Recent progress on the vanishing viscosity limit in the presence of a boundary

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The Setting

To keep things as simple as possible, we assume:

- $\Omega \subseteq \mathbb{R}^2$ is open, bounded, connected, and simply connected.
- $\partial\Omega$ is C^∞, $\boldsymbol{n} =$ outward unit normal.
- $\boldsymbol{\tau}$ is the unit tangent vector, with $(\boldsymbol{n}, \boldsymbol{\tau})$ in the standard orientation—that is, a rotation of (e_1, e_2).
- $u_0 \in C^\infty(\overline{\Omega})$ vector field, $\operatorname{div} u_0 = 0$, and $u_0 \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.

Working in 2D avoids certain technicalities related to uniqueness of solutions, time of existence, and energy equalities versus inequalities.

Most of what we say will apply, however, for all higher dimensions.

Incompressible Navier-Stokes and Euler equations

Navier-Stokes equations with no-slip boundary conditions:

$$(NS)\begin{cases}\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u & \text{on } \Omega,\\ \operatorname{div} u = 0,\ u(0) = u_0 & \text{on } \Omega,\\ u = 0 & \text{on } \partial\Omega.\end{cases}$$

Velocity $u = u^\nu$, pressure $p = p^\nu$.

Euler equations with no-penetration boundary conditions:

$$(E)\begin{cases}\partial_t \overline{u} + \overline{u}\cdot\nabla \overline{u} + \nabla \overline{p} = 0 & \text{on } \Omega,\\ \operatorname{div} \overline{u} = 0,\ \overline{u}(0) = u_0 & \text{on } \Omega,\\ \overline{u}\cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega.\end{cases}$$

The central problem

Classical vanishing viscosity limit:

$$(VV) \quad u \to \overline{u} \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ as } \nu \to 0.$$

Let $T = T(u_0) \geq 0$ be the maximal time for which (VV) holds.

Central question:

1. Is $T(u_0) > 0$ for all u_0 or
2. is $T(u_0) = 0$ for some u_0?

That is, does (VV) hold in general, or does it fail in at least one instance?

The fundamental difficulty

Letting $w = u - \overline{u}$, a standard energy argument gives:

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \nu \int_0^t \|\nabla w\|^2 = -\int_0^t \langle w \cdot \nabla \overline{u}, w\rangle + \int_0^t \langle \Delta \overline{u}, w\rangle + \nu \int_0^t \int_{\partial\Omega} (\nabla w \cdot \boldsymbol{n}) \cdot w.$$

All terms above are easily controllable except for the boundary integral: $w = -\overline{u} \not\equiv 0$ on $\partial\Omega$, so the boundary integral does not vanish.

The plan

I will focus on the central question. This means that I will not discuss recent results of several authors (including myself).

I plan to:

- Explain how boundary layer correctors have been used to explore this problem.
- Make a conjecture.
- *Time allowing*: Contrast the boundary layer corrector approach with the Prandtl boundary layer expansion.

Boundary layer correctors

A boundary layer corrector, z, is a velocity field on $[0, T] \times \Omega$ that has the following properties:

1. $z = -\overline{u}$ on $\partial\Omega$;
2. z and its various derivatives are sufficiently small, or blowup sufficiently slowly with ν, in various norms;
3. z is either supported in a boundary layer Γ_δ for some $\delta = \delta(\nu)$ or at least decays rapidly outside of Γ_δ;
4. z might also include an additional term, small in ν, supported throughout Ω.
5. z is typically (for us, always) divergence-free.

Note that the *Prandtl theory* does not use a boundary layer corrector in this sense.

Boundary layer corrector estimates

Various boundary layer correctors *have* been used very successfully for linear equations or situations where the nonlinearity of the equations is weakened (for instance, by different boundary conditions), or when there is some degree of analyticity to the data and boundary. Such correctors are usually derived from formal asymptotics and satisfy the resulting PDE, at least approximately.

A heat equation-based corrector employed by Gie 2015, for instance, does have better estimates on $\partial_t z - \nu \Delta z$. Gie utilizes these improved estimates to obtain bounds on the H^1 norms of solutions to the *Stokes equations*. There is no hope, however, to obtain such bounds for (NS).

In fact, the simple boundary layer corrector used by Kato in his seminal 1984 paper suffices to obtain all existing results (I know of) for necessary and/or sufficient conditions for (VV) to hold that rely on the use of boundary layer correctors.

Boundary layer corrector: inherent limitation

An inherent limitation to all existing correctors is that they remain uniformly bounded in $L^\infty([0, T] \times \Omega)$ over ν. We do not expect solutions to (NS) to remain so bounded, so the corrector is destined to miss much of the behavior in the boundary layer.

The Prandtl approach has some hope of doing a better job, but it likely suffers from the same limitation.

Kato's corrector

In 2D, Kato's corrector is obtained by cutting off the stream function for $-\overline{u}$ in the boundary layer, Γ_δ. Then:

$$\begin{aligned}
\|z^\tau(x_1,x_2)\|_{L^p(\Omega)} &\leq C\delta^{\frac{1}{p}}, & \|z^n(x_1,x_2)\|_{L^p(\Omega)} &\leq C\delta^{\frac{1}{p}+1}, \\
\|\partial_\tau z^\tau(x_1,x_2)\|_{L^p(\Omega)} &\leq C\delta^{\frac{1}{p}}, & \|\partial_n z^\tau(x_1,x_2)\|_{L^p(\Omega)} &\leq C\delta^{\frac{1}{p}-1}, \\
\|\partial_\tau z^n(x_1,x_2)\|_{L^p(\Omega)} &\leq C\delta^{\frac{1}{p}+1}, & \|\partial_n z^n(x_1,x_2)\|_{L^p(\Omega)} &\leq C\delta^{\frac{1}{p}}
\end{aligned}$$

for any $p \in [1,\infty]$.

- These estimates hold as well for $\partial_t z$ in place of z as long as δ is independent of time.
- Even if δ depends on time, other existing correctors do not do a better job in any critical way.
- Kato's corrector is supported in Γ_δ.

Using a boundary layer corrector

We pair the equation for $w = u - \overline{u}$ with $w - z$ (which is divergence-free and vanishes on the boundary) to obtain

$$\frac{1}{2}\|w(t)\|^2 + \nu \int_0^t \|\nabla w\|^2 \leq C(1+t)\nu^{\frac{1}{2}} + Ct\delta^{\frac{1}{2}} + \frac{C}{\delta}\int_0^t \int_{\Gamma_\delta} |u^\tau u^n| + \nu \int_0^t (\nabla u, \nabla z) + C\int_0^t \|w\|^2 .$$

Here and in what follows, $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$.

For (VV) to hold, it is sufficient that

$$\frac{1}{\delta}\int_0^t \int_{\Gamma_\delta} |u^\tau u^n| \to 0 \text{ as } \nu \to 0$$

and

$$\nu \int_0^t (\nabla u, \nabla z) \to 0 \text{ as } \nu \to 0.$$

Sufficient conditions only

If we assume that

$$\delta,\ \frac{\nu}{\delta} \to 0 \text{ as } \nu \to 0$$

then we can control the second integral, since (recall $w = u - \overline{u}$)

$$\begin{aligned}
\nu\,|(\nabla u, \nabla z)| &\leq \nu\,|(\nabla w, \nabla z)| + \nu\,|(\nabla \overline{u}, \nabla z)| \\
&\leq \frac{\nu}{2}\,\|\nabla w\|^2 + \frac{\nu}{2}\,\|\nabla z\|^2 + \nu\,\|\nabla \overline{u}\|_{L^\infty}\,\|\nabla z\|_{L^1} \\
&\leq \frac{\nu}{2}\,\|\nabla w\|^2 + C\frac{\nu}{\delta} + C\nu,
\end{aligned}$$

giving

$$\begin{aligned}
&\|w(t)\|^2 + \nu \int_0^t \|\nabla w\|^2 \\
&\qquad \leq \left(C\left((1+t)\nu^{\frac{1}{2}} + t\delta^{\frac{1}{2}} + \frac{\nu}{\delta}\right) + \frac{C}{\delta}\int_0^t \int_{\Gamma_\delta} |u^\tau u^n| \right) e^{Ct}.
\end{aligned}$$

The boundary integral

The boundary integral in the sufficient condition

$$\frac{1}{\delta}\int_0^t \int_{\Gamma_\delta} |u^\tau u^n| \to 0 \text{ as } \nu \to 0,$$

is similar to what appears in a paper of Temam and Wang 1998, though there it is in terms of $w^\tau w^n$. In this form, it is like a recent paper of Constantin, Elgindi, Ignatova, and Vicol.

So, for instance, when $\delta = \nu^\alpha$ for some $\alpha \in (0, 1)$, the above condition is sufficient to insure that (VV) holds. It is not possible to extend this to $\alpha = 1$, however, because then $\nu \int_0^t (\nabla u, \nabla z)$ cannot be controlled and so must be kept as a condition.

Xiaoming Wang 2001

Though the approach we have described is not quite that taken in Xiaoming Wang's 2001 paper, it is in the same spirit. He shows that

$$\frac{1}{\delta(\nu)} \int_0^t \int_{\Gamma_\delta} \|\partial_\tau u^n\|^2 \to 0 \text{ as } \nu \to 0 \text{ or}$$

$$\frac{1}{\delta(\nu)} \int_0^t \int_{\Gamma_\delta} \|\partial_\tau u^\tau\|^2 \to 0 \text{ as } \nu \to 0$$

is sufficient for (VV) to hold. He also shows that the existence of some such $\delta = \delta(\nu)$ for which one (and hence both) of these conditions holds is necessary.

What's nice about these conditions is that *if* the Prandtl theory holds, one would expect $\partial_\tau u^n$ and $\partial_\tau u^\tau$ to be smaller than $\partial_n u^\tau$, which would dominate ∇u.

The critical integrals

Now, suppose that $\delta = \nu$. In this case, the two time integrals in

$$\frac{1}{2}\|w(t)\|^2 + \nu\int_0^t \|\nabla w\|^2 \le C(1+t)\nu^{\frac{1}{2}} + Ct\delta^{\frac{1}{2}} + \frac{C}{\delta}\int_0^t\int_{\Gamma_\delta}|u^\tau u^n| + \nu\int_0^t(\nabla u, \nabla z) + C\int_0^t\|w\|^2$$

are both critical in the sense that they can be shown to be bounded by the basic energy inequality for the Navier-Stokes equations, but the energy inequality is insufficient to show that these integrals vanish with viscosity. (They are also critical in a scaling sense.)

Hence, we must assume that both integrals vanish.

It makes sense, however, to put them to eliminate z and put the condition on the two integrals in a common form.

Common form: First critical integral

We return to $(u \cdot \nabla u, z)$, from which the first integral came:

$$
\begin{aligned}
\left|\int_0^t (u\cdot\nabla z, u)\right| &= \left|\int_0^t (u\cdot\nabla u, z)\right| = \left|\int_0^t (u\cdot \operatorname{curl} u, z)\right| \\
&\le \|z\|_{L^\infty([0,t]\times\Omega)} \int_0^t \|u\|_{L^2(\Gamma_\delta)} \|\operatorname{curl} u\|_{L^2(\Gamma_\delta)} \\
&\le C\delta \int_0^t \|\nabla u\|_{L^2(\Gamma_\delta)} \|\operatorname{curl} u\|_{L^2(\Gamma_\delta)} \\
&\le C\delta^{1/2} \|\nabla u\|_{L^2([0,t];L^2(\Gamma_\delta))}\, \delta^{1/2} \|\operatorname{curl} u\|_{L^2([0,t];L^2(\Gamma_\delta))} \\
&\le C\left(\delta \int_0^t \|\operatorname{curl} u\|^2_{L^2(\Gamma_\delta)}\right)^{1/2} = C\left(\nu \int_0^t \|\operatorname{curl} u\|^2_{L^2(\Gamma_\nu)}\right)^{1/2}.
\end{aligned}
$$

Here, we did a fancy integration by parts and applied Poincare's inequality for a domain of width δ in the x_2-direction, along with the basic energy inequality for the Navier-Stokes equations.

Common form: Second critical integral

For the second integral,

$$
\begin{aligned}
\nu \int_0^t |(\nabla u, \nabla z)| &= \nu \int_0^t |(\operatorname{curl} u, \operatorname{curl} z)| \le \nu \int_0^t \|\nabla z\|_{L^2} \|\operatorname{curl} u\|_{L^2(\Gamma_\delta)} \\
&\le C\nu\delta^{-\frac{1}{2}} \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\delta)} = C\nu^{\frac{1}{2}} \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\delta)} \\
&\le C\nu^{\frac{1}{2}} \left(\int_0^t 1 \right)^{\frac{1}{2}} \left(\nu \int_0^t \|\operatorname{curl} u\|^2_{L^2(\Gamma_\nu)} \right)^{\frac{1}{2}} \\
&= Ct^{\frac{1}{2}} \left(\nu \int_0^t \|\operatorname{curl} u\|^2_{L^2(\Gamma_\nu)} \right)^{\frac{1}{2}}.
\end{aligned}
$$

Kato 1984: Necessary and sufficient conditions

Hence,

$$\nu \int_0^t \|\text{curl}\, u\|^2_{L^2(\Gamma_\nu)} \to 0 \text{ as } \nu \to 0$$

is a sufficient condition for (VV) to hold.

In fact, this is also a necessary condition, as shown by Kato in 1984. (Actually, Kato used ∇u instead of curl u; the “clever” integration by parts above is due to K 2007.)

When $\delta = c\nu$, the energy inequality we obtained holds in reverse *(I oversimplify!)*, which allowed Kato to show that the same condition is also necessary.

Kato uses the corrector to obtain a criterion *equivalent* to (VV) that depends only upon u. The corrector itself is meaningless and is discarded. This is in contrast to the Prandtl approach.

Other Dirichlet boundary conditions

Suppose instead of no-slip for Navier-Stokes, we fix $g \in C^\infty([0, T] \times \partial\Omega)$ vector field with $g \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ and use

$$u = g \text{ on } \partial\Omega.$$

(NS) is still well-posed.

Note:

- If $g = \overline{u}$ then (VV) holds because the boundary integral vanishes.
- In special cases, such as radially symmetric initial vorticity in a disk or shear flow, (VV) holds *regardless* of g.
- Kato's criteria and most (maybe all) of its descendants hold unchanged for nonzero g.

So what's so special about $g = 0$ (no-slip)?

If convergence is a matter of size

If (VV), for short time, is a matter of how large the solution to (NS) gets in the Kato boundary layer, then one might believe, as I do, that:

Conjecture

Given any u_0 and any $g \in C^\infty([0, T] \times \partial\Omega)$ with $g \cdot \boldsymbol{n} = 0$ there exists $T = T(g, u_0) > 0$ such that (VV) holds.

If (VV) for short time depends intimately on the structure of vorticity formulation in the boundary layer, then one might instead believe that:

Conjecture

Given any u_0 there exists $g \in C^\infty([0, T] \times \partial\Omega)$ with $g \cdot \boldsymbol{n} = 0$ such that (VV) fails for all $T > 0$.

Multiply connected domains

Suppose, now, that Ω is doubly connected (an annulus). Require that

- $g \cdot \boldsymbol{n} < 0$ (inflow) on one boundary component,
- $g \cdot \boldsymbol{n} > 0$ (outflow) on the other boundary component,
- $\int_{\partial\Omega} g \cdot \boldsymbol{n} = 0$.

Then (VV) is known to hold. This is a result of Gie, Hamouda, and Temam 2012, building on a result for a periodic channel by Temam and Xiaoming Wang 2002.

However, in this *non-characteristic* case, the boundary conditions for the Euler equations are different, with the full velocity specified for inflow, and only the normal component for outflow.

- So in this case, distinct g's will yield convergence to *distinct* solutions to (E).
- The behavior for no-penetration boundary conditions is hence very different from that of inflow/outflow.

(VV) is perhaps best appreciated in the context of boundary layer *expansions*. The mother of all such expansions was that performed formally by Prandtl in 1904.

Prandtl hypothesized that outside of a boundary layer, Γ_δ, of width $\delta = \delta(\nu)$, $u = \overline{u}$, while inside the boundary layer, he assumed (I simplify) that:

1. u varies more rapidly in the direction normal to the boundary.
2. The normal component of u remains small.

Prandtl equations

Assume:

- $\Omega = [0, 1] \times [0, \infty)$, periodic in x.
- Write $u = (u, v)$.
- $Y = \dfrac{y}{\delta}$.
- $u = U(t, x, Y)$, $v = \eta V(t, x, Y)$, $p = P(t, x, Y)$.

Using these assumptions in (NS), we find that terms balance when $\eta = \delta = \sqrt{\nu}$. The leading order terms form the Prandtl equations:

$$\begin{aligned} &\partial_t U + U\partial_x U + V\partial_Y V + \partial_x P = \partial_{YY} U, \\ &\partial_Y P = 0, \\ &\partial_x U + \partial_Y V = 0. \end{aligned}$$

The boundary conditions are $U = V = 0$ on $\partial\Omega$ and there are conditions at $Y = \infty$ to insure that (u, v) and p match the Eulerian velocity and pressure.

Well-posedness? Convergence?

Define the Prandtl velocity by

$$u^P(t, x, y) = (U(t, x, Y), \sqrt{\epsilon}V(t, x, Y)).$$

Guo and Nguyen 2011 (building on Gérard-Varet 2010 and Guo and Tice 2009):

> *The Prandtl equations cannot be well-posed in $L^\infty(0, T; H^1_{x,Y}(\Omega))$, where $H^1_{x,Y}(\Omega)$ is a weighted (decay at infinity) H^1 space. Convergence of $u - u^P$ to 0 cannot hold in general in a similar space (this is not quite what they prove).*

- Well-posedness and convergence are known to hold in certain cases—analytic data (Sammartino and Caflisch 1998) and monotonic initial vorticity (Oleinik 1966)–and initial vorticity vanishing near the boundary (Maekawa, 2014).

At best, then, we can expect the Prandtl equations to be well-posed in a fairly weak sense.

But if. . .

But suppose that the Prandtl equations are well-posed in, say, $L^\infty(0, T; L^2(\Omega))$ (we ignore issues of lack of decay at infinity). Then

$$\begin{aligned}
\|u - \overline{u}\|_{L^2(\Omega)} &\leq \|u - \overline{u}\|_{L^2(\Omega\setminus\Gamma_\delta)} + \|u - u^P\|_{L^2(\Gamma_\delta)} + \|u^P - \overline{u}\|_{L^2(\Gamma_\delta)} \\
&\leq \|u - \overline{u}\|_{L^2(\Omega\setminus\Gamma_\delta)} + \|u - u^P\|_{L^2(\Gamma_\delta)} \\
&\qquad + \|u^P\|_{L^2(\Gamma_\delta)} + \|\overline{u}\|_{L^2(\Gamma_\delta)} \\
&\leq \|u - \overline{u}\|_{L^2(\Omega\setminus\Gamma_\delta)} + \|u - u^P\|_{L^2(\Gamma_\delta)} + \|u^P\|_{L^2(\Gamma_\delta)} + C\nu^{\frac{1}{4}}.
\end{aligned}$$

Here, $\delta = \nu^{1/2}$, and we have used that $\overline{u} \in L^\infty([0, T] \times \Omega)$.

So...

From

$$\|u - \overline{u}\|_{L^2(\Omega)} \leq \|u - u^P\|_{L^2(\Gamma_\delta)} + \|u^P\|_{L^2(\Gamma_\delta)} + C\nu^{\frac{1}{4}}$$

we conclude that assuming

$$(*) \qquad \begin{cases} u \to \overline{u} & \text{in } L^\infty(0, T; L^2(\Omega \setminus \Gamma_\delta)), \\ u - u^P \to 0 & \text{in } L^\infty(0, T; L^2(\Gamma_\delta)), \end{cases}$$

(VV) holds if (and also only if)

$$(**) \qquad \|u^P\|_{L^2(\Gamma_\delta)} \to 0 \text{ as } \nu \to 0.$$

Maekawa 2014 showed that if the initial vorticity vanishes in a layer near the boundary then $(*)$ holds. It follows from his estimates that $(**)$ holds in that setting as well, so that (VV) holds.

It is quite possible, however, that $(*)$ could hold without $(**)$ holding. If so, it could be that u converges to something other than $\overline{u}$.

Thank you

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Some kind of convergence always happens

Theorem

There exists v in $L^\infty(0, T; H)$ such that a subsequence of $(u^\nu)_{\nu>0}$ converges weakly to v in $L^\infty(0, T; H)$.

Proof.

- $(u^\nu)_{\nu>0}$ is bounded in $L^\infty(0, T; H)$;
- $u^\nu = \nabla^\perp \psi^\nu$ for $\psi \in H_0^1(\Omega)$;
- Thus, $(\psi^\nu)_{\nu>0}$ is bounded in $L^\infty(0, T; H_0^1(\Omega))$;
- $H_0^1(\Omega)$ is weakly compact so $\exists$ a subsequence, (ψ^n), converging weakly in $L^\infty(0, T; H_0^1(\Omega))$ to some ψ lying in $L^\infty(0, T; H_0^1(\Omega))$.
- Let $u^n = \nabla^\perp \psi^n$, $v = \nabla^\perp \psi$. For any $h \in L^\infty(0, T; H)$,

$$(u^n, h) = (\nabla^\perp \psi^n, h) = -(\nabla \psi^n, h^\perp) = (\psi^n, -\operatorname{div} h^\perp) = (\psi^n, \operatorname{curl} h)$$
$$\to (\psi, \operatorname{curl} h) = (v, h).$$

□

Convergence possibilities

Thus, it may well be that:

1. (VV) holds or
2. (u^ν) converges to a solution to the Euler equations with different initial data or some sort of forcing term or
3. (u^ν) converges to something other than a solution to the Euler equations or
4. (u^ν) fails to converge to anything (as a full sequence).

As shown by Tosio Kato in 1984, however, if (u^ν) converges weakly to $\overline{u}$ in $L^\infty(0, T; L^2)$ then it converges strongly in that space.

The same does *not* hold if (u^ν) converges weakly to v in $L^\infty(0, T; L^2)$ and $v \not\equiv \overline{u}$.

Vortex sheet on the boundary

Similar, simple observations show that, for $v \in L^\infty(0, T; H^1(\Omega))$,

Theorem (K, 2007)

$$(u^\nu) \text{ converges weakly to } v \text{ in } L^\infty(0, T; L^2(\Omega))$$
$$\iff \operatorname{curl} u^\nu \to \operatorname{curl} v - (v \cdot \boldsymbol{\tau})\mu \text{ in } (H^1(\Omega))' \text{ uniformly on } [0, T],$$

where μ is arc length measure on the boundary.

- That is, if convergence occurs in the energy space if and only if a vortex sheet of a specific type forms on the boundary.
- If $v \equiv \overline{u}$ then weak convergence to v in $L^\infty(0, T; L^2(\Omega))$ can be replaced by strong convergence by the result of Tosio Kato 1984.