\lambda t} \|u_0\|_{H^{1/2}(\partial\Omega)} + \|\operatorname{div} u_0\| \right),$$

since

$$\|\operatorname{div} u(s)\| \le C \|\operatorname{div} u(s)\| \le C \|\operatorname{div} u^0\|$$

by the regularity of solutions to the heat equations with Dirichlet boundary conditions.

Well-posedness

The bounds on z_1 and z_2 allow us to treat $z = z_1 + z_2$ essentially the same as for the extended Navier-Stokes equations, and we get a similar well-posedness result.

Theorem (Us)

Let $d = 2, 3$, and $\Omega \subset \mathbb{R}^d$ be a C^2 domain. Let

$$u_0 \in L^2(\Omega), \quad \operatorname{div} u_0 \in L^2(\Omega), \quad u_0 \cdot n \in H^{1/2}(\partial\Omega), \quad f \in L^2(0, T; V').$$

There exists a weak solution, u, with initial data u_0 such that

$$u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1) \quad \text{and} \quad \operatorname{div} u \in C^\infty(\Omega \times (0, T)),$$

and this solution is unique if $d = 2$. Moreover, if u_0 lies in $H^1_0(\Omega)$ with $\operatorname{div} u_0 \in H^1(\Omega)$ *and $f \in L^2((0, T) \times \Omega)$ then*

$$u \in L^\infty(0, T; H^1_0) \cap L^2(0, T; H^2).$$

Concluding remarks

- Using “divergence lifting” rather than the commutator estimate allowed us to easily obtain existence and uniqueness results closely paralleling those of the classical Navier-Stokes equations.
- Divergence lifting also allowed us to say something about Lipschitz domains.
- So far, our efforts to use divergence lifting to reproduce the stability results of Liu, Liu, and Pego more simply, or to extend them to Lipschitz domains have not been successful.
- Stability for the Shirokoff-Rosales system also poses difficulties using either method.