Observations on the Vanishing Viscosity Limit

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Fixing notation

$$(NS)\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma := \partial\Omega, \\ u(0) = u_0. \end{cases}$$

$$(EE)\begin{cases} \partial_t \overline{u} + \overline{u}\cdot\nabla \overline{u} + \nabla p = \overline{f} & \text{in } \Omega, \\ \operatorname{div} \overline{u} = 0 & \text{in } \Omega, \\ \overline{u}\cdot\mathbf{n} = 0 & \text{on } \Gamma, \\ \overline{u}(0) = u_0. \end{cases}$$

Here, $u = u_\nu$, $\overline{u}$, f, and $\overline{f}$ are vector fields, while p and $\overline{p}$ are pressure (scalar) fields. (We are using here the notation of [Kato 1983].)

We assume Ω is bounded, connected, and simply connected; Γ has C^2 regularity; we write $\mathbf{n}$ for the outward unit normal vector.

The vanishing viscosity limit

- The limit,

$$(VV) \qquad u \to \overline{u} \text{ in } L^\infty(0, T; H),$$

  we refer to as the *classical vanishing viscosity limit*. Whether it holds in general, or fails in any one instance, is a major open problem in mathematical fluids mechanics.
- $H := \left\{ u \in L^2(\Omega)^d : \operatorname{div} u = 0 \text{ in } \Omega,\ u \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\}$,
  $V := \left\{ u \in (H_0^1(\Omega))^d : \operatorname{div} u = 0 \text{ in } \Omega \right\}$.
- For simplicity I will describe 2D only, and assume that $f \equiv 0$ and $u_0 \in H \cap C^{1,\alpha}$, $\alpha > 0$.
- Define the vorticity as the scalar curl of u, $\overline{u}$:

$$\omega := \partial_1 u^2 - \partial_2 u^1,$$
$$\overline{\omega} := \partial_1 \overline{u}^2 - \partial_2 \overline{u}^1.$$

Kato 1983

Kato showed that the following conditions are equivalent:

$$
\begin{aligned}
&u \to \overline{u} \text{ weakly in } L^2(\Omega) \text{ uniformly on } [0, T], \\
&u \to \overline{u} \text{ in } L^\infty(0, T; H), \\
&\nu \int_0^T \|\nabla u\|^2_{L^2(\Omega)} \, dt \to 0, \\
&\nu \int_0^T \|\nabla u\|^2_{L^2(\Gamma_{c\nu})} \, dt \to 0, \\
&u(T) \to \overline{u}(T) \text{ in } L^2(\Omega),
\end{aligned}
$$

where Γ_δ is the boundary strip of width $\delta > 0$.

In the fourth condition, ω can replace ∇u [K 2007].

Temam & Wang (1998), Wang (2001)

At the expense of increasing the size of the boundary layer slightly, one need only consider the tangential derivatives of the tangential components of the velocity or the tangential derivatives of the normal components of the velocity.

(VV) is equivalent to

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} u_{\boldsymbol{\tau}}(s)\|^2_{L^2(\Gamma_{\delta(\nu)})} \, ds \to 0 \text{ as } \nu \to 0,$$
$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} u_{\mathbf{n}}(s)\|^2_{L^2(\Gamma_{\delta(\nu)})} \, ds \to 0 \text{ as } \nu \to 0,$$

where

$$\delta(\nu) \to 0, \; \delta(\nu)/\nu \to \infty \text{ as } \nu \to 0.$$

So $\Gamma_{\delta(\nu)}$ is just slightly thicker than $\Gamma_{c\nu}$.

- (VV) is equivalent to

$$\omega \to \overline{\omega} - (\overline{u} \cdot \boldsymbol{\tau})\mu \text{ in } H^1(\Omega)^* \text{ uniformly on } [0, T],$$
$$\omega \to \overline{\omega} \text{ in } H^{-1}(\Omega) \text{ uniformly on } [0, T].$$

  Here, $\boldsymbol{\tau}$ is the unit tangent vector on Γ that is obtained by rotating **n** counterclockwise by 90 degrees.

- $H^1(\Omega)$ is the dual space to $H^1(\Omega)$, not to $H^1_0(\Omega)$.

- The first condition expresses that (VV) holds if and only if a vortex sheet of a specific type forms on the boundary.

- The second condition holds for Ω simply connected.

When the vanishing viscosity limit is known to hold

- All examples where (VV) is known to hold have some kind of symmetry—in geometry of the domain or the initial data—or have some degree of analyticity.

- In the case of radially symmetric initial vorticity in a disk:
  - $\omega = \omega_\nu$ remains bounded in $L^\infty([0, \infty); L^1(\Omega))$ uniformly over $\nu > 0$ [Lopes Filho, Mazzucato, Nussenzveig Lopes, Taylor 2008].
  - The vorticity is not bounded uniformly over $\nu > 0$ in $L^\infty([0, \infty); L^p(\Omega))$ for any $p > 1$, however, unless the initial velocity (and hence $\overline{u}$ for all time) vanishes on the boundary. (Equivalent to $u_0 \in V$ for radially symmetric initial vorticity.)

- This is, in fact, a more general phenomenon, as can be very simply shown.

L^p norms of the vorticity

Theorem

Assume that $\overline{u}$ *is not identically zero on* $[0, T] \times \Gamma$*. If* (VV) *holds then*

$$\limsup_{\nu \to 0^+} \|\omega\|_{L^\infty(0,T;L^p)} \to \infty \text{ for all } p \in (1, \infty].$$

Proof: We prove the contrapositive. Assume that the conclusion is not true. Then for some $q' \in (1, \infty]$ it must be that for some $C_0 > 0$ and $\nu_0 > 0$,

$$\|\omega\|_{L^\infty(0,T;L^{q'})} \leq C_0 \text{ for all } 0 < \nu \leq \nu_0.$$

Since Ω is a bounded domain we can assume that $q' \in (1, \infty)$.
Let q be Hölder conjugate to q' and $p = 2/q + 1 \in (1, 3)$. Note that

$$(p - 1)q = 2.$$

Trace lemma

So by a simple trace lemma ($p, q' \in (1, \infty)$ is important)

$$\begin{aligned}
\|u(t) - \overline{u}(t)\|_{L^p(\Gamma)} &\leq C \|u(t) - \overline{u}(t)\|_{L^2(\Omega)}^{1-\frac{1}{p}} \|\nabla u(t) - \nabla \overline{u}(t)\|_{L^{q'}(\Omega)}^{\frac{1}{p}} \\
&\leq C \|u(t) - \overline{u}(t)\|_{L^2(\Omega)}^{1-\frac{1}{p}} \left(\|\nabla u(t)\|_{L^{q'}} + \|\nabla \overline{u}(t)\|_{L^{q'}}\right)^{\frac{1}{p}} \\
&\leq C \|u(t) - \overline{u}(t)\|_{L^2(\Omega)}^{1-\frac{1}{p}} \left(C(q')\|\omega(t)\|_{L^{q'}} + \|\nabla \overline{u}(t)\|_{L^{q'}}\right)^{\frac{1}{p}} \\
&\leq C \|u(t) - \overline{u}(t)\|_{H}^{1-\frac{1}{p}}
\end{aligned}$$

for all $0 < \nu \leq \nu_0$.

Lemma

Let $p \in (1, \infty)$ and $q \in [1, \infty]$ be chosen arbitrarily, and let q' be Hölder conjugate to q. For any $v \in H$,

$$\|v\|_{L^p(\Gamma)} \leq C \|v\|_{L^{(p-1)q}(\Omega)}^{1-\frac{1}{p}} \|\nabla v\|_{L^{q'}(\Omega)}^{\frac{1}{p}}$$

From

$$\|u(t) - \overline{u}(t)\|_{L^p(\Gamma)} \leq C \|u(t) - \overline{u}(t)\|_H^{1-\frac{1}{p}}$$

holds for almost all $t \in [0, T]$, we have

$$\|u - \overline{u}\|_{L^\infty(0,T;L^p(\Gamma))} \leq C \|u - \overline{u}\|_{L^\infty(0,T;H)}^{1-\frac{1}{p}} \to 0$$

as $\nu \to 0$. But,

$$\|u - \overline{u}\|_{L^\infty(0,T;L^p(\Gamma))} = \|\overline{u}\|_{L^\infty(0,T;L^p(\Gamma))} \neq 0,$$

so $u(t) - \overline{u}(t) \not\to 0$ in H.

Consequence of vorticity being bounded in L^1

As an aside, control on the L^1 norm of the vorticity yields convergence as a true vortex sheet—that is, when (VV) holds.

Theorem (3D form)

Suppose that $u \to \overline{u}$ *in* $L^\infty(0, T; H)$ *and* $\operatorname{curl} u$ *is bounded in* $L^\infty(0, T; L^1(\Omega))$ *uniformly in* ν. *Then in 3D,*

$$\operatorname{curl} u \to \operatorname{curl} \overline{u} + (\overline{u} \times \mathbf{n})\mu \quad \text{weak}^* \ \text{ in } \mathcal{M}(\overline{\Omega}) \text{ uniformly on } [0, T],$$

by which we mean that for any φ *in* $C(\overline{\Omega})^3$,

$$(\operatorname{curl} u(t), \varphi) \to (\operatorname{curl} \overline{u}(t), \varphi) + \int_\Gamma (\overline{u}(t) \times \mathbf{n}) \cdot \varphi \text{ in } L^\infty([0, T]).$$

Vorticity in the boundary layer

The total mass of the vorticity (in fact, its L^1-norm) in any layer smaller than that of Kato goes to zero—and also for Kato's layer if (VV) holds. This is the content of the following theorem.

Theorem

For any positive function $\delta = \delta(\nu)$,

$$\|\omega\|_{L^2(0,T;L^1(\Gamma_{\delta(\nu)}))} \leq C \left(\frac{\delta(\nu)}{\nu}\right)^{1/2}.$$

If (VV) *holds and* $\limsup_{\nu \to 0^+} \frac{\delta(\nu)}{\nu} < \infty$ *then*

$$\|\omega\|_{L^2(0,T;L^1(\Gamma_{\delta(\nu)}))} \to 0 \text{ as } \nu \to 0.$$

Proof.

By the Cauchy-Schwarz inequality,

$$\|\omega\|_{L^1(\Gamma_{\delta(\nu)})} \leq \|1\|_{L^2(\Gamma_{\delta(\nu)})} \|\omega\|_{L^2(\Gamma_{\delta(\nu)})} \leq C\delta^{1/2} \|\omega\|_{L^2(\Gamma_{\delta(\nu)})}$$

so

$$\frac{C}{\delta} \|\omega\|^2_{L^1(\Gamma_{\delta(\nu)})} \leq \|\omega\|^2_{L^2(\Gamma_{\delta(\nu)})}$$

and

$$\frac{C\nu}{\delta} \|\omega\|^2_{L^2(0,T;L^1(\Gamma_{\delta(\nu)}))} \leq \nu \|\omega\|^2_{L^2(0,T;L^2(\Gamma_{\delta(\nu)}))}.$$

By the basic energy inequality for the Navier-Stokes equations, the right-hand side is bounded, and if (VV) holds, the right-hand side goes to zero by [Kato 1983]. □

Mass of vorticity

We can use this control on the vorticity in the boundary layer to make an observation concerning the optimal rate of convergence in (VV) for non-compatible initial data—that is, for $u_0 \notin V$.

Assume that

$$m := \int_\Omega \omega_0 = (\omega_0, 1) \neq 0.$$

(In particular, this means that $u_0 \notin V$.) The total mass of the vorticity of the Euler solution is conserved so

$$(\overline{\omega}(t), 1) = m \text{ for all } t \in \mathbb{R}.$$

The Navier-Stokes velocity, however, is in V for all positive time, so its total mass is zero; that is,

$$(\omega(t), 1) = 0 \text{ for all } t > 0.$$

Now assume that (VV) holds. Fix $\delta > 0$ and let φ_δ be a smooth cutoff function equal to 1 on Γ_δ and equal to 0 on $\Omega \setminus \Gamma_{2\delta}$. Then,

$$\begin{aligned}|(\omega, 1-\varphi_\delta) - m| \to |(\overline{\omega}, 1-\varphi_\delta) - m| &= |m - (\overline{\omega}, \varphi_\delta) - m| \\ &= |(\overline{\omega}, \varphi_\delta)| \leq \|\overline{\omega}\|_{L^\infty} \|\varphi_\delta\|_{L^1} \leq C\delta,\end{aligned}$$

the convergence being uniform on $[0, T]$. Convergence holds because $1 - \varphi_\delta \in H_0^1(\Omega)$ and (VV) holds, and we used $(\overline{\omega}(t), 1) = m$.

So the mass of vorticity must build up outside any layer of fixed width δ to be as least as large in magnitude as $|m| - C\delta$, counterbalancing the mass of vorticity with opposite sign in the layer itself.

To say anything for $\delta = \delta(\nu)$ we need to bring in the rate of convergence in (VV).

Rate of weak convergence

Theorem (Follows from K 2008)

Assume that (VV) holds with

$$\|u - \overline{u}\|_{L^\infty(0,T;H)} \leq F(\nu)$$

for some fixed $T > 0$. *Then*

$$\|(u(t) - \overline{u}(t), v)\|_{L^\infty([0,T])} \leq F(\nu)\,\|v\|_{L^2(\Omega)} \text{ for all } v \in L^2(\Omega)^d$$

and

$$\|(\omega(t) - \overline{\omega}(t), \varphi)\|_{L^\infty([0,T])} \leq F(\nu)\,\|\nabla\varphi\|_{L^2} \text{ for all } \varphi \in H_0^1(\Omega).$$

Theorem

Assume that (VV) holds with a rate of convergence, $F(\nu) = o(\nu^{1/2})$. Then in 2D the initial mass of the vorticity must be zero.

Proof: Returning to the derivation of $|(\omega, 1-\varphi_\delta) - m| \le C\delta$, we now make $\delta = \delta(\nu) \to 0$ as $\nu \to 0$, and choose φ_δ so that for some $\delta^* = \delta^*(\nu)$ to be chosen later with $0 < \delta^* < \delta$,

$$\varphi_\delta = 1 \text{ on } \Gamma_{\delta^*}, \quad \varphi_\delta = 0 \text{ on } \Gamma_\delta^C := \Omega \setminus \Gamma_\delta.$$

It follows from the rate of weak convergence theorem that

$$|(\omega, 1-\varphi_\delta) - m| \le C\left[\delta + (\delta - \delta^*)^{-\frac{1}{2}} F(\nu)\right].$$

For any measurable subset Ω' of Ω, define

$$\mathbf{M}(\Omega') := \int_{\Omega'} \omega,$$

the total mass of vorticity on Ω'. Then

$$\mathbf{M}(\Gamma_\delta^C) = (\omega, \mathbb{1}_{\Gamma_\delta^C}) = (\omega, 1 - \varphi_\delta) + \int_{\Gamma_\delta \setminus \Gamma_{\delta*}} \varphi_\delta \omega$$

so

$$\begin{aligned} \left|(\omega, 1 - \varphi_\delta) - \mathbf{M}(\Gamma_\delta^C)\right| &\leq \left|\int_{\Gamma_\delta \setminus \Gamma_{\delta*}} \varphi_\delta \omega\right| \\ &\leq \|\varphi_\delta\|_{L^2(\Gamma_\delta \setminus \Gamma_{\delta*})} \|\omega\|_{L^2(\Gamma_\delta \setminus \Gamma_{\delta*})} \\ &\leq C(\delta - \delta^*)^{\frac{1}{2}} \|\omega\|_{L^2(\Gamma_\delta)}. \end{aligned}$$

Thus,

$$
\begin{aligned}
M_\delta := \left| m - \mathbf{M}(\Gamma_\delta^C) \right| &\leq |m - (\omega, 1 - \varphi_\delta)| + \left| (\omega, 1 - \varphi_\delta) - \mathbf{M}(\Gamma_\delta^C) \right| \\
&\leq C \left[\delta + (\delta - \delta^*)^{-\frac{1}{2}} F(\nu) \right] + C(\delta - \delta^*)^{\frac{1}{2}} \|\omega\|_{L^2(\Gamma_\delta)} .
\end{aligned}
$$

Choosing $\delta(\nu) = \nu$, $\delta^*(\nu) = \nu/2$, we have

$$
M_\nu \leq C \left[\nu + \nu^{-\frac{1}{2}} o(\nu^{\frac{1}{2}}) \right] + C \nu^{\frac{1}{2}} \|\omega\|_{L^2(\Gamma_\nu)} \to 0
$$

uniformly over $[0, T]$.

Squaring, integrating in time, and applying Young's inequality gives

$$
\|M_\nu\|^2_{L^2([0,T])} = \int_0^T M_\nu^2 \leq CT(\nu^2 + o(1)) + C\nu \int_0^T \|\omega\|^2_{L^2(0,T;L^2(\Gamma_\nu))} \to 0
$$

as $\nu \to 0$ since $F(\nu) = o(\nu^{1/2})$ by assumption and since (VV) holds.

Then,

$$\begin{aligned}\|m - M(\Omega)\|_{L^2([0,T])} &\le \left\| m - M(\Gamma_\nu^C) \right\|_{L^2([0,T])} + \|M(\Gamma_\nu)\|_{L^2([0,T])} \\ &= \|M_\nu\|_{L^2([0,T])} + \|M(\Gamma_\nu)\|_{L^2([0,T])} \\ &\le \|M_\nu\|_{L^2([0,T])} + \|\omega\|_{L^2(0,T;L^1(\Gamma_\nu))} \to 0\end{aligned}$$

as $\nu \to 0$, since $\|M_\nu\|_{L^2([0,T])} \to 0$ as we just showed and $\|\omega\|_{L^2(0,T;L^1(\Gamma_\nu))} \to 0$ by our control on the L^1 norm in the boundary layer.

But $u(t)$ lies in V so $M(\Omega) = 0$ for all $t > 0$. Hence, the limit above is possible only if $m = 0$.

Thank you

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