SERFATI SOLUTIONS TO THE 2D EULER EQUATIONS ON EXTERIOR DOMAINS

DAVID M. AMBROSE, JAMES P. KELLIHER, MILTON C. LOPES FILHO, AND HELENA J. NUSSENZVEIG LOPES

Abstract. We prove existence and uniqueness of a weak solution to the incompressible 2D Euler equations in the exterior of a bounded smooth obstacle when the initial data is a bounded divergence-free velocity field having bounded scalar curl. This work completes and extends the ideas outlined by P. Serfati for the same problem in the whole-plane case. With non-decaying vorticity, the Biot-Savart integral does not converge, and thus velocity cannot be reconstructed from vorticity in a straightforward way. The key to circumventing this difficulty is the use of the Serfati identity, which is based on the Biot-Savart integral, but holds in more general settings.

1. Introduction

The incompressible Euler equations describe the velocity field, u, and pressure, p, of a constant-density, inviscid fluid. The equations (without forcing) can be written,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \\ u(0) = u^0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here, Ω is a domain with boundary (empty, if $\Omega = \mathbb{R}^2$) and $\boldsymbol{n}$ is the outward unit normal to the boundary. The initial velocity, u^0, and the solution, (u, p), are assumed to lie in appropriate function spaces. If Ω is unbounded, some condition at infinity must be imposed.

In two dimensions, the classical well-posedness result for finite-energy *weak* solutions with bounded initial vorticity (the scalar curl of the velocity) is that established by Yudovich in [36] (and extended by him in [37] to allow slightly unbounded vorticities). Yudovich's results are for a bounded domain, but his ideas were adapted to the full plane case, see [24]. Vishik, in [35], working in the full plane, established a slightly larger uniqueness class of unbounded vorticities. Each of these full-plane results, however, requires that the initial vorticity decay at infinity. This assumption is not natural from the physical point of view, as full plane flow is an approximate model for flow far from boundaries, where no decay of distant vorticity should be expected.

In 1995, Ph. Serfati stated and outlined proofs of existence and uniqueness of solutions for the incompressible 2D Euler equations in the full plane with each of the initial velocity and initial vorticity bounded [29]. We call such velocity fields, *Serfati velocity fields*. Once no decay of vorticity is assumed, uniform boundedness of vorticity

Date: (compiled on September 25, 2014).
2010 *Mathematics Subject Classification.* Primary 35Q31, 76B03.
Key words and phrases. Fluid mechanics, Euler equations.

no longer implies boundedness of velocity, so it makes sense to add this condition as an hypothesis.

Our purpose in the present paper is to extend Serfati's result to exterior domain flows. We also include a complete proof of Serfati's original result, following his ideas, as this is not available in the literature and because it will help to better organize the presentation of our own result.

Until Serfati's 1995 paper, all existence results in an unbounded domain, including the full plane, made key use of the Biot-Savart law to recover the velocity from the vorticity and, hence, to obtain strong a priori estimates for velocity from estimates for vorticity. This law can be expressed in the form,

$$K_\Omega[\omega] := \int_\Omega K_\Omega(\cdot, y)\omega(y)\, dy. \tag{1.2}$$

Here, K_Ω is the Biot-Savart kernel,

$$K_\Omega(x, y) = \nabla_x^\perp G_\Omega(x, y), \tag{1.3}$$

where G_Ω is the Green's function for the Dirichlet Laplacian on Ω. Above, $\nabla_x^\perp = (-\partial_{x_2}, \partial_{x_1})$. For the full plane, $K_\Omega(x, y) = K(x - y)$, where

$$K(x) := \frac{x^\perp}{2\pi\,|x|^2}. \tag{1.4}$$

When Ω is the domain exterior to a single, connected, bounded, domain then, given a scalar field, ω, $u = K_\Omega[\omega]$ is the unique divergence-free vector field on Ω, decaying at infinity, with $u \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, whose scalar curl (vorticity) is ω, and whose circulation about the boundary is $-\displaystyle\int_\Omega \omega$. (See Section 5.1 for more details.)

Convergence of the Biot-Savart integral requires, however, membership of ω in an appropriate space; for instance, $\omega \in L^1 \cap L^\infty$ would be sufficient. For ω only in L^∞, the Biot-Savart integral fails to converge. This is the heart of the difficulty in working with Serfati solutions.

Serfati's key insight, which we adopt, is to use, in place of the Biot-Savart law, the identity,

$$\begin{aligned} u^j(t, x) - (u^0)^j(x) &= \int_\Omega a(x - y) K_\Omega^j(x, y)(\omega(t, y) - \omega^0(y))\, dy \\ &\quad - \int_0^t \int_\Omega \nabla_y \nabla_y^\perp \left[(1 - a(x - y))K_\Omega^j(x, y)\right] \cdot (u \otimes u)(s, y)\, dy\, ds, \end{aligned} \tag{1.5}$$

$j = 1, 2$, for all (t, x) in $[0, T] \times \Omega$. Here, a is any radially symmetric, smooth, compactly supported cutoff function with $a = 1$ in a neighborhood of the origin. We call (1.5) the *Serfati identity*. (Actually, Serfati never derives or even states this identity, but rather states inequalities that follow from it.) Using the Serfati identity it is possible to deduce L^∞ estimates for velocity in terms of L^∞ bounds for *initial* velocity and *initial* vorticity, see Section 4.2 and Section 5.2.

This paper is organized as follows: We state our results in Section 2. In Section 3 we state the estimates on the Biot-Savart kernel that we will need in the proofs of existence and uniqueness, giving their proofs, which are quite lengthy and of a different flavor from the rest of this paper, in Appendix A. We give the proof of existence separately for the full plane in Section 4 and for an exterior domain in Section 5. In each of these sections, we start by deriving the Serfati identity for the given type of domain then give the existence proofs. We prove uniqueness in Section 6.1, extending the argument to give a type of continuous dependence on initial data in Section 6.2. Examples of Serfati velocities are given in Section 7.

In Appendix B, we show how to prepare a sequence of initial velocities that are smooth with compactly supported vorticity and that converge in an appropriate sense to a given bounded initial velocity having bounded vorticity. (This approximate sequence is employed in Sections 4.2 and 5.2 to obtain existence of solutions.)

2. Statement of results

The purpose of this section is to give precise statements of the main results in this work: existence, uniqueness, and a mild form of continuous dependence of solutions on initial data. We will treat two very different fluid domains—the full plane and domains exterior to a single obstacle. To be more precise, we will denote the fluid domain by Ω, be it all of $\mathbb{R}^2$ or the exterior of a single connected and simply connected bounded domain with a C^∞ boundary. In the latter case let $\boldsymbol{n}$ denote the unit exterior normal to Ω at the finite boundary Γ. (For notational convenience we set $\Gamma = \emptyset$ when considering full plane flow.) We let $\boldsymbol{\tau}$ denote the unit tangent vector, oriented so that

$$\boldsymbol{\tau} = -\boldsymbol{n}^\perp := -(-n_2, n_1) = (n_2, -n_1).$$

We begin with basic definitions concerning the type of velocity field we are interested in and the notion of weak solution of the Euler equations we will consider.

If u is a vector field on Ω, we write

$$\omega(u) := \operatorname{curl} u = \partial_1 u^2 - \partial_2 u^1$$

for the scalar curl (vorticity) of u. We write ω for $\omega(u)$ when u is understood.

Taking the scalar curl of the two dimensional incompressible Euler equations (1.1), we obtain the vorticity equation, or the vorticity formulation of the Euler equations:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ \operatorname{curl} u = 0 & \text{in } \Omega, \\ u \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \\ \omega(0) = \omega_0 = \operatorname{curl} u^0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

Definition 2.1. We say that a divergence-free vector field $u \in L^\infty(\Omega)$ with $u \cdot \boldsymbol{n} = 0$ on Γ and $\omega(u) \in L^\infty(\Omega)$ is a *Serfati velocity*. We denote by $S = S(\Omega)$ the Banach space of all Serfati velocity fields with the norm,

$$\|u\|_S = \|u\|_{L^\infty} + \|\omega(u)\|_{L^\infty}.$$

Remark 2.2. Since $u \in L^2_{loc}$ and u is divergence-free, the trace of its normal component, $u \cdot \boldsymbol{n}$, is well-defined and belongs to $H^{-1/2}(\Gamma)$ (see, for instance, Theorem I.1.2 of [34]).

Definition 2.3. Fix $T > 0$. Assume that $u \in L^\infty(0, T; S) \cap C([0, T] \times \Omega)$ and let $\omega = \omega(u)$. We say that u is a *Serfati solution* to the Euler equations without forcing and with initial velocity $u^0 = u|_{t=0}$ in S if the following conditions hold:

(1) The vorticity equation $\partial_t \omega + u \cdot \nabla \omega = 0$ (see $(2.1)_1$) holds in the sense of distributions.

(2) For any radially symmetric, smooth, compactly supported cutoff function a with $a = 1$ in a neighborhood of the origin the Serfati identity in (1.5) holds.

(3) If Ω is the exterior of a single obstacle then the circulation of velocity around the boundary is conserved in time.

Remark 2.4. Since any Serfati solution u is in $L^\infty(0,T;S) \cap C([0,T] \times \Omega)$, it follows that u is log-Lipschitz in space, uniformly over $(0,T)$; see Lemma B.3. Therefore, there exists a unique, continuous, measure-preserving flow map, $X\colon [0,T] \times \Omega \to \Omega$, for u; that is,

$$\begin{aligned} \partial_t X(t,x) &= u(t, X(t,x)), \quad && t \in (0,T), x \in \Omega \\ X(0,x) &= x, && x \in \Omega. \end{aligned}$$

In addition, the vorticity $\omega = \omega(u)$ is transported by this flow map, meaning that for all $t \in [0,T]$, $x \in \Omega$,

$$\omega(t, X(t,x)) = \omega^0(x).$$

Our main results are Theorems 2.5 and 2.8, in which we establish the existence, uniqueness, and a limited form of continuous dependence on initial data for Serfati solutions. We begin with the statement of existence and uniqueness.

Theorem 2.5. *Let $T > 0$. Assume that $u^0 \in S$. Then there exists a unique, Serfati solution u to the Euler equations as in Definition 2.3. Moreover, the flow map $X(t,\cdot) \in C^{\beta(t)}$, where $\beta(t) = e^{-\alpha|t|}$ and $\alpha = C\,\|u\|_{L^\infty(0,T;S)}$.*

Remark 2.6. It is shown in [19] that the solutions constructed in Theorem 2.5 are also distributional solutions of the velocity formulation of the Euler equations, (1.1). Moreover, there exists an associated pressure whose asymptotic behavior is $O(\log|x|)$ for large $|x|$ and whose gradient is bounded.

Remark 2.7. The Hölder regularity of the flow map in Theorem 2.5 is optimal, as shown by an explicit (compactly supported) example in [2].

The following is a statement that Serfati solutions depend continuously, in the L^∞-norm, on the (Serfati) initial data. We will need additional notation to state the result.

For any $p \in [1,\infty]$, $L^p_{uloc}(\Omega)$ is the uniformly local L^p space; that is, the space of all measurable functions whose norm,

$$\|f\|_{L^p_{uloc}(\Omega)} := \sup_{U \subset \Omega, |U| \le C_0} \|f\|_{L^p(U)},$$

is finite, where C_0 is an arbitrary fixed positive constant and $|U|$ is the Lebesgue measure of U. For any $p \in [2,\infty]$, let

$$S^p = \left\{ u \in (L^\infty(\Omega))^2 \colon \operatorname{div} u = 0,\, \omega(u) \in L^p_{uloc}(\Omega),\, u \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \right\}. \tag{2.2}$$

Then S^p is a Banach space under the norm $\|u\|_{S^p} = \|u\|_{L^\infty} + \|\omega(u)\|_{L^p_{uloc}}$. (We require $p \ge 2$ so that $u \cdot \boldsymbol{n}$ is well-defined, as in Remark 2.2.)

Theorem 2.8. *Let u_1, u_2 be Serfati solutions to the Euler equations for a fixed $T > 0$ and let $p \in (2,\infty]$. Let u_1^0, u_2^0 be the initial velocities with $u_1^0 - u_2^0 \in S^p$. For all sufficiently small $s_0 = \|u_1^0 - u_2^0\|_{S^p}$ there exist $C > 0$ such that*

$$\|u_1(t) - u_2(t)\|_{L^\infty} \le Ce^{Ct} s_0 - C(1+t)e^{Ct}(Cs_0 t)^{e^{-Ct(1+t)}} \log(Cs_0 t) \tag{2.3}$$

for all t in $[0, T]$, where C depends on T and p.

Remark 2.9. The last term in (2.3) goes to zero as $s_0 \to 0^+$ since $\lim_{r\to 0^+} r^\alpha \log r = 0$ for any $\alpha > 0$.

Remark 2.10. Stability in C^r for all $r < 1$ follows by interpolation from Theorem 2.8, though stability in S does not. In fact, we should not expect continuous dependence in the L^∞ norm of the vorticity, and hence not in S. For instance, a small initial perturbation of a vortex patch will displace the contour and result in a large discrepancy between perturbed and unperturbed solutions, relative to the L^∞-norm of vorticity, for any positive time.

3. Estimates for the Biot-Savart kernel

In Propositions 3.1 through 3.3, we state the estimates on the Biot-Savart kernel and its derivatives that will be needed in the proof of existence and uniqueness in Sections 4 through 6. We state these estimates in a manner that unifies, to the extent possible, the two cases of the full plane and an exterior domain. Their proofs, which are quite lengthy, are deferred to Appendix A.

Let Ω be the domain exterior to a bounded simply connected domain having C^∞ boundary. We recall the definitions of K_Ω in (1.3) and K in (1.4), and define the *hydrodynamic Biot-Savart kernel*,

$$J_\Omega(x, y) = K_\Omega(x, y) + \overline{K}_\Omega(x), \tag{3.1}$$

where $\overline{K}_\Omega$ is the unique divergence-free vector field tangential to $\partial\Omega$ having circulation one and decaying at infinity. (An explicit form for $\overline{K}_\Omega$ is given in (A.17).) The hydrodynamic Biot-Savart hernel was first introduced by C. C. Lin in [21], as the perpendicular gradient of the *hydrodynamic Green's function*. For more details on the hydrodynamic Biot-Savart law, see [22].

In the statement of the propositions that follow, Ω can be either the full plane or the domain exterior to a bounded simply connected domain having C^∞ boundary.

Proposition 3.1. *Let a be a cutoff function as in (2) of Definition 2.3, smooth, radially symmetric, and equal to 1 in a neighborhood of the origin. For $\lambda > 0$ set $a_\lambda(\cdot) = a(\cdot/\lambda)$.*

There exists $C > 0$ such that, for all $x \in \Omega$ and all $\lambda > 0$ we have the full-plane estimates,

$$\|a_\lambda(x - y)K(x - y)\|_{L^1_y(\mathbb{R}^2)} \le C\lambda, \tag{3.2}$$

$$\|\nabla_y\nabla_y((1 - a_\lambda(x - y))K(x - y))\|_{L^1_y(\mathbb{R}^2)} \le C\lambda^{-1}. \tag{3.3}$$

Moreover, there exists $C_0 > 0$ such that for all $\lambda > C_0$

$$\|a_\lambda(x - y)J_\Omega(x, y)\|_{L^1_y(\Omega)} \le C\lambda, \tag{3.4}$$

$$\|a_\lambda(x - y)K_\Omega(x, y)\|_{L^1_y(\Omega)} \le C(\lambda + \lambda^2). \tag{3.5}$$

and for all $\lambda > C_0$,

$$\|\nabla_y\nabla_y((1 - a_\lambda(x - y))J_\Omega(x, y))\|_{L^1_y(\Omega)} \le C\lambda^{-1}, \tag{3.6}$$

$$\|\nabla_y a_\lambda(x - y) \otimes \nabla_y J_\Omega(x, y)\|_{L^1_y(\Omega)} \le C\lambda^{-1}, \tag{3.7}$$

$$\|\nabla_y\nabla_y((1 - a_\lambda(x - y))K_\Omega(x, y))\|_{L^1_y(\Omega)} \le C. \tag{3.8}$$

Proposition 3.2. *Let $U \subseteq \Omega$ have measure $2\pi R^2$ for some $R < \infty$. Then for any p in $[1, 2)$,*

$$\begin{aligned} \|K(x-\cdot)\|^p_{L^p(U)} &\le \frac{R^{2-p}}{2-p}, \\ \|K_\Omega(x,y)\|^p_{L^p_y(U)} &\le C\frac{R^{2-p}}{2-p} + CR^2, \\ \|J_\Omega(x,y)\|^p_{L^p_y(U)} &\le C\frac{R^{2-p}}{2-p}. \end{aligned} \tag{3.9}$$

Proposition 3.3. *Let X_1 and X_2 be measure-preserving homeomorphisms of Ω. Let $\delta = \|X_1 - X_2\|_{L^\infty}$ and suppose $\delta < e^{-1}$. Then, for any measurable subset $U \subset \Omega$, with finite measure, there exists $C > 0$, depending only on Ω and the measure of U, such that*

$$\begin{aligned} \|K(x - X_1(z)) - K(x - X_2(z))\|_{L^1_z(U)} &\le -C\delta\log\delta, \\ \|K_\Omega(x, X_1(z)) - K_\Omega(x, X_2(z))\|_{L^1_z(U)} &\le -C\delta\log\delta. \end{aligned} \tag{3.10}$$

4. Existence in the full plane

Existence of weak solutions for the incompressible 2D Euler equations has been established under many different kinds of regularity assumptions. The proofs follow a standard strategy consisting in first generating a sequence of approximations, then establishing enough a priori estimates to show that the sequence is compact in an appropriate function space and, finally, passing to the limit in the weak form of the Euler equations. To obtain compactness, a priori estimates are needed for both velocity and vorticity. Whenever the function space is based on a rearrangement invariant space, the vorticity estimates are immediate, as future vorticity is simply a rearrangement of its initial values. One then establishes velocity estimates by integrating vorticity estimates using the Biot-Savart law, which relates vorticity to velocity through a Biot-Savart kernel (see [23] for details). For an unbounded fluid domain, this kernel has very mild decay at infinity. Hence, in order to ensure that the Biot-Savart law is well-defined it is necessary to impose decay of vorticity at infinity. It turns out that this is the only reason to impose decay of vorticity at infinity.

In the proof that we give in Section 4.2, we include only those aspects of the existence argument that are not standard. The approach taken in Section 8.2 of [25], see also Sections 5.1 and 5.2 of [6], can be used to fill in the rest of the argument. First, however, we derive the Serfati identity (1.5) in Section 4.1 for the full plane.

4.1. **The Serfati identity in the full plane.** It will be convenient to introduce the notation,

$$\begin{aligned} &v \ast\cdot\, w = v^i \ast w^i && \text{if } v \text{ and } w \text{ are vector fields,} \\ &A \ast\cdot\, B = A^{ij} \ast B^{ij} && \text{if } A,\ B \text{ are matrix-valued functions on } \mathbb{R}^2, \end{aligned}$$

where $\ast$ denotes convolution. We have adopted the convention that repeated indices are implicitly summed.

Let f be a scalar field and v a vector field. Then, using the notation introduced above, we have

$$f \ast \operatorname{curl} v = f \ast (\partial_1 v^2 - \partial_2 v^1) = \partial_1 f \ast v^2 - \partial_2 f \ast v^1 = \nabla^\perp f \ast\cdot\, v \tag{4.1}$$

and

$$\begin{aligned}\nabla^\perp f *\cdot \operatorname{div}(v\otimes v) &= -\partial_2 f * \partial_j(v^1 v^j) + \partial_j\partial_1 f * (v^2 v^j)\\ &= -\partial_j\partial_2 f * (v^1 v^j) + \partial_j\partial_1 f * (v^2 v^j)\\ &= \nabla\nabla^\perp f *\cdot (v\otimes v).\end{aligned}\tag{4.2}$$

Proposition 4.1. *Let u be a C^∞ classical solution to the Euler equations with initial vorticity, ω^0, compactly supported. Then, for any radially symmetric function $a \in C_c^\infty(\mathbb{R}^2)$ such that $a = 1$ in a neighborhood of the origin, the following identity holds true:*

$$\begin{aligned}u^j(t) - (u^0)^j &= (aK^j) * (\omega(t) - \omega^0)\\ &\quad - \int_0^t \left(\nabla\nabla^\perp\left[(1-a)K^j\right]\right) *\cdot (u\otimes u)(s)\,ds, \quad j = 1, 2.\end{aligned}\tag{4.3}$$

Remark 4.2. It is easy to check that (4.3) corresponds exactly to the Serfati identity (1.5) when $\Omega = \mathbb{R}^2$. (Recall that K is defined in (1.4).)

Proof. For classical solutions, the vorticity is transported by the flow, so since it is initially compactly supported it remains so for all time. This fact and the smoothness of the solution justify the calculations that follow.

For $j = 1, 2$, we have,

$$\partial_t u^j = \partial_t(K^j * \omega) = \partial_t(aK^j * \omega) + \partial_t((1-a)K^j * \omega).$$

We integrate in time to get

$$\begin{aligned}u(t,x) &= u^0(x) + \int_0^t \partial_s(K^j * \omega)(s,x)\,ds\\ &= u^0(x) + \int_0^t \partial_s\left[(aK^j) * \omega(s,x)\right]\,ds + \int_0^t ((1-a)K^j) * \partial_s\omega(s,x)\,ds\\ &= u^0(x) + (aK^j) * (\omega(t) - \omega^0)(x) + \int_0^t ((1-a)K^j) * \partial_s\omega(s,x)\,ds.\end{aligned}\tag{4.4}$$

We now treat the final integrand. We have:

$$\begin{aligned}((1-a)K^j) * \partial_s\omega &= -((1-a)K^j) * (u\cdot\nabla\omega) = -((1-a)K^j) * \operatorname{curl}(u\cdot\nabla u)\\ &= -\nabla^\perp((1-a)K^j) *\cdot (u\cdot\nabla u) = -\nabla^\perp((1-a)K^j) *\cdot (\operatorname{div} u\otimes u)\\ &= -\nabla\nabla^\perp((1-a)K^j) *\cdot (u\otimes u),\end{aligned}$$

where we used the vorticity equation $\partial_s\omega + u\cdot\nabla\omega = 0$, the identity $u\cdot\nabla\omega = \operatorname{curl}(u\cdot\nabla u)$, and (4.1, 4.2). Substituting this back into (4.4) yields (4.3). □

4.2. **Proof of existence in the full plane.** As we mentioned earlier, the proof of existence is mostly standard. We outline the steps, broadly following the approach used in Section 8.2 of [25], providing details only for the two nonstandard steps in the proof. The first nonstandard step is the estimation of the L^∞ norm of the velocity: this is where we employ the Serfati identity (as expressed in (4.3)) as described in Section 1. The second nonstandard step proving that the approximate sequence of velocities has a convergent subsequence. Whereas, in [25], equicontinuity in time is obtained, employing potential theory estimates that require the vorticity to decay at infinity, we instead use the Serfati identity once more to show that sequence of velocities is Cauchy.

Proof of existence in Theorem 2.5 for the full plane. Let $u^0 \in S$ and assume that u^0 does not vanish identically; otherwise, there is nothing to prove.

Step 1. Construct approximating sequence. We construct the sequence of approximations by generating a smooth sequence of vector fields which approximate the initial data and, afterwards, by exactly solving the Euler equations with the smooth data.

Let $(u_n^0)_{n=1}^\infty$ and $(\omega_n^0)_{n=1}^\infty$ be the approximating sequences to the initial velocity, u^0, and initial vorticity, ω^0, given by Proposition B.2. By hypothesis, u^0 is not identically zero, which means that u_n^0 does not vanish identically either. Let u_n be the classical, smooth solution to the Euler equations with initial velocity u_n^0, and with initial vorticity, ω_n^0. The existence and uniqueness of such solutions follows, for instance, from [26] and references therein. (See also Chapter 4 of [25] or Chapter 4 of [6].) Finally, let $\omega_n = \operatorname{curl} u_n$.

Step 2. Bound velocities in $L^\infty([0,T]\times\mathbb{R}^2)$. We begin with the a priori estimate,

$$\|\omega_n\|_{L^\infty(\mathbb{R}\times\mathbb{R}^2)} \le \|\omega_n^0\|_{L^\infty}, \tag{4.5}$$

on the vorticity, which can be deduced from the fact that the smooth vorticity is transported by a smooth, divergence-free vector field. We also have, by construction,

$$\|u_n^0\|_{L^\infty} \le C\|u^0\|_{L^\infty}, \qquad \|\omega_n^0\|_{L^\infty} \le C\|\omega^0\|_{L^\infty}. \tag{4.6}$$

Next, we will use the Serfati identity (4.3) for $\mathbb{R}^2$, with u_n, ω_n in place of u, ω.

Let a be any smooth, compactly supported cutoff function that is equal to 1 in a neighborhood of 0 (see item (2) of Definition 2.3). Fix $\lambda > 0$, to be specified later. Set

$$a_\lambda = a_\lambda(x) = a\left(\frac{x}{\lambda}\right). \tag{4.7}$$

From (4.3), for u_n, ω_n, it follows that

$$\begin{aligned}|u_n(t,x)| &\le \left|u_n^0(x)\right| + \left|(a_\lambda K^j) * (\omega_n(t) - \omega_n^0)\right| \\ &\quad + \int_0^t \left|\left(\nabla\nabla^\perp\left[(1-a_\lambda)K^j\right]\right) *\cdot (u_n\otimes u_n)(s)\right|\, ds.\end{aligned}$$

Applying Young's convolution inequality, followed by the localized estimates on the Biot-Savart kernel in $\mathbb{R}^2$ contained in Proposition 3.1, we conclude that

$$\begin{aligned}\|u_n(t)\|_{L^\infty} &\le \|u^0\|_{L^\infty} + (\|\omega_n(t)\|_{L^\infty} + \|\omega^0\|_{L^\infty})\|a_\lambda K\|_{L^1} \\ &\quad + \int_0^t \|\nabla\nabla^\perp[(1-a_\lambda)K]\|_{L^1}\, \|u_n(s)\|_{L^\infty}^2\, ds \\ &\le C\|u^0\|_{L^\infty} + C\lambda\|\omega^0\|_{L^\infty} + \frac{C}{\lambda}\int_0^t \|u_n(s)\|_{L^\infty}^2\, ds,\end{aligned} \tag{4.8}$$

where we also used (4.5, 4.6, 3.2, 3.3) in the last inequality.

Observe that we can choose $\lambda > 0$ arbitrarily, even allowing it to depend on t, for each fixed t. Let

$$\lambda = \lambda(t) = \left(\int_0^t \|u_n(s)\|_{L_\infty}^2\, ds\right)^{1/2}.$$

We obtain

$$\|u_n(t)\|_{L^\infty} \le C + C\left(\int_0^t \|u_n(s)\|_{L^\infty}^2 \, ds\right)^{1/2},$$

so that

$$\|u_n(t)\|_{L^\infty}^2 \le C + C\int_0^t \|u_n(s)\|_{L^\infty}^2 \, ds.$$

We conclude from Gronwall's lemma that

$$\|u_n(t)\|_{L^\infty} \le Ce^{Ct}. \tag{4.9}$$

Therefore, u_n lies in $L^\infty([0,T]\times\mathbb{R}^2)$ for any $T>0$, with a bound that is uniform in n. This, together with (4.5, 4.6), yields

$$\|u_n(t)\|_S \le C \tag{4.10}$$

for some $C = C(T, u^0) > 0$ and for all $0 \le t \le T$.

Step 3. Log-Lipschitz bound on modulus of continuity of (u_n) uniform in n. Recall the definition of the space of log-Lipschitz functions LL on a domain $U \subseteq \mathbb{R}^2$:

$$LL(U) = \left\{ f \in L^\infty(U) \;\middle|\; \sup_{x\neq y} \frac{|f(x)-f(y)|}{(1+\log^+|x-y|)|x-y|} < \infty \right\}, \tag{4.11}$$

where $\log^+(z) = \max\{-\log z, 0\}$. This is a Banach space under the norm given by

$$\|f\|_{LL} := \|f\|_{L^\infty} + \sup_{x\neq y} \frac{|f(x)-f(y)|}{(1+\log^+|x-y|)|x-y|}.$$

We have,

$$\|u_n(t)\|_{LL} \le C\|u^0\|_S.$$

This follows immediately from Lemma B.3 together with the a priori estimate (4.10) on $\|u_n\|_S$.

Step 4. Convergence of flow maps. Associated to each (smooth) u_n there is a unique (smooth) forward flow map, X_n. Much as in Lemma 8.2 of [25] or Chapter 5 of [6], we conclude that

$$|X_n(t,x_1) - X_n(t,x_2)| \le C\,|x_1 - x_2|^{e^{-\|u_n\|_{LL}|T|}},$$
$$\left|X_n^{-1}(t,y_1) - X_n^{-1}(t,y_2)\right| \le C\,|y_1 - y_2|^{e^{-\|u_n\|_{LL}|T|}}$$

and that

$$|X_n(t_1,x) - X_n(t_2,x)| \le \|u_n\|_{L^\infty([0,T]\times\mathbb{R}^2)}\,|t_1 - t_2| \le C\,|t_1 - t_2|\,,$$
$$\left|X_n^{-1}(t_1,y) - X_n^{-1}(t_2,y)\right| \le \|u_n\|_{L^\infty([0,T]\times\mathbb{R}^2)}\,|t_1 - t_2|^{e^{-\|u_n\|_{LL}|T|}} \le C\,|t_1 - t_2|^{e^{-\|u_n\|_{LL}|T|}}.$$

These estimates yield a subsequence that converges uniformly on any compact subset L of $[0,T]\times\mathbb{R}^2$. We relabel this subsequence, (X_n). Clearly, the limit flow map X also satisfies the Hölder estimates above.

Step 5. Convergence of vorticities: Define, a.e. $t\in[0,T]$, $\omega(t,x) := \omega^0(X^{-1}(t,x))$. Then $\omega_n \to \omega$ in $L^\infty(0,T;L^p_{loc}(\mathbb{R}^2))$ for all $p\in[1,\infty)$ follows from a simple adaptation of the proof for bounded vorticity on page 316 of [25], that $\omega_n(t) \to \omega(t)$ in $L^1(\mathbb{R}^2)$.

Step 6. Velocities are Cauchy in $C([0,T]\times L)$: We have established convergence of the flow maps (and its inverse maps) to a limiting flow map (and its inverse) and convergence of the vorticities to a limiting vorticity, which is transported by the limiting flow map. As shown in Step 3, we also have equicontinuity of (u_n) in space. We will now use the Serfati identity once more to show that the sequence, (u_n), is Cauchy in $C([0,T]\times L)$, for any compact subset, L, of $\mathbb{R}^2$.

Let x belong to L and let $L_\lambda = L + B_{c\lambda}(0)$, where a is supported in $B_c(0)$. From (4.3), for any *fixed* $\lambda > 0$,

$$|u_n(t,x) - u_m(t,x)| \le \left|u_n^0(x) - u_m^0(x)\right| + I_1 + I_2 + I_3, \tag{4.12}$$

where

$$\begin{aligned} I_1 &= \left|(a_\lambda K^j) * (\omega_n(t) - \omega_m(t))\right|, \ I_2 = \left|(a_\lambda K^j) * (\omega_n^0 - \omega_m^0)\right|, \\ I_3 &= \int_0^t \left|\left(\nabla\nabla^\perp\left[(1-a_\lambda)K^j\right]\right) *\cdot (u_n\otimes u_n - u_m\otimes u_m)(s)\right| ds. \end{aligned}$$

Fix q in $(2,\infty)$ and let p in $(1,2)$ be the Hölder exponent conjugate to q. Then from Proposition 3.2 and Young's convolution inequality,

$$\begin{aligned} I_1 &\le C\|a_\lambda(x-\cdot)K(x-\cdot)\|_{L^p(L_\lambda)}\|\omega_n(t)-\omega_m(t)\|_{L^q(L_\lambda)} \\ &\le \frac{C\lambda^{2-p}}{2-p}\|\omega_n(t)-\omega_m(t)\|_{L^q(L_\lambda)} \end{aligned}$$

and, similarly,

$$I_2 \le \frac{C\lambda^{2-p}}{2-p}\|\omega_n^0 - \omega_m^0\|_{L^q(L_\lambda)},$$

while

$$\begin{aligned} I_3 &\le \int_0^t \|\nabla\nabla((1-a_\lambda(x-\cdot))K(x,\cdot))\|_{L^1(\mathbb{R}^2)} \\ &\qquad\qquad \|(u_m\otimes u_m - u_n\otimes u_n)(s,\cdot)\|_{L^\infty(\mathbb{R}^2)}\, ds \\ &\le C\lambda^{-1}\int_0^t \|(u_m-u_n)(s,\cdot)\|_{L^\infty(\mathbb{R}^2)}\, ds \le Ct\lambda^{-1}. \end{aligned}$$

Here, we used (3.3) and the identity, $u_m\otimes u_m - u_n\otimes u_n = u_m\otimes(u_m-u_n)+u_n\otimes(u_m-u_n)$, with the uniform bound on the sequence, (u_k), in $L^\infty([0,T]\times\mathbb{R}^2)$.

Thus,

$$\begin{aligned} |u_n(t,x)-u_m(t,x)| &\le \left|u_n^0(x)-u_m^0(x)\right| + Ct\lambda^{-1} \\ &\quad + \frac{C\lambda^{2-p}}{2-p}\left[\|\omega_n(t,\cdot)-\omega_m(t,\cdot)\|_{L^q(L_\lambda)} + \left\|\omega_n^0-\omega_m^0\right\|_{L^q(L_\lambda)}\right]. \end{aligned}$$

For concreteness, we choose $p = 3/2$, so that $q = 3$. Taking the supremum over all (t,x) in $[0,T]\times L$ gives

$$\begin{aligned} \|u_n-u_m\|_{L^\infty([0,T]\times L)} &\le \left\|u_n^0-u_m^0\right\|_{L^\infty(L)} + Ct\lambda^{-1} \\ &\quad + C\lambda^{\frac12}\left[\|\omega_n-\omega_m\|_{L^\infty([0,T];L^3(L_\lambda))} + \left\|\omega_n^0-\omega_m^0\right\|_{L^3(L_\lambda)}\right]. \end{aligned}$$

Now, given any $\delta > 0$, let $\lambda = 1/\delta$. Then choose N large enough that

$$\|\omega_n-\omega_m\|_{L^\infty([0,T];L^3(L_\lambda))} + \left\|\omega_n^0-\omega_m^0\right\|_{L^3(L_\lambda)} < \delta$$

and $\|u_n^0 - u_m^0\|_{L^\infty(L)} < \delta$ for all $n, m > N$. It follows that

$$\|u_n - u_m\|_{L^\infty([0,T]\times L)} < \delta + C\delta + C\delta^{1/2}.$$

This shows that the sequence, (u_n), is Cauchy in $C([0,T] \times L)$ (without the need to take a further subsequence).

Step 7. Convergence to a solution: The convergence of (u_n) to a solution to $\partial_t\omega + u \cdot \nabla\omega = 0$ in $\mathcal{D}'$ is standard. That the Serfati identity (4.3) holds for u regardless of the choice of the cutoff function, a, follows from these same convergences and the observation that (u_n) is bounded in L^∞.

Step 8. Modulus of continuity of the velocity: The limit velocity $u(t)$ has a log-Lipschitz modulus of continuity; this follows either from Lemma B.3 or directly from the convergence of (u_n) with a uniform bound on the log-Lipschitz modulus of continuity on compact subsets. □

5. Existence in an exterior domain

The proof of existence in an exterior domain closely parallels that for the whole plane; in this section, we report only on the differences between the proofs. The derivation of the Serfati identity requires the majority of the effort, as it now requires us to treat boundary integrals. We give its derivation in Section 5.1, turning to the existence proof in Section 4.2.

Throughout this section Ω denotes the domain exterior to a bounded, C^∞, connected and simply connected obstacle.

The sequence of approximating solutions in an exterior domain that we employ in our proof of existence are those constructed by Kikuchi in [20], given in Theorem 5.1.

Theorem 5.1. *[Kikuchi, [20]] Fix $T > 0$. Let $u^0 \in C^\infty(\Omega)$ with $\omega(u^0)$ compactly supported (this is more regularity than Kikuchi requires). There exists a unique classical solution, (u, p), to the Euler equations without forcing, having u^0 as initial velocity, such that the vorticity is transported by the flow map, the circulation of $u(t)$ about $\partial\Omega$ is conserved over time, and $u(t, x) \to 0$ as $|x| \to \infty$. Moreover, $u \in C^1([0,T] \times \Omega))$ and $\nabla p \in C([0,T] \times \Omega))$.*

5.1. The Serfati identity in an exterior domain.

In this subsection we show that the alternate Serfati identity in (5.1) holds for any radially symmetric, smooth, compactly supported cutoff function a, with $a = 1$ in a neighborhood of the origin.

Recall the hydrodynamic Biot-Savart kernel J_Ω as defined in (3.1), and the divergence-free vector field, tangential to $\partial\Omega$, having circulation one around $\partial\Omega$ and decaying at infinity, $\overline{K}_\Omega$ (see (3.1)).

Proposition 5.2. *Let u be a C^∞ smooth solution to the Euler equations with initial vorticity ω^0, compactly supported, as given by Theorem 5.1. Let the function, a, be as* *in (2) of Definition 2.3. Then the Serfati identity, (1.5), holds, and we also have*

$$\begin{aligned} &u^j(t,x) \\ &\quad = (u^0)^j(x) + \int_\Omega a(x-y) J_\Omega^j(x,y)(\omega(t,y) - \omega^0(y))\, dy \\ &\qquad - \int_0^t \int_\Omega (u(s,y)\cdot\nabla_y)\, \nabla_y^\perp \left[(1-a(x-y))J_\Omega^j(x,y)\right] \\ &\qquad\qquad\qquad\qquad \cdot u(s,y)\, dy\, ds \\ &\qquad - \frac{\overline{K}_\Omega^j(x)}{2} \int_0^t \int_\Gamma |u(y(\sigma))|^2\ \nabla a(x-y(\sigma))\cdot\boldsymbol{\tau}\, d\sigma\, ds, \end{aligned} \tag{5.1}$$

where $y = y(\sigma)$ is a parameterization by arc length of $\partial\Omega$.

Proof. Denote the circulation of u about $\partial\Omega$ by

$$\Gamma(u) = \int_\Gamma u\cdot\boldsymbol{\tau},$$

and the mass of the corresponding vorticity $\omega = \omega(u)$ by

$$m(\omega) = \int_\Omega \omega.$$

For smooth solutions of the Euler equations in Ω, both of these quantities are conserved. Because ω^0 is compactly supported and $\omega(u)$ is transported by the flow map, $\omega(u)$ remains compactly supported for all time. This fact and the smoothness of the solution justify the calculations that follow.

Set

$$K_\Omega[\omega] = \int_\Omega K_\Omega(x,y)\omega(y)\, dy, \quad J_\Omega[\omega] = \int_\Omega J_\Omega(x,y)\omega(y)\, dy,$$

and note that both integrals converge, since ω is compactly supported.

Observe that

$$J_\Omega[\omega] = K_\Omega[\omega] + m(\omega)\overline{K}_\Omega.$$

Since u conserves circulation over time, $\overline{K}_\Omega$ has unit circulation, and J_Ω has zero circulation we have

$$\begin{aligned} u &= J_\Omega[\omega] + \Gamma(u^0)\overline{K}_\Omega(x) \\ &= K_\Omega[\omega] + [m(\omega^0) + \Gamma(u^0)]\overline{K}_\Omega(x). \end{aligned}$$

Hence,

$$\partial_t u^j(x) = \partial_t \int_\Omega J_\Omega^j(x,y)\omega(t,y)\, dy = \partial_t \int_\Omega K_\Omega^j(x,y)\omega(t,y)\, dy, \tag{5.2}$$

where we have used both the conservation of $m(\omega)$ and of circulation.

Starting with (5.2) and using the vorticity equation (2.1), we have,

$$\begin{aligned} \partial_t u^j(t,x) = \partial_t &\int_\Omega a(x-y) K_\Omega^j(x,y)\omega(t,y)\, dy \\ &- \int_\Omega (1-a(x-y)) K_\Omega^j(x,y)(u\cdot\nabla\omega)(t,y)\, dy, \end{aligned} \tag{5.3}$$

$j = 1, 2$. We rewrite the last term as before as

$$
\begin{aligned}
-&\int_\Omega (1 - a(x-y)) K_\Omega^j(x,y)(u \cdot \nabla \omega)(t,y)\, dy \\
&= -\int_\Omega (1 - a(x-y)) K_\Omega^j(x,y) \operatorname{curl}(u \cdot \nabla u)(t,y)\, dy \\
&= \int_\Omega (1 - a(x-y)) K_\Omega^j(x,y) \operatorname{div}\left[(u \cdot \nabla u)^\perp(t,y)\right]\, dy \\
&= -\int_\Omega \left[(u \cdot \nabla u)^\perp(t,y)\right] \cdot \nabla \left[(1 - a(x-y)) K_\Omega^j(x,y)\right]\, dy \\
&= \int_\Omega (u \cdot \nabla u)(t,y) \cdot \nabla^\perp \left[(1 - a(x-y)) K_\Omega^j(x,y)\right]\, dy.
\end{aligned}
\tag{5.4}
$$

The boundary integral above vanishes because $K_\Omega(x, \cdot) = 0$ on the boundary.

Let V be a vector field on Ω and recall the following identity:

$$
(u \cdot \nabla)(V \cdot u) = [(u \cdot \nabla)V] \cdot u + [(u \cdot \nabla)u] \cdot V.
$$

Integrating on Ω, we obtain

$$
\begin{aligned}
\int_\Omega [(u \cdot \nabla)u] \cdot V &= \int_\Omega (u \cdot \nabla)(V \cdot u) - \int_\Omega [(u \cdot \nabla)V] \cdot u \\
&= -\int_\Omega (u \cdot \nabla V) \cdot u,
\end{aligned}
\tag{5.5}
$$

the first integral vanishing in integrating by parts since $\operatorname{div} u = 0$ and $u \cdot \boldsymbol{n} = 0$.

Using (5.5) with $V = \nabla^\perp \left[(1 - a(x-y)) K_\Omega^j(x,y)\right]$, putting the resulting term back into (5.3), and integrating in time yields (1.5).

To obtain (5.1), we return to (5.2), writing,

$$
\partial_t u^j(x) = \partial_t \int_\Omega a(x-y) J_\Omega^j(x,y) \omega(y)\, dy + \int_\Omega (1 - a(x-y)) J_\Omega^j(x,y) \partial_t \omega(y)\, dy,
$$

$j = 1, 2$. Integrating the last term by parts as we did in (5.4), we now have the additional boundary integral (using $J_\Omega(x,y) = \overline{K}_\Omega(x)$ when y is on $\partial\Omega$):

$$
\begin{aligned}
&\int_\Omega (1 - a(x-y)) J_\Omega^j(x,y) \partial_t \omega(y)\, dy \\
&\quad = \int_\Omega (u \cdot \nabla u)(y) \cdot \nabla^\perp \left[(1 - a(x-y)) J_\Omega^j(x,y)\right]\, dy \\
&\qquad + (\overline{K}_\Omega^j(x) \int_\Gamma [u(y(\sigma)) \cdot \nabla u(y(\sigma))]^\perp \cdot \boldsymbol{n}\, (1 - a(x - y(\sigma)))\, d\sigma.
\end{aligned}
\tag{5.6}
$$

The first term on the right-hand side we integrate by parts once more, as we did in proving (1.5), the vanishing of $u \cdot \boldsymbol{n}$ on the boundary again being used to eliminate the boundary term. For the second term, which contains the boundary integral, we use the identity,

$$
\begin{aligned}
[(u \cdot \nabla)u] \cdot \boldsymbol{\tau} &= [(u \cdot \boldsymbol{n}\, \partial_{\boldsymbol{n}} + u \cdot \boldsymbol{\tau}\, \partial_{\boldsymbol{\tau}})u] \cdot \boldsymbol{\tau} \\
&= (u \cdot \boldsymbol{\tau}) \partial_{\boldsymbol{\tau}} (u \cdot \boldsymbol{\tau}) = u \partial_{\boldsymbol{\tau}} u = \frac{1}{2} \frac{d}{d\sigma} \left|u(y(\sigma)\right|^2.
\end{aligned}
$$

To make sense of $\partial_{\boldsymbol{n}}$, we extended $\boldsymbol{n}$ into a tubular neighborhood of the boundary. Since $u \cdot \boldsymbol{n} = 0$, the term containing $\partial_{\boldsymbol{n}}$ then vanished.

Integrating the boundary integral in (5.6) by parts gives

$$\begin{aligned}
&\int_\Gamma [u(y(\sigma)) \cdot \nabla u(y(\sigma))]^\perp \cdot \boldsymbol{n}\, (1 - a(x - y(\sigma)))\, d\sigma \\
&\quad = \frac{1}{2} \int_\Gamma \frac{d}{d\sigma} \left| u(y(\sigma)) \right|^2 \, (1 - a(x - y(\sigma)))\, d\sigma \\
&\quad = \frac{1}{2} \int_\Gamma \left| u(y(\sigma)) \right|^2 \, \frac{d}{d\sigma} a(x - y(\sigma))\, d\sigma \\
&\quad = -\frac{1}{2} \int_\Gamma \left| u(y(\sigma)) \right|^2 \, \nabla a(x - y(\sigma)) \cdot \frac{dy(\sigma)}{d\sigma}\, d\sigma.
\end{aligned}$$

This yields (5.1), since $\frac{dy(\sigma)}{d\sigma} = \boldsymbol{\tau}$. □

To control the boundary term in (5.1), we need control not just on the size of the integrands, but cancellation due to the velocity field itself. This is easily obtained from the simple bound in Proposition 5.3.

Proposition 5.3. *Let $\lambda > 0$ and set a_λ as in Proposition 3.1. Let u be a continuous vector field on Ω which is tangent to the boundary. Then there exists $C > 0$ such that*

$$\left| \frac{\overline{K}^j_\Omega(x)}{2} \int_\Gamma \left| u(y(\sigma)) \right|^2 \, \nabla a_\lambda(x - y(\sigma)) \cdot \boldsymbol{\tau}\, d\sigma \right| \le \frac{C}{\lambda} \left\| u \right\|^2_{L^\infty}.$$

Proof. This follows from the bound,

$$\left| \nabla a_\lambda(x - y(\sigma)) \right| = \left| \lambda^{-1} \nabla a((x - y(\sigma))\lambda^{-1}) \right| \le C\lambda^{-1},$$

of Proposition 3.1 and because $\overline{K}_\Omega \in L^\infty(\Omega)$, as we show in (A.18). □

5.2. Proof of existence in an exterior domain.

Proof of existence in Theorem 2.5 for an exterior domain. As in our proof of existence for the full plane in Section 4.2, we approximate the initial data employing Proposition B.2 and construct smooth solutions to the Euler equations using Theorem 5.1. The key bounds in (4.5), then, continue to hold on Ω:

$$\left\| \omega_n \right\|_{L^\infty(\mathbb{R} \times \Omega)} \le \left\| \omega^0_n \right\|_{L^\infty} \le C \left\| \omega^0 \right\|_{L^\infty}, \quad \left\| u^0_n \right\|_{L^\infty} \le C \left\| u^0 \right\|_{L^\infty}. \tag{5.7}$$

The proof proceeds in the identical manner to that of Section 4.2 with the exception of two steps in the proof, described below. It is important to observe, though, that the convergences obtained are for compact subsets of $\overline{\Omega}$ and $[0, T] \times \overline{\Omega}$.

As before, we denote the approximate solutions by u_n and ω_n.

Bound velocities in $L^\infty([0, T] \times \Omega)$: Let a be any cutoff function as in (2) of Definition 2.3. Let a_λ be as in Proposition 3.1.

From (5.1), substituting u_n and ω_n for u and ω, we have

$$\begin{aligned}
\left| u_n(t, x) \right| &\le \left| u^0_n(x) \right| + \left| \int_\Omega a_\lambda(x - y) J_\Omega(x, y)(\omega_n(t, y) - \omega^0_n(y))\, dy \right| \\
&\quad + \int_0^t \left| \int_\Omega \left| \nabla_y \nabla_y \left((1 - a_\lambda(\cdot - y)) J_\Omega(\cdot, y) \right) \right| \left| u_n(s, y) \right|^2 \, dy \right| ds \\
&\quad + \left| \frac{\overline{K}^j_\Omega(x)}{2} \int_0^t \int_\Gamma \left| u_n(s, y(\sigma)) \right|^2 \, \nabla a_\lambda(x - y(\sigma)) \cdot \boldsymbol{\tau}\, d\sigma\, ds \right|.
\end{aligned}$$

Applying Propositions 3.1 and 5.3 to (5.1), and using (5.7), it follows from Hölder's inequality that, for some constant $C > 0$, independent of n,

$$\|u_n(t)\|_{L^\infty} \le C + C\lambda + \frac{C}{\lambda}\int_0^t \|u_n(s)\|_{L^\infty}^2 \, ds \tag{5.8}$$

for all $\lambda > C_0$, with C_0 as in Proposition 3.1.

Observe that we can choose $\lambda > C_0$ arbitrarily, even allowing it to depend on time. Hence, we can let

$$\lambda = \lambda(t) = \max\left\{C_0 + 1, \left(\int_0^t \|u_n(s)\|_{L^\infty}^2 \, ds\right)^{1/2}\right\}.$$

The function $\lambda(t)$ is continuous and non-decreasing, with $\lambda(0) = C_0 + 1$. Suppose that there exists a finite time, T_n^*, at which $\displaystyle\int_0^{T_n^*} \|u_n(s)\|_{L^\infty}^2 \, ds = (C_0+1)^2$. Then it follows directly from (5.8) that u_n lies in $L^\infty([0, T_n^*]; L^\infty)$ with a norm bounded by $C(C_0+1)$. After that time, $\lambda(t) = \displaystyle\int_0^t \|u_n(s)\|_{L^\infty}^2 \, ds > C_0 + 1$, and we obtain

$$\|u_n(t)\|_{L^\infty} \le C + C\left(\int_0^t \|u_n(s)\|_{L^\infty}^2 \, ds\right)^{1/2},$$

so that

$$\|u_n(t)\|_{L^\infty}^2 \le C + C\int_0^t \|u_n(s)\|_{L^\infty}^2 \, ds.$$

We conclude from Gronwall's lemma that

$$\|u_n(t)\|_{L^\infty} \le \max\left\{Ce^{Ct}, C_0\right\} = Ce^{Ct}.$$

Thus, u_n lies in $L^\infty([0,T] \times \Omega)$ for any $T > 0$ with a bound that is uniform in n.

Velocities are Cauchy in $C([0,T] \times L)$: Let L be a compact subset of $\overline{\Omega}$. The only change to the proof of this step in Section 4.2 is that J_Ω is used in place of K in the expressions for I_1, I_2, and I_3 in (4.12), which also includes the additional term,

$$I_4 = \left|\frac{\overline{K}_\Omega^j(x)}{2}\int_0^t \int_\Gamma \left(|u_n(y(\sigma))|^2 - |u_m(y(\sigma))|^2\right) \nabla a_\lambda(x - y(\sigma)) \cdot \boldsymbol{\tau} \, d\sigma \, ds\right|.$$

Proposition 5.3 and the uniform bound on the sequence, (u_k), in $L^\infty([0,T] \times \Omega)$ give

$$I_4 \le \frac{Ct}{\lambda}.$$

The estimates on I_1, I_2, and I_3 are unchanged, though now they only hold for $\lambda > C_0$. But this is of no matter, since we take λ to infinity. □

6. Uniqueness and continuous dependence on initial data

Our proof of uniqueness, which assumes that the Serfati identity holds, derives from that of Serfati in [29] (who also assumes, implicitly, that the Serfati identity holds). We present the proof in Section 6.1. The continuous dependence on initial data of Theorem 2.8 is a modification of our uniqueness proof, and is presented in Section 6.2.

In this section, Ω can be either all of $\mathbb{R}^2$ or an exterior domain. In the proofs, we exploit a number of estimates from Section 3. The estimates are stated in terms of

K (see (1.4)) for the full plane and in terms of K_Ω (see (1.3)) for an exterior domain. When $\Omega = \mathbb{R}^2$, we have $K_\Omega(x,y) = K(x-y)$.

6.1. **Uniqueness.** We begin by introducing some notation. Let $C > 0$ and set $\mu\colon [0,\infty) \to [0,\infty)$,

$$\mu(r) = C \max\left\{-r\log r, e^{-1}\right\}. \tag{6.1}$$

Then μ is an Osgood modulus of continuity, by which we mean that

$$\int_0^1 \frac{ds}{\mu(s)} = \infty.$$

Let u_1, u_2 be two solutions as in Definition 2.3. Each u_j is log-Lipschitz and, hence, there exists $C > 0$ for which μ given in (6.1) serves as a common, strictly increasing, bounded modulus of continuity for both u_1 and u_2.

Next, we recall Osgood's lemma, which we state in Lemma 6.1 in the form given in Lemma 5.2.1 of [5].

Lemma 6.1 (Osgood's lemma)**.** *Let L be a measurable nonnegative function and γ a nonnegative locally integrable function, each defined on the interval $[t_0, t_1]$. Let $\mu\colon [0,\infty) \to [0,\infty)$ be a continuous nondecreasing function, with $\mu(0) = 0$ (hence, μ is a modulus of continuity) and $\mu > 0$ on $(0,\infty)$. Let $a\colon [0,\infty) \to [0,\infty)$ be nondecreasing, and assume that for all t in $[t_0, t_1]$,*

$$L(t) \le a(t) + \int_{t_0}^t \gamma(s)\mu(L(s))\,ds.$$

For all t in $[t_0, t_1]$,

$$\int_{a(t)}^{L(t)} \frac{ds}{\mu(s)} \le \int_{t_0}^t \gamma(s)\,ds.$$

If $a \equiv 0$ and μ is an Osgood modulus of continuity then $L \equiv 0$.

Let X_1, X_2 be the flow maps corresponding to u_1, u_2, and define

$$h(t) = \left\|X_1(t,\cdot) - X_2(t,\cdot)\right\|_{L^\infty}. \tag{6.2}$$

Our proof of uniqueness rests upon Proposition 6.2, which we prove first. Because we will also use Proposition 6.2 in Section 6.2 to prove continuous dependence on initial data, we do not assume that $u_1(0) = u_2(0)$.

Proposition 6.2. *Assume that u_1, u_2 are Serfati solutions to the Euler equations with vorticities, ω_1, ω_2 and initial vorticities, ω_1^0, ω_2^0, lying in S^p for $p \in (2,\infty]$, where S^p is defined in (2.2). Let $\overline{\gamma}$ be any Lipschitz function on Ω having finite-measure support.*

Let $h = h(t)$ be as in (6.2) and consider μ as in (6.1), a common modulus of continuity for u_1 and u_2. Then, for all x in Ω, we have

$$\begin{aligned}\Big|\int_\Omega \overline{\gamma}(y) K_\Omega^j(x,y)(\omega_1(t,y) - \omega_2(t,y))\,dy\Big| \\ \le C \max\left\{\|\omega_1^0\|_{L^\infty}, \|\omega_2^0\|_{L^\infty}\right\}\mu(h) + C_p \left\|\omega_1^0 - \omega_2^0\right\|_{L^p_{uloc}}.\end{aligned}$$

The constant, C, depends only on the Lipschitz constant and measure of the support of $\overline{\gamma}$, and C_p depends only on p and the measure of the support of $\overline{\gamma}$.

Proof. Assume first that $h < e^{-1}$.

Since for Serfati solutions, ω_j is transported by the flow, X_j, associated to u_j, $j = 1, 2$, we have

$$\begin{aligned}\int_\Omega &\overline{\gamma}(y) K_\Omega^j(x, y)(\omega_1(t, y) - \omega_2(t, y))\, dy \\ &= \int_\Omega \overline{\gamma}(y) K_\Omega^j(x, y) \left(\omega_1^0(X_1^{-1}(t, y)) - \omega_2^0(X_2^{-1}(t, y))\right)\, dy.\end{aligned}$$

Alternately making the changes of variable $y = X_1(t, z)$ and $y = X_2(t, z)$, this becomes, since X_1 and X_2 are measure-preserving,

$$\int_\Omega \overline{\gamma}(X_1(t, z)) K_\Omega^j(x, X_1(t, z))\omega_1^0(z)\, dz - \int_\Omega \overline{\gamma}(X_2(t, z)) K_\Omega^j(x, X_2(t, z))\omega_2^0(z)\, dz.$$

We can write this as

$$\int_\Omega \overline{\gamma}(y) K_\Omega^j(x, y)(\omega_1(t, y) - \omega_2(t, y))\, dy = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}I_1 &= \int_\Omega \left[\overline{\gamma}(X_1(t, z)) - \overline{\gamma}(X_2(t, z))\right] K_\Omega^j(x, X_2(t, z))\omega_1^0(z)\, dz, \\ I_2 &= \int_\Omega \overline{\gamma}(X_1(t, z)) \left[K_\Omega^j(x, X_1(t, z)) - K_\Omega^j(x, X_2(t, z))\right] \omega_1^0(z)\, dz, \\ I_3 &= \int_\Omega \overline{\gamma}(X_2(t, z)) K_\Omega(x, X_2(t, z))(\omega_1^0(z) - \omega_2^0(z))\, dz.\end{aligned}$$

Letting

$$U = \{z \colon \overline{\gamma}(X_1(t, z)) \neq \overline{\gamma}(X_2(t, z))\},$$

we have

$$\begin{aligned}|I_1| &\le \left\|\omega_1^0\right\|_{L^\infty} \sup_{z \in \Omega} |\overline{\gamma}(X_1(t, z)) - \overline{\gamma}(X_2(t, z))| \int_U \left|K_\Omega^j(x, X_2(t, z))\right|\, dz \\ &\le C \left\|\omega_1^0\right\|_{L^\infty} h \int_U \left|K_\Omega^j(x, X_2(t, z))\right|\, dz,\end{aligned}$$

where C is the Lipschitz constant for $\overline{\gamma}$. But,

$$U \subseteq X_1(t, \operatorname{supp} \overline{\gamma}) \cup X_2(t, \operatorname{supp} \overline{\gamma})$$

has measure bounded in time, since X_1 and X_2 are measure-preserving, and

$$\int_U \left|K_\Omega^j(x, X_2(t, z))\right|\, dz = \int_{X_2(t, U)} \left|K_\Omega^j(x, y)\right|\, dy \le C$$

by Proposition 3.2 and using $|X_2(t, U)| = |U|$. Hence,

$$|I_1| \le C\|\omega_1^0\|_{L^\infty} h.$$

Applying Proposition 3.3, we can easily bound I_2 by

$$\begin{aligned}|I_2| &\le \|\overline{\gamma}\|_{L^\infty} \|\omega_1^0\|_{L^\infty} \|K_\Omega^j(x, X_1(t, z)) - K_\Omega^j(x, X_2(t, z))\|_{L^1(X_1^{-1}(t, \operatorname{supp} \overline{\gamma}))} \\ &\le -C\|\omega_1^0\|_{L^\infty} h \log h,\end{aligned}$$

noting that we used $h < e^{-1}$.

For I_3, we have

$$\begin{aligned}
|I_3| &\le \|\overline{\gamma}(X_2(t,z))K_\Omega(x, X_2(t,z))\|_{L^{p'}_z} \left\|\omega_1^0 - \omega_2^0\right\|_{L^p(\operatorname{supp}\overline{\gamma}\circ X_2(t,\cdot))} \\
&= \|\overline{\gamma}(w)K_\Omega(x,w)\|_{L^{p'}_w} \left\|\omega_1^0 - \omega_2^0\right\|_{L^p(\operatorname{supp}\overline{\gamma}\circ X_2(t,\cdot))} \\
&\le \|\overline{\gamma}(w)K_\Omega(x,w)\|_{L^{p'}_w} \left\|\omega_1^0 - \omega_2^0\right\|_{L^p_{uloc}(\Omega)} \\
&\le C_p \left\|\omega_1^0 - \omega_2^0\right\|_{L^p_{uloc}(\Omega)},
\end{aligned}$$

where $1/p' + 1/p = 1$. In the final inequality, we used Proposition 3.2.

Combining the bounds for I_1, I_2, and I_3 gives

$$\begin{aligned}
&\left|\int_\Omega \overline{\gamma}(y)K^j_\Omega(x,y)(\omega_1(t,y) - \omega_2(t,y))\,dy\right| \\
&\qquad \le -C\|\omega_1^0\|_{L^\infty} h\log h + C_p \left\|\omega_1^0 - \omega_2^0\right\|_{L^p_{uloc}(\Omega)} \\
&\qquad = C\|\omega_1^0\|_{L^\infty}\mu(h) + C_p \left\|\omega_1^0 - \omega_2^0\right\|_{L^p_{uloc}(\Omega)}.
\end{aligned}$$

For $h \ge e^{-1}$, we apply, as above, Proposition 3.2 to conclude that

$$\begin{aligned}
&\left|\int_\Omega \overline{\gamma}(y)K^j_\Omega(x,y)(\omega_1(t,y) - \omega_2(t,y))\,dy\right| \le C\max\left\{\|\omega_1^0\|_{L^\infty}, \|\omega_2^0\|_{L^\infty}\right\} \\
&\qquad = C\mu(e^{-1})\max\left\{\|\omega_1^0\|_{L^\infty}, \|\omega_2^0\|_{L^\infty}\right\} \le C\mu(h)\max\left\{\|\omega_1^0\|_{L^\infty}, \|\omega_2^0\|_{L^\infty}\right\},
\end{aligned}$$

and the proof is complete. ☐

Proof of uniqueness in Theorem 2.5. Assume now that $u_1(0) = u_2(0)$. We will assume that the cutoff function, a, of (1.5) is equal to 0 outside of $B_{e^{-1}}$. The choice of e^{-1} is convenient because of the estimates in Proposition 3.3. We will also assume that the cutoff function is such that C_0 of Proposition 3.1 is less than 1; thus, the estimates in (3.6) through (3.8) hold for $\lambda = 1$.

Let X_j be the flow map for u_j, $j = 1, 2$. Set μ to be as in (6.1), a common modulus of continuity for u_1 and u_2.

We will establish uniqueness by showing that $X_1 = X_2$. Let t lie in $[0,T]$. Our approach is to bound the quantity,

$$M(t) = \int_0^t P(s)\,ds, \tag{6.3}$$

where

$$P(s) = \|u_2(s, X_2(s,\cdot)) - u_1(s, X_1(s,\cdot))\|_{L^\infty}.$$

We do this by obtaining, through a long series of estimates, the inequality

$$M(t) \le \int_0^t \nu(M(s))\,ds, \tag{6.4}$$

where

$$\nu(r) = C\left[(1+t)\mu(r) + r\right]. \tag{6.5}$$

The modulus of continuity μ is Osgood and $\mu(r) >> r$ near $r = 0$, so that ν is also an Osgood modulus of continuity. Hence, applying Lemma 6.1 to (6.4) gives $M \equiv 0$.

Then letting $h(t)$ be as in (6.2), it follows that

$$\begin{aligned} h(t) &= \left\| \int_0^t u_1(s, X_1(s,\cdot)) - u_2(s, X_2(s,\cdot))\, ds \right\|_{L^\infty} \\ &\le \int_0^t \|u_1(s, X_1(s,\cdot)) - u_2(s, X_2(s,\cdot))\|_{L^\infty}\, ds \\ &= M(t). \end{aligned} \tag{6.6}$$

Hence, $X_1 \equiv X_2$ so that $u_1 \equiv u_2$, and uniqueness holds.

(It is easy to see that $h(t)$ and $M(t)$ are continuous and bounded, because of the boundedness of u_1 and u_2. Hence the the inequality in (6.4, 6.6) and the inequalities that follow all contain finite quantities.)

We now proceed to prove (6.4). We start by obtaining a bound on the quantity, $|u_1(t, X_1(t,x)) - u_2(t, X_2(t,x))|$, to obtain a bound on $P(t)$, which we will transform to the bound on $M(t)$ in (6.4).

By the triangle inequality,

$$\begin{aligned} &|u_1(t, X_1(t,x)) - u_2(t, X_2(t,x))| \\ &\qquad \le |u_2(t, X_1(t,x)) - u_2(t, X_2(t,x))| \\ &\qquad\qquad + |u_1(t, X_1(t,x)) - u_2(t, X_1(t,x))| \\ &\qquad =: A_1 + A_2. \end{aligned} \tag{6.7}$$

We easily bound A_1 by

$$A_1 \le \mu(|X_1(t,x) - X_2(t,x)|) \le \mu(h(t)). \tag{6.8}$$

We obtain a bound for A_2 by subtracting (1.5) for u_2 from (1.5) for u_1:

$$\begin{aligned} A_2 \le &\left| \int_\Omega a(X_1(t,x) - y) K_\Omega(X_1(t,x), y)(\omega^1(t,y) - \omega^2(t,y))\, dy \right| \\ &+ \int_0^t \int_\Omega |\nabla_y \nabla_y \left((1 - a(X_1(t,x) - y)) K_\Omega(X_1(t,x), y)\right)| \\ &\qquad\qquad |u_1 \otimes u_1 - u_2 \otimes u_2|\,(s,y)\, dy\, ds \\ =:\ & B_1 + B_2. \end{aligned} \tag{6.9}$$

Because $\overline{\gamma}(y) := a(X_1(t,x) - y)$ is Lipschitz-continuous and has finite-measure support with Lipschitz constant and measure independent of t and x, we can apply Proposition 6.2 to conclude that

$$B_1 \le C\|\omega^0\|_{L^\infty} \mu(h(t)) \tag{6.10}$$

for some constant C depending only upon the cutoff function a.

For B_2, we have simply,

$$\begin{aligned} B_2 \le \int_0^t &\|\nabla\nabla((1 - a(X_1(t,x) - \cdot)) K_\Omega(X_1(t,x), \cdot))\|_{L^1} \\ &\qquad \|(u_2 \otimes u_2 - u_1 \otimes u_1)(s,\cdot)\|_{L^\infty}\, ds. \end{aligned} \tag{6.11}$$

The L^1-norm in the integrand above is finite and bounded uniformly in x by Proposition 3.1. Using,

$$u_2 \otimes u_2 - u_1 \otimes u_1 = u_2 \otimes (u_2 - u_1) + u_1 \otimes (u_2 - u_1),$$

because u_j lies in $L^\infty([0,T]\times\Omega)$, we have

$$\begin{aligned}
B_2 &\le C\int_0^t \|u_2(s)-u_1(s)\|_{L^\infty}\,ds \\
&= C\int_0^t \|u_2(s,X_1(s,\cdot))-u_1(s,X_1(s,\cdot))\|_{L^\infty}\,ds \\
&\le C\int_0^t \|u_2(s,X_1(s,\cdot))-u_2(s,X_2(s,\cdot))\|_{L^\infty}\,ds \\
&\qquad + C\int_0^t \|u_2(s,X_2(s,\cdot))-u_1(s,X_1(s,\cdot))\|_{L^\infty}\,ds \\
&\le C\int_0^t \mu(h(s))\,ds + C\int_0^t \|u_2(s,X_2(s,\cdot))-u_1(s,X_1(s,\cdot))\|_{L^\infty}\,ds.
\end{aligned}$$

Here, we used $|u_2(s,X_2(s,\cdot))-u_2(s,X_1(s,\cdot))|\le\mu(|X_2(s,\cdot)-X_1(s,\cdot)|)\le\mu(h(s))$.

What we have shown is that

$$\begin{aligned}
&|u_1(t,X_1(t,x))-u_2(t,X_2(t,x))| \\
&\qquad\le C\int_0^t \mu(h(s))\,ds + C\mu(h(t)) + C\int_0^t P(s)\,ds.
\end{aligned}$$

Taking the supremum over all x in $\mathbb{R}^2$ and using (6.3), we conclude that

$$P(t)\le C\int_0^t \mu(h(s))\,ds + C\mu(h(t)) + CM(t).$$

But $h(t)\le M(t)$ by (6.6), and μ is nondecreasing so $\mu(h(t))\le\mu(M(t))$ and $\mu(h(s))\le\mu(M(s))$. Thus,

$$M'(t)=P(t)\le C\int_0^t \mu(M(s))\,ds + C\mu(M(t)) + CM(t).$$

Since M is increasing, we can write

$$M'(t)\le C(1+t)\mu(M(t)) + CM(t).$$

For our purposes, it is easier to weaken this inequality slightly to

$$M'(s)\le C(1+t)\mu(M(s)) + CM(s) = \nu(M(s)) \tag{6.12}$$

for all s in $(0,t)$, where ν is given in (6.5).

In integral form, using $M(0)=0$, (6.12) becomes

$$M(t)\le\int_0^t \nu(M(s))\,ds.$$

That $M\equiv 0$ follows from Lemma 6.1, and since $h(t)\le M(t)$, $h\equiv 0$ as well, which proves uniqueness. □

Our proof of uniqueness above differs from Serfati's proof in [29] in two key respects. First, we bound, in effect, the quantity $h(t)$ defined in (6.2), whereas Serfati bounds the quantity $\int_0^t |h'(s)|\,ds$, which is more difficult to deal with rigorously. Second, we also bound the terms involving the Biot-Savart law differently, via Proposition 6.2, so as to obtain a unified argument that applies both to the full plane and to an exterior domain.

6.2. **Continuous dependence on initial data.** In this subsection, we modify slightly the proof of uniqueness in the previous section to obtain the limited continuity on initial data stated in Theorem 2.8.

Proof of Theorem 2.8. We follow the same steps as in the proof of uniqueness in Section 6.1, and use the same definitions made in that proof. Now, however, u_1 and u_2 are the unique solutions for *different* initial data. This leads to the bound,

$$\begin{aligned} &|u_1(t,X_1(t,x)) - u_2(t,X_2(t,x))| \\ &\qquad \le \left|u_1^0(x) - u_2^0(x)\right| + A_1 + A_2 \le \left|u_1^0(x) - u_2^0(x)\right| + A_1 + B_1 + B_2, \end{aligned}$$

where A_1, A_2, B_1, and B_2 are the same as in Section 6.1.

We bound A_1 and B_2 exactly as in (6.8, 6.11), for the initial data was not used in their derivations. As in the proof of uniqueness, we bound the term B_1 using Proposition 6.2, but now an additional term,

$$C_p\|\omega_1^0 - \omega_2^0\|_{L^p_{loc}(\Omega)},$$

appears because the initial vorticities differ.

The net effect is that the bound in (6.12) becomes

$$\begin{aligned} M'(s) &\le \left\|u_1^0 - u_2^0\right\|_{L^\infty} + C_p\|\omega_1^0 - \omega_2^0\|_{L^p_{loc}(\Omega)} \\ &\qquad + C(1+t)\mu(M(s)) + CM(s) \\ &\le Cs_0 + C(1+t)\mu(M(s)) + CM(s) \\ &= Cs_0 + \nu(M(s)), \end{aligned} \tag{6.13}$$

where ν is as in (6.5). In integral form, this is

$$M(t) \le Cs_0 t + \int_0^t \nu(M(s))\, ds,$$

since still $M(0) = 0$.

Lemma 6.1 tells us that $M(t) \le \Gamma(t)$, where $\Gamma(t)$ is defined by

$$\int_{Cs_0 t}^{\Gamma(t)} \frac{ds}{\nu(s)} = t.$$

It follows from (6.13) that

$$P(t) = M'(t) \le Cs_0 + C(1+t)\mu(\Gamma(t)) + C\int_0^t P(s)\, ds,$$

so by Gronwall's lemma we conclude that

$$P(t) \le C\left[s_0 + (1+t)\mu(\Gamma(t))\right] e^{Ct}.$$

Since $P(t) = \|\partial_t(X_2 - X_1)\|_{L^\infty}$, we have

$$\begin{aligned} |X_2 - X_1|\,(t,x) &= \left|\int_0^t \partial_s(X_2 - X_1)(s,x)\right| \le \int_0^t |\partial_s(X_2 - X_1)(s,x)| \\ &\le \int_0^t P(s)\, ds = M(t) \le \Gamma(t). \end{aligned}$$

So, one obtains continuous dependence of the flow maps with respect to initial data.

We can turn this into continuous dependence of velocity, as

$$\begin{aligned}\|u_1(t) - u_2(t)\|_{L^\infty} &= \|u_1(t, X_1(t,\cdot)) - u_2(t, X_1(t,\cdot))\|_{L^\infty} \\ &\le \|u_1(t, X_1(t,\cdot)) - u_2(t, X_2(t,\cdot))\|_{L^\infty} \\ &\qquad + \|u_2(t, X_2(t,\cdot)) - u_2(t, X_1(t,\cdot))\|_{L^\infty} \\ &\le P(t) + \|\mu(|X_2(t,\cdot) - X_1(t,\cdot)|)\|_{L^\infty} \\ &\le C\left[s_0 + (1+t)\mu(\Gamma(t))\right] e^{Ct} + \mu(\Gamma(t)).\end{aligned} \tag{6.14}$$

To be explicit, for a fixed t and all sufficiently small s_0, we will have $\nu(s) = C[-(1+t)s\log s + s]$. Calculating, we have

$$\begin{aligned}t &= \int_{Cs_0t}^{\Gamma(t)} \frac{ds}{\nu(s)} = -C\int_{Cs_0t}^{\Gamma(t)} \frac{ds}{s((1+t)\log s - 1)} \\ &= -C\int_{\log(Cs_0t)}^{\log\Gamma(t)} \frac{dr}{(1+t)r - 1} \\ &= -\frac{C}{1+t}\left[\log((1+t)\log\Gamma(t) - 1) - \log((1+t)\log(Cs_0t) - 1)\right] \\ &= \frac{C}{1+t}\log\frac{(1+t)\log(Cs_0t) - 1}{(1+t)\log\Gamma(t) - 1}.\end{aligned}$$

Simplifying yields the following equation:

$$\frac{(1+t)\log\Gamma(t) - 1}{(1+t)\log(Cs_0t) - 1} = e^{-Ct(1+t)},$$

which leads to

$$\begin{aligned}\log\Gamma(t) &= \frac{1}{1+t} + e^{-Ct(1+t)}\left[\log(Cs_0t) - \frac{1}{1+t}\right] \\ &= C_t + e^{-Ct(1+t)}\log(Cs_0t),\end{aligned}$$

where

$$C_t = \frac{1 - e^{-Ct(1+t)}}{1+t},$$

which we note is greater than 0. Thus,

$$\Gamma(t) = e^{C_t}(Cs_0t)^{e^{-Ct(1+t)}}.$$

The following then holds:

$$\begin{aligned}\mu(\Gamma(t)) &= -C\Gamma(t)\log\Gamma(t) \\ &= -e^{C_t}(Cs_0t)^{e^{-Ct(1+t)}}\left[C_t + e^{-Ct(1+t)}\log(Cs_0t)\right].\end{aligned}$$

Hence, from (6.14),

$$\begin{aligned}\|u_1(t) - u_2(t)\|_{L^\infty} &\le Ce^{Ct}s_0 + \mu(\Gamma(t))\left[1 + C(1+t)e^{Ct}\right] \\ &= Ce^{Ct}s_0 - e^{C_t}(Cs_0t)^{e^{-Ct(1+t)}}\left[C_t + e^{-Ct(1+t)}\log(Cs_0t)\right]\left[1 + C(1+t)e^{Ct}\right] \\ &\le Ce^{Ct}s_0 - C(1+t)e^{Ct}(Cs_0t)^{e^{-Ct(1+t)}}\log(Cs_0t),\end{aligned}$$

which is (2.3). The final inequality was obtained by keeping only the dominant terms. □

7. Examples of Serfati vorticities

It is natural to ask which bounded vorticities in the plane, or in an exterior domain, are the curl of some bounded velocity, or, in other words, to characterize vorticities which give rise to Serfati velocities; we call these *Serfati vorticities*. This turns out to be a surprisingly subtle issue, which will be addressed in [4]. We discuss it briefly here for the sake of completeness. Let us start with some observations.

- Any Yudovich velocity (velocity having bounded and integrable vorticity) is Serfati in the whole plane or exterior domain.
- Periodic vorticities, with integral zero on the period, are Serfati vorticities.
- Any linear combination of Serfati velocities is Serfati; that is, S is a vector space. In particular, adding a bounded, compactly supported, function to a periodic vorticity whose integral vanishes on the period gives rise to a Serfati vorticity.
- Take
$$u(x) = \frac{x^\perp}{|x|} \text{ on } \Omega = \left\{x \in \mathbb{R}^2 \colon\ |x| > 1\right\}.$$
Then $\omega(u) = \operatorname{curl}(u)(x) = |x|^{-1}$, which is bounded but does not decay fast enough to belong to $L^p(\Omega)$ for any $p \le 2$. Hence ω does not decay fast enough for the Biot-Savart law (in the exterior of the unit disk) to converge. Nonetheless, u is bounded with bounded vorticity and, hence, Serfati. Treated as a stationary solution to the Euler equations, the corresponding pressure satisfies $\nabla p = \widehat{r}/r$ so that $p = \log r$, in accordance with Remark 2.6. This example also gives rise, by composition with a conformal map, to an example in the exterior of a general, smooth, connected domain conformally equivalent to the disk.
- To any vorticity that is the characteristic function of an infinite strip in $\mathbb{R}^2$ there corresponds a Serfati velocity. Note that this vorticity does not decay at infinity.

  For example, suppose the strip is $\{(x_1, x_2) \colon 0 < x_2 < 1\}$. Then the velocity u can be chosen to vanish on $x_2 \ge 1$, equal $(1 - x_2)\widehat{\boldsymbol{i}}$ on the strip, and equal $\widehat{\boldsymbol{i}}$ below the strip. Treated as a stationary solution to the Euler equations, the gradient of the corresponding pressure is zero, so again p is in accordance with Remark 2.6.
- If ω is Serfati in $\mathbb{R}^2$ and is supported away from Ω^C, then ω corresponds to a Serfati velocity in Ω. To see this, cut off the stream function for ω so that the resulting velocity field, u, is tangent to $\partial\Omega$; in fact, u vanishes on $\partial\Omega$.
- Consider the strip $\mathsf{S} = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2 < x_2 < 3\}$ (of course one can consider an arbitrary strip with arbitrary inclination). Then
$$u(x) = \begin{cases} (1,0) & \text{if } x_2 > 3, \\ (x_2 - 2, 0) & \text{if } 2 < x_2 < 3, \\ (0,0) & \text{if } x_2 < 2 \end{cases}$$
is a Serfati velocity in the exterior of the unit disk. Indeed, it is divergence-free, tangent to the boundary of the disk, and its curl is $\omega = -\chi_{\mathsf{S}}$, hence bounded and non-decaying at infinity. Similar constructions hold for arbitrary Ω as long as the strip is placed at a distance away from Ω^C. This gives rise to a family of

examples—just vary the size of the strip, the constant flow outside the strip, and the linear interpolation.

On the other hand, consider the following very simple solution to the Euler equations:

$$u(t,x) = (t,0), \quad p(t,x) = -x_1$$

for all (t,x) in $\mathbb{R} \times \mathbb{R}^2$. (This is a special case of an example in [18].) Then u lies in $C([0,T];S)$ for all $T > 0$ and $\nabla p = (-1,0)$ lies in $L^\infty(\mathbb{R} \times \mathbb{R}^2)$. Nonetheless, u is not a weak solution as defined in Definition 2.3. To see this, observe first that $u^0 = 0$ and the vorticity, ω, of u vanishes on $\mathbb{R} \times \mathbb{R}^2$. This leaves only the term,

$$\begin{aligned}\int_0^t \int_\Omega ((s,0) \cdot \nabla_y)\, \nabla_y^\perp \left[(1 - a(x-y))K_\Omega^j(x,y)\right] \cdot (s,0)\, dy\, ds \\ = \int_0^t (-s^2) \int_\Omega \partial_{y_1} \partial_{y_2} \left[(1 - a(x-y))K_\Omega^j(x,y)\right]\, dy\, ds = 0,\end{aligned}$$

on the right-hand side of (1.5). Hence, for $j = 1$, the right-hand side of (1.5) is zero while the left-hand side is t, so the Serfati identity is not satisfied. Hence, requiring that the Serfati identity hold selects certain solutions to the Euler equations whose velocity lies in the Serfati space.

We observe also that while ∇p is bounded, p is not sublinear. The pressure does not satisfy property (2.20) of [33] that is imposed to ensure uniqueness of solutions to the Euler equations for Serfati velocities in the whole plane. (This example is discussed further in [19], where it is shown that sublinear growth of the pressure is equivalent to the Serfati identity and that, specifically in the full plane, these two equivalent conditions reflect the solution being expressed in an inertial reference frame.)

Finally, it is proved in [4] that vorticities that are identically (nonzero) constants *are not* Serfati, since the associated velocities grow linearly at infinity. Any vortex patch whose support contains disks of arbitrary radius is also not Serfati. The vortex patch consisting of a semi-infinite strip such as, for example, the characteristic function of the set $\{(0,\infty) \times (0,1\}) \subset \mathbb{R}^2$ is not a Serfati vorticity.

8. Non-decaying vorticity: Comparison with other approaches

In this section we discuss other works in the literature concerning bounded vorticity, bounded velocity solutions to the Euler equations *having non-decaying vorticity*. More precisely, we will compare the approach in our work to the approaches in [32, 33, 10].

In [32], Taniuchi establishes existence of solutions for initial velocity in S. Actually, he does so for slightly more general initial velocity in which the vorticity can be "slightly unbounded," a local version, with nondecaying initial data, of Yudovich's space defined in [37], but we will discuss his argument, and that of [33], only as it relates to initial data in S. Taniuchi employs a sequence of smooth solutions with velocities in S proven to exist in another 1995 paper of Serfati [30]. Key to Taniuchi's argument is the identity for these smooth solutions from [30],

$$\nabla p = \frac{1}{2\pi}(\nabla(a \log|\cdot|)) * \partial_i \partial_j u^i u^j + \frac{1}{2\pi}(\partial_i \partial_j \nabla(1-a)\log|\cdot|) * u^i u^j, \tag{8.1}$$

where a lies in $C_c^\infty(\mathbb{R}^2)$ with $a = 1$ near the origin.

Taniuchi uses (8.1) to prove that his smooth solutions are mild solutions to the Euler equations in the sense that

$$u(t) = u(s) - \int_s^t \mathcal{P}(u \cdot \nabla u)(\tau)\, d\tau, \tag{8.2}$$

where $\mathcal{P}$ is the Helmholtz operator on $\mathbb{R}^2$, defined in terms of Riesz transforms. Making a Littlewood-Paley decomposition, Taniuchi uses (8.2) in somewhat the same way that we, following Serfati, use (1.5) to obtain a uniform bound on the L^∞-norm of the approximating velocities. Taniuchi establishes a uniform-in-viscosity bound using the vorticity equation for the Navier-Stokes equations, including the case of zero viscosity, to show that $\omega(t)$ remains bounded in L^∞. (We use the transport of vorticity by the flow map for the approximate solutions to show this.) Using these uniform bounds, he ultimately obtains convergence of a subsequence to a solution of the Euler equations having sublinear growth of the pressure at infinity. This solution, however, is not shown to satisfy the Serfati identity, (1.5) (it is shown in [19], however, that it does.)

In [33], the authors establish a type of continuous dependence on initial data (including uniqueness as an important special case) for solutions to the Euler equations lying in S. They start with the solutions to the Euler equations constructed in [32], first showing that the pressure satisfies

$$p = \sum_{j,k=1}^{2} R_j R_k (u^j u^k),$$

with p lying in BMO, where the R_j are Riesz transforms. (That this might be the key to uniqueness is suggested by the result of [12, 18] on uniqueness of unbounded solutions to the Navier-Stokes equations.) This identity, along with the estimates established in [32], is sufficient for the authors to apply an adaptation of the fundamental uniqueness argument of Vishik in [35] to prove uniqueness (and continuous dependence on initial data) assuming that pressure grows sublinearly at infinity.

Vishik's uniqueness argument, like ours or Serfati's, does not employ an energy argument. Vishik employs in a critical way the $B^0_{\infty,1}$-norm (and ultimately a borderline Besov space norm he defines) of the difference, w, between velocities. We, on the other hand, employ the L^∞-norm of the flow map associated with w (and so also w itself). Since the $B^0_{\infty,1}$-norm of w is defined in terms of the L^∞-norms of the Littlewood-Paley operators applied to w, these are perhaps not so far apart in spirit, though the proofs are radically different.

Properties of the flow map are used in [33] only for a smoothed version of the velocity field (suppressing high frequencies using a Littlewood-Paley operator) and no vorticity is assumed to be transported by the flow. This brings up the question of whether it is possible to establish the existence of a flow map for the solutions constructed in [29, 33]. That this is so is proven indirectly in [19], by showing that the solutions we constructed in Section 4 have sublinear growth of the pressure. Since Taniuchi's uniqueness proof only relies upon this fact, Taniuchi's solutions are the same as our own, which were constructed so that the vorticity is transported by the flow map.

The recent paper [7] also works in larger spaces than S (spaces much like those of [32, 33]) and employs paradifferential calculus. Like [32], their existence argument uses the smooth non-decaying solutions constructed by Serfati in [30], though the proof differs from that of [32].

There is, in effect, another proof of uniqueness for Serfati initial velocity in [8], where the short-time vanishing viscosity limit of solutions to the Navier-Stokes equations to a solution to the Euler equations is proved. (The short-time result in [8] is improved to arbitrarily large finite time in [9], but with the additional assumption that the initial velocity is in L^2. This last assumption is subsequently dropped in [10].) The uniqueness of the solutions to the Euler equations then follows since the solutions to the Navier-Stokes equations in this setting were shown to be unique in [13]. Cozzi's approach departs significantly both from our approach and that of Vishik's as employed in [33]. Letting w be the difference between the Navier-Stokes and Euler solutions, she uses the mild formulation of the solutions to control the low frequencies of w, the boundedness of vorticity to control the high frequencies, and controls the middle frequencies by reducing the problem to proving the vanishing viscosity limit in the homogenous Besov space, $\dot{B}^0_{\infty,\infty}$ It is easier to obtain the vanishing viscosity limit in this space because Calderon-Zygmund operators are bounded on $\dot{B}^0_{\infty,\infty}$ but not on L^∞.

We stress that none of the approaches to existence or uniqueness in [35, 32, 33, 7, 8, 9, 10] is adaptable to an exterior domain because of their use of Littlewood-Paley theory and paradifferential calculus.

In addition to the works we have discussed we would like to mention other instances of the use of Serfati's idea to obtain L^∞ estimates; namely, [27, 28, 14, 15, 1].

Finally, we conclude by noting that, in [11], Gallay and Slijepčević study the related problem of long-time dynamics of viscous flows where no decay of velocity is assumed. This is an interesting problem also in the inviscid case.

Appendix A. Estimates for the Biot-Savart kernel

In this section, we derive the estimates on the Biot-Savart kernel and its derivatives stated in Propositions 3.1 through 3.3. We follow the basic approach of employing a conformal map developed in [16, 17], but must extend the methods considerably to deal with higher derivatives. Because of the use of a conformal map this approach is specific to 2D domains. (The exterior of multiply connected domains could be treated as in [17], at the expense of considerable extra complexity.)

We give the proofs of Propositions 3.1 through 3.3 first for the full plane in Section A.1, then for the exterior of the unit disk in Section A.2, and finally for a domain exterior to an obstacle—a general smooth, connected and simply connected, bounded domain with C^∞ boundary—in Section A.3.

The estimates in the full plane are the simplest, as the Biot-Savart kernel, which has an explicit form, has the greatest degree of symmetry. For the exterior of the unit disk, the Biot-Savart kernel can also be written explicitly, but the presence of the boundary induces a type of distortion that complicates the estimates considerably. It is this case that will consume most of our efforts. The exterior of an obstacle can be treated by employing a conformal map provided by the Riemann mapping theorem. Because this conformal map is well-behaved we can transfer all of the key estimates for the exterior of the unit disk to apply to the exterior of the obstacle as well.

A.1. The Biot-Savart kernel in the full plane.

In this subsection we obtain the estimates in Propositions 3.1 through 3.3 that apply specifically to the full plane: (3.2), (3.3), $(3.9)_1$, and $(3.10)_1$. As we will see in Section A.2, the Biot-Savart kernel, K, for the full plane appears in the expressions for the Biot-Savart kernel, K_Ω, for the exterior of the unit disk. Not surprisingly, then, the estimates developed in this subsection are fundamental to the estimates in Section A.2.

Proof of Proposition 3.1 for the full plane. We can easily prove (3.2) by integrating using polar coordinates centered at x:

$$\|a_\lambda(x-y)K(x-y)\|_{L^1_y(\mathbb{R}^2)} \le \frac{2\pi}{2\pi}\int_0^{C\lambda} \frac{r\,dr}{r} = C\lambda,$$

where C is given in terms of the size of the support of a.

For (3.3), we need first to make several estimates. We begin by computing the first and second-order derivatives of

$$-K^\perp(z) = N(z) \equiv \frac{z}{2\pi|z|^2}.$$

We have

$$\partial_{z_p} N^j(z) = \frac{\delta_{jp}}{2\pi|z|^2} - \frac{z_j z_p}{\pi|z|^4}, \tag{A.1}$$

and

$$\partial_{z_m}\partial_{z_p} N^j(z) = -\frac{z_m\delta_{jp} + z_p\delta_{jm} + z_j\delta_{mp}}{\pi|z|^4} + 4\frac{z_j z_m z_p}{\pi|z|^6}. \tag{A.2}$$

It is clear, then, that there exists $C > 0$ such that

$$\begin{aligned} \left|\partial_{y_p}[K^j(x-y)]\right| &\le C\,|x-y|^{-2}, \\ \left|\partial_{y_m}\partial_{y_p}[K^j(x-y)]\right| &\le C\,|x-y|^{-3}. \end{aligned} \tag{A.3}$$

We have,

$$\begin{aligned} &\nabla_y\nabla_y((1-a_\lambda(x-y))K^j(x-y)) \\ &\quad= \nabla_y\left[((1-a_\lambda(x-y)))\nabla_y K^j(x-y) - \nabla_y a_\lambda(x-y)K^j(x-y)\right] \\ &\quad= ((1-a_\lambda(x-y)))\nabla_y\nabla_y K^j(x-y) - 2\nabla_y a_\lambda(x-y)\otimes\nabla_y K^j(x-y) \\ &\qquad\quad - \nabla_y\nabla_y a_\lambda(x-y)K^j(x-y). \end{aligned}$$

Suppose that a is supported on B_c, the ball of radius $c > 0$ centered at the origin, with $a \equiv 1$ on $B_{c'}$, the ball centered at the origin with radius c' saitisfying $0 < c' < c$, and let

$$A_\lambda(x) = \{y \in \Omega \colon c'\lambda < |x-y| < c\lambda\}. \tag{A.4}$$

Then

$$|\nabla_y a_\lambda(x-y)| \le C\lambda^{-1} \text{ and } |\nabla_y\nabla_y a_\lambda(x-y)| \le C\lambda^{-2}, \tag{A.5}$$

with each function supported on y in $A_\lambda(x)$.

Continuing to estimate the term $|\nabla_y\nabla_y((1-a_\lambda(x-y))K^i(x-y))|$, we write

$$\left|\nabla_y\nabla_y((1-a_\lambda(x-y))K^i(x-y))\right| \le (f_1+f_2+f_3)(x,y),$$

where

$$\begin{aligned} f_1 = f_1(x,y) &= \left|((1-a_\lambda(x-y)))\nabla_y\nabla_y K^j(x-y)\right|, \\ f_2 = f_2(x,y) &= 2\left|\nabla_y a_\lambda(x-y)\otimes\nabla K^j(x-y)\right|, \\ f_3 = f_3(x,y) &= \left|\nabla_y\nabla_y a_\lambda(x-y)K^j(x-y)\right|. \end{aligned}$$

Observe that $f_j \geq 0$, $j = 1, 2, 3$; $f_1(x, y)$ is supported on $|x - y| \geq c'\lambda$; $f_2(x, y)$ and $f_3(x, y)$ are supported on $y \in A_\lambda(x)$. Thus, combining the bounds we obtained we find

$$\begin{aligned} f_1(x,y) &\leq \frac{C}{|x-y|^3} \quad , \quad f_2(x,y) \leq \frac{C}{\lambda\,|x-y|^2}, \\ f_3(x,y) &\leq \frac{C}{\lambda^2\,|x-y|}. \end{aligned}$$

The necessary estimates for f_1, f_2, f_3 can be easily derived:

$$\begin{aligned} \|f_1(x,y)\|_{L^1_y(\Omega)} &\leq 2\pi C \int_{c'\lambda}^{\infty} \frac{r\,dr}{r^3} = \frac{C}{\lambda}, \\ \|f_2(x,y)\|_{L^1_y(\Omega)} &\leq \frac{2\pi C}{\lambda} \int_{c'\lambda}^{c\lambda} \frac{r\,dr}{r^2} = \frac{C}{\lambda}\left[\log(c\lambda) - \log(c'\lambda)\right] = \frac{C}{\lambda}, \\ \|f_3(x,y)\|_{L^1_y(\Omega)} &\leq \frac{2\pi C}{\lambda^2} \int_{c'\lambda}^{c\lambda} \frac{r\,dr}{r} = \frac{C}{\lambda^2}\left[c\lambda - c'\lambda\right] = \frac{C}{\lambda}. \end{aligned}$$

Together these bounds yield (3.3), establishing the estimates for the full plane. □

Proof of Proposition 3.2 for the full plane. Since $|K(x - y)|$ is a strictly decreasing function of the distance from x, it follows that $\|K(x - \cdot)\|_{L^p(U)}$ is maximized over all subsets $U \subset \mathbb{R}^2$ with $|U| = 2\pi R^2$ when $U = B_R(x)$, the ball of radius R centered at x. Thus,

$$\begin{aligned} \|K(x-\cdot)\|^p_{L^p(U)} \leq \|K(x-\cdot)\|^p_{L^p(B_R(x))} &= 2\pi \int_0^R \frac{r\,dr}{(2\pi)^p r^p} \\ &= (2\pi)^{1-p} \frac{R^{2-p}}{2-p}, \end{aligned}$$

giving $(3.9)_1$. □

Lemma A.1 is used in our proof of Proposition 3.3 for the full plane, below.

Lemma A.1. *For any $p, q \geq 1$ with $p^{-1} + q^{-1} = 1$,*

$$|K(x-y_1) - K(x-y_2)| \leq \frac{2^{\frac{1}{p}}\,|y_1-y_2|^{\frac{1}{q}}}{2\pi \min(|x-y_1|\,,|x-y_2|)^{2-\frac{1}{p}}}.$$

Proof. Before we begin, we mention the following identity, which we will use:

$$2\pi|K(z_1) - K(z_2)| = \frac{|z_1 - z_2|}{|z_1||z_2|},$$

for any z_1 and z_2. This identity may be verified by a direct calculation.

Now, let $a = |x - y_1|$, $b = |x - y_2|$, and let θ be the angle between $x - y_1$ and $x - y_2$. Then

$$
\begin{aligned}
2\pi \left| K(x - y_1) - K(x - y_2) \right| &= \frac{|y_1 - y_2|}{|x - y_1|\,|x - y_2|} = \frac{|y_1 - y_2|^{\frac{1}{p}}\,|y_1 - y_2|^{\frac{1}{q}}}{|x - y_1|\,|x - y_2|} \\
&= \frac{(a^2 + b^2 - 2ab\cos\theta)^{\frac{1}{2p}}}{ab}\,|y_1 - y_2|^{\frac{1}{q}} \\
&= (a^{2-2p}b^{-2p} + a^{-2p}b^{2-2p} - 2a^{1-2p}b^{1-2p}\cos\theta)^{\frac{1}{2p}}\,|y_1 - y_2|^{\frac{1}{q}} \\
&\le \left((ab)^{-2p}(a^2 + b^2 + 2ab)\right)^{\frac{1}{2p}}\,|y_1 - y_2|^{\frac{1}{q}} = \left(\frac{(a+b)^2}{(ab)^2(ab)^{2p-2}}\right)^{\frac{1}{2p}}\,|y_1 - y_2|^{\frac{1}{q}} \\
&= \left(\frac{(a^{-1} + b^{-1})^2}{(ab)^{2p-2}}\right)^{\frac{1}{2p}}\,|y_1 - y_2|^{\frac{1}{q}} \le \left(\frac{(2\min(a,b)^{-1})^2}{(\min(a,b)^2)^{2p-2}}\right)^{\frac{1}{2p}}\,|y_1 - y_2|^{\frac{1}{q}} \\
&= \left(\frac{4}{\min(a,b)^{4p-2}}\right)^{\frac{1}{2p}}\,|y_1 - y_2|^{\frac{1}{q}},
\end{aligned}
$$

from which the result follows. □

Proof of Proposition 3.3 for the full plane. Set $A = A(z) = K(x - X_1(z)) - K(x - X_2(z))$. It follows from Lemma A.1 that, for any p, $q > 1$, with $p^{-1} + q^{-1} = 1$,

$$
\begin{aligned}
\|A\|_{L^1_z(U)} &\le C \left\| \frac{|X_1(z) - X_2(z)|^{\frac{1}{q}}}{\min(|x - X_1(z)|, |x - X_2(z)|)^{2-\frac{1}{p}}} \right\|_{L^1_z(U)} \\
&\le C\delta^{\frac{1}{q}} \sum_{j=1}^{2} \left\| \frac{1}{|x - X_j(z)|^{2-\frac{1}{p}}} \right\|_{L^1_z(U)} = C\delta^{\frac{1}{q}} \sum_{j=1}^{2} \left\| \frac{1}{|x - y|^{2-\frac{1}{p}}} \right\|_{L^1_y(X_j(U))} \\
&= C\delta^{\frac{1}{q}} \sum_{j=1}^{2} \|K(x - y)\|^{2-\frac{1}{p}}_{L_y^{2-\frac{1}{p}}(X_j(U))}.
\end{aligned}
$$

Let $R > 0$ be such that $|U| = 2\pi R^2$ and apply $(3.9)_1$ of Proposition 3.2 to obtain

$$
\|A\|_{L^1_z(U)} \le CpR^{\frac{1}{p}}\delta^{1-\frac{1}{p}} \le Cp\max\{1, R\}\delta^{1-\frac{1}{p}}.
$$

Whenever $\delta < e^{-1}$, this bound is minimized, relative to p, when $p = -\log\delta$, giving

$$
\|A\|_{L^1_z(U)} \le C\max\{1, R\}(-\log\delta)\delta^{1+\frac{1}{\log\delta}} = C\max\{1, R\}e(-\log\delta)\delta,
$$

which is $(3.10)_1$. □

A.2. **The Biot-Savart kernel exterior to the unit disk.** In this subsection we prove the estimates in Propositions 3.1 through 3.3 that apply to an exterior domain in the special case where Ω is the domain exterior to the (closed) unit disk. These estimates are those in (3.4) through (3.7), $(3.9)_{2,3}$, and (3.10).

Let

$$
\Omega = \overline{B}^C \equiv \mathbb{R}^2 \setminus \overline{B_1(0)}.
$$

Let $K_{\overline{B}^C} = K_{\overline{B}^C}(x, y) \equiv \nabla_x^\perp G_{\overline{B}^C}(x, y)$, where $G_{\overline{B}^C}$ is the Green's function for this domain. With K as in (1.4), the Biot-Savart kernel for all of $\mathbb{R}^2$, it is classical that

$$
K_\Omega(x, y) = K_{\overline{B}^C}(x, y) = K(x - y) - K(x - y^*), \tag{A.6}
$$

with $y^* = y/|y|^2$.

We will prove the estimates in

Our next lemma gives us some limited control over how much $K(x-y^*)$ differs from $K(x-y)$.

Lemma A.2. *Let $x \in \mathbb{R}^2$ such that $|x| > 1$. Then*

$$\frac{|x-y|}{|x-y^*|} \le \max\{2, 2R\} \le 2(1+R) \tag{A.7}$$

for all y in Ω such that $|x-y| \le R$. Also,

$$\frac{1}{|x-y^*|} \le 2 \tag{A.8}$$

for all $y \in \Omega$ with $|x-y| \ge 1$.

Proof. We assume without loss of generality that $x = (a, 0)$ lies along the positive x-axis. Let $y \in \Omega \cap B_R(x)$ with $|y| = r$ and set θ be the angle between y and x. Assume that $x \ne y$. Then, fixing a and r, let

$$k(\theta) := \frac{|x-y|^2}{|x-y^*|^2} = \frac{a^2 + r^2 - 2ar\cos\theta}{a^2 + \frac{1}{r^2} - 2\frac{a}{r}\cos\theta}.$$

Direct calculations show that the only solutions to $k'(\theta) = 0$ are $\theta = 0$ and $\theta = \pi$, and that $k''(0) > 0$ while $k''(\pi) < 0$. Thus, k is maximized when $\theta = \pi$. (The maximum may occur for y on $\partial B_1(0)$.) We then write

$$k(\pi) = \frac{a^2 + r^2 + 2ar}{a^2 + r^{-2} + 2ar^{-1}} = \left(\frac{a+r}{a+r^{-1}}\right)^2.$$

If y lies along the negative real axis, then a and r must be less than R, so that $a + r \le 2R$. Then, since also $a + r^{-1} \ge 1$, we have that $k(\pi) \le 4R^2$. If y lies along the positive real axis then $r < a$, so $k(\pi) < (2a/a)^2 = 4$. This gives (A.7).

Similarly, letting $m(\theta) = |x-y^*|^2 = a^2 + r^{-2} - 2ar^{-1}\cos\theta$ for fixed a and r, we have $m'(\theta) = 2a\sin(\theta)/r$ and $m''(\theta) = 2a\cos(\theta)/r$. Thus, the minimum of $m(\theta)$ occurs at $\theta = 0$, where $m(0) = a^2 + r^{-2} - 2ar^{-1} = (a - r^{-1})^2$. But if $|x-y| = M \ge 1$ then $r = a + M$, so that

$$m(0)^{\frac{1}{2}} = a - \frac{1}{a+M} \ge 1 - \frac{1}{a+1} = \frac{a}{a+1}.$$

Thus,

$$\frac{1}{|x-y^*|} \le \frac{1}{m(0)^{\frac{1}{2}}} = 1 + \frac{1}{a} \le 2,$$

since $|a| \ge 1$. This is (A.8). □

Lemma A.3. *For all $R \ge 2$,*

$$\inf\{|x|\,|y| : x, y \in \Omega, |x-y| = R\} = R - 1.$$

Proof. Begin by observing that, using Lagrange multipliers,

$$\min_{\{|x-y|^2 = R^2,\, x,y \in \Omega\}} \{|x|^2|y|^2\}$$

is attained when either x and y are linearly dependent or when one of x or y is on the boundary $\partial\Omega$. In the latter case, assuming without loss of generality that $|y| = 1$, we have

$$|x||y| = |x| \geq |x - y| - |y| = R - 1,$$

as desired. Otherwise, if x and y are linearly dependent then $x = \beta y$, $\beta \in \mathbb{R}$ and the result follows easily from $|x - y| = R$. □

In the proof of existence we make use of a modified Biot-Savart kernel, the hydrodynamic Biot-Savart kernel (3.1). In the case of the exterior of the unit disk this kernel is given by:

$$J_\Omega(x, y) = J_{\overline{B}^C}(x, y) \equiv K_{\overline{B}^C}(x, y) + K(x). \tag{A.9}$$

Lemma A.4. *Let*

$$L(x, y) = K(x - y^*) - K(x). \tag{A.10}$$

There exists a constant $C > 0$ such that, for all x, y in $\overline{B}^C$, we have

$$|J_\Omega(x, y)| \leq \frac{C}{|x - y|}, \tag{A.11}$$

$$|L(x, y)| \leq \frac{C}{|x - y|}. \tag{A.12}$$

Proof. We have,

$$\begin{aligned}
|L(x, y)| &= \frac{1}{2\pi}\left|\frac{x^\perp}{|x|^2} - \frac{x^\perp - (y^*)^\perp}{|x - y^*|^2}\right| = \frac{1}{2\pi}\frac{|y^*|}{|x|\,|x - y^*|} \\
&= \frac{1}{2\pi}\frac{1}{|y|\,|x|\,|x - y^*|} \leq \frac{C(1 + |x - y|)}{\max\{1, |x - y| - 1\}\,|x - y|} \\
&= \frac{C(1 + s)}{\max\{1, s - 1\}\,|x - y|},
\end{aligned}$$

where $s = |x - y|$. In the last inequality, we used Lemmas A.2 and A.3.

Let $g(s) = (1 + s)/\max\{1, s - 1\}$. When $s \leq 2$, $g(s) \leq 1 + s \leq 3$, and when $s > 2$,

$$g(s) = \frac{1 + s^{-1}}{1 - s^{-1}} < \frac{2}{\frac{1}{2}} = 4.$$

Hence, $|L(x, y)| \leq C/|x - y|$. But $J_\Omega(x, y) = K(x - y) + L(x, y)$ and $|K(x - y)| = C/|x - y|$, so the same inequality applies to J. □

***Proof of Proposition 3.1** for $\overline{B}^C$.* Due to Lemma A.4, (3.4) follows directly from (3.2).

To establish (3.6), we need only establish it with L of (A.10) in place of J_Ω, for then we can add that bound to (3.3).

We have,

$$\begin{aligned}
\partial_{y_n}\partial_{y_j}(K^i(x - y^*)) &\equiv \partial_n\partial_j(K^i(x - y^*)) = -\partial_n(\partial_{y_k^*}K^i(x - y^*)\partial_j y_k^*) \\
&= -\partial_{y_k^*}K^i(x - y^*)\partial_n\partial_j y_k^* - \partial_n\partial_{y_k^*}K^i(x - y^*)\partial_j y_k^* \\
&= -\partial_{y_k^*}K^i(x - y^*)\partial_n\partial_j y_k^* + \partial_{y_m^*}\partial_{y_k^*}K^i(x - y^*)\partial_n y_m^*\partial_j y_k^*.
\end{aligned}$$

But,

$$\partial_j y_k^* = \partial_j \frac{y_k}{|y|^2} = -2\frac{y_k}{|y|^3}\partial_j |y| + \frac{\delta_{jk}}{|y|^2} = -2\frac{y_j y_k}{|y|^4} + \frac{\delta_{jk}}{|y|^2},$$

so

$$\begin{aligned}
\partial_n \partial_j y_k^* &= \partial_n \left(-2\frac{y_j y_k}{|y|^4} + \frac{\delta_{jk}}{|y|^2} \right) \\
&= 8\frac{y_j y_k}{|y|^5}\partial_n |y| - 2\frac{\partial_n y_j y_k}{|y|^4} - 2\frac{y_j \partial_n y_k}{|y|^4} - 2\frac{\delta_{jk}}{|y|^3}\partial_n |y| \\
&= 8\frac{y_j y_k y_n}{|y|^6} - 2\frac{\delta_{jn} y_k}{|y|^4} - 2\frac{\delta_{nk} y_j}{|y|^4} - 2\frac{\delta_{jk} y_n}{|y|^4}.
\end{aligned}$$

Thus,

$$|\partial_n \partial_j y_k^*| \le C\,|y|^{-3}, \quad |\partial_n y_m^* \partial_j y_k^*| \le C\,|y|^{-4}.$$

Hence,

$$\left|\partial_{y_n} \partial_{y_j} (K^i(x - y^*))\right| \le C \frac{\left|\partial_{y_k^*} K^i(x - y^*)\right|}{|y|^3} + \frac{\left|\partial_{y_m^*} \partial_{y_k^*} K^i(x - y^*)\right|}{|y|^4}.$$

Clearly, from (A.1, A.2) we obtain that

$$\begin{aligned}
\left|\partial_{y_k^*} K^i(x - y^*)\right| &\le C\,|x - y^*|^{-2}, \\
\left|\partial_{y_m^*} \partial_{y_k^*} K^i(x - y^*)\right| &\le C\,|x - y^*|^{-3},
\end{aligned}$$

so that

$$\begin{aligned}
\left|\partial_{y_n} (K^i(x - y^*))\right| &\le \frac{C}{|x - y^*|^2\,|y|^2}, \\
\left|\partial_{y_n} \partial_{y_j} (K^i(x - y^*))\right| &\le \frac{C}{|x - y^*|^2\,|y|^3} + \frac{C}{|x - y^*|^3\,|y|^4}.
\end{aligned} \tag{A.13}$$

Then,

$$\begin{aligned}
&\nabla_y \nabla_y ((1 - a_\lambda(x - y)) L^i(x, y)) \\
&\quad = \nabla_y \left[((1 - a_\lambda(x - y))) \nabla_y L^i(x, y) - \nabla_y a_\lambda(x - y) L^i(x, y) \right] \\
&\quad = ((1 - a_\lambda(x - y))) \nabla_y \nabla_y K^i(x - y^*) - 2\nabla_y a_\lambda(x - y) \otimes \nabla K^i(x - y^*) \\
&\qquad\quad - \nabla_y \nabla_y a_\lambda(x - y) L^i(x - y^*).
\end{aligned}$$

It is only the one, final, term in which L appears in place of K.

Thus,

$$\left|\nabla_y \nabla_y ((1 - a_\lambda(x - y)) L^i(x, y))\right| \le (f_1 + f_2 + f_3)(x, y), \tag{A.14}$$

where

$$\begin{aligned}
f_1 &= f_1(x, y) = \left|((1 - a_\lambda(x - y))) \nabla_y \nabla_y K^i(x - y^*)\right|, \\
f_2 &= f_2(x, y) = 2\left|\nabla_y a_\lambda(x - y) \otimes \nabla_y K^i(x - y^*)\right|, \\
f_3 &= f_3(x, y) = \left|\nabla_y \nabla_y a_\lambda(x - y) L^i(x, y)\right|.
\end{aligned}$$

Define A_λ as in (A.4). Observe, then, that f_1 is supported on $|x-y| \geq C_1\lambda$, while $f_2(x,y)$ and $f_3(x,y)$ are supported on $y \in A_\lambda(x)$. Because of $(A.13)_2$, it is natural to decompose f_1 as $f_1 = f_{1,1} + f_{1,2}$ in such a way that

$$|f_{1,1}(x,y)| \leq \frac{C}{|x-y^*|^2\,|y|^3}, \quad |f_{1,2}(x,y)| \leq \frac{C}{|x-y^*|^3\,|y|^4}.$$

From (A.5), (A.12), and $(A.13)_1$, we obtain

$$|f_2(x,y)| \leq \frac{C}{\lambda\,|x-y^*|^2\,|y|^2}, \quad |f_3(x,y)| \leq \frac{C}{\lambda^2\,|x-y|}.$$

Set $F_{1,j} = \|f_{1,j}(x,\cdot)\|_{L^1}$, $j=1,2$ and $F_j(x) = \|f_j(x,\cdot)\|_{L^1}$, $j=2,3$.

The bound on F_3 is very simple and applies without restriction on $\lambda > 0$:

$$F_3(x) \leq \frac{C}{\lambda^2}\int_{c'\lambda}^{c\lambda} \frac{r\,dr}{r} = C\frac{(c-c')\lambda}{\lambda^2} \leq \frac{C}{\lambda}.$$

To bound $F_{1,1}$, $F_{1,2}$, and F_2, fix x in Ω and let

$$U_\lambda(x) = \{y \in \Omega\colon\ |x-y| > \lambda\}.$$

Without loss of generality assume that $c' = 1$ (see (A.4)). With this choice of c', we can set $C_0 = 2$ (at the end of the proof we will reduce this to $C_0 = 1$). Our goal now is to show that (3.6) through (3.8) hold for all $\lambda > C_0$.

Assume first that $\lambda > C_0\,|x|$.

For all y in $U_\lambda(x)$, we have $|y| > \lambda - |x| > (C_0 - 1)\,|x| \geq C_0 - 1$ and hence $|x-y^*| > |x| - (C_0-1)^{-1} \geq 1 - (C_0-1)^{-1} =: \alpha > 0$. Thus,

$$\begin{aligned} F_{1,1}(x) &= \int_{U_\lambda(x)} f_1(x,y)\,dy \leq \int_{U_\lambda(x)} \frac{C}{|x-y^*|^2\,|y|^3}\,dy \leq \frac{C}{\alpha^2}\int_{U_\lambda(x)} \frac{1}{|y|^3}\,dy \\ &\leq C\int_{|y|>\lambda-|x|} \frac{1}{|y|^3}\,dy = C\int_{\lambda-|x|}^\infty \frac{r\,dr}{r^3} = \frac{C}{\lambda-|x|}. \end{aligned}$$

But, $|x| < \lambda/C_0$ so $\lambda - |x| > \lambda(1-C_0^{-1})$. Hence,

$$F_{1,1}(x) \leq \frac{1}{1-C_0^{-1}}\frac{1}{\lambda} = \frac{C}{\lambda}.$$

A similar estimate for $F_{1,2}$ gives $F_{1,2}(x) \leq C\lambda^{-2}$.

To estimate F_2, first observe that for all y in $A_\lambda(x)$, $C_1\lambda < \lambda - |x| < |y| < c\lambda + |x| < C_2\lambda$, where $C_1 = 1 - C_0^{-1}$ and $C_2 = c + C_0^{-1}$. (The values of C_1 and C_2 come from our assumption that $\lambda > C_0\,|x|$.) Then, since $f_2(x,\cdot)$ is supported in $A_\lambda(x) \subseteq U_\lambda(x)$, and $|x-y^*| > \alpha$ for all y in $U_\lambda(x)$, as we observed above, we have

$$F_2(x) \leq \frac{C}{\lambda}\int_{A_\lambda(x)} \frac{dy}{|y|^2} \leq \frac{C}{\lambda}\int_{C_1\lambda}^{C_2\lambda} \frac{r\,dr}{r^2} = C\frac{\log(C_2\lambda) - \log(C_1\lambda)}{\lambda} = \frac{C}{\lambda}.$$

Together, these bounds give (3.6, 3.7) when $\lambda > C_0\,|x|$.

Now assume that $|x| > 2$ and that $\lambda > 0$. Then $|x-y^*| \geq |x| - 1 \geq \frac{1}{2}\,|x|$, so we can simply estimate,

$$\begin{aligned} F_{1,1}(x) &\leq \frac{C}{|x|^2}\int_1^\infty \frac{r\,dr}{r^3} = \frac{C}{|x|^2} \leq \frac{C}{|x|}, \\ F_{1,2}(x) &\leq \frac{C}{|x|^3}\int_1^\infty \frac{r\,dr}{r^4} = \frac{C}{|x|^3} \leq \frac{C}{|x|}. \end{aligned}$$

For F_2, since $A_\lambda(x)$ is contained in the annulus centered at the origin of inner radius 1 and outer radius $c\lambda + |x|$, we have

$$\begin{aligned}
F_2(x) &\le \frac{C}{\lambda\,|x|^2}\int_1^{c\lambda+|x|}\frac{r\,dr}{r^2} = \frac{C\log(c\lambda+|x|)}{\lambda\,|x|^2} \le \frac{C\log(2\max\{c\lambda,|x|\})}{\lambda\,|x|^2} \\
&\le C\max\left\{\frac{\log(2c\lambda)}{\lambda}\frac{1}{|x|^2}, \frac{\log(2\,|x|)}{|x|}\frac{1}{\lambda\,|x|}\right\} \\
&\le C\max\left\{\frac{1}{|x|^2}, \frac{1}{\lambda\,|x|}\right\} \le C\max\left\{\frac{1}{|x|}, \frac{1}{\lambda}\right\}.
\end{aligned}$$

Now if $\lambda \le C_0\,|x|$ then $|x|^{-1} \le C\lambda^{-1}$, and these bounds, along with the earlier bound for $\lambda > C_0\,|x|$, give (3.6, 3.7) for all $\lambda > 0$ when $|x| > 2$.

On the other hand, if $1 < |x| < 2$ then the restriction that $\lambda > C_0\,|x|$ is satisfied if $\lambda > 2C_0$. Relabeling $2C_0$ to be C_0, this gives the stated result for all x in Ω when $\lambda > C_0$.

The bounds in (3.5, 3.8) now follow immediately from (3.4, 3.6) and the observations that

$$\|a_\lambda(x-y)K(x)\|_{L^1_y(\Omega)} \le \frac{1}{2\pi\,|x|}\,\|a_\lambda(x-y)\|_{L^1_y(\Omega)} \le C\lambda^2,$$

since $|x| \ge 1$, and

$$\begin{aligned}
&\|\nabla_y\nabla_y((1-a_\lambda(x-y))K(x))\|_{L^1_y(\Omega)} \\
&\qquad = \|\nabla_y\nabla_y((1-a_\lambda(x-y))\otimes K(x)\|_{L^1_y(\Omega)} \\
&\qquad \le \frac{C}{\lambda^2}\left(\int_{\operatorname{supp} a_\lambda} 1\right)|K(x)| \le C.
\end{aligned}$$

□

Proof of Proposition 3.2 *for* $\overline{B}^C$. From (A.11), $|J_\Omega(x,y))| \le C\,|K(x-y)|$. Hence, the bound on J_Ω in $(3.9)_3$ follows from $(3.9)_1$, which we proved in Section A.1. The bound on K_Ω in $(3.9)_2$ follows from the bound on J_Ω combined with (3.1) and the boundedness of $\overline{K}_\Omega$ on Ω. □

Proof of Proposition 3.3 *for* $\overline{B}^C$. We start by using the expression for K_Ω in (A.6) to split the left-hand side of $(3.10)_2$ into two terms using the triangle inequality:

$$\begin{aligned}
&\|K_\Omega(x,X_1(z)) - K_\Omega(x,X_2(z))\|_{L^1_z(U)} \\
&\qquad \le \|K(x-X_1(z)) - K(x-X_2(z))\|_{L^1_z(U)} \\
&\qquad\qquad + \|K(x-(X_1(z))^*) - K(x-(X_2(z))^*)\|_{L^1_z(U)} \\
&\qquad \le -C\delta\log\delta + \|K(x-(X_1(z))^*) - K(x-(X_2(z))^*)\|_{L^1_z(U)}.
\end{aligned} \tag{A.15}$$

In the last inequality, we bounded the first of the two L^1 norms using $(3.10)_1$.

We now bound the the remaining L^1 norm in (A.15).

We first observe that for all $x, y \in \Omega$,

$$|x^* - y^*| = \frac{|x-y|}{|x|\,|y|} \le |x-y|,$$

so

$$\|(X_1)^* - (X_2)^*\|_{L^\infty} \le \|X_1 - X_2\|_{L^\infty} = \delta.$$

It also follows from Lemma A.2 that for all $x, y \in \Omega$,

$$\frac{1}{|x-y^*|} \le \frac{2}{|x-y|}(1+|x-y|) \le \frac{2}{|x-y|} + 2.$$

With these two observations, we now proceed as in the proof of Proposition 3.3 for the full plane in Section A.1, setting $A = A(z) = K(x-(X_1(z))^*) - K(x-(X_2(z))^*)$. It follows from Lemma A.1 that, for any p, $q > 1$, with $p^{-1}+q^{-1} = 1$,

$$\begin{aligned}
\|A\|_{L^1_z(U)} &\le \left\| \frac{|(X_1(z))^* - (X_2(z))^*|^{\frac{1}{q}}}{\min(|x-(X_1(z))^*|\,, |x-(X_2(z))^*|)^{2-\frac{1}{p}}} \right\|_{L^1_z(U)} \\
&\le \delta^{\frac{1}{q}} \sum_{j=1}^{2} \left\| \frac{1}{|x-(X_j(z))^*|^{2-\frac{1}{p}}} \right\|_{L^1_z(U)} \\
&= \delta^{1-\frac{1}{p}} \sum_{j=1}^{2} \int_U \frac{dz}{|x-(X_j(z))^*|^{2-\frac{1}{p}}} \\
&\le \delta^{1-\frac{1}{p}} \sum_{j=1}^{2} \int_U 2^{2-\frac{1}{p}} 2^{2-\frac{1}{p}} \left(\frac{1}{|x-X_j(z)|^{2-\frac{1}{p}}} + 1 \right) dz \\
&= 4^{2-\frac{2}{p}} \delta^{1-\frac{1}{p}} \sum_{j=1}^{2} \left(\int_U \frac{dz}{|x-X_j(z)|^{2-\frac{1}{p}}} + |U| \right).
\end{aligned}$$

Let $R > 0$ be such that $|U| = 2\pi R^2$ and apply $(3.9)_1$ of Proposition 3.2 to obtain

$$\begin{aligned}
\|A\|_{L^1_z(U)} &\le 4^{2-\frac{2}{p}} p R^{\frac{1}{p}} \delta^{1-\frac{1}{p}} + 4^{2-\frac{2}{p}} 2\pi R^2 \delta^{1-\frac{1}{p}} \\
&\le C \max\left\{1, R^2\right\} \delta^{1-\frac{1}{p}}(p+1) \\
&= C\delta^{1-\frac{1}{p}}(p+1),
\end{aligned}$$

where C depends only on the measure of U. For $\delta < e^{-1}$, we set $p = -\log\delta$, giving

$$\begin{aligned}
\|A\|_{L^1_z(U)} &\le C\delta^{1+\frac{1}{\log\delta}}(-\log\delta + 1) \\
&= Ce\delta(-\log\delta + 1) \\
&\le -C\delta\log\delta.
\end{aligned}$$

We have now bounded both L^1-norms in (A.15) by $-C\delta\log\delta$. Combining the two bounds gives $(3.10)_2$. □

A.3. **The Biot-Savart kernel exterior to a single obstacle.** In the previous subsection, we obtained estimates on K_Ω and J_Ω for the exterior of the unit disk. In this subsection, we extend these same estimates—those in(3.4) through (3.7), $(3.9)_{2,3}$, and (3.10)—to the domain, Ω, exterior to a bounded simply connected domain having C^∞ boundary.

Denote by B the open ball of radius one centered at the origin. We assume without loss of generality that $B \subseteq \Omega^C$ (else a translation and dilation would make it so). As in [17, 16], we have a C^∞-diffeomorphism (biholomorphishm when treated as a map from and to domains in the complex plane or Riemann sphere), $T \colon \Omega \to \mathbb{R}^2 \setminus \overline{B}$, that extends smoothly to the boundary; see also [3]. By (2.3) of [16], DT and DT^{-1} are both bounded above so, as observed in [17], T is bi-Lipschitz.

We then have

$$\begin{aligned} K_\Omega(x,y) &= K_{\overline{B}^C}(T(x),T(y))DT(x), \\ J_\Omega(x,y) &= J_{\overline{B}^C}(T(x),T(y))DT(x) = K_\Omega(x,y) + \overline{K}_\Omega(x), \end{aligned} \tag{A.16}$$

where

$$\overline{K}_\Omega(x) = K(T(x))DT(x). \tag{A.17}$$

Then because T is Lipschitz and K is bounded on Ω,

$$\left\|\overline{K}_\Omega\right\|_{L^\infty} \le C. \tag{A.18}$$

We will also need an estimate on D^2T.

Lemma A.5. *For some constant, C_1,*

$$\left|D^2T(y)\right| \le C_1 \left|y\right|^{-3}.$$

Proof. Viewing z as a complex variable, it is established in (2.1, 2.2) of [16] that $T(z) = \beta z + h(z)$ for some nonzero real constant, β, and bounded function, h, holomorphic on Ω (as a subset of the Riemann sphere) with $h'(z) = O(z^{-2})$ as $|z| \to \infty$. It follows that h is analytic at the point at infinity in the Riemann sphere and that $h''(z) = O(|z|^{-3})$. If $h(z) = h(x_1, x_2) = u(x_1, x_2) + iv(x_1, x_2)$ then

$$T''(z) = h''(z) = \frac{\partial^2 u}{\partial x_1^2} + i\frac{\partial^2 v}{\partial x_2^2} = \frac{\partial^2 v}{\partial x_1 \partial x_2} - i\frac{\partial^2 u}{\partial x_1 \partial x_2}.$$

Therefore, $D^2T(y) = O(|y|^{-3})$ as $|y| \to \infty$. The results follows, then, since T is C^∞ up to the boundary. □

In Section A.2, we obtained bounds on J_Ω and K_Ω for Ω the exterior of a unit disk. We now show that these bounds continue to hold for an exterior domain.

Proof of Propositions 3.1 through 3.3 for an exterior domain. Both (3.4) and (3.5) of Proposition 3.1 for an exterior domain follow by making the change of variables, $z = T(y)$, using the boundedness of DT on Ω, and the fact that the estimates in Proposition 3.1 for the unit disk are uniform in x. The proofs of Propositions 3.2 and 3.3 for an exterior domain as well as the bound on the hydrodynamic Biot-Savart kernel, (A.11) of Lemma A.4, require only the boundedness of DT. Also, (3.8) follows from (3.6) (which we establish below) using the same argument as used for the exterior of the unit disk along with the boundedness of DT.

It remains to prove (3.6) and (3.7).

Now,

$$\begin{aligned} &\nabla_y \nabla_y ((1 - a_\lambda(x-y)) J^i_\Omega(x,y)) \\ &\quad = \nabla_y \left[((1 - a_\lambda(x-y)))\nabla_y J^i_\Omega(x,y) - \nabla_y a_\lambda(x-y) J^i_\Omega(x,y)\right] \\ &\quad = \Lambda_1 + \Lambda_2 + \Lambda_3, \end{aligned}$$

where

$$\begin{aligned} \Lambda_1 &= ((1 - a_\lambda(x-y)))\nabla_y \nabla_y J^i_\Omega(x,y), \\ \Lambda_2 &= -2\nabla_y a_\lambda(x-y) \otimes \nabla_y J^i_\Omega(x,y), \\ \Lambda_3 &= -\nabla_y \nabla_y a_\lambda(x-y) J^i_\Omega(x,y). \end{aligned}$$

By virtue of (A.5) and (A.11) we can bound the L^1 norm of Λ_3 as we did for the function, f_3, in the proof of Proposition 3.1 for $\overline{B}^C$ to conclude that $\|\Lambda_3(x,y)\|_{L^1_y(\Omega)} \le C\lambda^{-1}$.

To treat Λ_1 and Λ_2 we first introduce some notation for differentials. Since J_Ω is a function of two variables, we will write D_2 to mean the differential with respect to the second variable. (So far we have been following the convention common in fluid mechanics of writing ∇ in place of D, even for a 2-tensor.)

For Λ_1 and Λ_2, we calculate,

$$\begin{aligned}
\nabla_y J^i_\Omega(x,y) &= \nabla_y (J_{\overline{B}^C}(T(x),T(y))DT(x)) \\
&= D_2 J_{\overline{B}^C}(T(x),T(y))DT(x)DT(y), \\
\nabla_y \nabla_y J^i_\Omega(x,y) &= D_2^2 J_{\overline{B}^C}(T(x),T(y))DT(x)(DT(y))^2 \\
&\quad + D_2 J_{\overline{B}^C}(T(x),T(y))DT(x)D^2T(y).
\end{aligned}$$

Making the change of variables, $z = T(y)$, the annulus, A_λ, of (A.4) becomes $B_\lambda := \{z \in \Omega\colon c'\lambda < |x - T^{-1}(z)| < c\lambda\}$. Because T is bi-Lipschitz, it follows easily that B_λ is contained in the annulus, A'_λ, centered at u of inner radius, $c'/\left\|DT\right\|_{L^\infty}$, and outer radius, $c\left\|DT^{-1}\right\|_{L^\infty}$.

Thus, the common support of Λ_2 and Λ_3 is distorted by making the change of variable, $z = T(y)$, and its center is moved, but the bounds in (A.5) still apply. This allows us to conclude that Λ_2 and the term,

$$((1 - a_\lambda(x-y)))D_2^2 J_{\overline{B}^C}(T(x),T(y))DT(x)(DT(y))^2,$$

are bounded in the L^1 norm by $C\lambda^{-1}$ for all $\lambda > C_0$.

What remains is to bound the L^1 norm of

$$\Lambda_4 = ((1 - a_\lambda(x-y))D_2 J_{\overline{B}^C}(T(x),T(y))DT(x)D^2T(y).$$

Using $(A.3)_1$, $(A.13)_1$, the bi-Lipschitzness of T, and Lemma A.5,

$$|\Lambda_4(x,y))| \le \left[\frac{C}{|x-y|^2} + \frac{C}{|T(x) - T(y)^*|^2\,|y|^2}\right]\frac{1}{|y|^3}.$$

Now, on $U_\lambda(x) := B^C_{c'\lambda} \cap \Omega$, the support of $(1 - a_\lambda(x - \cdot))$, $|x-y|^{-2} < (c'\lambda)^{-2}$, so

$$\int_{U_\lambda(x)} \frac{C}{|x-y|^2\,|y|^3}\,dy \le \frac{C}{c'\lambda^2}\int_\Omega \frac{dy}{|y|^3} \le \frac{C}{\lambda^2} \le \frac{C}{C_0\lambda} = \frac{C}{\lambda}$$

for all $\lambda > C_0$. The integral above was finite since Ω does not include the origin.

Making the change of variables, $z = T(y)$, we have

$$\frac{C}{|T(x) - T(y)^*|^2\,|y|^5} = \frac{C}{|T(x) - z^*|^2\,|T^{-1}(z)|^5} \le \frac{C}{|T(x) - z^*|^2\,|z|^5},$$

since T is bi-Lipschitz. Note that $T(x)$ lies in $\overline{B}^C$, and in calculating the L^1-norm, z is integrated over $\overline{B}^C$, while the change of variables has unit Jacobian determinant. Hence, we can bound the L^1-norm of this term just as we did $F_{1,1}$ or $F_{1,2}$ in the proof of Proposition 3.1 for $\overline{B}^C$. This leads to

$$\int_{U_\lambda(x)} \frac{C}{|T(x) - T(y)^*|^2\,|y|^5}\,dy < \frac{C}{\lambda}$$

for all $\lambda > C_0$, so that also,

$$\|\Lambda_4(x,y)\|_{L^1_y(\Omega)} < \frac{C}{\lambda}$$

for all $\lambda > C_0$.

Combining these bounds, we obtain (3.6, 3.7) for an exterior domain. $\square$

Appendix B. Approximating the initial data

In our proof of existence of weak solutions to the Euler equations in Sections 4 and 5 we employed a sequence of smooth compactly supported initial velocities that converged to a given initial velocity in the Serfati space, S, of Definition 2.1. In this appendix, we detail the construction of that sequence.

In essence, our approach is very simple and entirely standard: apply a cutoff function to the mollified stream function for the velocity and let the support of the cutoff function increase to fill all of Ω. For the full plane, this is, in fact, all that is required.

For the exterior of a single obstacle, however, there are two technical hurdles which require some work to overcome; namely, the low regularity of the space, S, and the presence of a boundary. In the absence of a boundary, one could simply employ convolution to smooth the stream function. Both these issues are dealt with in Lemma B.1, where we construct a sequence of smooth stream functions converging to the stream function for u. We then cut off this sequence in Proposition B.2 to construct our approximate sequence of initial velocities.

Lemma B.1. *Let $u \in S$. There exist $\psi \in C^1(\overline{\Omega})$ and $(\psi_n)_{n=1}^\infty \in C^\infty(\overline{\Omega})$ such that the following hold:*

(1) *$u = \nabla^\perp \psi$;*
(2) *$\left\|\nabla^\perp \psi_n\right\|_S$ is uniformly bounded with respect to n;*
(3) *there exists $C > 0$ such that $|\psi_n(x)| \le C|x|$ for all $n \in \mathbb{N}$;*
(4) *$\psi_n = 0$ on $\partial\Omega$;*
(5) *for any p in $[1,\infty)$, $\Delta\psi_n \to \Delta\psi$ in $L^p_{loc}(\Omega)$ as $n \to \infty$;*
(6) *$\nabla\psi_n \to \nabla\psi$ in $L^\infty(\Omega)$.*

Proof. It follows from Lemma B.3 that u is uniformly continuous (in fact, log-Lipschitz). Because of this, the stream function, ψ, for u, which satisfies $u = \nabla^\perp\psi$, $\omega = -\Delta\psi$, $\psi = 0$ on Γ, lies in $C^1(\Omega)$. We can explicitly construct this stream function, as follows. Let $x \in \Omega$. Let $T\colon \Omega \to \mathbb{R}^2 \setminus B$ be the conformal map defined in Section A.3. (Here, $B = B_1(0)$ is the unit disk centered at the origin.) Let γ_x be the curve whose image, under T, is the ray joining $T(x)/|T(x)|$ and $T(x)$. Then set

$$\psi = \psi(x) = -\int_{\gamma_x} u^\perp \circ T^{-1} DT^{-1} \cdot d\mathbf{s}. \tag{B.1}$$

Since $\operatorname{div} u = 0$ it follows that $u = \nabla^\perp\psi$. Also, as u is bounded, and we also know that DT^{-1} is bounded and T grows at most linearly, we find that $\psi(x)$ grows at most linearly.

We now begin the construction of the approximate stream functions, ψ_n. First fix $\bar{x} \in \Gamma$. Let U be a Möbius transformation that takes the unit circle to the real axis, with the unit disk mapping to the lower half-plane, and with $U(\bar{x}) = (0,0)$. Let $\Phi = U \circ T$. Observe that Φ is a bi-holomorphism of Ω to the upper half-plane that extends smoothly up to the boundary.

Now let $r = r_{\bar{x}} > 0$ be small enough that $\Phi(B_r(\bar{x}) \cap \Gamma)$ is an interval around $(y_1, y_2) = (0, 0)$ on the y_1-axis and $\Phi(B_r(\bar{x}) \cap \Omega)$ is an open set contained in $y_2 > 0$. Let $s > 0$ be such that $B_s(0) \subset \Phi(B_r(\bar{x}))$. We introduce $\phi = \phi(y)$, $y \in B_s(0)$, as

$$\phi(y) = \begin{cases} \psi(\Phi^{-1}(y_1, y_2)), & y_2 \geq 0, \\ -\psi(\Phi^{-1}(y_1, -y_2)), & \text{otherwise.} \end{cases} \tag{B.2}$$

Let $\eta \in C_c^\infty(\mathbb{R}^2)$ taking values in $[0, 1]$ have total mass 1 and be supported in the unit ball. Assume that $\eta(y_1, -y_2) = \eta(y_1, y_2)$ and set, for each $\lambda > 0$, $\eta_\lambda(y) = \lambda^{-2}\eta(\lambda^{-1}y)$. Define $\phi_\lambda = \eta_\lambda * \phi$ on $B_{s/2}(0)$, for $\lambda < s/2$; because η_λ is supported on $B_\lambda(0)$, this convolution is well-defined. Finally, let

$$\psi_\lambda^{\bar{x}} = \psi_\lambda^{\bar{x}}(x) = \phi_\lambda(\Phi(x))$$

for $x \in \Phi^{-1}(B_{s/2}(0))$.

Because Φ is a bi-holomorhpishm,

$$\begin{aligned} \Delta\psi_\lambda^{\bar{x}} &= \Delta\left((\eta_\lambda * \phi)(\Phi(x))\right) = \left|\Phi'(x)\right|^2 (\Delta(\eta_\lambda * \phi))(\Phi(x)) \\ &= \left|\Phi'(x)\right|^2 (\eta_\lambda * \Delta\phi)(\Phi(x)) \\ &= \left|\Phi'(x)\right|^2 (\eta_\lambda * (\left|(\Phi^{-1})'\right|^2 \Delta\psi)(\Phi(x)). \end{aligned}$$

Thus,

$$\|\Delta\psi_\lambda^{\bar{x}}\|_{L^\infty} \leq C \|\Delta\psi\|_{L^\infty} \leq C.$$

This construction defines a smooth approximation of ψ locally, near the boundary. Moreover, the approximation vanishes on the portion of the boundary where it is defined, due to the fact that we performed an odd extension followed by even mollification. The next step is to use the compactness of the boundary Γ to introduce approximations everywhere on the boundary. Denote $U^{\bar{x}} \equiv \Phi^{-1}(B_{s/2}(0))$, and recall that Φ is also $\bar{x}$-dependent. Since $\bar{x} \in U^{\bar{x}}$ it follows that

$$\Gamma \subset \cup_{\bar{x} \in \Gamma} U^{\bar{x}}$$

is a cover of Γ by open sets. By compactness of Γ we can find a finite subcover $U^1, U^2, \ldots, U^N$, $\lambda_0 > 0$, and corresponding approximations $\psi_\lambda^1, \psi_\lambda^2, \ldots, \psi_\lambda^N$, $\lambda < \lambda_0$, such that each approximation ψ_λ^i vanishes on $U^i \cap \Gamma$ and is smooth in U^i.

It then follows easily that

$$\left\|\nabla\psi_\lambda^i\right\|_{L^\infty} \leq C, \quad \left\|\Delta\psi_\lambda^i\right\|_{L^\infty} \leq C \tag{B.3}$$

and that

$$\Delta\psi_\lambda^i \to \Delta\psi \text{ in } L^p(U^i) \text{ and } \nabla\psi_\lambda^i \to \nabla\psi \text{ in } L^\infty(U^i).$$

Let U^0 be such that

$$\Omega \subset \cup_{i=0}^N U^i, \quad U^0 \cap \Gamma = \emptyset.$$

Consider a partition of unity ρ^i, $i = 0, 1, \ldots, N$, so that

$$\sum_{i=0}^N \rho^i = 1; \quad 0 \leq \rho^i \leq 1; \quad \text{supp } \rho^i \subset U^i; \quad \rho^i \in C^\infty(U^i).$$

We introduce, finally,

$$\psi_\lambda = \psi_\lambda(x) \equiv \sum_{i=1}^N \rho^i \psi_\lambda^i(x) + \rho^0(x)(\eta_\lambda * \psi)(x),$$

noting that the last convolution is defined for small λ since ρ^0 is supported in U^0.

Consider $\lambda = 1/n$, $n \in \mathbb{N}$. By construction all the desired properties hold for the corresponding ψ_n. In particular, we note that

$$\begin{aligned}\Delta \psi_n = \sum_{i=1}^N &\left(\Delta \rho^i \psi_n^i + \rho^i \Delta \psi_n^i + \nabla \rho^i \cdot \nabla \psi_n^i\right) \\ &+ \Delta \rho^0 \eta_n * \psi + \rho^0 \eta_n * \Delta \psi + \nabla \rho^0 \cdot \eta_n * \nabla \psi,\end{aligned}$$

in light of (B.3), shows that $\nabla^\perp \psi_n$ is bounded in S, since $\operatorname{curl} \nabla^\perp = -\Delta$. □

In Proposition B.2, we cut off the sequence of smooth stream functions constructed in Lemma B.1 to construct our approximate sequence of initial velocities.

Proposition B.2. *Let u lie in S with $\omega = \operatorname{curl} u$. There exists a sequence, $(u_n)_{n=1}^\infty$, of approximations to u with the properties that:*

1. *$u_n = K_\Omega[\omega_n]$ and lies in $C_c^\infty(\Omega)$, where $\omega_n = \operatorname{curl} u_n$ lies in $C_c^\infty(\Omega)$;*
2. *$u_n \to u$ uniformly on any compact subset, L, of Ω;*
3. *for any p in $[1, \infty)$, $\omega_n \to \omega$ in $L^p(L)$ for any compact subset, L, of Ω at a rate that depends only on p and L;*
4. *u_n is bounded in S uniformly in n.*

Proof. Let ψ and (ψ_n) be as given by Lemma B.1.

Suppose that $\overline{\Omega^C} \subseteq B_a(0)$, $a > 0$, and let h be a cutoff function equal to 1 on $B_a(0)$ and equal to zero outside of $B_{2a}(0)$. Define $\phi_n \colon \Omega \to [0, 1]$ for $n > 1$ by

$$\phi_n(x) = h(ax/n).$$

Then, defining $\dot{B}_n := B_n(0) \cap \Omega$, ϕ_n is supported on $\dot{B}_{2n}$ and is equal to 1 on $\dot{B}_n$.

Let

$$\overline{\psi}_n = \phi_n \psi_n, \quad u_n = \nabla^\perp \overline{\psi}_n, \quad \omega_n = \operatorname{curl} u_n = \Delta \overline{\psi}_n$$

and note that $\omega_n, u_n \in C_c^\infty(\Omega)$ with $u_n = K_\Omega[\omega_n]$, giving (1).

Let L be a compact subset of Ω. Then

$$\|u_n - u\|_{L^\infty(L)} = \|\phi_n \nabla^\perp \psi_n + \psi_n \nabla^\perp \phi_n - \nabla^\perp \psi\|_{L^\infty(L)}.$$

For all sufficiently large n, $\phi_n = 1$ on L so

$$\|u_n - u\|_{L^\infty(L)} \le \|\nabla \psi_n - \nabla \psi\|_{L^\infty(L)} \to 0$$

because of (6) of Lemma B.1. This gives (2).

For (3), we calculate,

$$\|\omega_n - \omega\|_{L^p(L)} = \|\psi_n \Delta \phi_n + \phi_n \Delta \psi_n + 2 \nabla \phi_n \cdot \nabla \psi_n - \Delta \psi\|_{L^p(L)}.$$

For all sufficiently large n, $\phi_n = 1$ on L so

$$\begin{aligned}\|\omega_n - \omega\|_{L^p(L)} &\le \|\psi_n\|_{L^p(L)} \|\Delta \phi_n\|_{L^\infty(L)} + 2 \|\nabla \phi_n\|_{L^\infty(L)} \|u_n\|_{L^p(L)} \\ &\quad + \|\Delta \psi_n - \Delta \psi\|_{L^p(L)} \\ &= \|\Delta \psi_n - \Delta \psi\|_{L^p(L)} \to 0\end{aligned}$$

by (5) of Lemma B.1. This gives (3).

For (4), we have

$$\begin{aligned}
&\|u_n\|_{L^\infty(\Omega)} \\
&\qquad \le \|\phi_n \nabla^\perp \psi_n + \psi_n \nabla^\perp \phi_n\|_{L^\infty(\Omega)} \\
&\qquad \le \|\phi_n\|_{L^\infty(\Omega)} \|\nabla^\perp \psi_n\|_{L^\infty(\Omega)} + \|\psi_n\|_{L^\infty(\dot{B}_{2n})} \|\nabla^\perp \phi_n\|_{L^\infty(\Omega)} \\
&\qquad \le C + Cnn^{-1} = C.
\end{aligned} \tag{B.4}$$

Here, we used $\|\nabla \phi_n\|_{L^\infty(\Omega)} \le Cn^{-1}$. Also,

$$\begin{aligned}
\|\omega_n\|_{L^\infty(\Omega)} &= \|\psi_n \Delta \phi_n + \phi_n \Delta \psi_n + 2 \nabla \phi_n \cdot \nabla \psi_n\|_{L^\infty(\Omega)} \\
&\le \|\psi_n\|_{L^\infty(\dot{B}_{2n})} \|\Delta \phi_n\|_{L^\infty(\Omega)} + \|\phi_n\|_{L^\infty(\Omega)} \|\Delta \psi_n\|_{L^\infty(\Omega)} \\
&\qquad + 2 \|\nabla \phi_n\|_{L^\infty(\Omega)} \|u_n\|_{L^\infty(\Omega)} \\
&\le Cnn^{-2} + C + Cn^{-1} \le C.
\end{aligned}$$

Together with (B.4), this yields (4). $\square$

Recall the definition of the log-Lipschitz space $LL(\Omega)$ in(4.11).

Lemma B.3. *Suppose $u \in S$. Then $u \in LL$ with $\|u\|_{LL} \le C \|u\|_S$.*

Proof. Let $\mathcal{E}$ be the extension operator from Ω to $\mathbb{R}^2$ defined by Stein in Theorem 5' p. 181 of [31]. This operator has the property that it continuously extends functions on all Sobolev spaces on Ω to the corresponding space on $\mathbb{R}^2$. Let ψ be a stream function for u and extend ψ using $\mathcal{E}$ to all of $\mathbb{R}^2$, also calling the extended stream function ψ. (If $\Omega = \mathbb{R}^2$ we need not perform this extension.)

Let ϕ be a smooth cutoff function supported in $B_2(0)$ with $\phi \equiv 1$ on $B_1(0)$ and let $\phi_x(\cdot) := \phi(\cdot - x)$. Let $\overline{u} = \nabla^\perp(\phi_x \psi)$ and let $\overline{\omega} = \operatorname{curl} \overline{u}$.

Applying Morrey's inequality gives, for any $|y| < 1$ and $p \ge p_0$,

$$|u(x+y) - u(y)| = |\overline{u}(x+y) - \overline{u}(y)| \le C_{p_0} \|\nabla \overline{u}\|_{L^p(\mathbb{R}^2)} |y|^{1-\frac{2}{p}}.$$

Because $\overline{\omega}$ is compactly supported, $\overline{u} = K * \overline{\omega}$. Thus, we can apply the Calderon-Zygmund inequality to obtain

$$\begin{aligned}
|u(x+y) - u(y)| &\le C \inf_{p \ge p_0} \left\{ p \|\overline{\omega}\|_{L^p(\mathbb{R}^2)} |y|^{1-\frac{2}{p}} \right\} \\
&= C \inf_{p \ge p_0} \left\{ p \|\overline{\omega}\|_{L^p(B_2(x))} |y|^{1-\frac{2}{p}} \right\} \le C \|\overline{\omega}\|_{L^\infty(\mathbb{R}^2)} \inf_{p \ge p_0} \left\{ p |y|^{1-\frac{2}{p}} \right\} \\
&= -C \|\overline{\omega}\|_{L^\infty(\mathbb{R}^2)} |y| \log |y|
\end{aligned}$$

for all sufficiently small y.

But,

$$\begin{aligned}
\|\overline{\omega}\|_{L^\infty(\mathbb{R}^2)} &= \|\phi_x \omega - \nabla^\perp \phi_x \cdot u\|_{L^\infty(B_2(x))} \\
&\le \|\omega\|_{L^\infty(B_2(x))} + C \|u\|_{L^\infty(B_2(x))} \le C \|u\|_S.
\end{aligned}$$

$\square$

Acknowledgements

We gratefully acknowledge support from the following funding agencies: DMA was supported by the National Science Foundation through grants DMS-1008387 and DMS-1016267. JPK was supported by the National Science Foundation through grants DMS-1212141 and DMS-1009545. MCLF's research was supported by CNPq grants # 200434/2011-0 and # 303089/2010-5. HJNL was supported by CAPES grant BEX 6649/10-6, CNPq grant # 306331/2010-1, and FAPERJ grant E-26/103.197/2012.

We thank Elaine Cozzi and Yasushi Taniuchi for useful conversations. We also thank Alexis Vasseur for bringing the example in Section 7 of a non-Serfati solution to our attention.

References

[1] Hammadi Abidi, Taoufik Hmidi, and Sahbi Keraani. On the global well-posedness for the axisymmetric Euler equations. *Math. Ann.*, 347(1):15–41, 2010. 26

[2] H. Bahouri and J.-Y. Chemin. Équations de transport relatives á des champs de vecteurs non-lipschitziens et mécanique des fluides. *Arch. Rational Mech. Anal.*, 127(2):159–181, 1994. 4

[3] S. Bell and S. Krantz. Smoothness to the boundary of conformal maps. *Rocky Mt. J. Math.*, 17:23–40, 1987. 35

[4] M. Benvenutti and M. C. Lopes Filho. Characterization of Serfati vorticities. *In preparation*. 23, 24

[5] Jean-Yves Chemin. A remark on the inviscid limit for two-dimensional incompressible fluids. *Comm. Partial Differential Equations*, 21(11-12):1771–1779, 1996. 16

[6] Jean-Yves Chemin. *Perfect incompressible fluids*, volume 14 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie. 6, 8, 9

[7] Qionglei Chen, Changxing Miao, and Xiaoxin Zheng. The bi-dimensional Euler equations in bmo-type space. *arXiv:1311.0934v1*, 2013. 25, 26

[8] Elaine Cozzi. Vanishing viscosity in the plane for nondecaying velocity and vorticity. *SIAM J. Math. Anal.*, 41(2):495–510, 2009. 26

[9] Elaine Cozzi. A finite time result for vanishing viscosity in the plane with nondecaying vorticity. *Commun. Math. Sci.*, 8(4):851–862, 2010. 26

[10] Elaine Cozzi. Vanishing viscosity in the plane for non decaying velocity and vorticity II. *To appear in The Pacific Journal of Mathematics*, 2014. 24, 26

[11] Thierry Gallay and Siniša Slijepčević. Energy Bounds for the Two-Dimensional Navier-Stokes Equations in an Infinite Cylinder. *Comm. Partial Differential Equations*, 39(9):1741–1769, 2014. 26

[12] Yoshikazu Giga, Katsuya Inui, Jun Kato, and Shin'ya Matsui. Remarks on the uniqueness of bounded solutions of the Navier-Stokes equations. In *Proceedings of the Third World Congress of Nonlinear Analysts, Part 6 (Catania, 2000)*, volume 47, pages 4151–4156, 2001. 25

[13] Yoshikazu Giga, Katsuya Inui, and Shin'ya Matsui. On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data. In *Advances in fluid dynamics*, volume 4 of *Quad. Mat.*, pages 27–68. Dept. Math., Seconda Univ. Napoli, Caserta, 1999. 26

[14] Taoufik Hmidi and Sahbi Keraani. Incompressible viscous flows in borderline Besov spaces. *Arch. Ration. Mech. Anal.*, 189(2):283–300, 2008. 26

[15] Taoufik Hmidi and Mohamed Zerguine. Inviscid limit for axisymmetric Navier-Stokes system. *Differential Integral Equations*, 22(11-12):1223–1246, 2009. 26

[16] D. Iftimie, M. C. Lopes Filho, and H. J. Nussenzveig Lopes. Two dimensional incompressible ideal flow around a small obstacle. *Comm. Partial Differential Equations*, 28(1-2):349–379, 2003. 26, 35, 36

[17] D. Iftimie, M. C. Lopes Filho, H. J. Nussenzveig Lopes, and F. Sueur. Weak vorticity formulation for the incompressible 2D Euler equations in domains with boundary. *arXiv:1305.0905v1.*, 2013. 26, 35

[18] Jun Kato. The uniqueness of nondecaying solutions for the Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, 169(2):159–175, 2003. 24, 25

[19] James P Kelliher. Bounded vorticity, bounded velocity solutions to the 2D Euler equations. *arXiv:1404.3404*, 2014. 4, 24, 25

[20] Keisuke Kikuchi. Exterior problem for the two-dimensional Euler equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 30(1):63–92, 1983. 11

[21] C. C. Lin. On the motion of vortices in two dimensions I and II. *Proc. Natl. Acad. Sci.*, 27:570 – 575 and 575 – 577, 1941. 5

[22] M. C. Lopes Filho and H. J. Nussenzveig Lopes. Vortex dynamics on a domain with holes. *Theor. Comput. Fluid Dyn.*, 24:51–57, 2010. 5

[23] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and E. Tadmor. Approximate solutions of the incompressible euler equations with no concentrations. *Ann. Inst. Henri Poincaré, Analyse non linéaire*, 17:371–412, 2000. 6

[24] A. Majda. Vorticity and the mathematical theory of incompressible fluid flow. *Comm. Pure Appl. Math.*, 39:S187–S220, 1986. 1

[25] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002. 6, 7, 8, 9

[26] F. J. McGrath. Nonstationary plane flow of viscous and ideal fluids. *Arch. Rational Mech. Anal.*, 27:329–348, 1967. 8

[27] Okihiro Sawada and Yasushi Taniuchi. On the Boussinesq flow with nondecaying initial data. *Funkcial. Ekvac.*, 47(2):225–250, 2004. 26

[28] Okihiro Sawada and Yasushi Taniuchi. A remark on L^∞ solutions to the 2-D Navier-Stokes equations. *J. Math. Fluid Mech.*, 9(4):533–542, 2007. 26

[29] Philippe Serfati. Solutions C^∞ en temps, n-log Lipschitz bornées en espace et équation d'Euler. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(5):555–558, 1995. 1, 15, 20, 25

[30] Philippe Serfati. Structures holomorphes à faible régularité spatiale en mécanique des fluides. *J. Math. Pures Appl. (9)*, 74(2):95–104, 1995. 24, 25

[31] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970. 41

[32] Yasushi Taniuchi. Uniformly local L^p estimate for 2-D vorticity equation and its application to Euler equations with initial vorticity in **bmo**. *Comm. Math. Phys.*, 248(1):169–186, 2004. 24, 25, 26

[33] Yasushi Taniuchi, Tomoya Tashiro, and Tsuyoshi Yoneda. On the two-dimensional Euler equations with spatially almost periodic initial data. *J. Math. Fluid Mech.*, 12(4):594–612, 2010. 24, 25, 26

[34] Roger Temam. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition. 3

[35] Misha Vishik. Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type. *Ann. Sci. École Norm. Sup. (4)*, 32(6):769–812, 1999. 1, 25, 26

[36] V. I. Yudovich. Non-stationary flows of an ideal incompressible fluid. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 3:1032–1066 (Russian), 1963. 1

[37] V. I. Yudovich. Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid. *Math. Res. Lett.*, 2(1):27–38, 1995. 1, 24

Department of Mathematics, Drexel University, 3141 Chestnut Street, Philadelphia, PA 19104
E-mail address: ambrose@math.drexel.edu

Department of Mathematics, University of California, Riverside, 900 University Ave., Riverside, CA 92521
E-mail address: kelliher@math.ucr.edu

Instituto de Matemática, Universidade Federal do Rio de Janeiro, Cidade Universitária – Ilha do Fundão, Caixa Postal 68530, 21941-909 Rio de Janeiro, RJ – Brazil
E-mail address: mlopes@im.ufrj.br

Instituto de Matemática, Universidade Federal do Rio de Janeiro, Cidade Universitária – Ilha do Fundão, Caixa Postal 68530, 21941-909 Rio de Janeiro, RJ – Brazil
E-mail address: hlopes@im.ufrj.br