

# THE AGGREGATION EQUATION WITH NEWTONIAN POTENTIAL: THE VANISHING VISCOSITY LIMIT

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ABSTRACT. The viscous and inviscid aggregation equation with Newtonian potential models a number of different physical systems and has close analogs in 2D incompressible fluid mechanics. We consider a slight generalization of these equations in the whole space establishing well-posedness of the viscous and inviscid equations, spatial decay of the viscous solutions, and the convergence of viscous solutions to the inviscid solution as the viscosity goes to zero.

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## 1. INTRODUCTION

In this work we study on  $\mathbb{R}^d$ ,  $d \geq 2$ , the (viscous or inviscid) aggregation equation with Newtonian potential,

$$(AG_\nu) \quad \begin{cases} \partial_t \rho^\nu + \operatorname{div}(\rho^\nu \mathbf{v}^\nu) = \nu \Delta \rho^\nu, \\ \mathbf{v}^\nu = -\nabla \Phi * \rho^\nu, \\ \rho^\nu(0) = \rho_0. \end{cases}$$

Here,  $\nu \geq 0$  is the viscosity and  $\Phi$  is the fundamental solution of the Laplacian, or Newtonian potential (so  $\Delta \Phi = \delta$  and  $\operatorname{div} \mathbf{v}^\nu = -\rho^\nu$ ). The density is  $\rho^\nu$ , the velocity is  $\mathbf{v}^\nu$ , and  $\rho_0$  is the initial density.

Many variations on these equations are considered in the literature, primarily by using potentials other than the Newtonian or by using more general diffusive terms. We restrict our attention to the Newtonian potential with linear diffusion, for we will be concerned with

analyzing the viscous ( $\nu > 0$ ) and inviscid ( $\nu = 0$ ) aggregation equation using techniques adapted from the study of 2D fluid mechanics.

The aggregation equation models numerous physical problems. For the Newtonian potential, as in  $(AG_\nu)$ , this includes chemotaxis, where  $(AG_\nu)$  for  $\nu > 0$  is a limiting case of the Keller-Segel system (see Section 5.2 of [23]) and has been extensively studied. In this context,  $\rho^\nu$  measures the density of cells (bacteria or cancer cells, for instance) and  $\mathbf{v}^\nu$  is the gradient of the concentration of a chemoattractant. References most closely related to the approach to the aggregation equation taken in this paper include [2, 3, 4, 15, 16, 22, 23].

We will, in fact, consider a slightly more general set of equations of the form

$$(GAG_\nu) \quad \begin{cases} \partial_t \rho^\nu + \mathbf{v}^\nu \cdot \nabla \rho^\nu = \sigma_2 (\rho^\nu)^2 + \nu \Delta \rho^\nu, \\ \mathbf{v}^\nu = \sigma_1 \nabla \Phi * \rho^\nu, \\ \rho^\nu(0) = \rho_0, \end{cases}$$

where  $\sigma_1, \sigma_2$  are constants with  $\sigma_1 \neq 0$ . When  $\sigma_1 = -1, \sigma_2 = 1$ ,  $(GAG_\nu)$  reduces to  $(AG_\nu)$  since then  $\operatorname{div}(\rho^\nu \mathbf{v}^\nu) = \mathbf{v}^\nu \cdot \nabla \rho^\nu + \operatorname{div} \mathbf{v}^\nu \rho^\nu = \mathbf{v}^\nu \cdot \nabla \rho^\nu - (\rho^\nu)^2$ .

The special case  $\sigma_1 = 1, \sigma_2 = -1$  has been used to model type-II superconductivity when  $\nu = 0$  (see [22] and the references therein). In such applications,  $\rho_0$  is not assumed to have a distinguished sign, so there is no real difference between studying  $\sigma_1 = -1, \sigma_2 = 1$  and  $\sigma_1 = 1, \sigma_2 = -1$ .

At least one other special case of  $(GAG_\nu)$  has been studied in the literature:  $(GAG_0)$  with  $\sigma_1 = -1, \sigma_2 = 0$  are derived from  $(AG_0)$  by making a transformation of variables in (1.6) of [2]. This transformation applies only in the special case of aggregation patch initial data (analogous to a vortex patch for fluids) for  $(AG_0)$ . Although this transformation only works for aggregation patch initial data the authors of [2] go on to use this special case of  $(GAG_0)$  throughout their analysis of aggregation patches. A general well-posedness result is not needed in [2] and hence not established there, but such a result was one of our motivations for studying  $(GAG_\nu)$ , the parameters  $\sigma_1, \sigma_2$  merely interpolating between  $(AG_0)$  and the equations studied in [2] (when  $\nu = 0$ ).

We will study  $(GAG_\nu)$  in all of  $\mathbb{R}^d$ , but we note that much of what we find extends naturally to a bounded domain if, as is typically done, one uses no-flux boundary conditions,  $\nabla \rho^\nu \cdot \mathbf{n} = 0$ . This is because such boundary conditions eliminate all troublesome boundary integrals. The situation is the same for 2D incompressible fluids using no-flux conditions on the vorticity, though for fluids such conditions have no real physical meaning.

We will find establishing the existence of weak viscous solutions to  $(GAG_\nu)$  no more difficult than doing the same for  $(AG_\nu)$  except for keeping track of the constants  $\sigma_1$  and  $\sigma_2$ . In most applications compact support of the initial density would suffice, but such compact support is not conserved for viscous solutions. We will find that density having spatial decay of a specific type is conserved, however, and we will find it convenient to work in a space having such decay. Roughly speaking, this space, which we call  $L_N^2$  (see Definition 2.1), consists of  $L^2$  densities having sufficiently rapid algebraic decay at infinity.

In outline, our proof of existence of solutions to  $(GAG_\nu)$  proceeds as follows. We first define in Section 2 the spaces  $L_N^2$  and  $H_N^1$  in analogy with  $L^2(\mathbb{R}^d)$  and  $H^1(\mathbb{R}^d)$  and prove that  $H_N^1$  is compactly embedded in  $L_M^2$  for all  $M < N$ . We show existence of solutions to a linearized version of  $(GAG_\nu)$  in Section 3 using an abstract functional analytic approach due to J.-L. Lions. We expend most of our effort showing that such solutions lie in  $L_N^2 \cap L^\infty$  for all time and then obtaining bounds over time on all  $L^p$  norms of the density that are uniform in viscosity. In Section 4 we use a sequence of these linearized solutions to approximate a solution to the viscous aggregation equation taking advantage of the compact embedding of

$H_N^1$  in  $L_M^2$  to obtain a solution via a compactness argument. (We address uniqueness of solutions only later in Theorem 7.5.)

Our viscous existence results suit our needs in later sections when we examine the vanishing viscosity limit; in particular, we need uniform-in-viscosity bounds on  $L^p$ -norms of the density and on the time of existence of solutions to obtain the limit. Much more is known, however, about the existence time of solutions and how it relates to the initial mass of the density, at least for  $(AG_\nu)$  for nonnegative  $\rho_0$  (as summarized in Sections 5.2, 5.3 of [23]). See also the proof of existence of global-in-time renormalized solutions to  $(GAG_\nu)$  for the special case  $\sigma_1 = 1$ ,  $\sigma_2 = -1$  in [22] assuming only that  $\rho_0 \in L^1$ .

We return in Section 5 to the well-posedness of weak solutions to the generalized inviscid aggregation equation. We adapt the approach of Marchioro and Pulvirenti in [21] (which originates in their earlier text [20]) to prove existence and uniqueness of Lagrangian solutions to the 2D Euler equations, combining it with some ideas from Chapter 8 of Majda and Bertozzi's [19]. Marchioro and Pulvirenti's argument is both economical and elegant, but both of these virtues are impacted by the need to handle non-divergence free vector fields. Fundamentally, this is because the Jacobian of the transformation induced by the flow map is no longer 1 but involves  $\rho_0$ . This introduces into the argument new terms that require us to assume some regularity of the initial density. Only after proving existence with such regularity can a limiting argument be made to treat initial densities lying in  $L_N^2 \cap L^\infty$ . (Our proofs of existence and uniqueness are simplified by adding the assumption that  $\rho_0$  is compactly supported.) We will also adapt Marchioro and Pulvirenti's approach to establishing higher regularity of inviscid solutions, as their approach translates with only minor difficulties to non-divergence free vector fields.

The varying effects of  $\sigma_1$  and  $\sigma_2$  begin to become apparent in Section 6 when we examine the behavior of the total mass of the density,  $m(\rho^\nu) := \int_{\mathbb{R}^d} \rho^\nu$ . We will find that  $m(\rho^\nu)$  is conserved only when  $\sigma_1 + \sigma_2 = 0$ . The mass is particularly important in 2D where the energy of the solutions is infinite. The lack of finite energy was no real obstacle for studying weak solutions because only the  $L^\infty$  norm of the velocity played a role in the estimates. But when treating uniqueness of regular viscous solutions and proving that the vanishing viscosity limit holds, the  $L^2$  norm of the velocity play an important role.

As is the case with the 3D Navier-Stokes equations, the uniqueness of weak solutions is an open problem: we will content ourselves with proving uniqueness of solutions having sufficient regularity. Even for regular solutions uniqueness is not a simple matter, for one needs to control both the density and velocity of the difference between two solutions. Uniqueness will follow, however, from the vanishing viscosity arguments we make beginning in Section 7, where we show that

$$(VV) : \quad \mathbf{v}^\nu \rightarrow \mathbf{v}^0 \text{ in } L^\infty(0, T; H^1), \quad \rho^\nu \rightarrow \rho^0 \text{ in } L^\infty(0, T; L^2) \text{ as } \nu \rightarrow 0$$

for a  $T > 0$  as stated in Theorem 4.5. When  $d = 3$ ,  $\mathbf{v}^\nu$  and  $\mathbf{v}^0$  both lie in  $L^2(\mathbb{R}^d)$ . This is no longer (in general) the case when  $d = 2$ . When  $\sigma_1 + \sigma_2 = 0$ , however, because the total mass of the densities  $\rho^\nu$  and  $\rho^0$  are conserved over time, the infinite parts of the energies cancel, giving  $\mathbf{v}^\nu - \mathbf{v}^0 \in L^2(\mathbb{R}^2)$ . In both of the cases  $d \geq 3$  or  $d = 2$  with  $\sigma_1 + \sigma_2 = 0$  (VV) holds, as we show in Section 7.

In Section 8 we consider the remaining case where  $d = 2$  but  $\sigma_1 + \sigma_2 \neq 0$ . In this case the total mass of the densities are not conserved over time and the infinite parts of the energies do not cancel. We will nonetheless be able to isolate the infinite parts of the energy and use them to define a spatially smooth corrector  $\theta^\nu$  that lies in weak- $L^2$  and all higher  $L^p$  spaces

and show that in place of  $(VV)$  we have

$$(VV)' : \quad \mathbf{v}^\nu - \mathbf{v}^0 - \boldsymbol{\theta}^\nu \rightarrow 0 \text{ in } L^\infty(0, T; H^1), \quad \rho^\nu \rightarrow \rho^0 \text{ in } L^\infty(0, T; L^2) \text{ as } \nu \rightarrow 0, \\ \boldsymbol{\theta}^\nu \rightarrow 0 \text{ in } L^\infty(0, T; C^k) \text{ for all } k \geq 0.$$

As can be seen from  $(VV)$ ,  $(VV)'$  both the velocity and density converge strongly in the vanishing viscosity limit. Indeed, the arguments in Sections 7 and 8 involve showing the simultaneous convergence of both the velocities and the densities.

In Section 9 we use the results from Sections 7 and 8 along with uniform bounds in viscosity on Hölder norms of solutions to  $(GAG_\nu)$  to prove that the vanishing viscosity limit holds in the  $L^\infty$ -norm of the density for more regular initial data. In Section 10 we make some concluding remarks.

We follow the convention that  $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^d)}$ . We write  $\langle \cdot, \cdot \rangle$  for the  $L^2$ -inner product and  $(\cdot, \cdot)$  for the pairing in the duality between  $H^1(\mathbb{R}^d)$  and  $H^{-1}(\mathbb{R}^d)$ .

We make the convention that  $C$  stands for an unspecified positive constant that is independent of any significant parameters. Its value may vary from expression to expression. If its explicit dependence upon certain parameters is significant we write  $C(a_1, \dots, a_n)$ . In particular,

$$C_0(t) \text{ is a positive, continuous, nondecreasing function of } t \in [0, \infty).$$

We will find various uses for the following cutoff function:

**Definition 1.1.** *Let  $a$  be a radially symmetric function in  $C^\infty(\mathbb{R}^d)$  taking values in  $[0, 1]$ , supported in  $B_2(0)$  with  $a \equiv 1$  on  $B_1(0)$  and with  $a(x)$  nonincreasing in  $|x|$ . For any  $R \geq 1$  define  $a_R(\cdot) = a(\cdot/R)$ . Note that  $a_R(x)$  is nondecreasing in  $R$  for any fixed  $x \in \mathbb{R}^d$ . We also define  $b_R(x) = a_{2R}(x) - a_R(x) \geq 0$ , noting that  $b_R$  is supported on the annulus of inner, outer radii  $R, 4R$ .*

For any  $p_1, p_2 \in [1, \infty]$ ,  $p_1 \neq p_2$ , we define  $\|f\|_{L^{p_1} \cap L^{p_2}} = \|f\|_{L^{p_1}} + \|f\|_{L^{p_2}}$  for the Banach space  $L^{p_1} \cap L^{p_2}$ .

If a displayed equation with equation number  $(m.n)$  consists of multiple equalities then  $(m.n)_k$  refers to the  $k$ -th equality in the equations. For example, (4.3)<sub>2</sub> refers to the equation  $\mathbf{v}_n = \sigma_1 \nabla \Phi * \rho_{n-1}$ .

We will use many times the following simple form of a classical result:

**Lemma 1.2** (Grönwall's inequality). *Fix  $T > 0$ . Let  $L, \alpha$ , and  $\beta$  be nonnegative continuous functions on the interval  $[0, T]$  with  $\alpha$  non-decreasing and assume that  $M \in L^1((0, T))$ . If*

$$L(t) + \int_0^t M(s) ds \leq \alpha(t) + \int_0^t \beta(s)L(s) ds \text{ for all } t \in [0, T]$$

then

$$L(t) + \int_0^t M(s) ds \leq \alpha(t) \exp \int_0^t \beta(s) ds \text{ for all } t \in [0, T].$$

## 2. MEASURING PERSISTENCE OF SPATIAL DECAY

The main issue we will face in obtaining weak solutions to  $(GAG_\nu)$  is not regularity, which can be dealt with in a very classical manner, but rather spatial decay. The difficulty is that even if we assume compact support of the initial density—as we will in Theorem 5.7 for the inviscid solutions—diffusion ensures that compact support is lost for all positive time. We will

find, however, that algebraic spatial decay as characterized by the space  $L_N^2$  of Definition 2.1 will persist. We explore in this section key properties of this space that we will use in the following sections.

**Definition 2.1.** Fix a real number  $N \geq 0$ , let  $p \in [1, \infty]$ , and let  $a, b_R$  be as in Definition 1.1. For any integer  $k \geq 0$ , define the function space  $W_N^{k,p}(\mathbb{R}^d)$  to be the subspace of  $W^{k,p}(\mathbb{R}^d)$  with the norm,

$$\|f\|_{W_N^{k,p}(\mathbb{R}^d)} := \|af\|_{W^{k,p}(\mathbb{R}^d)} + \sup_{R \geq 1} R^N \|b_R f\|_{W^{k,p}(\mathbb{R}^d)}.$$

Let  $H_N^k(\mathbb{R}^d) := W_N^{k,2}(\mathbb{R}^d)$ ,  $L_N^p(\mathbb{R}^d) := W_N^{0,p}(\mathbb{R}^d)$ .

**Lemma 2.2.** Each of the following holds for all  $d \geq 2$ :

- (1) Fix  $q \in [1, 2]$  and let  $n_q = d \frac{2-q}{2q}$ . If  $f \in L_N^2(\mathbb{R}^d)$  for some  $N > n_q$  then  $f \in L^q(\mathbb{R}^d)$  with  $\|f\|_{L^q} \leq C \|f\|_{L_N^2}$ ,  $f \in L_{N-n_q}^q(\mathbb{R}^d)$  with  $\|f\|_{L_{N-n_q}^q} \leq C \|f\|_{L_N^2}$ , and

$$\|(1 - a_R)f\|_{L^q} \leq CR^{-(N-n_q)} \|f\|_{L_N^2} \text{ for all } R \geq 1. \quad (2.1)$$

- (2) We have,

$$\|\nabla \Phi * f\|_{LL} \leq C \|f\|_{L^1 \cap L^\infty} \leq C \|f\|_{L_N^2 \cap L^\infty}, \quad (2.2)$$

the first inequality holding for all  $f \in L^1 \cap L^\infty$ , the second holding for all  $f \in L_N^2 \cap L^\infty$  for some  $N > d/2$ . In (2.2),  $LL = LL(\mathbb{R}^d)$  is the space of bounded log-Lipschitz vector fields with

$$\|g\|_{LL} := \|g\|_{L^\infty} + \sup \left\{ \frac{|g(x) - g(y)|}{-|x - y| \log|x - y|} : x, y \in \mathbb{R}^d, 0 < |x - y| \leq e^{-1} \right\}.$$

- (3) For any  $p \in (d/(d-1), \infty]$  there exists  $C(p) > 0$  such that for all  $f \in L^1 \cap L^p$ ,

$$\|\nabla \Phi * f\|_{L^p} \leq C \|f\|_{L^1 \cap L^p}. \quad (2.3)$$

- (4) For any  $p \in (d, \infty]$  there exists  $C(p) > 0$  such that for all  $f \in L^1 \cap L^\infty$ ,

$$\|\nabla \Phi * f\|_{L^\infty} \leq C(p) \|f\|_{L^\infty}^{\frac{p-2}{p}} \|f\|_{L^\infty}^{\frac{p-2}{p}} + C \|f\|_{L^1}. \quad (2.4)$$

- (5) If  $f \in L_N^2 \cap L^\infty$  for some  $N > 1 + d/2$  then for all  $R \geq 1$ ,

$$\|b_R(\nabla \Phi * f)\|_{L^\infty} \leq C \left( \|f\|_{L_N^2} + \|f\|_{L^\infty}^{\frac{d-1}{d+1}} \|f\|_{L_N^2}^{\frac{2}{d+1}} \right) R^{-1} \leq C \|f\|_{L_N^2 \cap L^\infty} R^{-1}. \quad (2.5)$$

(Hence,  $\nabla \Phi * f \in L_1^\infty$  with  $\|\nabla \Phi * f\|_{L_1^\infty} \leq C \|f\|_{L_N^2 \cap L^\infty}$ .)

*Proof.* (1) Fix  $q \in [1, 2]$ . Then for any  $R \geq 1$ ,

$$\|b_R f\|_{L^q} \leq \|1\|_{L^{\frac{2q}{2-q}}(\text{supp } b_R)} \|b_R f\| \leq CR^{\frac{2-q}{2q}} R^{-N} \|f\|_{L_N^2} = CR^{-(N-n_q)} \|f\|_{L_N^2}. \quad (2.6)$$

This along with  $\|af\|_{L^q} \leq C \|af\|$  shows that  $f \in L_{N-n_q}^q$  with  $\|f\|_{L_{N-n_q}^q} \leq C \|f\|_{L_N^2}$ .

Observe that

$$a_{2^n} = (a_{2^n} - a_{2^{n-1}}) + (a_{2^{n-1}} - a_{2^{n-2}}) + \cdots + (a_2 - a_1) + a_1 = a + \sum_{k=0}^{n-1} b_{2^k}.$$

Then, using (2.6),

$$\|a_{2^n} f\|_{L^q} \leq \|af\|_{L^q} + \sum_{k=0}^{n-1} \|b_{2^k} f\|_{L^q} \leq C \|f\|_{L_N^2} + C \sum_{k=0}^{n+1} 2^{-k(N-n_q)} \|f\|_{L_N^2} \leq C \|f\|_{L_N^2},$$

since  $N > n_q$ . But  $a_{2^n}$  increases monotonically to 1. Thus  $f \in L^q$  with  $\|f\|_{L^q} \leq C \|f\|_{L_N^2}$  by the monotone convergence theorem. Similarly,

$$\|(1 - a_{2^n})f\|_{L^q} \leq C \sum_{k=n}^{\infty} 2^{-k(N-n_q)} \|f\|_{L_N^2} \leq C 2^{-n(N-n_q)} \|f\|_{L_N^2}.$$

Given  $R > 0$ , choose  $n$  such that  $2^n \leq R < 2^{n+1}$ . Then

$$\|(1 - a_R)f\|_{L^q} \leq \|(1 - a_{2^n})f\|_{L^q} \leq C 2^{-n(N-n_q)} \|f\|_{L_N^2} \leq C R^{-(N-n_q)} \|f\|_{L_N^2}.$$

So we see that more generally (2.1) holds.

(2) In 2D, the first inequality in (2.2) is Lemma 8.1 of [19]. It can be proved in all dimensions in a manner very similar to that of Theorem 3.1 of [25], so we suppress the proof. The second inequality in (2.2) follows from (1).

(3) We have

$$\|\nabla\Phi * \rho\|_{L^p} \leq \|a\nabla\Phi\|_{L^1} \|\rho\|_{L^p} + \|(1-a)\nabla\Phi\|_{L^p} \|\rho\|_{L^1} < \infty$$

for all  $p > d/(d-1)$ , giving (2.3).

(4) Observe that for any  $p \in (d, \infty]$ ,

$$\begin{aligned} \|\nabla\Phi * f\|_{L^\infty} &\leq \|(a\nabla\Phi) * f\|_{L^\infty} + \|((1-a)\nabla\Phi) * f\|_{L^\infty} \\ &\leq C \|a\nabla\Phi\|_{L^{p'}} \|f\|_{L^p} + C \|(1-a)\nabla\Phi\|_{L^\infty} \|f\|_{L^1}, \end{aligned}$$

where  $p' = p/(p-1) \in [1, d/(d-1))$ . But by Lebesgue interpolation,  $\|f\|_{L^p} \leq \|f\|_{L^\infty}^{\frac{2}{p}} \|f\|_{L^\infty}^{\frac{p-2}{p}}$ , and (2.4) follows.

(5) Assume that  $|x| \geq 1$ , and let  $R = |x|/8$  so that  $x \notin \text{supp } a_{2R}$ . We write

$$|\nabla\Phi * f(x)| \leq |\nabla\Phi * (a_R f)(x)| + |\nabla\Phi * ((1 - a_R)f)(x)|$$

and bound the two terms separately. For the first term,

$$|\nabla\Phi * (a_R f)(x)| \leq C \int_{\text{supp } a_R} |x-y|^{1-d} a_R(y) |f(y)| dy \leq C |x|^{1-d} \|a_R f\|_{L^1} \leq C \|f\|_{L_N^2} |x|^{1-d},$$

since  $|x-y| \geq 3|x|/4$  for  $y \in \text{supp } a_R$ . For the other term, we use (2.1) and (2.4) to obtain

$$\begin{aligned} |\nabla\Phi * ((1 - a_R)f)(x)| &\leq C \|(1 - a_R)f\|_{L^\infty}^{\frac{2}{p}} \|f\|_{L^\infty}^{\frac{p-2}{p}} + C \|(1 - a_R)f\|_{L^1} \\ &\leq C (R^{-N})^{\frac{2}{p}} \|f\|_{L_N^2}^{\frac{2}{p}} \|f\|_{L^\infty}^{\frac{p-2}{p}} + C R^{-(N-\frac{d}{2})} \|f\|_{L_N^2}. \end{aligned}$$

But  $N - d/2 > 1$  and  $2N/p > 2(1 + d/2)/p = (2 + d)/p \geq 1$  as long as  $p \in (d, 2 + d]$ . Choosing  $p = d + 1$  yields (2.5).  $\square$

**Remark 2.3.** An implication of (2.1) is that we could replace  $b_R$  with  $1 - a_R$  in Definition 2.1. The compact support of  $b_R$ , however, makes it more convenient in most applications.

**Lemma 2.4.** Let  $N > 0$ . Then  $H_N^1(\mathbb{R}^d)$  is continuously and compactly embedded in  $L^2(\mathbb{R}^d)$  and in  $L_M^2(\mathbb{R}^d)$  for any  $M < N$ . Moreover, if  $(f_n)$  is a bounded sequence in  $H_N^1(\mathbb{R}^d)$  then there exists  $f \in L_N^2(\mathbb{R}^d)$  such that a subsequence of  $(f_n)$  converges to  $f$  in  $L_M^2(\mathbb{R}^d)$  for all  $M < N$ .

*Proof.* Let  $(f_n)$  be a bounded sequence in  $H_N^1$ . We will construct a subsequence  $(f_{n_k})$  and a function  $f \in L_N^2(\mathbb{R}^d)$  for which  $f_{n_k} \rightarrow f$  in  $L_M^2(\mathbb{R}^d)$  for any  $M < N$ , giving the compact embedding of  $H_N^1(\mathbb{R}^d)$  in  $L_M^2(\mathbb{R}^d)$  and hence also in  $L^2(\mathbb{R}^d)$ .

Because  $\text{supp } a_k, k = 1, 2, \dots$  is bounded,  $H^1(\text{supp } a_k)$  is compactly embedded in  $L^2(\text{supp } a_k)$ . We can thus extract a subsequence of  $(f_n)$ , which we relabel as  $(f_n^{(1)})_{n=1}^\infty$ , and a  $g_1 \in L^2(\text{supp } a_1)$  such that  $f_n^{(1)}|_{\text{supp } a_1} \rightarrow g_1$  in  $L^2(\text{supp } a_1)$  as  $n \rightarrow \infty$ . We continue this process inductively, constructing sequences  $(f_n^{(k)})_{n=1}^\infty$  and  $(g_k)$  with  $g_k \in L^2(\text{supp } a_k)$  and  $f_n^{(k)}|_{\text{supp } a_k} \rightarrow g_k$  in  $L^2(\text{supp } a_k)$  as  $n \rightarrow \infty$ . At each step, we choose the subsequence  $(f_n^{(k)})_{n=1}^\infty$  from  $(f_n^{(k-1)})_{n=1}^\infty$ .

We have  $g_j|_{\text{supp } a_k} = g_k$  for  $j > k$  since

$$\|g_j - g_k\|_{L^2(\text{supp } a_k)} \leq \lim_{n \rightarrow \infty} \left[ \|g_k - f_n^{(j)}\|_{L^2(\text{supp } a_k)} + \|g_j - f_n^{(j)}\|_{L^2(\text{supp } a_k)} \right] = 0.$$

Hence, we can define a function  $f$  pointwise on  $\mathbb{R}^d$  by  $f(x) = \lim_{k \rightarrow \infty} g_k(x)$ . Observe that  $\|(1 - a_k)f_n\| < Ck^{-N}$  for all  $n$  follows from (2.1). Hence, also  $\|(1 - a_k)f\| < Ck^{-N}$ , so  $f \in L_N^2$ .

We now construct a subsequence of  $(f_n)$  as follows. We set the first term of the subsequence equal to  $f_{j_1}^{(1)}$ , where  $j_1$  is chosen sufficiently large to ensure that  $\|f_{j_1}^{(1)} - f\|_{L^2(\text{supp } a_1)} < 1$ . Proceeding inductively, we set the  $k$ -th term of the subsequence equal to  $f_{j_k}^{(k)}$ , where  $j_k$  satisfies  $j_{k-1} < j_k$  and  $\|f_{j_k}^{(k)} - f\|_{L^2(\text{supp } a_k)} < 1/k$ . Relabeling this sequence as  $(f_n)$ , we have that  $\|f_n - f\|_{L^2(\text{supp } a_n)} < 1/n$  for all  $n$ .

Although we have established that the limiting function  $f$  is in  $L_N^2$ , we cannot conclude that  $f_n \rightarrow f$  in  $L_N^2$ . We can, however, show that  $f_n \rightarrow f$  in  $L_M^2$  for any  $M < N$ , as follows.

Fix  $M < N$ , and observe that

$$\|f_n - f\|_{L_M^2} = \|a(f - f_n)\| + \sup_{R \geq 1} R^M \|b_R(f - f_n)\| \leq \|a(f - f_n)\| + C \sup_{k \in \mathbb{N}} k^M \|b_k(f - f_n)\|.$$

Now let  $\varepsilon > 0$ . For any fixed  $k_0 \in \mathbb{N}$  we can choose  $N_0 > 0$  sufficiently large that  $\|f - f_n\|_{L^2(\text{supp } a_{2k})} < k_0^{-M} \varepsilon$  for any  $n \geq N_0$  and any  $k \leq k_0$ . Then for all  $n \geq N_0$ ,

$$\begin{aligned} \|f_n - f\|_{L_M^2} &\leq C \sup_{k \leq k_0} k^M \|f - f_n\|_{L^2(\text{supp } a_{2k})} + C \sup_{k > k_0} k^{M-N} k^N \|(1 - a_k)(f - f_n)\| \\ &\leq C k_0^M k_0^{-M} \varepsilon + C k_0^{M-N} k^N k^{-N} \leq C(\varepsilon + k_0^{M-N}). \end{aligned}$$

We used here that  $\|(1 - a_k)(f - f_n)\| \leq \|(1 - a_k)f\| + \|(1 - a_k)f_n\| \leq Ck^{-N}$ . Choosing  $k_0$  large enough that  $k_0^{M-N} < \varepsilon$  we have  $\|f_n - f\|_{L_M^2} < C\varepsilon$  for all  $n \geq N_0$ . Since this holds for all  $\varepsilon > 0$ , it follows that  $\|f_n - f\|_{L_M^2} \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

**Corollary 2.5.** *Let  $\alpha \geq 0$  and assume that  $f \in L_N^2(\mathbb{R}^d)$  for some integer  $N > \alpha + d/2$ . Then  $|x|^\alpha f(x) \in L^1(\mathbb{R}^d)$  with  $\||x|^\alpha f\|_{L^1} \leq C \|f\|_{L_N^2}$ .*

*Proof.* For any  $R \geq 1$  we have

$$R^{N-\alpha} \|b_R |x|^\alpha f\| \leq (4R)^\alpha R^{N-\alpha} \|b_R f\| \leq 4^\alpha R^N \|b_R f\| \leq CR^N R^{-N} \|f\|_{L_N^2} = C \|f\|_{L_N^2}.$$

This along with  $\|a(x)|x|^\alpha f(x)\| \leq C \|af\|$  shows that  $|x|^\alpha f(x) \in L_{N-\alpha}^2$  with  $\||x|^\alpha f(x)\|_{L_{N-\alpha}^2} \leq C \|f\|_{L_N^2}$ . That  $|x|^\alpha f(x) \in L^1(\mathbb{R}^d)$  with  $\||x|^\alpha f\|_{L^1} \leq C \|f\|_{L_N^2}$  then follows from (1) of Lemma 2.2.  $\square$

Lemma 2.6 shows that time-continuity in  $L^2$  with boundedness in  $L_N^2$  implies time-continuity in all lower Lebesgue norms. We note that this result follows for all  $L^q$ -norms with  $q \in (1, 2]$  easily by (1) of Lemma 2.2 and Lebesgue space interpolation, so it is the  $L^1$  case that is most important to us.

**Lemma 2.6.** *If  $f \in C([0, T]; L^2) \cap L^\infty(0, T; L_N^2)$  for  $N > d/2$  then  $f \in C([0, T]; L^q)$  for all  $q \in [1, 2]$ .*

*Proof.* Let  $\varepsilon > 0$ . For any  $t \in [0, T]$  let  $\delta > 0$  be small enough that  $\|f(s) - f(t)\|_{L^2} < \varepsilon$  for all  $s \in [0, T] \cap (t - \delta, t + \delta)$ . Fix  $q \in [1, 2]$  and let  $n_q = d \frac{2-q}{2q}$ . Then by (1) of Lemma 2.2, for any  $R \geq 1$ ,

$$\begin{aligned} \|f(s) - f(t)\|_{L^q} &\leq \|a_R(f(s) - f(t))\|_{L^q} + \|(1 - a_R)(f(s) - f(t))\|_{L^q} \\ &\leq \|f(s) - f(t)\|_{L^q(B_{2R}(0))} + CR^{-(N-n_q)} \|f\|_{L_N^2} \\ &\leq CR^{n_q} \|f(s) - f(t)\| + CR^{-(N-n_q)} \|f\|_{L_N^2} \leq CR^{n_q} \varepsilon + CR^{-(N-n_q)}. \end{aligned}$$

Set  $R = \varepsilon^{-\frac{1}{N}}$ . Then

$$\|f(s) - f(t)\|_{L^q} \leq C\varepsilon^{1 - \frac{n_q}{N}},$$

from which  $f \in C([0, T]; L^q)$  follows, since  $n_q \leq n_1 = d/2 < N$ .  $\square$

### 3. THE LINEAR VISCOUS PROBLEM

In this section, we investigate solutions to the linear parabolic problem,

$$\begin{cases} \partial_t \xi + \mathbf{v}_f \cdot \nabla \xi = \sigma_2 f \xi + \nu \Delta \xi + g, \\ \xi(0) = f(0), \end{cases} \quad (3.1)$$

where  $f, g$  are given functions of space and time and  $\mathbf{v}_f := \sigma_1 \nabla \Phi * f$ . In Section 4, we will use a sequence of solutions to (3.1) to obtain existence of a solution to the nonlinear problem in  $(GAG_\nu)$ . In the limit we will have  $f = \xi = \rho^\nu$ , so that we will want  $f$  and  $\xi$  to exist in the same function spaces.

**Definition 3.1.** *Fix  $N > 1 + d/2$  and define the solution space*

$$Y_N := \{h \in C([0, T]; L^2) \cap L^2(0, T; H_N^1) \cap L^\infty(0, T; L_N^2) : \partial_t h \in L^2(0, T; H^{-1})\}.$$

*(We place no norm on  $Y_N$ , however.) Assume that  $f \in Y_N \cap L^\infty([0, T] \times \mathbb{R}^d)$ , and let  $\mathbf{v}_f = \sigma_1 \nabla \Phi * f$ . Assume that  $g \in L^2(0, T; L_N^2)$ . We say that  $\xi \in Y_N$  is a weak solution to the linear problem (3.1) on the interval  $[0, T]$  if  $\xi(0) = f(0)$  and*

$$\begin{aligned} (\partial_t \xi(t), \varphi) + \nu (\nabla \xi(t), \nabla \varphi) - (\sigma_1 + \sigma_2) \langle f(t) \xi(t), \varphi \rangle - \langle \mathbf{v}_f(t) \xi(t), \nabla \varphi \rangle &= \langle g, \varphi \rangle \\ \text{for a.e. } t \in [0, T] \text{ for all } \varphi \in H_N^1. \end{aligned} \quad (3.2)$$

*Equality in (3.2) is to hold in the sense of distributions on  $(0, T)$ .*

**Remark 3.2.** *By  $\partial_t \xi$  in Definition 3.1 we mean the weak time derivative of  $f$  (see 5.9.2 and Appendix E.5 of [11]). Thus,  $\partial_t \xi \in L^2(0, T; H^{-1})$  means that  $\int_0^T \partial_t \xi(t) v(t) dt = - \int_0^T \xi(t) v'(t) dt$  for all real-valued  $v \in C_c^\infty((0, T))$ , the integrations being over Banach space  $(H^{-1})$  valued functions. We will show in Theorem 3.3 that we can also treat  $\partial_t \xi$  as a distributional derivative. Also, by the initial condition  $\xi(0) = f(0)$  we mean that  $\xi(t) \rightarrow f(0)$  in  $L^2$  as  $t \rightarrow 0^+$ , which makes sense because  $\xi \in C([0, T]; L^2)$ .*

In many of the proofs that follow we will want to make an energy argument by applying (3.2) with a  $\varphi$  that lies in the solution space. In other cases we wish to apply (3.2) with a test function in  $C_c^\infty((0, T) \times \mathbb{R}^d)$  to prove the existence of a solution. This can be justified in an entirely standard way. For completeness, we give the proof below.

**Theorem 3.3.** *Let*

$$Y := \{h \in C([0, T]; L^2) \cap L^2(0, T; H^1) : \partial_t h \in L^2(0, T; H^{-1})\}$$

and assume that  $f$ ,  $g$ , and  $\mathbf{v}_f$  are as in Definition 3.1. The function  $\xi \in Y_N$  with  $\xi(0) = f(0)$  is a weak solution as in Definition 3.1 if and only if any of the following hold:

- (1) (3.2) holds for all  $\varphi \in C_c^\infty(\mathbb{R}^d)$ .  
(2) For all  $\varphi \in C_c^\infty((0, T) \times \mathbb{R}^d)$  we have

$$\begin{aligned} - \int_0^T (\partial_t \xi, \varphi) + \int_0^T \int_{\mathbb{R}^d} (\xi \mathbf{v}_f \cdot \nabla \varphi + (\sigma_1 + \sigma_2) f \xi \varphi - \nu \nabla \xi \cdot \nabla \varphi) &= - \int_0^T \int_{\mathbb{R}^d} g \varphi \text{ or} \\ \int_0^T \int_{\mathbb{R}^d} (\xi \partial_t \varphi + \xi \mathbf{v}_f \cdot \nabla \varphi + (\sigma_1 + \sigma_2) f \xi \varphi - \nu \nabla \xi \cdot \nabla \varphi) &= - \int_0^T \int_{\mathbb{R}^d} g \varphi. \end{aligned} \quad (3.3)$$

That is, (3.1) holds on  $(0, T) \times \mathbb{R}^d$  in the sense of distributions.

- (3) For all  $\varphi \in C([0, T]; L^2) \cap L^2(0, T; H^1)$

$$- \int_0^T (\partial_t \xi, \varphi) + \int_0^T \int_{\mathbb{R}^d} (\xi \mathbf{v}_f \cdot \nabla \varphi + (\sigma_1 + \sigma_2) f \xi \varphi - \nu \nabla \xi \cdot \nabla \varphi) = - \int_0^T \int_{\mathbb{R}^d} g \varphi.$$

- (4) For all  $\varphi \in Y$  we have

$$\begin{aligned} \int_0^T (\xi, \partial_t \varphi) + \int_0^T \int_{\mathbb{R}^d} (\xi \mathbf{v}_f \cdot \nabla \varphi + (\sigma_1 + \sigma_2) f \xi \varphi - \nu \nabla \xi \cdot \nabla \varphi) \\ = \int_{\mathbb{R}^d} (\xi(T) \varphi(T) - f(0) \varphi(0)) - \int_0^T \int_{\mathbb{R}^d} g \varphi. \end{aligned}$$

Finally, if  $\xi$  is a weak solution as in Definition 3.1 then (1)-(4) hold with any  $t \in [0, T]$  in place of  $T$ .

*Proof.* We give the proof for  $g \equiv 0$  as forcing plays no significant role in the proofs. Assume first that  $\xi$  is a solution as given in Definition 3.1. Then (1) follows since  $C_c^\infty(\mathbb{R}^d) \subseteq H_N^1$ .

Next we prove (2). Let  $\varphi(t, x) = \varphi_1(t) \varphi_2(x)$ , where  $\varphi_1 \in \mathcal{D}(0, T)$  and  $\varphi_2 \in \mathcal{D}(\mathbb{R}^d)$ . In light of Remark 3.2 we see that integrating (3.2) in time gives either equality in (3.3). Because  $\mathcal{D}'((0, T) \times \mathbb{R}^d) = \mathcal{D}'((0, T)) \otimes \mathcal{D}'(\mathbb{R}^d)$  by the Schwartz kernel theorem, it follows that (3.3) holds for all  $\varphi \in \mathcal{D}((0, T) \times \mathbb{R}^d)$ . This establishes (2), since  $C_c^\infty((0, T) \times \mathbb{R}^d) = \mathcal{D}((0, T) \times \mathbb{R}^d)$  as sets.

To prove (3), we first show that (3.3)<sub>1</sub> holds for any  $\varphi$  in  $C^\infty([0, T]; C_c^\infty(\mathbb{R}^d))$ , which is clearly dense in  $C([0, T]; L^2) \cap L^2(0, T; H^1)$  (and in  $Y$ ). So let  $\varphi \in C^\infty([0, T]; C_c^\infty(\mathbb{R}^d))$ . Letting  $h_\varepsilon$  be as in Lemma 3.8, define  $\varphi_\varepsilon(t, x) = h_\varepsilon(t) \varphi(t, x)$ . Since  $\varphi_\varepsilon \in C_c^\infty((0, T) \times \mathbb{R}^d)$ , (3.3)<sub>1</sub> holds for  $\varphi_\varepsilon$  by (2). But

$$\left| \int_0^T (\partial_t \xi, \varphi) - \int_0^T (\partial_t \xi, \varphi_\varepsilon) \right| \leq \left( \int_0^{3\varepsilon/2} + \int_{T-3\varepsilon/2}^T \right) \|\partial_t \xi(t)\|_{H^{-1}} \|(\varphi - \varphi_\varepsilon)(t)\|_{H^1} dt \rightarrow 0$$

by the continuity of the Lebesgue integral. The same kind of bound holds for the other terms in (3.3)<sub>1</sub>. Thus, (3.3)<sub>1</sub> holds for  $\varphi$ .

Now let  $(\varphi_n)$  be a sequence in  $C^\infty([0, T]; C_c^\infty(\mathbb{R}^d))$  converging to  $\varphi$  in  $C([0, T]; L^2) \cap L^2(0, T; H^1)$ . Then (3.3)<sub>1</sub> holds for each  $\varphi_n$ , and taking advantage of  $f$  and, by (2) of Lemma 2.2,  $\mathbf{v}_f$  both lying in  $L^\infty((0, T) \times \mathbb{R}^d)$ , it is easy to see that (3) holds in the limit as  $n \rightarrow \infty$ . Then (4) follows the same way, except that we use Lemma 3.9 for the time integral.

Also note that (1)-(4) clearly hold with any  $t \in [0, T]$  in place of  $T$ .

We now prove the reverse implications. That (1) implies that  $\xi$  is a weak solution follows from the density of  $C_c^\infty(\mathbb{R}^d)$  in  $H_N^1$ . That (2) implies (1) follows from applying either form of (3.3) with a test function of the form  $\varphi(t, x) = \varphi_1(t)\varphi_2(x)$ , since (3.2) is to hold in the sense of distributions in time. Finally, that (3) implies (2) and that (4) implies (3) follows by handling the time integral using Lemma 3.9 as we did in the proof of (4).  $\square$

To obtain the existence of solutions, we use the following extremely general result due to Lions and Magenes [17]. We quote the result as it appears in Theorem 10.9 of [5].

**Theorem 3.4.** [*J.-L. Lions*] *Let  $H$  be a Hilbert space with the subspace  $V$  continuously and densely embedded in  $H$ . Fix  $T > 0$  and suppose that  $a(t; u, v): V \times V \rightarrow \mathbb{R}$  is a bilinear form satisfying for some constants  $M, \alpha, C > 0$ :*

- (1) *for every  $u, v \in V$  the function  $t \mapsto a(t; u, v)$  is measurable;*
- (2)  *$|a(t; u, v)| \leq M \|u\|_V \|v\|_V$  for a.e.  $t \in [0, T]$  for all  $u, v \in V$ ;*
- (3)  *$|a(t; v, v)| \geq \alpha \|v\|_V^2 - C \|v\|_H^2$  for a.e.  $t \in [0, T]$  for all  $v \in V$ .*

*Given  $g \in L^2(0, T; V^*)$  and  $u_0 \in H$ , there exists a unique  $u$  with*

$$u \in L^2(0, T; V) \cap C([0, T]; H) \text{ and } \partial_t u \in L^2(0, T; V^*)$$

*such that  $u(0) = u_0$  and*

$$(\partial_t u(t), v) + a(t; u(t), v) = (g(t), v) \text{ for a.e. } t \in [0, T] \text{ for all } v \in V.$$

We will apply Theorem 3.4 with  $H = L^2(\mathbb{R}^d)$ ,  $V = H^1(\mathbb{R}^d)$ , and

$$a(t; u, v) = \nu \langle \nabla u, \nabla v \rangle - (\sigma_1 + \sigma_2) \langle f u, v \rangle - \langle \mathbf{v}_f, u \nabla v \rangle, \quad (3.4)$$

noting that formally,

$$\langle \partial_t \xi(t), \varphi \rangle + a(t; \xi(t), \varphi) = \langle \partial_t \xi(t) - \nu \Delta \xi(t) - \sigma_2 f \xi(t) + \mathbf{v}_f \cdot \nabla \xi(t), \varphi \rangle,$$

in accordance with (3.1).

**Theorem 3.5.** *Let  $\nu > 0$  and  $T > 0$  and let  $N > 1 + d/2$ . There exists a unique weak solution to (3.1) as in Definition 3.1. Moreover, the norms on  $\xi$  in  $L^2(0, T; H_N^1)$  and in  $L^\infty(0, T; L_N^2)$  can be bounded strictly in terms of  $\|f(0)\|_{L_N^2}$ ,  $\|f\|_{L^2(0, T; L_N^2 \cap L^\infty)}$ ,  $\|\xi\|_{L^2(0, T; L^2)}$ ,  $\|g\|_{L^2(0, T; L_N^2)}$ , and  $\nu^{-1}$ .*

*Proof.* Let  $H = L^2(\mathbb{R}^d)$ ,  $V = H^1(\mathbb{R}^d)$ , and define the bilinear form on  $V \times V$  as in (3.4). Then the existence and uniqueness of a weak solution  $\xi$  as in Definition 3.1 but with  $\xi \in L^2(0, T; H^1) \cap C([0, T]; L^2)$  and test functions in  $H^1$  follows from Theorem 3.4 once we verify the three required properties of  $a$  as follows:

- (1) This follows from  $f \in Y_N \cap L^\infty((0, t) \times \mathbb{R}^d)$  and (2.2).
- (2) We have

$$\begin{aligned} |a(t; u, v)| &\leq \nu \|\nabla u\| \|\nabla v\| + |\sigma_1 + \sigma_2| \|f\|_{L^\infty} \|u\| \|v\| + \|\mathbf{v}_f\|_{L^\infty} \|u\| \|\nabla v\| \\ &\leq M \|u\|_V \|v\|_V. \end{aligned}$$

(3) We have,

$$|a(t; v, v)| = |\nu \|\nabla v\|^2 - (\sigma_1 + \sigma_2)\langle f v, v \rangle - \langle \mathbf{v}_f, v \nabla v \rangle|.$$

But

$$\langle \mathbf{v}_f, v \nabla v \rangle = \frac{1}{2} \langle \mathbf{v}_f, \nabla |v|^2 \rangle = -\frac{1}{2} \langle \operatorname{div} \mathbf{v}_f, |v|^2 \rangle = -\frac{\sigma_1}{2} \langle f, |v|^2 \rangle.$$

(A density argument is used to integrate by parts here. We simply note that we have sufficient regularity and decay of  $f$  and  $v$  so that the first and last expressions make sense.) Thus

$$\begin{aligned} |a(t; v, v)| &= \left| \nu \|\nabla v\|^2 - \frac{\sigma_1 + 2\sigma_2}{2} \langle f v, v \rangle \right| \\ &\geq \nu \|\nabla v\|^2 - \left| \frac{\sigma_1 + 2\sigma_2}{2} \right| \|f\|_{L^\infty} \|v\|^2 = \nu \|\nabla v\|^2 - C \|v\|^2 \\ &= \nu (\|\nabla v\|^2 + \|v\|^2) - (C + \nu) \|v\|^2 = \nu \|v\|_V^2 - (C + \nu) \|v\|_H^2. \end{aligned}$$

We now show that, in fact,  $\xi \in Y_N$  (we note that it is the inability to verify (3) for  $H = L_N^2$ ,  $V = H_N^1$  that requires us to explicitly show this). By (4) of Theorem 3.3 applied with the test function  $\varphi = b_R^2 \xi \in Y$ , we have

$$\begin{aligned} &\int_0^t (\langle \partial_t \xi, b_R^2 \xi \rangle + \langle \xi \mathbf{v}_f, \nabla (b_R^2 \xi) \rangle + (\sigma_1 + \sigma_2) \langle f, b_R^2 \xi^2 \rangle - \nu \langle \nabla \xi, \nabla (b_R^2 \xi) \rangle) \\ &= \|b_R \xi(t)\|^2 - \|b_R f(0)\|^2 - \int_0^t \langle g, b_R^2 \xi \rangle. \end{aligned} \tag{3.5}$$

By Lemma 3.9, we have

$$\int_0^t \langle \partial_t \xi, b_R^2 \xi \rangle = - \int_0^t \langle \partial_t \xi, b_R^2 \xi \rangle + \|b_R \xi(t)\|^2 - \|b_R f(0)\|^2.$$

Thus (3.5) can be rewritten as

$$\int_0^t (\langle \partial_t \xi, b_R^2 \xi \rangle - \langle \xi \mathbf{v}_f, \nabla (b_R^2 \xi) \rangle - (\sigma_1 + \sigma_2) \langle f, b_R^2 \xi^2 \rangle + \nu \langle \nabla \xi, \nabla (b_R^2 \xi) \rangle) = \int_0^t \langle g, b_R^2 \xi \rangle.$$

Now observe that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} b_R^2 \xi \partial_t \xi &= \frac{1}{2} \left( \|b_R \xi(t)\|^2 - \|b_R f(0)\|^2 \right), \\ |\langle \xi \mathbf{v}_f, \nabla (b_R^2 \xi) \rangle| &= |\langle \mathbf{v}_f, 2b_R \xi^2 \nabla b_R + b_R^2 \xi \nabla \xi \rangle| \\ &\leq 2 \|b_R \mathbf{v}_f\|_{L^\infty} \|\nabla b_R\|_{L^\infty} \|\xi\|^2 + \|b_R \mathbf{v}_f\|_{L^\infty} \|\xi\| \|b_R \nabla \xi\| \\ &\leq C \nu^{-1} R^{-2} \|\xi\|^2 + \frac{\nu}{4} \|b_R \nabla \xi\|^2, \\ |\langle f, b_R^2 \xi^2 \rangle| &\leq \|f\|_{L^\infty} \|b_R \xi\|^2, \\ \nu \langle \nabla \xi, \nabla (b_R^2 \xi) \rangle &= \nu \langle \nabla \xi, 2b_R \xi \nabla b_R + b_R^2 \nabla \xi \rangle = \nu \|b_R \nabla \xi\|^2 + 2\nu \langle b_R \xi \nabla \xi, \nabla b_R \rangle, \\ |2\nu \langle b_R \xi \nabla \xi, \nabla b_R \rangle| &\leq 2\nu \|\nabla b_R\|_{L^\infty} \|\xi\| \|b_R \nabla \xi\| \leq C \nu R^{-1} \|\xi\| \|b_R \nabla \xi\| \\ &\leq C \nu R^{-2} \|\xi\|^2 + \frac{\nu}{4} \|b_R \nabla \xi\|^2, \\ \langle g, b_R^2 \xi \rangle &\leq \|b_R g\| \|b_R \xi\| \leq (1/2) \|b_R g\|^2 + (1/2) \|b_R \xi\|^2. \end{aligned}$$

We used (5) of Lemma 2.2 to bound  $b_R \mathbf{v}_f$  in  $L^\infty$ , and we used Young's inequality in the second inequality above and in the estimates for the last two terms. In the one term involving a time derivative we applied Lemma 3.9, using  $(\partial_t \xi, b_R^2 \xi) = (\partial_t (b_R^2 \xi), \xi)$ . Thus,

$$\begin{aligned} \|b_R \xi(t)\|^2 + \int_0^t \nu \|b_R \nabla \xi\|^2 &\leq \|b_R f(0)\|^2 + \int_0^t \|b_R g\|^2 \\ &+ \int_0^t \|b_R \xi\|^2 + \int_0^t \left( \frac{C}{\nu R^2} \|\xi\|^2 + C \|b_R \xi\|^2 \right). \end{aligned}$$

Applying Lemma 1.2 (Grönwall's lemma) gives

$$\|b_R \xi(t)\|^2 + \nu \int_0^t \|b_R \nabla \xi\|^2 \leq (CR^{-2N} + C_0(t)\nu^{-1}tR^{-2}) e^{C_0(t)t}, \quad (3.6)$$

where we used  $f(0) \in L_N^2$ ,  $g \in L^2(0, T; L_N^2)$  for some  $N > 1 + d/2$  with  $2N$  an integer. It follows from (3.6) that

$$\|b_R \xi(t)\|^2 + \nu \int_0^t \|b_R \nabla \xi\|^2 \leq C_0(t)R^{-2k}\nu^{-k}e^{C_0(t)t} \quad (3.7)$$

for  $k = 1$  and all  $\nu \leq \nu_0$  for any fixed  $\nu_0 > 0$ .

We now proceed by induction. Let  $S(k)$  be the statement that (3.7) holds. We have shown that  $S(k)$  is true for  $k = 1$ ; now suppose that it holds up to some  $k - 1 < N$ .

We refine the estimates,

$$\begin{aligned} |\langle \xi \mathbf{v}_f, \nabla (b_R^2 \xi) \rangle| &\leq C\nu^{-1}R^{-2} \|\xi\|_{L^2(\text{supp } \nabla b_R)}^2 + \frac{\nu}{4} \|b_R \nabla \xi\|^2, \\ |2\nu \langle b_R \xi \nabla \xi, \nabla b_R \rangle| &\leq C\nu R^{-2} \|\xi\|_{L^2(\text{supp } \nabla b_R)}^2 + \frac{\nu}{4} \|b_R \nabla \xi\|^2. \end{aligned}$$

But since  $b_{2R}(x) + b_R(x) + b_{R/2}(x) = 1$  on the support of  $\nabla b_R$ , we can write

$$\begin{aligned} \|\xi\|_{L^2(\text{supp } \nabla b_R)}^2 &\leq \|(b_{2R} + b_R + b_{R/2})\xi\|^2 \\ &\leq \nu^{-(k-1)} \left( C_0(t)(R/2)^{-2(k-1)} e^{C_0(t)t} + C_0(t)R^{-2(k-1)} e^{C_0(t)t} + C_0(t)(2R)^{-2(k-1)} e^{C_0(t)t} \right) \\ &\leq C_0(t)R^{-2(k-1)} \nu^{-(k-1)} e^{C_0(t)t}. \end{aligned}$$

This follows from (3.7) applied with  $R$  as well as with  $R$  replaced by both  $R/2$  and  $2R$ . With these refinements the argument that led to (3.6) now gives

$$\|b_R \xi(t)\|^2 + \nu \int_0^t \|b_R \nabla \xi\|^2 \leq (CR^{-2N} + C_0(t)\nu^{-k}tR^{-2k}) e^{C_0(t)t}.$$

So (3.7) holds for  $k$  as well by induction. We must stop at  $k = N$ , however, the bounds  $\|b_R f(0)\|^2$ ,  $\|b_R g\|_{L^2(0, T; L^2)}^2 \leq CR^{-2N}$  being the limiting factors.

This shows that  $\xi \in L^\infty(0, T; L_N^2) \cap L^2(0, T; H_N^1)$ .

Finally, the dependence of the various constants  $C_0(t)$  on the data occurs only through  $\|f(0)\|_{L_N^2}$  and  $\|f\|_{L^2(0, T; L_N^2 \cap L^\infty)}$ . Hence, the norms on  $\xi$  in  $L^2(0, T; H_N^1)$  and in  $L^\infty(0, T; L_N^2)$  can be bounded strictly in terms of  $\|f(0)\|_{L_N^2}$ ,  $\|f\|_{L^2(0, T; L_N^2 \cap L^\infty)}$ ,  $\|\xi\|_{L^2(0, T; L^2)}$ , and  $\|g\|_{L^2(0, T; L_N^2)}$ .  $\square$

**Theorem 3.6.** *Fix an integer  $k \geq 0$  and define the space*

$$Y_N^k := \{h \in C([0, T]; H^k) \cap L^2(0, T; H_N^{k+1}) \cap L^\infty(0, T; H_N^k) : \partial_t h \in L^2(0, T; H^{k-1})\}$$

(but place no norm on it). Assume that  $f \in Y_N^k$ ,  $g \in L^2(0, T; H_N^k)$ . Let  $\xi$  be the unique weak solution to (3.1) given by Theorem 3.5. Then  $\xi \in Y_N^k$ .

*Proof.* This regularity result for  $L^2$ -based spaces rather than  $L_N^2$ -based spaces is classical, based on a sequence of smooth Galerkin approximations to the solution. We give only a formal bootstrapping argument to explain how to obtain the  $L_N^2$ -based result from the  $L^2$ -based result.

Taking  $\partial_i$  of (3.1) for any  $i = 1, \dots, d$  gives formally

$$\partial_t \partial_i \xi + \mathbf{v}_f \cdot \nabla \partial_i \xi = \sigma_2 f \partial_i \xi + \nu \Delta \partial_i \xi + G, \quad (3.8)$$

with  $\partial_i \xi(0) = \partial_i f(0)$ , where

$$G := \sigma_2 \partial_i f \xi - \partial_i \mathbf{v}_f \cdot \nabla \xi + \partial_i g.$$

That is,  $\partial_i \xi$  also satisfies (3.1) with different forcing and initial data.

But  $G \in L^2(0, T; L_N^2)$ , so by Theorem 3.5 there exists a unique weak solution, which we will call  $\gamma_i$ , to (3.1) with forcing function  $G$  and initial data  $\partial_i f(0)$ . Now,  $\gamma_i$  is a weak solution to (3.1) and  $\partial_i \xi$  is formally a solution to (3.1) with the same initial data and forcing as  $\gamma_i$ . In fact,  $\gamma_i = \partial_i \xi$  follows from the Galerkin-approximation argument referred to earlier. Hence,  $\partial_i \xi \in Y_N$ . Because this holds for all  $i$  we have  $\nabla \xi \in Y_N$  and thus  $\xi \in C([0, T]; H^1) \cap L^2(0, T; H_N^2) \cap L^\infty(0, T; H_N^1)$ . Since (3.1) holds in the sense of distributions by (2) of Theorem 3.3, it follows that  $\partial_t \xi \in L^2(0, T; H^{k-1})$  so that  $\xi \in Y_N^k$ . This gives the result for  $k = 1$ . The result follows for any  $k$  by repeating this same process  $k - 1$  more times.  $\square$

In Theorem 3.7 we obtain uniform-in-viscosity bounds on the norms of  $\xi$  in  $L^\infty([0, T]; L^q)$  over time for sufficiently regular solutions. These bounds can be obtained for weak solutions as well but only with considerable additional technical difficulties due to the lack of a priori knowledge that the solution is continuous over time in the  $L^q$ -norm. Because our use of Theorem 3.7 is only to show the analogous result for nonlinear solutions in the next section, we limit ourselves to regular solutions, which is all we will need.

**Theorem 3.7.** *Let  $\nu, T > 0$  and assume that  $f \in Y_N^k$  for some  $N > 1 + d/2$ ,  $k > d/2$ . Let  $\xi$  be the regular solution to (3.1) given by Theorem 3.6 without forcing ( $g \equiv 0$ ). For any  $q \in [1, \infty]$  we have*

$$\|\xi(t)\|_{L^q} \leq \|f(0)\|_{L^q} \exp \left( \left| \frac{\sigma_1}{q} + \sigma_2 \right| \int_0^t \|f(s)\|_{L^\infty} ds \right). \quad (3.9)$$

Further, for  $q = 2$  we have

$$\|\xi(t)\|^2 + 2\nu \int_0^t \|\nabla \xi\|^2 \leq \|f(0)\|^2 \exp \left( \left| \sigma_1 + 2\sigma_2 \right| \int_0^t \|f(s)\|_{L^\infty} ds \right). \quad (3.10)$$

*Proof.* By Theorem 3.6 and Sobolev embedding  $\xi \in C([0, T]; L^q)$  for all  $q \in [2, \infty)$ , while by Lemma 2.6  $\xi \in C([0, T]; L^q)$  for all  $q \in [1, 2)$ . Assume that  $q$  is a rational number in  $(1, \infty)$  with  $q = m_1/m_2$  in lowest terms for  $m_1$  even. This insures that  $\xi^q \geq 0$ . The conclusions we reach for such rational  $q$ 's will hold for all  $q \in [1, \infty)$  by the continuity of Lebesgue norms.

If done formally, the argument we will make is very simple: multiply (3.1) by  $\varphi = \xi^{q-1}$ , integrate over space and time, perform several integrations by parts, and in the end obtain a bound on  $\|\xi(t)\|_{L^q}$ . In fact, for  $q \geq 2$  there is little more to the argument since we have restricted ourselves to sufficiently regular solutions. One of these integrations by parts, however, introduces a factor of  $\xi^{q-2}$  which is singular when  $q < 2$ . We will remove this

singularity by multiplying  $\xi^{q-1}$  by a factor that vanishes when  $\xi$  is near zero. This factor will be derived from a function  $\lambda_\varepsilon \in C^\infty(\mathbb{R})$  parameterized by  $\varepsilon \in (0, 1/2)$  and defined so that

$$\lambda_\varepsilon(x) = \begin{cases} (3/2)\varepsilon, & x < \varepsilon, \\ x, & x > 2\varepsilon \end{cases}$$

and so that  $\lambda'_\varepsilon, \lambda''_\varepsilon \geq 0$  with  $\lambda'_\varepsilon \leq C$  where  $C$  is independent of  $\varepsilon$ .

So instead of simply using  $\varphi = \xi^{q-1}$  we use  $\varphi := \lambda'_\varepsilon(\xi^q)\xi^{q-1}$ . We can write this as  $\varphi = f_\varepsilon(\xi)\xi^q$ , where  $f_\varepsilon(x) := \lambda'_\varepsilon(x^q)/x$ . Then  $f_\varepsilon \in C^\infty(\mathbb{R})$  with

$$\begin{aligned} \|f_\varepsilon\|_{L^\infty} &\leq ((3/2)\varepsilon)^{-1} \|\lambda'_\varepsilon\|_{L^\infty} \leq C\varepsilon^{-1}, \\ \|f'_\varepsilon\|_{L^\infty} &= \left\| \frac{x\lambda''_\varepsilon(x^q)qx^{q-1} - \lambda'_\varepsilon(x^q)}{x^2} \right\|_{L^\infty} \leq \left(\frac{2}{3\varepsilon}\right)^2 \left( \|q\lambda''_\varepsilon(x^q)x^q\|_{L^\infty} + \|\lambda'_\varepsilon(x^q)\|_{L^\infty} \right) \\ &\leq C\varepsilon^{-2} (Cq(2\varepsilon)^q + C) \leq C(\varepsilon^{-2} + \varepsilon^{q-2}). \end{aligned}$$

It follows immediately from this that  $\varphi \in L^\infty(0, T; L^1 \cap L^\infty)$  since  $\xi^q$  belongs in this same space. For time continuity, we have

$$\begin{aligned} \|\varphi(t) - \varphi(s)\|_{L^r} &\leq \|f_\varepsilon(\xi(s))(\xi^q(t) - \xi^q(s))\|_{L^r} + \|\xi^q(t)(f_\varepsilon(\xi(t)) - f_\varepsilon(\xi(s)))\|_{L^r} \\ &\leq \|f_\varepsilon\|_{L^\infty} \|\xi^q(t) - \xi^q(s)\|_{L^r} + \|f'_\varepsilon\|_{L^\infty} \|\xi^q(t)\|_{L^\infty} \|\xi(t) - \xi(s)\|_{L^r}. \end{aligned}$$

We conclude that  $\varphi \in C([0, T]; L^r)$  for all  $r \in [1, \infty)$ .

Then  $\nabla\varphi = (q-1)\lambda'_\varepsilon(\xi^q)\xi^{q-2}\nabla\xi + q\lambda''_\varepsilon(\xi^q)\xi^{2(q-1)}\nabla\xi \in L^2(0, T; L^2_N)$  since  $\nabla\xi \in L^2(0, T; L^2_N)$  and the singularity in  $\xi^{q-2}$  is removed because  $\lambda'_\varepsilon(x) = 0$  for  $x < \varepsilon$ . Hence,  $\varphi \in L^2(0, T; H^1)$  and thus has sufficient regularity to apply in (3) of Theorem 3.3. This gives

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} \partial_t \xi \lambda'_\varepsilon(\xi^q) \xi^{q-1} &= \int_0^t \int_{\mathbb{R}^d} \xi \mathbf{v}_f \cdot \nabla(\lambda'_\varepsilon(\xi^q) \xi^{q-1}) \\ &+ \int_0^t \int_{\mathbb{R}^d} ((\sigma_1 + \sigma_2) f \xi \lambda'_\varepsilon(\xi^q) \xi^{q-1} - \nu \nabla \xi \cdot \nabla(\lambda'_\varepsilon(\xi^q) \xi^{q-1})). \end{aligned} \quad (3.11)$$

We were able to replace the pairing in the first integral by a spatial integral because  $\partial_t \xi \in L^2(0, T; H^{k-1})$  by Theorem 3.6 and  $\varphi = \lambda'_\varepsilon(\xi^q)\xi^{q-1} \in L^\infty(0, T; L^2)$ . Because of the regularity and decay given by Theorem 3.6, we can easily integrate all but the time integral by parts, as follows:

$$\begin{aligned} \int_{\mathbb{R}^d} \xi \mathbf{v}_f \cdot \nabla(\lambda'_\varepsilon(\xi^q) \xi^{q-1}) &= - \int_{\mathbb{R}^d} \operatorname{div}(\xi \mathbf{v}_f) \lambda'_\varepsilon(\xi^q) \xi^{q-1} \\ &= -\sigma_1 \int_{\mathbb{R}^d} f \lambda'_\varepsilon(\xi^q) \xi^q - \int_{\mathbb{R}^d} (\mathbf{v}_f \cdot \nabla \xi) \lambda'_\varepsilon(\xi^q) \xi^{q-1}, \\ - \int_{\mathbb{R}^d} (\mathbf{v}_f \cdot \nabla \xi) \lambda'_\varepsilon(\xi^q) \xi^{q-1} &= -\frac{1}{q} \int_{\mathbb{R}^d} \mathbf{v}_f \cdot \nabla(\lambda_\varepsilon(\xi^q)) = \frac{1}{q} \int_{\mathbb{R}^d} \operatorname{div}(\mathbf{v}_f) \lambda_\varepsilon(\xi^q) \\ &= \frac{\sigma_1}{q} \int_{\mathbb{R}^d} f \lambda_\varepsilon(\xi^q) \\ \int_{\mathbb{R}^d} f \xi \lambda'_\varepsilon(\xi^q) \xi^{q-1} &= \int_{\mathbb{R}^d} f \lambda'_\varepsilon(\xi^q) \xi^q, \\ -\nu \int_{\mathbb{R}^d} \nabla \xi \cdot \nabla(\lambda'_\varepsilon(\xi^q) \xi^{q-1}) &= -(q-1)\nu \int_{\mathbb{R}^d} \lambda'_\varepsilon(\xi^q) \xi^{q-2} |\nabla \xi|^2 \\ &- q\nu \int_{\mathbb{R}^d} \lambda''_\varepsilon(\xi^q) \xi^{2(q-1)} |\nabla \xi|^2 \leq 0. \end{aligned}$$

Note that  $f\lambda_\varepsilon(\xi^q) \in L^\infty(0, T; L^1 \cap L^\infty)$  since  $f$  lies in this same space and  $\lambda_\varepsilon(\xi^q) \in L^\infty(0, T; L^\infty)$ . In the last inequality we used  $\lambda'_\varepsilon, \lambda''_\varepsilon \geq 0$  to conclude that the two integrals were  $\leq 0$ . For  $q = 2$ , though, these terms would simplify to

$$-\nu \int_{\mathbb{R}^d} \lambda'_\varepsilon(\xi^2) |\nabla \xi|^2 - 2\nu \int_{\mathbb{R}^d} \lambda''_\varepsilon(\xi^2) \xi^2 |\nabla \xi|^2. \quad (3.12)$$

We will return to this issue at the end of the proof.

From (3.11) we now have

$$\int_0^t \int_{\mathbb{R}^d} \partial_t \xi \lambda'_\varepsilon(\xi^q) \xi^{q-1} \leq \int_0^t \int_{\mathbb{R}^d} \left( \sigma_2 f \lambda'_\varepsilon(\xi^q) \xi^q + \frac{\sigma_1}{q} f \lambda_\varepsilon(\xi^q) \right). \quad (3.13)$$

We will now take  $\lim_{\varepsilon \rightarrow 0}$  of each term in (3.13). The two terms on the right-hand side are easy: Since  $\lambda_\varepsilon(\xi^q) \neq \xi^q$  only on the set  $E_\varepsilon(t) := \{x \in \mathbb{R}^d : \xi^q(t, x) < 2\varepsilon\}$ , we have

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^d} f \lambda_\varepsilon(\xi^q) - \int_0^t \int_{\mathbb{R}^d} f \xi^q \right| &\leq \int_0^t \int_{E_\varepsilon(t)} |f(\lambda_\varepsilon(\xi^q) - \xi^q)| \\ &\leq 4\varepsilon \int_0^t \int_{\mathbb{R}^d} |f| \leq 4\varepsilon \|f\|_{L^1((0, T) \times \mathbb{R}^d)} \leq C\varepsilon, \end{aligned}$$

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^d} f \lambda'_\varepsilon(\xi^q) \xi^q - \int_0^t \int_{\mathbb{R}^d} f \xi^q \right| &\leq \int_0^t \int_{\mathbb{R}^d} |f \xi^q (\lambda'_\varepsilon(\xi^q) - 1)| \\ &\leq \int_0^t \int_{E_\varepsilon(t)} |f \xi^q| |\lambda'_\varepsilon(\xi^q) - 1| \leq C\varepsilon \int_0^t \int_{\mathbb{R}^d} |f|, \end{aligned}$$

since  $\lambda'_\varepsilon \leq C$ ,  $\xi^q \leq 2\varepsilon$  on  $E_\varepsilon(t)$ , and  $\lambda'_\varepsilon(\xi^q) = 1$  on  $\mathbb{R}^d \setminus E_\varepsilon(t)$ . Hence as  $\varepsilon \rightarrow 0$ ,

$$\int_0^t \int_{\mathbb{R}^d} f \lambda_\varepsilon(\xi^q) \rightarrow \int_0^t \int_{\mathbb{R}^d} f \xi^q, \quad \int_0^t \int_{\mathbb{R}^d} f \lambda'_\varepsilon(\xi^q) \xi^q \rightarrow \int_0^t \int_{\mathbb{R}^d} f \xi^q.$$

This leaves the time integral. We have,

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} \partial_t \xi \lambda'_\varepsilon(\xi^q) \xi^{q-1} &= \lim_{R \rightarrow \infty} \int_0^t \int_{\mathbb{R}^d} a_R \partial_t \xi \lambda'_\varepsilon(\xi^q) \xi^{q-1} = \frac{1}{q} \lim_{R \rightarrow \infty} \int_0^t \int_{\mathbb{R}^d} \partial_t (a_R \lambda_\varepsilon(\xi^q)) \\ &= \frac{1}{q} \lim_{R \rightarrow \infty} (\|a_R \lambda_\varepsilon(\xi(t)^q)\|_{L^1} - \|a_R \lambda_\varepsilon(f(0)^q)\|_{L^1}) = \frac{1}{q} \lim_{R \rightarrow \infty} \int_{\mathbb{R}^d} a_R (\lambda_\varepsilon(\xi(t)^q) - \lambda_\varepsilon(f(0)^q)) \\ &= \frac{1}{q} \int_{\mathbb{R}^d} (\lambda_\varepsilon(\xi(t)^q) - \lambda_\varepsilon(f(0)^q)). \end{aligned}$$

The first equality holds by the dominated convergence theorem, and the third follows by integrating by parts in time. For the final equality, we used that

$$\begin{aligned} |\lambda_\varepsilon(\xi(t, x)^q) - \lambda_\varepsilon(f(0, x)^q)| &= \frac{|\lambda_\varepsilon(\xi(t, x)^q) - \lambda_\varepsilon(f(0, x)^q)|}{|\xi(t, x)^q - f(0, x)^q|} |\xi(t, x)^q - f(0, x)^q| \\ &\leq \|\lambda'_\varepsilon\|_{L^\infty} |\xi(t, x)^q - f(0, x)^q| = C |\xi(t, x)^q - f(0, x)^q| \end{aligned}$$

to conclude that  $\lambda_\varepsilon(\xi(t)^q) - \lambda_\varepsilon(f(0)^q) \in L^1(\mathbb{R}^d)$  (even though neither term alone lies in  $L^1(\mathbb{R}^d)$ ). This allowed us to apply the dominated convergence theorem to take  $R \rightarrow \infty$ . This same bound then also allows us to apply the dominated convergence theorem to take  $\varepsilon \rightarrow 0$ , so that

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \int_{\mathbb{R}^d} \partial_t \xi \lambda'_\varepsilon(\xi^q) \xi^{q-1} = \frac{1}{q} \int_{\mathbb{R}^d} (\xi(t)^q - f(0)^q) = \frac{1}{q} (\|\xi(t)\|_{L^q}^q - \|f(0)\|_{L^q}^q).$$

Therefore, in the limit as  $\varepsilon \rightarrow 0$ , (3.13) becomes

$$\frac{1}{q} (\|\xi(t)\|_{L^q}^q - \|f(0)\|_{L^q}^q) \leq \int_0^t \int_{\mathbb{R}^d} \left( \frac{\sigma_1}{q} + \sigma_2 \right) f \xi^q.$$

Rearranging and applying Lemma 1.2 (Grönwall's lemma) gives

$$\|\xi(t)\|_{L^q}^q \leq \|f(0)\|_{L^q}^q \exp \left( |\sigma_1 + q\sigma_2| \int_0^t \|f(s)\|_{L^\infty} ds \right), \quad (3.14)$$

from which (3.9) follows for all  $q \in [1, \infty]$ .

The result for  $q = 2$  in (3.10) can be obtained by keeping the terms in (3.12). It can also be obtained directly by a simplification of the argument we just gave, since there is no need to introduce either  $a_R$  or  $\lambda_\varepsilon$  when  $q = 2$ .  $\square$

We used the following lemmas in the proofs above.

**Lemma 3.8.** *Let  $T > 0$  and fix  $t_0 \in [0, T]$ . We can define functions  $h_\varepsilon \in \mathcal{D}((0, T))$  parameterized by  $\varepsilon \in (0, t_0/2)$  such that  $h_\varepsilon = 1$  on  $[3\varepsilon/2, t_0 - 3\varepsilon/2]$ ,  $h_\varepsilon = 0$  on  $[0, \varepsilon/2] \cup [t_0 - \varepsilon/2, T]$ ,  $h_\varepsilon \geq 0$ , and  $h'_\varepsilon(\cdot) \rightarrow \delta(\cdot) - \delta(\cdot - t_0)$  as  $\varepsilon \rightarrow 0$ , the convergence being as Radon measures on  $[0, T]$ .*

*Proof.* Let  $\eta \in C_c^\infty(\mathbb{R})$  be supported on  $(-1/2, 1/2)$ , nonnegative with  $\int_{\mathbb{R}} \eta = 1$ , and define  $\eta_\varepsilon(\cdot) = \varepsilon^{-d} \eta(\cdot/\varepsilon)$ . Set

$$h_\varepsilon(t) := \int_0^t (\eta_\varepsilon(s - \varepsilon) - \eta_\varepsilon(s - t_0 + \varepsilon)) ds.$$

It is easy to see that  $h_\varepsilon$  has all the stated properties.  $\square$

**Lemma 3.9.** *For all  $\xi, \varphi \in Y$ ,*

$$\int_0^T (\partial_t \xi, \varphi) = - \int_0^T (\xi, \partial_t \varphi) + (\xi(T), \varphi(T)) - (\xi(0), \varphi(0)).$$

*Proof.* Fix  $\xi \in Y$  and assume at first that  $\varphi \in C^\infty([0, T]; C_c^\infty(\mathbb{R}^d))$ . Let  $h_\varepsilon$  be as in Lemma 3.8. Then  $h_\varepsilon \varphi \in \mathcal{D}((0, T) \times \mathbb{R}^d)$  so

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^T (\partial_t \xi, h_\varepsilon \varphi) &= - \lim_{\varepsilon \rightarrow 0} \int_0^T (\xi, \partial_t (h_\varepsilon \varphi)) \\ &= - \lim_{\varepsilon \rightarrow 0} \int_0^T h_\varepsilon(t) (\xi(t), \partial_t \varphi(t)) dt - \lim_{\varepsilon \rightarrow 0} \int_0^T h'_\varepsilon(t) (\xi(t), \varphi(t)) dt \\ &= - \int_0^T (\xi(t), \partial_t \varphi(t)) dt + (\xi(T), \varphi(T)) - (\xi(0), \varphi(0)). \end{aligned}$$

In the last step we used the dominated convergence theorem for the first integral, since

$$|h_\varepsilon(t) (\xi(t), \partial_t \varphi(t))| \leq \|\xi(t)\|_{H^1} \|\partial_t \varphi(t)\|_{H^{-1}} \in L^1([0, T])$$

and in the second integral we used the continuity of  $(\xi(t), \varphi(t))$  in time. The result then follows from the density of  $C^\infty([0, T]; C_c^\infty(\mathbb{R}^d))$  in  $Y$ .  $\square$

## 4. THE NONLINEAR VISCOUS PROBLEM

Definition 4.1 gives our definition of a weak solution to the generalized aggregation equation. This definition applies for both viscous and inviscid solutions. In this section we treat viscous solutions, leaving inviscid solutions to Section 5.

**Definition 4.1.** Fix  $N > 1 + d/2$  and let  $Y_N$  be as in Definition 3.1. Let  $\nu \geq 0$  and  $\rho_0 \in L^2_N \cap L^\infty$ . We say that  $\rho^\nu \in Y_N$  is a weak solution to the generalized aggregation equations (GAG $_\nu$ ) on the interval  $[0, T]$  with initial density  $\rho_0$  if  $\rho^\nu(0) = \rho_0$  with

$$\begin{aligned} (\partial_t \rho^\nu(t), \varphi) + \nu \langle \nabla \rho^\nu(t), \nabla \varphi \rangle - (\sigma_1 + \sigma_2) \langle (\rho^\nu(t))^2, \varphi \rangle - \langle \rho^\nu(t) \mathbf{v}^\nu(t), \nabla \varphi \rangle = 0 \\ \text{for a.e. } t \in [0, T] \text{ for all } \varphi \in H^1_N. \end{aligned} \quad (4.1)$$

Equality in (4.1) is to hold in the sense of distributions on  $(0, T)$ .

**Theorem 4.2.** The function  $\rho^\nu \in Y_N \cap L^\infty([0, T] \times \mathbb{R}^d)$  with  $\rho^\nu(0) = \rho_0$  is a weak solution as in Definition 4.1 if and only if any of the following hold:

- (1) (4.1) holds for all  $\varphi \in C_c^\infty(\mathbb{R}^d)$ .
- (2) For all  $\varphi \in C_c^\infty((0, T) \times \mathbb{R}^d)$  we have

$$\begin{aligned} - \int_0^T (\partial_t \rho^\nu, \varphi) + \int_0^T \int_{\mathbb{R}^d} (\rho^\nu \mathbf{v}^\nu \cdot \nabla \varphi + (\sigma_1 + \sigma_2) (\rho^\nu)^2 \varphi - \nu \nabla \rho^\nu \cdot \nabla \varphi) = 0 \text{ or} \\ \int_0^T \int_{\mathbb{R}^d} (\rho^\nu \partial_t \varphi + \rho^\nu \mathbf{v}^\nu \cdot \nabla \varphi + (\sigma_1 + \sigma_2) (\rho^\nu)^2 \varphi - \nu \nabla \rho^\nu \cdot \nabla \varphi) = 0. \end{aligned}$$

That is, (GAG $_\nu$ ) holds on  $(0, T) \times \mathbb{R}^d$  in the sense of distributions.

- (3) For all  $\varphi \in C([0, T]; L^2) \cap L^2(0, T; H^1)$

$$- \int_0^T (\partial_t \rho^\nu, \varphi) + \int_0^T \int_{\mathbb{R}^d} (\rho^\nu \mathbf{v}^\nu \cdot \nabla \varphi + (\sigma_1 + \sigma_2) (\rho^\nu)^2 \varphi - \nu \nabla \rho^\nu \cdot \nabla \varphi) = 0.$$

- (4) For all  $\varphi \in Y$  we have

$$\begin{aligned} \int_0^T (\rho^\nu, \partial_t \varphi) + \int_0^T \int_{\mathbb{R}^d} (\rho^\nu \mathbf{v}^\nu \cdot \nabla \varphi + (\sigma_1 + \sigma_2) (\rho^\nu)^2 \varphi - \nu \nabla \rho^\nu \cdot \nabla \varphi) \\ = \int_{\mathbb{R}^d} (\rho^\nu(T) \varphi(T) - \rho_0 \varphi(0)). \end{aligned} \quad (4.2)$$

Finally, if  $\rho^\nu$  is a weak solution as in Definition 4.1 then (1)-(4) hold with any  $t \in [0, T]$  in place of  $T$ .

*Proof.* This follows from Theorem 3.3, since  $\rho^\nu$  is a weak solution to (3.1) as in Definition 3.1 with  $f = \rho^\nu$  and  $\rho_0 = f(0)$ .  $\square$

**Remark 4.3.** In Theorem 4.2 we used the assumption that  $\rho^\nu$  lies not just in  $Y^N$ , but also in  $L^\infty([0, T] \times \mathbb{R}^d)$ , so that it can serve as a valid  $f$  in Definition 3.1. It is only in showing that (4) follows from  $\rho^\nu$  being a weak solution to the nonlinear problem that this additional assumption is required, however.

To establish the existence of solutions as in Definition 4.1 we will construct a sequence of approximations as follows:

$$\begin{aligned} \rho_0(t, x) &= \rho_0(x), \\ \mathbf{v}_n &= \sigma_1 \nabla \Phi * \rho_{n-1}, \\ \partial_t \rho_n + \mathbf{v}_n \cdot \nabla \rho_n &= \sigma_2 \rho_{n-1} \rho_n + \nu \Delta \rho_n, \\ \rho_n(0) &= \rho_0 \end{aligned} \quad (4.3)$$

for  $n = 1, 2, \dots$ . In (4.3)<sub>2,3,4</sub>  $\rho_n$  is a regular solution to the linear problem as given by Theorem 3.6 with  $f = \rho_{n-1}$ . Note that  $\operatorname{div} \mathbf{v}_n = \sigma_1 \rho_{n-1}$ .

**Proposition 4.4.** *Fix  $T > 0$  with  $T < (|\sigma_2| \|\rho_0\|_{L^\infty})^{-1}$  or  $T < \infty$  if  $\sigma_2 = 0$ . Assume that  $\rho_0 \in H_N^k$  for some  $N > 1 + d/2$ ,  $k > d/2$ . Let  $\nu > 0$ ,  $n \geq 0$ ,  $t \in [0, T]$ . We have*

$$\|\rho_n(t)\|_{L^\infty} \leq \frac{\|\rho_0\|_{L^\infty}}{1 - |\sigma_2| \|\rho_0\|_{L^\infty} t}. \quad (4.4)$$

When  $\sigma_2 \neq 0$  we have

$$\begin{aligned} \|\rho_n(t)\|_{L^q} &\leq \|\rho_0\|_{L^q} (1 - |\sigma_2| \|\rho_0\|_{L^\infty} t)^{-|\frac{\sigma_1}{q\sigma_2} + 1|} \quad \forall q \in [1, \infty), \\ \|\rho_n(t)\|^2 + 2\nu \int_0^t \|\nabla \rho_n\|^2 &= \|\rho_0\|^2 (1 - |\sigma_2| \|\rho_0\|_{L^\infty} t)^{-|\frac{\sigma_1}{\sigma_2} + 2|}. \end{aligned} \quad (4.5)$$

When  $\sigma_2 = 0$  we have

$$\begin{aligned} \|\rho_n(t)\|_{L^q} &\leq \|\rho_0\|_{L^q} \exp(|\sigma_1| q^{-1} \|\rho_0\|_{L^\infty} t) \quad \forall q \in [1, \infty), \\ \|\rho_n(t)\|^2 + 2\nu \int_0^t \|\nabla \rho_n\|^2 &= \|\rho_0\|^2 \exp(|\sigma_1| \|\rho_0\|_{L^\infty} t). \end{aligned} \quad (4.6)$$

*Proof.* We proceed by induction. Let  $S(k)$  be the statement that (4.4) holds for  $n = k$ . Certainly,  $S(0)$  holds trivially. Assume that  $S(n-1)$  holds. We will use this to show that  $S(n)$  holds.

From (3.9) (with  $\xi = \rho_n$ ,  $f = \rho_{n-1}$ ) we have

$$\|\rho_n(t)\|_{L^q} \leq \|\rho_0\|_{L^q} \exp\left(q^{-1} \sigma_1 + \sigma_2 \int_0^t \|\rho_{n-1}(s)\|_{L^\infty} ds\right). \quad (4.7)$$

Now, by the induction hypothesis,

$$\begin{aligned} \int_0^t \|\rho_{n-1}(s)\|_{L^\infty} ds &\leq \int_0^t \frac{\|\rho_0\|_{L^\infty}}{1 - |\sigma_2| \|\rho_0\|_{L^\infty} s} ds \\ &= \begin{cases} -|\sigma_2|^{-1} \log(1 - |\sigma_2| \|\rho_0\|_{L^\infty} t), & \sigma_2 \neq 0, \\ \|\rho_0\|_{L^\infty} t, & \sigma_2 = 0. \end{cases} \end{aligned}$$

Taking the limit as  $q \rightarrow \infty$  of both sides of (4.7), it follows by the continuity of Lebesgue norms that for  $\sigma_2 \neq 0$ ,

$$\|\rho_n(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} \exp(-\log(1 - |\sigma_2| \|\rho_0\|_{L^\infty} t)) = \|\rho_0\|_{L^\infty} (1 - |\sigma_2| \|\rho_0\|_{L^\infty} t)^{-1},$$

and  $\|\rho_n(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}$  if  $\sigma_2 = 0$ . Thus,  $S(n-1) \implies S(n)$ , and we see by induction that in fact  $S(n)$  holds for all  $n$ .

Returning to (4.7), we see that (4.5)<sub>1</sub> and (4.6)<sub>1</sub> hold.

The bounds in (4.5)<sub>2</sub>, (4.6)<sub>2</sub> follow by the same argument specifically for  $q = 2$  and using the energy bound in (3.10).  $\square$

**Theorem 4.5.** *Fix  $T > 0$  with  $T < (|\sigma_2| \|\rho_0\|_{L^\infty})^{-1}$  or  $T < \infty$  if  $\sigma_2 = 0$ . (Note that  $[0, T]$  is within the time of existence for the inviscid problem—see Theorem 5.7). Let  $\nu > 0$  and assume that  $\rho_0 \in L^\infty \cap L_N^2$  for some  $N > 1 + d/2$ . Then there exists a weak solution to  $(GAG_\nu)$  as in Definition 4.1 on the time interval  $[0, T]$  with*

$$\|\rho^\nu(t)\|_{L^\infty} \leq \frac{\|\rho_0\|_{L^\infty}}{1 - |\sigma_2| \|\rho_0\|_{L^\infty} t}. \quad (4.8)$$

When  $\sigma_2 \neq 0$ , we have

$$\begin{aligned} \|\rho^\nu(t)\|_{L^q} &\leq \|\rho_0\|_{L^q} (1 - |\sigma_2| \|\rho_0\|_{L^\infty} t)^{-|\frac{\sigma_1}{q\sigma_2}+1|} \quad \forall q \in [1, \infty), \\ \|\rho^\nu(t)\|^2 + 2\nu \int_0^t \|\nabla \rho^\nu\|^2 &\leq \|\rho_0\|^2 (1 - |\sigma_2| \|\rho_0\|_{L^\infty} t)^{-|\frac{\sigma_1}{\sigma_2}+2|}. \end{aligned} \quad (4.9)$$

When  $\sigma_2 = 0$ , we have

$$\begin{aligned} \|\rho^\nu(t)\|_{L^q} &\leq \|\rho_0\|_{L^q} \exp(|\sigma_1| q^{-1} \|\rho_0\|_{L^\infty} t) \quad \forall q \in [1, \infty), \\ \|\rho^\nu(t)\|^2 + 2\nu \int_0^t \|\nabla \rho^\nu\|^2 &\leq \|\rho_0\|^2 \exp(|\sigma_1| \|\rho_0\|_{L^\infty} t). \end{aligned} \quad (4.10)$$

Further, if  $\rho_0 \in H_N^k$  for a positive integer  $k$  then  $\rho^\nu \in Y_N^k$ . (The space  $Y_N^k$  is defined in Theorem 3.6.)

*Proof.* Because of the bounds in Proposition 4.4, we can make a standard argument to prove the existence of solutions along the same lines as that for the existence of solutions to the Navier-Stokes equations (for instance, see pages 72-73 of [9]). For completeness, we give a full argument here.

Let  $\rho_{0,j}(t, x) = \eta_{1/j} * \rho_0(x)$ ,  $j = 1, 2, \dots$ , where  $\eta$  is a Friedrich's mollifier and note that  $\rho_{0,j} \in Y_N^m$  for all  $m \geq 0$ . Let  $(\rho_n^{(j)})$  be the sequence of linear solutions defined in (4.3), but with the starting solution being  $\rho_{0,j}$  instead of  $\rho_0$ . Note that  $\rho_n^{(j)} \in Y_N^m$  for all  $m \geq 0$  as well by Theorem 3.6. Then define  $\rho_n := \rho_n^{(n)}$ .

By Proposition 4.4 and Theorem 3.5 we see that  $(\rho_n)$  is bounded in  $L^\infty(0, T; L_N^2) \cap L^2(0, T; H_N^1) \subseteq L^2(0, T; H^1)$ . But  $L^2(0, T; H^1)$  is weakly compact, so  $(\rho_n)$  converges weakly to some  $\rho$  in  $L^2(0, T; H^1)$ . The compact embedding given by Lemma 2.4 implies that some subsequence of  $(\rho_n)$ , which we relabel as  $(\rho_n)$ , converges strongly to  $\rho$  in  $L^2(0, T; L_M^2)$  for all  $M < N$ ; hence also  $\rho_n(t) \rightarrow \rho(t)$  in  $L_M^2$  for all  $M < N$  for almost all  $t \in [0, T]$ . Moreover, by the uniqueness of limits, Lemma 2.4 also gives that  $\rho(t) \in L_N^2$  for almost all  $t$  in  $[0, T]$ .

Let  $\varphi \in C_c^\infty((0, T) \times \mathbb{R}^d)$ . Then using (2) of Theorem 3.3, since each  $\rho_n$  is a weak solution as in Definition 3.1 with  $f = \rho_{n-1}$ , we have

$$0 = \lim_{n \rightarrow \infty} \int_0^T \int_{\mathbb{R}^d} (\rho_n \partial_t \varphi + \rho_n \mathbf{v}_{\rho_{n-1}} \cdot \nabla \varphi + (\sigma_1 + \sigma_2) \rho_{n-1} \rho_n \varphi - \nu \nabla \rho_n \cdot \nabla \varphi).$$

Now,  $\rho_n$  and  $\rho_{n-1}$  are each bounded in  $L^1 \cap L^\infty$  by Proposition 4.4, and  $\mathbf{v}_{\rho_{n-1}}$  is bounded in  $L^\infty$  as well by (2) of Lemma 2.2. This allows us to apply the dominated convergence theorem to the first three terms. We can then take the limit as  $n$  approaches infinity, using  $\rho_n(t), \rho_{n-1}(t) \rightarrow \rho(t)$  in  $L_M^2$  (and thus in  $L^2$ ) and for almost all  $t \in [0, T]$ . By (1) and (4) of Lemma 2.2,  $\mathbf{v}_{\rho_{n-1}}(t) \rightarrow \mathbf{v}(t)$  in  $L^\infty$  for almost all  $t \in [0, T]$ , and we conclude that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \int_0^T \int_{\mathbb{R}^d} (\rho_n \partial_t \varphi + \rho_n \mathbf{v}_{\rho_{n-1}} \cdot \nabla \varphi + (\sigma_1 + \sigma_2) \rho_{n-1} \rho_n \varphi - \nu \nabla \rho_n \cdot \nabla \varphi) \\ &= \int_0^T \int_{\mathbb{R}^d} (\rho \partial_t \varphi + \rho \mathbf{v} \cdot \nabla \varphi + (\sigma_1 + \sigma_2) \rho^2 \varphi - \nu \nabla \rho \cdot \nabla \varphi). \end{aligned}$$

Here, we also used that  $\rho_n \rightarrow \rho$  weakly in  $L^2(0, T; H^1)$ .

By Proposition 4.4,  $(\rho_n)$  is uniformly bounded in  $L^\infty([0, T] \times \mathbb{R}^d)$ . Hence  $\rho \in L^\infty([0, T] \times \mathbb{R}^d)$ . Since  $\rho_n(0) \rightarrow \rho_0$  in  $L^2$ , it follows from Theorem 4.2 that  $\rho^\nu = \rho$  is a weak solution to

$(GAG_\nu)$  as in Definition 4.1. Then for all  $q \geq 2$ ,

$$\begin{aligned} \|\rho(t)\|_{L^q} &\leq \|\rho(t) - \rho_n(t)\|_{L^q} + \|\rho_n(t)\|_{L^q} \\ &\leq \|\rho(t) - \rho_n(t)\|_{L^q}^{\frac{2}{q}} (\|\rho(t)\|_{L^\infty} + \|\rho_n(t)\|_{L^\infty})^{\frac{q-2}{q}} + \|\rho_n(t)\|_{L^q}. \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$  and using the bounds in (4.5) and (4.6) gives (4.9) and (4.10) for  $q \in (2, \infty)$  and then, by the continuity of Lebesgue norm, we obtain (4.8). For  $q = 2$ , (4.9) and (4.10) follow directly from (4.5) and (4.6), using also that the convergence of  $\rho_n$  to  $\rho$  weakly in  $L^2(0, T; H^1)$  implies that  $\|\rho_n\|_{L^2(0, T; H^1)} \rightarrow \|\rho\|_{L^2(0, T; H^1)}$ . For  $q \in [1, 2)$  we apply (1) of Lemma 2.2 to show that  $\|\rho(t) - \rho_n(t)\|_{L^q} \rightarrow 0$  and so obtain (4.9) and (4.10).

That  $\rho_0 \in Y_N^k$  implies  $\rho^\nu \in Y_N^k$  follows similarly using Theorem 3.6.  $\square$

Uniqueness of solutions to  $(GAG_\nu)$ , even for higher regularity solutions, is not an entirely simple matter. It will follow as a consequence of our proofs of the vanishing limit in Sections 7 and 8 (see Theorem 7.5).

## 5. THE INVISCID PROBLEM

Establishing existence and uniqueness of weak solutions in Eulerian variables as formulated in Definition 4.1 is quite difficult. This is in contrast to the 2D Euler equations, for which existence of solutions for bounded initial vorticity can be established quite easily using a sequence of solutions to the Navier-Stokes equations. This argument for the 2D Euler equations is possible because the  $L^p$ -norms of the Navier-Stokes vorticity  $\omega$  can be bounded uniformly over viscosity in the whole plane (or in a bounded domain if  $\omega = u \cdot \mathbf{n} = 0$  is used as a boundary condition; see, for instance, Section 4.1 of [18] for the bounded domain argument). Then a weak solution to the Euler equations can be obtained using the velocity formulation. A velocity formulation of  $(GAG_\nu)$  is possible as we briefly describe in Section 10, but a long detour into developing properties of the pressure is required in order to properly develop a weak formulation.

Instead, we will work with Lagrangian solutions, adapting the economical and elegant proofs for 2D solutions to the Euler equations given by Marchioro and Pulvirenti in [21], which originates in their earlier text [20]. We will also use some of the ideas from Chapter 8 of [19].

In [3], the authors obtain well-posedness in the special case of  $(AG_0)$ . Their approach could in principle be extended to the more general equations in  $(GAG_0)$ . In brief, the authors of [3] first construct smooth solutions then use an approximating sequence of such solutions to obtain a Lagrangian solution by demonstrating convergence of the flow maps (as in [19]). This approach is reversed in [21], where Lagrangian solutions are first constructed by obtaining the convergence of a sequence of flow maps for approximating linearizations of the 2D Euler equations. A very simple argument then shows that regularity of the initial data is propagated over time.

Considerable complications arise when adapting Marchioro and Pulvirenti's arguments for  $(GAG_0)$  because the underlying velocity field is not divergence-free (analogous complications are dealt with in [3]). This requires the assumption of some regularity on the initial data to obtain weak solutions, removing this assumption a posteriori via a separate argument similar to the proof of existence in Chapter 8 of [19].

Before giving the proof of well-posedness of inviscid solutions, let us motivate our definition of a Lagrangian solution by observing formally that if  $\rho = \rho^0$  solves  $(GAG_0)$  and  $X$  is the

flow map for  $\mathbf{v} = \mathbf{v}^0$  then

$$\frac{d}{dt}\rho(t, X(t, x)) = \sigma_2 \rho(t, X(t, x))^2.$$

Integrating along flow lines gives

$$\rho(t, X(t, x)) = \frac{\rho_0(x)}{1 - \sigma_2 t \rho_0(x)}. \quad (5.1)$$

This motivates the following definition of a Lagrangian solution to  $(GAG_0)$ :

**Definition 5.1.** Fix  $T > 0$ . Let  $X: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  with  $X(t, \cdot)$  a homeomorphism for all  $t \in [0, T]$  and let  $\rho_0 \in L^\infty(\mathbb{R}^d)$ . Define  $\rho: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  by

$$\rho(t, x) = \frac{\rho_0(X^{-1}(t, x))}{1 - \sigma_2 t \rho_0(X^{-1}(t, x))} \quad (5.2)$$

and let  $\mathbf{v} := \sigma_1 \nabla \Phi * \rho$ . Here,  $X^{-1}$  is defined by  $X^{-1}(t, X(t, x)) = x$  for all  $(t, x) \in [0, T] \times \mathbb{R}^d$ . Then  $X$  or more fully  $(X, \rho, \mathbf{v})$  is a Lagrangian solution to the inviscid generalized aggregation equations  $(GAG_0)$  with initial density  $\rho_0$  if  $X$  is the flow map for  $\mathbf{v}$ ; that is, if

$$X(t, x) = x + \int_0^t \mathbf{v}(s, X(s, x)) ds$$

for all  $t \geq 0$ ,  $x \in \mathbb{R}^d$ .

The form of  $\rho$  in (5.1) or (5.2) also yields a sharp time of existence for our Lagrangian solutions. If we do not consider the sign of  $\rho_0$  we obtain an upper limit on the existence time that is the same as that for viscous solutions in Theorem 4.5. Hence we should expect that if say,  $\sigma_2 < 0$  and  $\rho_0 > 0$ , so that the inviscid solution exists for all time, then the existence time for viscous solutions might be considerably longer than the bound given in Theorem 4.5. An open question is whether, for all sufficiently small viscosity, viscous solutions to  $(GAG_\nu)$  exist for as long as the inviscid solution exists, as was established for the 3D Navier-Stokes and Euler equations in [8]. Because we do not assume that the initial data has a distinguished sign that is compatible with the signs of  $\sigma_1, \sigma_2$ , we do not have a maximum or comparison principle. This is what makes this a difficult problem. (Issues of existence times of viscous solutions in relation to the total mass of  $\rho_0$  have been well-studied for  $(AG_\nu)$ : see Sections 5.2 and 5.3 of [23].)

We will need the following simple proposition that shows that the Lebesgue norms of the density of Lagrangian solutions depend only upon the initial density and time.

**Proposition 5.2.** Let  $(X, \rho, \mathbf{v})$  be a Lagrangian solution as in Definition 5.1 with initial density  $\rho_0 \in L^1 \cap L^\infty$ . If  $T < (|\sigma_2| \|\rho_0\|_{L^\infty})^{-1}$  or  $T < \infty$  if  $\sigma_2 = 0$ , then for all  $t \in [0, T]$ , we have

$$\|\rho(t)\|_{L^\infty} \leq \frac{\|\rho_0\|_{L^\infty}}{1 - |\sigma_2| \|\rho_0\|_{L^\infty} t}, \quad (5.3)$$

and for all  $q \in [1, \infty)$ ,

$$\|\rho(t)\|_{L^q} \leq \begin{cases} \|\rho_0\|_{L^q} (1 - |\sigma_2| \|\rho_0\|_{L^\infty} t)^{-\frac{\sigma_1}{q\sigma_2} - 1}, & \sigma_2 \neq 0, \\ \|\rho_0\|_{L^q} \exp(|\sigma_1| q^{-1} \|\rho_0\|_{L^\infty} t), & \sigma_2 = 0. \end{cases} \quad (5.4)$$

Moreover,  $\|v(t)\|_{L^\infty} \leq C_0(T)$  for all  $t \in [0, T]$ , and if  $\rho_0$  is compactly supported, then there exists  $R = R(T)$  such that  $\rho(t)$  remains supported in  $B_{R(T)}(0)$  for all  $t \in [0, T]$ .

*Proof.* The bound in (5.3) follows immediately from (5.2). For  $q \in [1, \infty)$ , we have

$$\|\rho(t)\|_{L^q} = \left( \int_{\mathbb{R}^d} \left| \frac{\rho_0(y)}{1 - \sigma_2 t \rho_0(y)} \right|^q |\det \nabla X(t, y)| dy \right)^{\frac{1}{q}},$$

where we made the change of variables  $x = X(t, y)$ . But

$$\begin{aligned} \partial_s \det \nabla X(s, y) &= \operatorname{div} \mathbf{v}(s, X(s, y)) \det \nabla X(s, y) = \sigma_1 \rho(s, X(s, y)) \det \nabla X(s, y) \\ &= \frac{\sigma_1 \rho_0(y)}{1 - \sigma_2 s \rho_0(y)} \det \nabla X(s, y) \end{aligned} \quad (5.5)$$

(see, for instance, page 3 of [7]) so that

$$J(t, y) := \det \nabla X(t, y) = \begin{cases} (1 - \sigma_2 t \rho_0(y))^{-\frac{\sigma_1}{\sigma_2}}, & \sigma_2 \neq 0, \\ e^{\sigma_1 \rho_0(y) t}, & \sigma_2 = 0. \end{cases} \quad (5.6)$$

Hence when  $\sigma_2 \neq 0$ ,

$$\|\rho(t)\|_{L^q} = \left( \int_{\mathbb{R}^d} |\rho_0(y)|^q (1 - \sigma_2 t \rho_0(y))^{-\frac{\sigma_1}{\sigma_2} - q} dy \right)^{\frac{1}{q}} \leq \|\rho_0\|_{L^q} (1 - |\sigma_2| \|\rho_0\|_{L^\infty} t)^{-\frac{\sigma_1}{q \sigma_2} - 1}.$$

This, with the analogous bound for  $\sigma_2 = 0$ , gives (5.4). Then  $\|v(t)\|_{L^\infty} \leq C_0(T)$  for all  $t \in [0, T]$  follows from (5.3), (5.4) for  $q = 1$ , and (2.2). The bound on the velocity then immediately yields the bound on the compact support of  $\rho(t)$ .  $\square$

In what follows, we make use of three lemmas that appear at the end of this section as well as Lemma 9.5, which we defer to Section 9 because we use Littlewood-Paley theory in its proof.

We start by showing that a Lagrangian solution—if it exists—maintains the Hölder regularity of the initial density.

**Theorem 5.3.** *Let  $\rho_0 \in C^{k, \alpha}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$  for some integer  $k \geq 0$  and  $\alpha \in (0, 1)$ . Fix  $T > 0$  with  $T < (|\sigma_2| \|\rho_0\|_{L^\infty})^{-1}$  or  $T < \infty$  if  $\sigma_2 = 0$ . Assume that  $\rho$  is a Lagrangian solution to (GAG<sub>0</sub>) on the interval  $[0, T]$ . Then  $\rho \in L^\infty(0, T; C^{k, \alpha})$ . Moreover,*

$$\|\rho(t)\|_{C^{k, \alpha}} \leq C(t, |\sigma_1|, \|\rho_0\|_{L^1}, \|\rho_0\|_{C^{k, \alpha}}), \quad (5.7)$$

and when  $k \geq 1$ ,  $\rho$  is a classical solution with  $\rho \in C^k(0, T; C^{k, \alpha})$ .

*Proof.* It follows (as in Theorem 5.1.1 of [7]) that  $x \mapsto X(t, x) - x$  has norm 1 in  $C^{\theta(t)}$ , where  $\theta(t) = e^{-c_0 t}$ ,  $c_0 = C_0(T) \|\rho_0\|_{L^1 \cap L^\infty}$ , and this same bound holds for the inverse flow map. Thus, by Lemma 5.9,

$$\rho(t) \in C^{\alpha \theta(t)},$$

with

$$\begin{aligned} \|\rho(t)\|_{C^{\alpha \theta(t)}} &= \|\rho(t)\|_{L^\infty} + \|\rho(t)\|_{\dot{C}^{\alpha \theta(t)}} \leq C_0(T) \|\rho_0\|_{L^\infty} + C_0(T) \|\rho_0\|_{\dot{C}^\alpha} \|X^{-1}(t)\|_{\dot{C}^{\theta(t)}}^\alpha \\ &\leq C_0(T) \|\rho_0\|_{L^\infty} + C_0(T) \|\rho_0\|_{\dot{C}^\alpha} = C_0(T) \|\rho_0\|_{C^\alpha}. \end{aligned}$$

Hence by Lemma 9.5,  $\mathbf{v}(t) = \sigma_1 \nabla \Phi * \rho(t) \in C^{1, \alpha \theta(t)}$  with

$$\|\nabla \mathbf{v}(t)\|_{C^{\alpha \theta(t)}} \leq C \|\rho(t)\|_{L^1} + C \|\rho(t)\|_{C^{\alpha \theta(t)}} \leq C_0(T) (\|\rho_0\|_{L^1} + \|\rho_0\|_{C^\alpha}), \quad (5.8)$$

where we also used Proposition 5.2.

Because  $\mathbf{v}$  is differentiable, it follows that

$$\nabla X(t, x) = I + \int_0^t \nabla \mathbf{v}(s, X(s, x)) \cdot \nabla X(s, x) ds$$

so that

$$\|\nabla X(t)\|_{L^\infty} \leq 1 + \int_0^t \|\nabla \mathbf{v}(s)\|_{L^\infty} \|\nabla X(s)\|_{L^\infty} ds. \quad (5.9)$$

Applying Lemma 1.2 (Grönwall's lemma) and using (5.8), we see that  $\nabla X(t) \in L^\infty$  with

$$\|\nabla X(t)\|_{L^\infty} \leq C(T, |\sigma_1|, \|\rho_0\|_{L^1}, \|\rho_0\|_{C^\alpha}) = C_0(T).$$

Hence, by Lemma 5.9 we actually have that  $\rho(t) \in C^\alpha$ , with

$$\|\rho(t)\|_{C^\alpha} \leq C_0(T) \|\rho_0\|_{C^\alpha} \|\nabla X(t)\|_{L^\infty}^\alpha \leq C_0(T).$$

By Lemma 9.5, then,  $\mathbf{v}(t) \in C^{1,\alpha}$ .

For higher regularity,  $k \geq 1$ , we observe that for all  $x, y \in \mathbb{R}^d$ ,

$$\frac{1}{1 - \sigma_2 t \rho_0(X^{-1}(t, x))} - \frac{1}{1 - \sigma_2 t \rho_0(X^{-1}(t, y))} = \frac{\sigma_2 t (\rho_0(X^{-1}(t, x)) - \rho_0(X^{-1}(t, y)))}{(1 - \sigma_2 t \rho_0(X^{-1}(t, x)))(1 - \sigma_2 t \rho_0(X^{-1}(t, y)))}.$$

Thus, when  $\nabla X^{-1} \in L^\infty(\mathbb{R}^d)$  and  $\rho_0 \in C^\alpha(\mathbb{R}^d)$  we have  $(1 - \sigma_2 t \rho_0(X^{-1}(t, x)))^{-1} \in C^\alpha(\mathbb{R}^d)$  as well. This observation combined with the fact that  $C^\alpha$  is a Banach algebra allows us to apply a bootstrap argument analogous to that in [21] to obtain regularity for  $k \geq 1$ .

Finally, when  $k \geq 1$  we have  $\partial_t \rho = -\mathbf{v} \cdot \nabla \rho + \sigma_2 \rho^2$  exists for all time so  $\rho$  is differentiable in time. Therefore,  $\rho$  is a classical solution to  $(GAG_0)$ .  $\square$

We show in Theorem 5.4 that a Lagrangian solution exists. It will be convenient in Theorem 5.4 to assume the initial density is compactly supported.

**Theorem 5.4.** *Fix  $T > 0$  with  $T < (|\sigma_2| \|\rho_0\|_{L^\infty})^{-1}$  or  $T < \infty$  if  $\sigma_2 = 0$ . Let  $\rho_0 \in L^\infty(\mathbb{R}^d)$  be compactly supported. There exists a Lagrangian solution  $\rho$  to  $(GAG_0)$  as in Definition 5.1.*

*Proof.* We first prove the existence of a Lagrangian solution assuming that  $\rho_0 \in C^{k,\alpha}(\mathbb{R}^d)$  for some integer  $k \geq 1$  and  $\alpha \in (0, 1)$  with  $\rho_0$  compactly supported.

We define sequences  $(\rho_n)_{n=0}^\infty$ ,  $(\mathbf{v}_n)_{n=1}^\infty$ , and  $(X_n)_{n=0}^\infty$  as follows:

$$\begin{aligned} \rho_0(t, \cdot) &= \rho_0(x), \\ X_0(t, x) &= x, \end{aligned}$$

with the iteration, for  $n = 1, 2, \dots$ ,

$$\begin{aligned} \mathbf{v}_n &= \sigma_1 \nabla \Phi * \rho_{n-1}, \\ \partial_t X_n(t, x) &= \mathbf{v}_n(t, X_n(t, x)), \\ \rho_n(t, X_n(t, x)) &= \frac{\rho_0(x)}{1 - \sigma_2 t \rho_0(x)}. \end{aligned} \quad (5.10)$$

Thus,  $\mathbf{v}_n$  is the unique curl-free vector field whose divergence is  $\rho_{n-1}$ , and  $X_n$  is the (non-measure-preserving) flow map for  $\mathbf{v}_n$ . To see that  $X_n$  exists, note that  $\mathbf{v}_n(t)$  has a log-Lipschitz MOC  $\mu$  that depends only upon  $\|\rho_{n-1}(t)\|_{L^1 \cap L^\infty}$  by (2.2). But the coarse bound

$$\|\rho_n(t)\|_{L^1 \cap L^\infty} \leq \frac{\|\rho_0\|_{L^1 \cap L^\infty}}{1 - |\sigma_2| t \|\rho_0\|_{L^\infty}} \leq C_0(T) \text{ for all } n \quad (5.11)$$

follows from (5.10)<sub>3</sub>. Hence the MOC, which we can write as

$$\mu(r) = \begin{cases} -C_0 r \log r & \text{if } r < e^{-1}, \\ C_0 e^{-1} & \text{if } r \geq e^{-1}, \end{cases} \quad (5.12)$$

applies uniformly over  $n$  and all  $t \in [0, T]$ . The existence and uniqueness of the flow map  $X_n$  is then classical. (See, for instance, Section 5.2 of [7].) We also define the inverse flow map  $(X^n)^{-1}$  by

$$(X^n)^{-1}(t, X_n(t, x)) = x,$$

so that

$$\rho_n(t, x) = \frac{\rho_0((X^n)^{-1}(t, x))}{1 - \sigma_2 t \rho_0((X^n)^{-1}(t, x))}. \quad (5.13)$$

The proof of existence proceeds by showing that the iteration in (5.10) converges to a Lagrangian solution. First observe that

$$\|\mathbf{v}_n(t)\|_{L^\infty} \leq C \|\rho_{n-1}(s)\|_{L^1 \cap L^\infty} \leq C_0(T), \quad \text{supp } \rho_n(t) \subseteq B_{R(T)}(0) \text{ for all } t \in [0, T] \quad (5.14)$$

for some  $R(T) < \infty$  and all  $n$  follows, as in the proof of Proposition 5.2, from the bound on the density in (5.11), which yields a bound on the  $LL$ -norm of the velocity from (2.2), which then yields the bound on the compact support of  $\rho_n(t)$ . It follows as in the proof of Theorem 5.3 that  $\|\rho_n(t)\|_{C^{\alpha\theta(t)}} \leq C_0(T) \|\rho_0\|_{C^\alpha}$ . Thus by Lemma 9.5,

$$\begin{aligned} \|\nabla \mathbf{v}_n(s)\|_{L^\infty} &\leq C \|\rho_{n-1}(s)\|_{L^1} + C(\theta(s)) \|\rho_{n-1}(s)\|_{C^{\alpha\theta(s)}} \leq C_0(T) + C_0(T) \|\rho_0\|_{C^\alpha} \\ &\leq C_0(T). \end{aligned}$$

Also

$$\partial_t \nabla X_n(t, x) = \nabla \mathbf{v}_n(t, X_n(t, x)) \nabla X_n(t, x).$$

Integrating in time, taking the  $L^\infty$ -norm, and applying Lemma 1.2 (Grönwall's lemma) gives

$$\|\nabla X_n(t, \cdot)\|_{L^\infty}, \|\nabla (X^n)^{-1}(t, \cdot)\|_{L^\infty} \leq \exp \int_0^t \|\nabla \mathbf{v}_n(s)\|_{L^\infty} ds \leq C_0(T).$$

The bound on  $\nabla (X^n)^{-1}$  does not follow as immediately as that on  $\nabla X_n$  because the flow is not autonomous. For the details see for instance the proof of Lemma 8.2 p. 318-319 of [19].

Moreover, a direct calculation gives

$$\nabla \rho_n(t, x) = \frac{(1 + \sigma_2 t) \nabla \rho_0((X^n)^{-1}(t, x)) \nabla (X^n)^{-1}(t, x) + \sigma_2 \rho_0((X^n)^{-1}(t, x))}{1 - \sigma_2 t \rho_0((X^n)^{-1}(t, x))}$$

so that

$$\|\nabla \rho_n(t)\|_{L^\infty} \leq C_0(T) + C_0(T) \|\nabla \rho_0\|_{L^\infty} \|\nabla (X^n)^{-1}(t)\|_{L^\infty} \leq C_0(T).$$

Hence for all  $n$  and all  $t \in [0, T]$  we have

$$\|\nabla X_n(t, \cdot)\|_{L^\infty}, \|\nabla (X^n)^{-1}(t, \cdot)\|_{L^\infty}, \|\nabla \rho_n(t)\|_{L^\infty} \leq C_0(T). \quad (5.15)$$

Define, for  $n \geq 1$ ,

$$h_n(t) = \|X_n(t, \cdot) - X_{n-1}(t, \cdot)\|_{L^\infty}.$$

We will show that  $h_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Fix  $n \geq 2$ . We have

$$\begin{aligned} |X_n(t, x) - X_{n-1}(t, x)| &\leq \int_0^t |\mathbf{v}_n(s, X_n(s, x)) - \mathbf{v}_n(s, X_{n-1}(s, x))| ds \\ &\quad + \int_0^t |\mathbf{v}_n(s, X_{n-1}(s, x)) - \mathbf{v}_{n-1}(s, X_{n-1}(s, x))| ds \\ &=: I_1 + I_2. \end{aligned}$$

Since  $\mathbf{v}_n$  is Lipschitz uniformly in  $n$ ,  $I_1$  can be bounded as

$$I_1 \leq \int_0^t \|\nabla \mathbf{v}_n(s)\|_{L^\infty} |X_n(s, x) - X_{n-1}(s, x)| ds \leq C_0(T) \int_0^t h_n(s) ds. \quad (5.16)$$

For  $I_2$ , we first note that the bound on the support of each  $\rho_i$  in (5.14) gives  $\|\rho_{n-1}(s) - \rho_n(s)\|_{L^1} \leq C_0(T) \|\rho_{n-1}(s) - \rho_n(s)\|_{L^\infty}$ . Taking advantage of (2.2) then we have

$$\begin{aligned} &|\mathbf{v}_n(s, X_{n-1}(s, x)) - \mathbf{v}_{n-1}(s, X_{n-1}(s, x))| \\ &= |\sigma_1| |(\nabla \Phi * \rho_{n-1})(s, X_{n-1}(s, x)) - (\nabla \Phi * \rho_{n-2})(s, X_{n-1}(s, x))| \\ &\leq |\sigma_1| \|\nabla \Phi * \rho_{n-1}(s) - \nabla \Phi * \rho_{n-2}(s)\|_{L^\infty} \leq C \|(\rho_{n-1} - \rho_{n-2})(s)\|_{L^1 \cap L^\infty} \\ &\leq C_0(T) \|(\rho_{n-1} - \rho_{n-2})(s)\|_{L^\infty} = C_0(T) \sup_{x \in \mathbb{R}^d} |\rho_{n-1}(s, X_{n-2}(s, x)) - \rho_{n-2}(s, X_{n-2}(s, x))|. \end{aligned}$$

Now,

$$\begin{aligned} &|\rho_{n-1}(s, X_{n-2}(s, x)) - \rho_{n-2}(s, X_{n-2}(s, x))| \\ &= \left| \rho_{n-1}(s, X_{n-2}(s, x)) - \frac{\rho_0(x)}{1 - \sigma_2 s \rho_0(x)} \right| \\ &= |\rho_{n-1}(s, X_{n-2}(s, x)) - \rho_{n-1}(s, X_{n-1}(s, x))| \\ &\leq \|\nabla \rho_{n-1}(s)\|_{L^\infty} \|X_{n-2}(s, \cdot) - X_{n-1}(s, \cdot)\|_{L^\infty} = \|\nabla \rho_{n-1}(s)\|_{L^\infty} h_{n-1}(s) \\ &\leq C_0(T) h_{n-1}(s), \end{aligned} \quad (5.17)$$

where we used (5.15) in the last inequality. We conclude that

$$I_2 \leq C_0(T) \int_0^t h_{n-1}(s) ds. \quad (5.18)$$

Thus, on  $[0, T]$  we have,

$$h_n(t) \leq C_0(T) \int_0^t (h_{n-1}(s) + h_n(s)) ds. \quad (5.19)$$

For any  $k \geq 2$  define

$$\delta^k(t) = \sup_{n \geq k-1} h_n(t).$$

Then by (5.19), for all  $j \geq 0$ ,

$$h_{k+j}(t) \leq C_0(T) \int_0^t (\max\{h_{k+j-1}(s), h_{k+j}(s)\}) ds \leq C_0(T) \int_0^t \delta^k(s) ds$$

and hence

$$\delta^{k+1}(t) \leq C_0(T) \int_0^t \delta^k(s) ds.$$

Iterating this inequality we obtain

$$\delta^k(t) \leq \frac{(C_0(T)T)^{k-1}}{(k-2)!} \quad (5.20)$$

for all  $t \leq T$ , where we used the bound

$$\delta^2(t) = \sup_{n \geq 1} \|X_n(t) - X_{n-1}(t)\|_{L^\infty} \leq CT \sup_{n \geq 1} \|\mathbf{v}_n(t)\| \leq CT \|\rho_0\|_{L^1 \cap L^\infty}.$$

The bound in (5.20) guarantees that  $\delta^N$  is Cauchy on  $[0, T]$ , so that  $(X_n)$  converges in  $L^\infty((0, T) \times \mathbb{R}^d)$  to some  $X$ . It follows as well from (5.17) that  $(\rho_n)$  is Cauchy in  $L^\infty((0, T) \times \mathbb{R}^d)$ , since we have, for  $m, n \in \mathbb{N}$ ,

$$\begin{aligned} |\rho_n(s, X_m(s, x)) - \rho_m(s, X_m(s, x))| &= |\rho_n(s, X_m(s, x)) - \rho_n(s, X_n(s, x))| \\ &\leq \|\nabla \rho_n(s)\|_{L^\infty} \|X_m - X_n\|_{L^\infty((0, T) \times \mathbb{R}^d)}. \end{aligned}$$

Therefore,  $\rho_n$  converges in  $L^\infty((0, T) \times \mathbb{R}^d)$  to some  $\rho$ . But then

$$\rho(t, x) = \lim_{n \rightarrow \infty} \rho_n(t, x) = \lim_{n \rightarrow \infty} \frac{\rho_0(X_n^{-1}(t, x))}{1 - \sigma_2 t \rho_0(X_n^{-1}(t, x))} = \frac{\rho_0(X^{-1}(t, x))}{1 - \sigma_2 t \rho_0(X^{-1}(t, x))},$$

since  $\rho_0$  is continuous by assumption. Setting  $\mathbf{v} := \sigma_1 \nabla \Phi * \rho$ , we conclude from (2.2) and the uniform bound in  $n$  on the supports of  $\rho_n$  that

$$\|\mathbf{v}_n - \mathbf{v}\|_{L^\infty(0, T; L^\infty(\mathbb{R}^d))} \leq C(T) \|\rho_n - \rho\|_{L^\infty(0, T; L^\infty(\mathbb{R}^d))}.$$

This convergence along with the convergence of  $(X_n)$  to  $X$  in  $L^\infty((0, T) \times \mathbb{R}^d)$  allow us to conclude that for any  $(t, x)$  in  $[0, T] \times \mathbb{R}^d$ ,

$$X(t, x) = \lim_{n \rightarrow \infty} X_n(t, x) = x + \lim_{n \rightarrow \infty} \int_0^t \mathbf{v}_n(s, X_n(s, x)) ds = x + \int_0^t \mathbf{v}(s, X(s, x)) ds.$$

Thus,  $\mathbf{v}$  is the velocity whose flow map is  $X$ . We conclude that  $(X, \rho, \mathbf{v})$  is a Lagrangian solution.

Now assume only that  $\rho_0 \in L^\infty(\mathbb{R}^d)$  and is compactly supported. Because the constant  $C_0(T)$  in (5.20) depends in part on  $\|\nabla \rho_0\|_{L^\infty}$ , we cannot directly use it to prove the existence of weak solutions for initial densities lacking regularity. Instead we let  $\rho_{0, \varepsilon} = \eta_\varepsilon * \rho_0$  where  $\eta_\varepsilon$  is a Friedrich's mollifier, noting that  $\rho_{0, \varepsilon}$  is compactly supported. We can then let  $(X_\varepsilon, \rho_\varepsilon, \mathbf{v}_\varepsilon)$  be the Lagrangian solution with initial density  $\rho_{0, \varepsilon}$ , which we know exists by the preceding argument. Observe that  $\rho_{0, \varepsilon} \rightarrow \rho_0$  in  $L^p$  for all  $p \in [1, \infty)$ , so, by (3) of Lemma 2.2,  $\mathbf{v}_{0, \varepsilon} := \nabla \Phi * \rho_{0, \varepsilon} \rightarrow \nabla \Phi * \rho_0 := \mathbf{v}_0$  in  $L^p$  for all  $p \in \left(\frac{d}{d-1}, \infty\right)$ .

As in Lemma 8.2 of [19], the family  $(X_\varepsilon)$  is equicontinuous in compact subsets of space and time, because their common modulus of continuity depends only upon  $\|\rho_\varepsilon\|_{L^\infty(0, T; L^1 \cap L^\infty)}$ , which is uniformly bounded over  $\varepsilon$  by Proposition 5.2. Therefore, there exists some flow map  $X$  and some subsequence of  $(X_\varepsilon)$  that we relabel as  $(X_n)$  (abusing notation) with  $X_n \rightarrow X$  uniformly on compact subsets of  $[0, T] \times \mathbb{R}^d$ .

Fix  $t \in [0, T]$ . Defining  $\rho$  by the expression in (5.2), we see that

$$\rho(t, x) - \rho_n(t, x) = \frac{\rho_0(X^{-1}(t, x)) - \rho_0(X_n^{-1}(t, x))}{(1 - \sigma_2 t \rho_0(X^{-1}(t, x)))(1 - \sigma_2 t \rho_0(X_n^{-1}(t, x)))}.$$

The denominator is bounded below up to time  $T$ , so, for any  $p \in [1, \infty)$ ,

$$\|\rho(t, x) - \rho_n(t, x)\|_{L_x^p} \leq C_0(T) \|\rho_0(X^{-1}(t, x)) - \rho_0(X_n^{-1}(t, x))\|_{L_x^p} \rightarrow 0$$

by the continuity of the  $L^p$ -norm under translation for  $p \in [1, \infty)$ . (For an explicit proof, see Lemma 8.2 of [14].) It follows by (3) of Lemma 2.2 that  $\mathbf{v}_n(t)$  is Cauchy and therefore converges to some  $\mathbf{v}(t)$  in  $L^p(\mathbb{R}^d)$  for any  $p \in \left(\frac{d}{d-1}, \infty\right)$ . But again by (3) of Lemma 2.2,  $\mathbf{v}_n(t) = \sigma_1 \nabla \Phi * \rho_{n-1}(t) \rightarrow \sigma_1 \nabla \Phi * \rho(t)$  in  $L^p(\mathbb{R}^d)$  for any  $p \in \left(\frac{d}{d-1}, \infty\right)$  so  $\mathbf{v} = \sigma_1 \nabla \Phi * \rho(t)$ . In particular,  $\mathbf{v}(0) = \mathbf{v}_0$  so  $\rho(0) = \rho_0$ .

Finally, for any  $(t, x)$  in a fixed compact subset  $K$  of  $[0, T] \times \mathbb{R}^d$ ,

$$X(t, x) = \lim_{n \rightarrow \infty} X_n(t, x) = x + \lim_{n \rightarrow \infty} \int_0^t \mathbf{v}_n(s, X_n(s, x)) ds = x + \int_0^t \mathbf{v}(s, X(s, x)) ds$$

by the dominated convergence theorem (for any fixed  $x \in \mathbb{R}^d$ ,  $|\mathbf{v}_n(\cdot, X_n(\cdot, x))| \leq C_0(T)$  and so is dominated in  $L^1((0, T))$ ) and the continuity of  $\mathbf{v}_n$  in space. Thus,  $X$  is the flow map for  $\mathbf{v}$ , and we can conclude that  $(X, \rho, \mathbf{v})$  is a Lagrangian solution as in Definition 5.1.  $\square$

**Theorem 5.5.** *If  $(X, \rho, \mathbf{v})$  is a Lagrangian solution as in Definition 5.1 then  $\rho$  is a weak (Eulerian) solution as in Definition 4.1 for  $\nu = 0$ . As a partial converse, suppose that  $\rho$  is a weak solution as in Definition 4.1 for  $\nu = 0$  with  $\rho \in C^k(0, T; C^{k, \alpha})$  for some  $k \geq 1$ . Let  $\mathbf{v} = \nabla \Phi * \rho$ ,  $X$  be the flow map for  $\mathbf{v}$  and let*

$$r(t, x) := \frac{\rho_0(X^{-1}(t, x))}{1 - \sigma_2 t \rho_0(X^{-1}(t, x))}, \quad (5.21)$$

so that  $(X, r, \mathbf{v})$  is a Lagrangian solution as in Definition 5.1. Then  $r$  is also a weak solution.

*Proof.* Let  $(X, \rho, \mathbf{v})$  be a Lagrangian solution. If we make the change of variables  $x = X(t, y)$  then

$$\rho(t, x) = \frac{\rho_0(y)}{1 - \sigma_2 t \rho_0(y)}, \quad \partial_t \rho(t, x) = \frac{\sigma_2 \rho_0(y)^2}{(1 - \sigma_2 t \rho_0(y))^2}. \quad (5.22)$$

Using (5.5) and (5.6) we have for any  $\varphi \in C_c^\infty((0, T) \times \mathbb{R}^d)$

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} (\rho(t, x) \partial_t \varphi(t, x) + \rho(t, x) (\mathbf{v} \cdot \nabla \varphi)(t, x) + (\sigma_1 + \sigma_2) \rho(t, x)^2 \varphi(t, x)) dx dt \\ &= \int_0^T \int_{\mathbb{R}^d} \frac{\rho_0(y)}{1 - \sigma_2 t \rho_0(y)} (\partial_t \varphi + \mathbf{v} \cdot \nabla \varphi)(t, X(t, y)) J(t, y) dy dt \\ & \quad + (\sigma_1 + \sigma_2) \int_0^T \int_{\mathbb{R}^d} \left( \frac{\rho_0(y)}{1 - \sigma_2 t \rho_0(y)} \right)^2 \varphi(t, X(t, y)) J(t, y) dy dt. \end{aligned}$$

But

$$(\partial_t \varphi + \mathbf{v} \cdot \nabla \varphi)(t, X(t, y)) = \frac{d}{dt} \varphi(t, X(t, y))$$

so

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} \frac{\rho_0(y)}{1 - \sigma_2 t \rho_0(y)} (\partial_t \varphi + \mathbf{v} \cdot \nabla \varphi)(t, X(t, y)) J(t, y) dy dt \\ &= - \int_0^T \int_{\mathbb{R}^d} \frac{\partial}{\partial t} \left( \frac{\rho_0(y)}{1 - \sigma_2 t \rho_0(y)} J(t, y) \right) \varphi(t, X(t, y)) dy dt. \end{aligned}$$

Then for  $\sigma_2 \neq 0$ ,

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{\rho_0(y)}{1 - \sigma_2 t \rho_0(y)} J(t, y) \right) &= \frac{\partial}{\partial t} \left( \rho_0(y) (1 - \sigma_2 t \rho_0(y))^{-\frac{\sigma_1 + \sigma_2}{\sigma_2}} \right) \\ &= (\sigma_1 + \sigma_2) \rho_0(y)^2 (1 - \sigma_2 t \rho_0(y))^{-\frac{\sigma_1 + 2\sigma_2}{\sigma_2}} \\ &= (\sigma_1 + \sigma_2) \left( \frac{\rho_0(y)}{1 - \sigma_2 t \rho_0(y)} \right)^2 J(t, y). \end{aligned}$$

We see then that

$$\int_0^T \int_{\mathbb{R}^d} (\rho(t, x) \partial_t \varphi(t, x) + \rho(t, x) (\mathbf{v} \cdot \nabla \varphi)(t, x) + (\sigma_1 + \sigma_2) \rho(t, x)^2 \varphi(t, x) dx dt) = 0,$$

meaning by Theorem 4.2 that  $\rho$  is a weak solution as in Definition 4.1. That this also holds for  $\sigma_2 = 0$  follows similarly or simply by taking the limit as  $\sigma_2 \rightarrow 0$  in the calculations above.

Now let  $\rho$  be a weak solution as in Definition 4.1 with  $\rho \in C^k(0, T; C^{k, \alpha})$  for some  $k \geq 1$ , and let  $X$  be the flow map for  $\mathbf{v} := \nabla \Phi * \rho$ , which we note exists and is unique. Then define  $r$  as in (5.21). Making the change of variables  $x = X(t, y)$  as above, we see that for any  $\varphi \in C_c^\infty((0, T) \times \mathbb{R}^d)$ ,

$$\int_0^T \int_{\mathbb{R}^d} (r(t, x) \partial_t \varphi(t, x) + r(t, x) (\mathbf{v} \cdot \nabla \varphi)(t, x) + (\sigma_1 + \sigma_2) r(t, x)^2 \varphi(t, x) dx dt) = 0,$$

meaning that, by Theorem 4.2,  $r$  is an Eulerian solution as in Definition 4.1. Note that to justify the change of variables we needed higher time regularity of  $\rho$  than that of a weak solution.  $\square$

Theorems 5.3 to 5.5 taken together show that, as long as the initial density has some regularity, we are assured of at least one solution to  $(GAG_\nu)$ , and that solution is both a Lagrangian and an Eulerian (weak) solution. Moreover, the solution maintains the spatial regularity it had at time zero. Next, we prove uniqueness of Lagrangian solutions (and so of our Eulerian solutions as well for higher regularity).

**Theorem 5.6.** *Fix  $T > 0$  with  $T < (|\sigma_2| \|\rho_0\|_{L^\infty})^{-1}$  or  $T < \infty$  if  $\sigma_2 = 0$  and assume that  $\rho_0 \in L^\infty(\mathbb{R}^d)$  and is compactly supported. Then there exists at most one Lagrangian solution to  $(GAG_0)$  as in Definition 5.1 having the same initial density.*

*Proof.* Suppose that  $\rho_1, \rho_2$  are two Lagrangian solutions as in Definition 5.1 having the same initial density,  $\rho_0$ . Define

$$h(t) = \|X_2(t, \cdot) - X_1(t, \cdot)\|_{L^\infty},$$

where  $X_j$  is the flow map for  $\mathbf{v}_j := \sigma_j \nabla \Phi * \rho_j$ . Then

$$\begin{aligned} |X_2(t, x) - X_1(t, x)| &\leq \int_0^t |\mathbf{v}_2(s, X_2(s, x)) - \mathbf{v}_2(s, X_1(s, x))| ds \\ &\quad + \int_0^t |\mathbf{v}_2(s, X_1(s, x)) - \mathbf{v}_1(s, X_1(s, x))| ds \\ &=: I_1 + I_2. \end{aligned}$$

By Proposition 5.2 and (2.2),  $\mathbf{v}_1(t), \mathbf{v}_2(t)$  have a log-Lipschitz MOC  $\mu$  as in (5.12). Then  $I_1$  can be bounded as

$$I_1 \leq \int_0^t \mu(|X_2(s, x) - X_1(s, x)|) ds \leq \int_0^t \mu(h(s)) ds.$$

(We used that  $\mu$  is nondecreasing in this bound.)

For  $I_2$ , we set  $z = X_1(s, x)$  and write

$$\begin{aligned}
& \mathbf{v}_2(s, X_1(s, x)) - \mathbf{v}_1(s, X_1(s, x)) \\
&= \sigma_1[\nabla\Phi * \rho_2(s)](z) - \sigma_1[\nabla\Phi * \rho_1(s)](z) \\
&= \sigma_1 \int_{\mathbb{R}^d} \nabla\Phi(z - y) \rho_2(s, y) dy - \sigma_1 \int_{\mathbb{R}^d} \nabla\Phi(z - y) \rho_1(s, y) dy \\
&= \sigma_1 \int_{\mathbb{R}^d} \nabla\Phi(z - y) \frac{\rho_0((X_2)^{-1}(s, y))}{1 - \sigma_2 s \rho_0((X_2)^{-1}(s, y))} dy \\
&\quad - \sigma_1 \int_{\mathbb{R}^d} \nabla\Phi(z - y) \frac{\rho_0((X_1)^{-1}(s, y))}{1 - \sigma_2 s \rho_0((X_1)^{-1}(s, y))} dy \\
&= \sigma_1 \int_{\mathbb{R}^d} \nabla\Phi(z - X_2(s, y)) \frac{\rho_0(y)}{1 - \sigma_2 s \rho_0(y)} \det \nabla X_2(s, y) dy \\
&\quad - \sigma_1 \int_{\mathbb{R}^d} \nabla\Phi(z - X_1(s, y)) \frac{\rho_0(y)}{1 - \sigma_2 s \rho_0(y)} \det \nabla X_1(s, y) dy.
\end{aligned} \tag{5.23}$$

Using the expression for  $\det \nabla X_j$  as in (5.6),

$$\begin{aligned}
& \mathbf{v}_2(s, X_1(s, x)) - \mathbf{v}_1(s, X_1(s, x)) \\
&= \sigma_1 \int_{\mathbb{R}^d} (\nabla\Phi(z - X_2(s, y)) - \nabla\Phi(z - X_1(s, y))) \rho_0(y) (1 - \sigma_2 s \rho_0(y))^{-\frac{\sigma_1}{\sigma_2} - 1} dy
\end{aligned} \tag{5.24}$$

when  $\sigma_2 \neq 0$ , and

$$\begin{aligned}
& \mathbf{v}_2(s, X_1(s, x)) - \mathbf{v}_1(s, X_1(s, x)) \\
&= \sigma_1 \int_{\mathbb{R}^d} (\nabla\Phi(z - X_2(s, y)) - \nabla\Phi(z - X_1(s, y))) \rho_0(y) e^{\sigma_1 \rho_0(y) s} dy
\end{aligned}$$

when  $\sigma_2 = 0$ . In both cases, by Proposition 5.2 and Lemma 5.11, we have

$$\begin{aligned}
I_2 &\leq C_0(T) \int_0^t \|(\nabla\Phi(z - X_2(s, y)) - \nabla\Phi(z - X_1(s, y))) \rho_0(y)\|_{L^1_y(\text{supp } \rho_0(y))} ds \\
&\leq C_0(T) \int_0^t \mu(h(s)) ds.
\end{aligned}$$

Combining these estimates gives

$$h(t) \leq C(T, |\sigma_1|, |\sigma_2|, \|\rho_0\|_{L^1 \cap L^\infty}, |\text{supp } \rho_0|) \int_0^t \mu(h(s)) ds \tag{5.25}$$

up to time  $T$ . Uniqueness of the Lagrangian solutions then follows immediately from Osgood's lemma, since  $\mu$  is an Osgood modulus of continuity.  $\square$

Taken together, Theorems 5.3 to 5.6 give the following:

**Theorem 5.7.** *Fix  $T > 0$  with  $T < (|\sigma_2| \|\rho_0\|_{L^\infty})^{-1}$  or  $T < \infty$  if  $\sigma_2 = 0$ . Let  $\rho_0 \in L^\infty(\mathbb{R}^d)$  be compactly supported. There exists a unique Lagrangian solution to  $(GAG_0)$  as in Definition 5.1, and (5.3) and (5.4) hold. Moreover, if also  $\rho_0 \in C^{k, \alpha}(\mathbb{R}^d)$  for some  $k \geq 0$  and  $\alpha \in (0, 1)$ , then  $\rho \in L^\infty(0, T; C^{k, \alpha})$ , and (5.7) holds. When  $k \geq 1$ ,  $\rho \in C^k(0, T; C^{k, \alpha})$  is also the unique classical, Eulerian solution.*

**Remark 5.8.** *It is easy to see that the solution as given in Theorem 5.7 can be extended uniquely up to any  $t < T^*$ , where  $T^*$  is the first time at which the denominator in (5.2) reaches zero for some  $x \in \mathbb{R}^d$ .*

We used the following lemmas above.

**Lemma 5.9.** *Let  $\alpha, \beta \in (0, 1)$ . Then for any real number  $C_1$ ,*

$$\begin{aligned} \left\| \frac{f \circ g}{1 - C_1(f \circ g)} \right\|_{\dot{C}^{\alpha\beta}} &\leq \|(1 - C_1(f \circ g))^{-1}\|_{L^\infty}^2 \|f\|_{\dot{C}^\alpha} \|g\|_{\dot{C}^\beta}^\alpha, \\ \left\| \frac{f \circ g}{1 - C_1(f \circ g)} \right\|_{\dot{C}^\alpha} &\leq \|(1 - C_1(f \circ g))^{-1}\|_{L^\infty}^2 \|f\|_{\dot{C}^\alpha} \|\nabla g\|_{L^\infty}^\alpha. \end{aligned} \quad (5.26)$$

Here,  $\dot{C}^\alpha$  is the homogeneous Hölder space.

*Proof.* For (5.26)<sub>1</sub>, by a simple calculation we have for any  $x$  and  $y$  in  $\mathbb{R}^d$ ,

$$\frac{f(g(x))}{1 - C_1(f(g(x)))} - \frac{f(g(y))}{1 - C_1(f(g(y)))} = \frac{f(g(x)) - f(g(y))}{(1 - C_1(f(g(x))))(1 - C_1(f(g(y))))},$$

so that

$$\begin{aligned} \left\| \frac{f \circ g}{1 - C_1(f \circ g)} \right\|_{\dot{C}^{\alpha\beta}} &\leq \|(1 - C_1(f \circ g))^{-1}\|_{L^\infty}^2 \sup_{x \neq y} \frac{|f(g(x)) - f(g(y))|}{|g(x) - g(y)|^\alpha} \left( \frac{|g(x) - g(y)|}{|x - y|^\beta} \right)^\alpha \\ &\leq \|(1 - C_1(f \circ g))^{-1}\|_{L^\infty}^2 \|f\|_{\dot{C}^\alpha} \|g\|_{\dot{C}^\beta}^\alpha. \end{aligned}$$

The same argument setting  $\beta = 1$  yields (5.26)<sub>2</sub>. □

The ideas in the two lemmas that follow originated for 2D in [1].

**Lemma 5.10.** *For  $x, y \in \mathbb{R}^2$ , we have*

$$|\Phi(x) - \Phi(y)| = \frac{1}{2\pi} \frac{|x - y|}{|x||y|}.$$

For  $x, y \in \mathbb{R}^d$ ,  $d \geq 2$ .

$$|\nabla\Phi(x) - \nabla\Phi(y)| \leq \frac{C_d}{|x||y|} \left( \frac{1}{|x|} + \frac{1}{|y|} \right)^{d-2} |x - y|.$$

*Proof.* This result is well-known for  $d = 2$ . So assume that  $d \geq 3$ . We have, for any  $x, y \in \mathbb{R}^d$ ,

$$\begin{aligned} \left| \frac{x}{|x|^d} - \frac{y}{|y|^d} \right|^2 &= \left| \frac{|y|^d x - |x|^d y}{|x|^d |y|^d} \right|^2 = \frac{|x|^2 |y|^{2d} + |y|^2 |x|^{2d} - 2x \cdot y |x|^d |y|^d}{|x|^{2d} |y|^{2d}} \\ &= |x|^2 |y|^2 \frac{|y|^{2(d-1)} + |x|^{2(d-1)} - 2x \cdot y |x|^{d-2} |y|^{d-2}}{|x|^{2d} |y|^{2d}} = \frac{||x|^{d-2} x - |y|^{d-2} y|^2}{|x|^{2(d-1)} |y|^{2(d-1)}}. \end{aligned}$$

Hence,

$$\begin{aligned}
\left| \frac{x}{|x|^d} - \frac{y}{|y|^d} \right| &= \frac{||x|^{d-2}x - |y|^{d-2}y|}{|x|^{d-1}|y|^{d-1}} = \frac{||x|^{d-2}(x-y) + (|x|^{d-2} - |y|^{d-2})y|}{|x|^{d-1}|y|^{d-1}} \\
&\leq \frac{|x-y|}{|x||y|^{d-1}} + \frac{||x|^{d-2} - |y|^{d-2}|}{|x|^{d-1}|y|^{d-2}} \\
&= \frac{|x-y|}{|x||y|^{d-1}} + \frac{(|x|-|y|)(|x|^{d-3} + |x|^{d-4}|y| + |x||y|^{d-4} + |y|^{d-3})}{|x|^{d-1}|y|^{d-2}} \\
&\leq \frac{|x-y|}{|x||y|^{d-1}} + |x-y| \sum_{j=2}^{d-1} \frac{1}{|x|^j|y|^{d-j}}.
\end{aligned}$$

But,

$$\sum_{j=2}^{d-1} \frac{1}{|x|^j|y|^{d-j}} = \frac{1}{|x||y|} \sum_{j=1}^{d-2} \frac{1}{|x|^j|y|^{d-2-j}} \leq \frac{C_d}{|x||y|} \left( \frac{1}{|x|} + \frac{1}{|y|} \right)^{d-2}$$

and  $(|x||y|^{d-1})^{-1}$  is bounded by the same quantity. Therefore,

$$|\nabla\Phi(x) - \nabla\Phi(y)| = C_d \left| \frac{x}{|x|^d} - \frac{y}{|y|^d} \right| \leq \frac{C_d}{|x||y|} \left( \frac{1}{|x|} + \frac{1}{|y|} \right)^{d-2} |x-y|.$$

□

**Lemma 5.11.** *Let  $X_1$  and  $X_2$  be homeomorphisms of  $\mathbb{R}^d$ ,  $d \geq 2$ . Let  $\delta = \|X_1 - X_2\|_{L^\infty}$  and suppose  $\delta < e^{-1}$ . Then, for any measurable subset  $U \subset \Omega$ , with finite measure, there exists  $C > 0$ , depending only on  $\Omega$ , the measure of  $U$ , and  $d$ , such that*

$$\|\nabla\Phi(X_1(x) - z) - \nabla\Phi(X_2(x) - z)\|_{L^1_x(U)} \leq -C\delta \log \delta \max_{j=1,2} \{\|\det \nabla X_j^{-1}\|_{L^\infty}\}. \quad (5.27)$$

*Proof.* Set  $A = \nabla\Phi(x-y_1) - \nabla\Phi(x-y_2)$  and let  $p, q > 1$ , with  $p^{-1} + q^{-1} = 1$ . Let  $a = |x-y_1|$ ,  $b = |x-y_2|$ , and note that  $|y_1 - y_2| \leq a + b$ . Then from Lemma 5.10,

$$\begin{aligned}
A &\leq \frac{C_d}{ab} \left( \frac{1}{a} + \frac{1}{b} \right)^{d-2} |y_1 - y_2|^{\frac{1}{p}} |y_1 - y_2|^{\frac{1}{q}} \leq \frac{C_d}{(ab)^{1-\frac{1}{p}}} \left( \frac{1}{a} + \frac{1}{b} \right)^{d-2} \left( \frac{a+b}{ab} \right)^{\frac{1}{p}} |y_1 - y_2|^{\frac{1}{q}} \\
&= \frac{C_d}{(ab)^{1-\frac{1}{p}}} \left( \frac{1}{a} + \frac{1}{b} \right)^{d+\frac{1}{p}-2} |y_1 - y_2|^{\frac{1}{q}} \leq \frac{C_d}{2} \left( \frac{1}{a} + \frac{1}{b} \right)^{d-\frac{1}{p}} |y_1 - y_2|^{\frac{1}{q}} \\
&\leq C_d 2^{d-2-\frac{1}{p}} \left( \frac{1}{a^{d-\frac{1}{p}}} + \frac{1}{b^{d-\frac{1}{p}}} \right) |y_1 - y_2|^{\frac{1}{q}}.
\end{aligned}$$

In the final two inequalities we used  $(ab)^{-1/2} \leq (1/2)(a^{-1} + b^{-1})$  followed by  $(c+d)^r \leq 2^{r-1}(c^r + d^r)$  for any  $c, d, r > 0$ .

It follows that

$$\begin{aligned}
\|A\|_{L^1_x(U)} &\leq C\delta^{\frac{1}{q}} \sum_{j=1}^2 \left\| (X_j(x) - z)^{-(d-\frac{1}{p})} \right\|_{L^1_x(U)} = C\delta^{\frac{1}{q}} \sum_{j=1}^2 \int_{\mathbb{R}^d} \frac{|\det \nabla X_j^{-1}(w)|}{|w-z|^{d-\frac{1}{p}}} dw \\
&\leq C\delta^{\frac{1}{q}} \max_{j=1,2} \{\|\det \nabla X_j^{-1}\|_{L^\infty}\} \sum_{j=1}^2 \left\| \frac{1}{|w-z|^{d-\frac{1}{p}}} \right\|_{L^1_w(X_j(U))}.
\end{aligned}$$

But, as in the proof of Proposition 3.2 of [1], the above norm is maximized when  $U$  is a ball centered at  $z$  (of radius  $R$ , depending on the measure of  $U$ ). This gives

$$\begin{aligned} \|A\|_{L_x^1(U)} &\leq C\delta^{\frac{1}{q}} \max_{j=1,2} \{\|\det \nabla X_j^{-1}\|_{L^\infty}\} \sum_{j=1}^2 \int_0^R \frac{r^{d-1}}{r^{d-\frac{1}{p}}} dr = C\delta^{1-\frac{1}{p}} \max_{j=1,2} \{\|\det \nabla X_j^{-1}\|_{L^\infty}\} pR^{\frac{1}{p}} \\ &\leq C\delta^{1-\frac{1}{p}} p \max\{1, R\} \max_{j=1,2} \{\|\det \nabla X_j^{-1}\|_{L^\infty}\}. \end{aligned}$$

This is minimized, relative to  $p$ , when  $p = -\log \delta$ , giving

$$\|A\|_{L_x^1(U)} \leq C \max\{1, R\} e(-\delta \log \delta),$$

which is (5.27).  $\square$

## 6. TOTAL MASS AND INFINITE ENERGY

In dimensions three and higher,  $\rho^\nu \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  is enough to guarantee  $\mathbf{v}^\nu \in L^2(\mathbb{R}^d)$ . In 2D, however, this is no longer true: the 2D velocity for any  $\nu \geq 0$  will generically have infinite energy even if it has finite energy at time zero (see, for example, Proposition 3.1.1 of [7]). When dealing only with existence of solutions to  $(GAG_\nu)$  the infinite energy of 2D velocities is a minor issue. We will need to face this issue directly, however, in Section 7 when we take the vanishing viscosity limit.

For  $f \in L^1(\mathbb{R}^d)$  define the total mass of  $f$  by

$$m(f) := \int_{\mathbb{R}^d} f. \quad (6.1)$$

The total mass of the viscous solutions evolves over time as described in Theorem 6.1.

**Theorem 6.1.** *Assume that  $\rho_0 \in L_N^2 \cap L^\infty$  for some  $N > 1 + d/2$ . Let  $\rho^\nu$  be a weak solution to  $(GAG_\nu)$  as in Definition 4.1 for some  $\nu \geq 0$  given by Theorem 4.5 or Theorem 5.7. Then up to the time of existence,*

$$m(\rho^\nu) = m(\rho_0) + (\sigma_1 + \sigma_2) \int_0^t \|\rho^\nu(s)\|^2 ds. \quad (6.2)$$

*Proof.* Apply the test function,  $\varphi = a_R(x)$  for  $R > 0$  in (4.2). This gives for any  $t \in [0, T]$

$$\int_0^t \int_{\mathbb{R}^d} (\rho^\nu \mathbf{v}^\nu \cdot \nabla a_R + (\sigma_1 + \sigma_2)(\rho^\nu)^2 a_R - \nu \nabla \rho^\nu \cdot \nabla a_R) dx dt = \int_{\mathbb{R}^d} \rho(t) a_R - \int_{\mathbb{R}^d} \rho_0 a_R.$$

Taking  $R \rightarrow \infty$  and using that  $\rho^\nu$  lies in  $L^\infty(0, T; L^q)$  for all  $q \in [1, \infty]$ ,  $\nabla \rho^\nu \in L^2(0, T; L^2)$ , and  $\mathbf{v}^\nu \in L^\infty((0, T) \times \mathbb{R}^d)$  yields (6.2).  $\square$

**Remark 6.2.** *When  $\sigma_1 + \sigma_2 = 0$ , as happens for  $(AG_\nu)$ , (6.2) shows that total mass is conserved.*

In recovering the velocity from its divergence the total mass of the density plays an important, if so far hidden, role in 2D: in short, if the total mass of the density is zero and has sufficient spatial decay, then the velocity will lie in  $L^2$ . We prove this along with other useful bounds on the velocity in Lemma 6.3.

**Lemma 6.3.** *Let  $\rho \in L^1 \cap L^\infty(\mathbb{R}^d)$ . If  $d \geq 3$  then for all  $p \in (d/(d-2), \infty]$ ,*

$$\|\Phi * \rho\|_{L^p} \leq C(p) \|\rho\|_{L^1 \cap L^p}, \quad \|\nabla \Phi * \rho\| \leq C \|\rho\|_{L^1 \cap L^\infty}^{\frac{1}{2}} \|\rho\|_{L^1}^{\frac{1}{2}}. \quad (6.3)$$

Let  $d = 2$  and assume that  $\rho \in L_N^2$  for some  $N > 1 + d/2$  with  $m(\rho) = 0$ . Then

$$\|\nabla\Phi * \rho\| \leq C \|\rho\|_{L_N^2 \cap L^\infty}. \quad (6.4)$$

*Proof.* Let  $a$  be as in Definition 1.1. For  $d \geq 3$ ,

$$\|\Phi * \rho\|_{L^p} \leq \|a\Phi\|_{L^1} \|\rho\|_{L^p} + \|(1-a)\Phi\|_{L^p} \|\rho\|_{L^1} < \infty$$

for all  $p > d/(d-2)$ . In particular,  $\|\Phi * \rho\|_{L^\infty} \leq C \|\rho\|_{L^1 \cap L^\infty}$ . Hence, for  $d \geq 3$  we can apply Lemma 6.4. This gives

$$\|\nabla\Phi * \rho\|^2 = - \int_{\mathbb{R}^d} (\Phi * \rho)(\Delta\Phi * \rho) \leq \|\Phi * \rho\|_{L^\infty} \|\Delta\Phi * \rho\|_{L^1} = \|\Phi * \rho\|_{L^\infty} \|\rho\|_{L^1}$$

and leads to (6.3).

Now assume that  $d = 2$  with  $\rho \in L_N^2$  and  $m(\rho) = 0$ , and assume that  $N > 1 + d/2$ . Fix  $x \in \mathbb{R}^2$  with  $|x| \geq 4$  and let  $R = |x|/4$ .

By (1) of Lemma 2.2 and because  $m(\rho) = 0$  we have

$$\left| \int_{\mathbb{R}^2} a_R \rho \right| = \left| \int_{\mathbb{R}^2} (1 - a_R) \rho \right| \leq C \|\rho\|_{L_N^2} R^{-(N-1)}.$$

Letting

$$\alpha_R := \frac{\int_{\mathbb{R}^2} a_R \rho}{\int_{\mathbb{R}^2} a_R} = \frac{C}{R^2} \int_{\mathbb{R}^2} a_R \rho$$

we can write

$$|\nabla\Phi * \rho(x)| \leq |\nabla\Phi * (a_R(\rho - \alpha_R))(x)| + |\nabla\Phi * (a_R \alpha_R)(x)| + |\nabla\Phi * ((1 - a_R)\rho)(x)|. \quad (6.5)$$

Because  $a_R(\rho - \alpha_R)$  has total mass zero we have

$$\begin{aligned} |\nabla\Phi * (a_R(\rho - \alpha_R))(x)| &= \frac{1}{2\pi} \int_{\text{supp } a_R} \frac{x-y}{|x-y|^2} a_R(y) (\rho(y) - \alpha_R) dy \\ &= \frac{1}{2\pi} \int_{\text{supp } a_R} \left[ \frac{x-y}{|x-y|^2} - \frac{x}{|x|^2} \right] a_R(y) (\rho(y) - \alpha_R) dy \\ &\leq \frac{1}{2\pi} \int_{\text{supp } a_R} \frac{|y|}{|x||x-y|} a_R(y) |\rho(y) - \alpha_R| dy. \end{aligned}$$

In the last inequality we used Lemma 5.10.

Since  $|x| = 4R$  we have  $|x-y| \geq (1/2)|x|$  for all  $y \in \text{supp } a_R$ . Hence in the final integrand above we have both  $|y|(|x||x-y|)^{-1} \leq C|y||x|^{-2}$  and  $|y|(|x||x-y|)^{-1} \leq CR|x|^{-2}$ . We conclude that

$$\begin{aligned} |\nabla\Phi * (a_R(\rho - \alpha_R))(x)| &\leq C \int_{\text{supp } a_R} \frac{|y|}{|x|^2} a_R(y) |\rho(y)| dy + C \int_{\text{supp } a_R} \frac{R}{|x|^2} a_R(y) |\alpha_R| dy \\ &\leq \frac{C}{|x|^2} \left( \| |y| \rho(y) \|_{L_y^1} + |\alpha_R| R^3 \right) \leq \frac{C}{|x|^2} \|\rho\|_{L_N^2} + C |\alpha_R| R, \end{aligned}$$

since  $\| |y| \rho(y) \|_{L_y^1} \leq C \|\rho\|_{L_N^2}$  by Corollary 2.5 and  $N > 1 + d/2$ .

It also follows from  $|x| = 4R$  that

$$\|\nabla\Phi(x - \cdot)\|_{L^\infty(\text{supp } a_R)} = \sup_{y \in \text{supp } a_R} \frac{1}{2\pi} \left| \frac{x-y}{|x-y|^2} \right| \leq \frac{C}{|x|}.$$

Thus

$$|\nabla\Phi * (a_R\alpha_R)(x)| \leq \|\nabla\Phi(x - \cdot)\|_{L^\infty(\text{supp } a_R)} \|a_R\alpha_R\|_{L^1} \leq \frac{C}{|x|} |\alpha_R| R^2 = C|\alpha_R|R.$$

For the final term in (6.5) we use (2.1) to give

$$\begin{aligned} |\nabla\Phi * ((1 - a_R)\rho)(x)| &\leq \|\nabla\Phi\|_{L^\infty(\text{supp}(1 - a_R)(x - \cdot))} \|((1 - a_R)\rho)\|_{L^1} \leq \frac{C}{R} \|\rho\|_{L_N^2} R^{-(N-1)} \\ &= C \|\rho\|_{L_N^2} |x|^{-N} \leq C \|\rho\|_{L_N^2} |x|^{-2} \end{aligned}$$

since  $N > 1 + d/2 = 2$ . Collecting these bounds we conclude that for  $|x| \geq 4$ ,

$$|\nabla\Phi * \rho(x)| \leq \frac{C}{|x|^2} \left( \|\rho\|_{L_N^2 \cap L^\infty} + |\alpha_R| |x| \right).$$

Note that these bounds were all for a fixed  $x$  with  $|x| \geq 4$  and so for fixed  $R \geq 1$ . It remains to bound  $\alpha_R$ , though. We do this using again that the total mass of  $\rho$  is zero so

$$|\alpha_R| = \frac{C}{R^2} \left| \int_{\mathbb{R}^2} (1 - a_R)\rho \right| \leq \frac{C}{R^2} \|(1 - a_R)\rho\|_{L^1} \leq CR^{-N-1} \|\rho\|_{L_N^2} \leq C \|\rho\|_{L_N^2} |x|^{-3}$$

by virtue of (2.1) and because  $N + 1 > 1 + d/2 + 1 > 3$ . Applying (2.2) for the case  $|x| < 4$  it follows that

$$|\nabla\Phi * \rho(x)| \leq \frac{C}{1 + |x|^2} \left( \|\rho\|_{L_N^2 \cap L^\infty} + \|\rho\|_{L_N^2} |x|^{-2} \right) \leq \frac{C}{1 + |x|^2} \|\rho\|_{L_N^2 \cap L^\infty}.$$

Hence,  $\nabla\Phi * \rho \in L^p(\mathbb{R}^2)$  for all  $p \in (1, \infty]$ , and in particular (6.4) holds.  $\square$

We used the following technical lemma in the proof of Lemma 6.3 above and will use it again in Section 8.

**Lemma 6.4.** *Let  $\varphi \in L^{p_1} \cap L^\infty(\mathbb{R}^d)$  with  $\nabla\varphi \in L^{p_2} \cap L^\infty(\mathbb{R}^d)$  and  $\Delta\varphi \in L^1 \cap L^\infty(\mathbb{R}^d)$ . If  $1/p_1 + 1/p_2 \geq (d-1)/d$  then  $\nabla\varphi \in L^2(\mathbb{R}^d)$ . If  $1/p_1 + 1/p_2 > (d-1)/d$  then*

$$\|\nabla\varphi\|^2 = - \int_{\mathbb{R}^d} \varphi \Delta\varphi. \quad (6.6)$$

*Proof.* Let  $a_R$  be as in Definition 1.1. Assume first that  $\varphi$  is also in  $C^\infty(\mathbb{R}^d)$ . Then

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla\varphi|^2 &= \lim_{R \rightarrow \infty} \int_{\mathbb{R}^d} a_R \nabla\varphi \cdot \nabla\varphi = - \lim_{R \rightarrow \infty} \int_{\mathbb{R}^d} \text{div}(a_R \nabla\varphi) \varphi \\ &= - \lim_{R \rightarrow \infty} \int_{\mathbb{R}^d} a_R \Delta\varphi \varphi - \lim_{R \rightarrow \infty} \int_{\mathbb{R}^d} (\nabla a_R \cdot \nabla\varphi) \varphi \\ &= - \int_{\mathbb{R}^d} \Delta\varphi \varphi - \lim_{R \rightarrow \infty} \int_{\mathbb{R}^d} (\nabla a_R \cdot \nabla\varphi) \varphi. \end{aligned}$$

For the first equality, the properties of  $a$  allow us to apply the monotone convergence theorem (we may obtain  $\infty$ , though). The one limit we evaluated is valid because  $a_R \Delta\varphi \rightarrow \Delta\varphi$  in  $L^1(\mathbb{R}^d)$ . For the remaining limit define  $p$  so that  $1 = \frac{1}{p} + \frac{1}{p_1} + \frac{1}{p_2}$ . Then  $1 \geq \frac{1}{p} + \frac{d-1}{d}$  so  $p \geq d$ . Applying Hölder's inequality gives

$$\left| \int_{\mathbb{R}^d} (\nabla a_R \cdot \nabla\varphi) \varphi \right| \leq \|\nabla a_R\|_{L^\infty} \|1\|_{L^p(\text{supp } a_R)} \|\nabla\varphi\|_{L^{p_2}} \|\varphi\|_{L^{p_1}} \leq \frac{C}{R} R^{\frac{d}{p}} = CR^{\frac{d}{p}-1}.$$

We conclude that the remaining limit vanishes from which (6.6) follows.  $\square$

To treat densities in  $\mathbb{R}^2$  having nonzero total mass, as we will need to do in Section 8, we will subtract from the associated velocity field a radially symmetric velocity field,  $\boldsymbol{\tau}_0$ . We do this in analogy with the definition of the stationary solution to the Euler equations used to obtain the radial-energy decomposition of a 2D velocity field in [7, 19].

**Definition 6.5.** Fix a radially symmetric function  $g_0 \in C_c^\infty(\mathbb{R}^2)$  having total mass 1. We will abuse notation by writing both  $g_0(x)$  and  $g_0(r)$ , where  $x \in \mathbb{R}^2$  and  $r = |x|$ . Define

$$\boldsymbol{\tau}_0(x) = f(r)x, \quad f(r) := \frac{1}{r^2} \int_0^r \eta g_0(\eta) d\eta.$$

Being a radially directed vector field,  $\boldsymbol{\tau}_0$  is a gradient. We see that

$$\begin{aligned} \operatorname{div} \boldsymbol{\tau}_0 &= 2f + x^i \partial_i f = 2f + x^i \frac{x^i}{r} \partial_r f = 2f + r \partial_r f \\ &= 2f - 2r \frac{1}{r^3} \int_0^r \eta g_0(\eta) d\eta + r \frac{r g_0(r)}{r^2} = 2f - 2f + g_0(r) = g_0(r). \end{aligned}$$

Hence also  $\boldsymbol{\tau}_0 = \nabla \Phi * g_0$ .

## 7. THE VANISHING VISCOSITY LIMIT FOR $(GAG_\nu)$ FOR VELOCITIES IN $L^2$

In this section we consider the vanishing viscosity limit (VV) (see Section 1) for any  $\sigma_1, \sigma_2$  when  $d \geq 3$  and for  $\sigma_1 + \sigma_2 = 0$  when  $d = 2$ . In both of these cases,  $\mathbf{v}^\nu - \mathbf{v}^0$  remains in  $L^2(\mathbb{R}^d)$ . In Section 8 we consider the general situation in 2D.

**Remark 7.1.** We assume throughout this section, Section 8, and Section 9 the following:

- (1) The initial density  $\rho_0$  lies at least in  $L_N^2 \cap L^\infty$  for some  $N > 1 + d/2$ , where  $d \geq 2$ .
- (2)  $\rho^0$  is the unique weak inviscid solution to  $(GAG_0)$  given by Theorem 5.7.
- (3)  $\rho^\nu, \nu > 0$ , is any choice of weak solution to  $(GAG_\nu)$  given by Theorem 4.5. (We will see in Theorem 7.5 that there is a unique solution for any  $\nu > 0$  if also  $\rho_0 \in H^2(\mathbb{R}^d)$ .)
- (4)  $T > 0$  is a uniform time of existence for all  $\nu \geq 0$  guaranteed by Theorems 4.5 and 5.7.

**Proposition 7.2.** Assume that  $\rho_0 \in L_N^2 \cap L^\infty$  and let  $\mu = \rho^\nu - \rho^0$ . For all  $t \in [0, T]$ ,

$$\begin{aligned} m(\mu(t)) &= (\sigma_1 + \sigma_2) \int_0^t \langle \mu(s), \rho^0(s) + \rho^\nu(s) \rangle ds, \\ |m(\mu(t))| &\leq |\sigma_1 + \sigma_2| \int_0^t (\|\rho^0(s)\| + \|\rho^\nu(s)\|) \|\mu(s)\| ds. \end{aligned}$$

*Proof.* This follows from Theorem 6.1. □

The total mass of  $\rho^\nu - \rho^0$  is zero at time zero, but there is no reason to expect, based upon Proposition 7.2, that  $m(\mu(t))$  remains zero. Proposition 7.2 does show, however, that as  $\nu \rightarrow 0$ ,  $m(\mu)$  vanishes if  $\|\mu\|$  vanishes. This will be very useful to us in Section 8.

**Theorem 7.3.** Assume that  $\rho_0 \in C^{1,\alpha}(\mathbb{R}^d)$  for some  $\alpha > 0$  is compactly supported. When  $d = 2$  assume that  $\sigma_1 + \sigma_2 = 0$ . For all  $\nu \leq 1$  and  $t \in [0, T]$ ,

$$\|(\mathbf{v}^\nu - \mathbf{v}^0)(t)\|_{H^1}^2 + \|(\rho^\nu - \rho^0)(t)\|^2 + \nu \int_0^t \|(\rho^\nu - \rho^0)(s)\|^2 ds \leq C_0(t) t \nu e^{C_0(t)}.$$

*Proof.* Define

$$\mu = \rho^\nu - \rho^0, \quad \mathbf{w} = \mathbf{v}^\nu - \mathbf{v}^0, \quad \eta = \rho^0 + \rho^\nu. \quad (7.1)$$

Then  $\operatorname{div} \mathbf{w} = \sigma_1 \mu$  and  $\mathbf{w} \in L^2(\mathbb{R}^d)$  for all  $t \in [0, T]$  by Lemma 6.3 (for  $d = 2$  this uses  $m(\mu) = 0$  by Proposition 7.2).

Taking  $(GAG_\nu) - (GAG_0)$  in the equivalent weak form given in (4) of Theorem 4.2 gives for any  $t \in [0, T]$

$$\begin{aligned} \int_0^t (\mu, \partial_t \varphi) + \int_0^t \int_{\mathbb{R}^d} (\rho^0 \mathbf{w} \cdot \nabla \varphi + \mu \mathbf{v}^\nu \cdot \nabla \varphi + (\sigma_1 + \sigma_2) \mu \eta \varphi - \nu \nabla \rho^\nu \cdot \nabla \varphi) \\ = \int_{\mathbb{R}^d} \mu(t) \varphi(t) \end{aligned} \quad (7.2)$$

for any  $\varphi \in Y$ . Choosing  $\varphi = \mu \in Y$  we write (7.2) as

$$\|\mu(t)\|^2 - \int_0^t (\partial_t \mu, \mu) = \int_0^t \langle \rho^0 \mathbf{w}, \nabla \mu \rangle + \langle \mu \mathbf{v}^\nu, \nabla \mu \rangle + (\sigma_1 + \sigma_2) \langle \mu \eta, \mu \rangle - \nu \langle \nabla \rho^\nu, \nabla \mu \rangle. \quad (7.3)$$

Employing Lemma 1.2 of Section III.1.4 of [24] (or Lemma 3.9) we have

$$\int_0^t (\partial_t \mu, \mu) = \frac{1}{2} \int_0^t \partial_t (\mu, \mu) = \frac{1}{2} \|\mu(t)\|^2 - \frac{1}{2} \|\mu(0)\|^2 = \frac{1}{2} \|\mu(t)\|^2.$$

For the other terms,

$$\begin{aligned} \langle \rho^0 \mathbf{w}, \nabla \mu \rangle &= -\langle \operatorname{div}(\rho^0 \mathbf{w}), \mu \rangle = -\langle \rho^0 \operatorname{div} \mathbf{w} + \mathbf{w} \cdot \nabla \rho^0, \mu \rangle \\ &= -\sigma_1 \langle \rho^0, \mu^2 \rangle - \langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle \leq C_0(t) \|\mu\|^2 - \langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle, \\ \langle \mu \mathbf{v}^\nu, \nabla \mu \rangle &= \frac{1}{2} \langle \mathbf{v}^\nu, \nabla \mu^2 \rangle = -\frac{1}{2} \langle \operatorname{div} \mathbf{v}^\nu, \mu^2 \rangle = -\frac{\sigma_1}{2} \langle \rho^\nu, \mu^2 \rangle \leq \frac{\sigma_1}{2} \|\rho^\nu\|_{L^\infty} \|\mu\|^2 \\ &\leq C_0(t) \|\mu\|^2, \\ (\sigma_1 + \sigma_2) \langle \mu \eta, \mu \rangle &\leq |\sigma_1 + \sigma_2| \|\rho^0 + \rho^\nu\|_{L^\infty} \|\mu\|^2 \leq C_0(t) \|\mu\|^2, \\ -\nu \langle \nabla \rho^\nu, \nabla \mu \rangle &= -\nu \langle \nabla \mu, \nabla \mu \rangle - \nu \langle \nabla \rho^0, \nabla \mu \rangle \\ &\leq -\nu \langle \nabla \mu, \nabla \mu \rangle + \frac{\nu}{2} \|\nabla \rho^0\|^2 + \frac{\nu}{2} \|\nabla \mu\|^2 \leq C_0(t) \nu - \frac{\nu}{2} \|\nabla \mu\|^2. \end{aligned}$$

To estimate the term  $-\langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle$  we consider the cases  $d = 2$  and  $d \geq 3$  separately. Note that  $\|\mathbf{w}\|_{L^{\frac{2d}{d-2}}} \leq C \|\mu\|$  when  $d \geq 3$  by the Hardy-Littlewood-Sobolev inequality. Therefore,

$$\begin{aligned} -\langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle &\leq \|\mathbf{w} \cdot \nabla \rho^0\| \|\mu\| \leq \|\mathbf{w}\|_{L^{\frac{2d}{d-2}}} \|\nabla \rho^0\|_{L^d} \|\mu\| \\ &\leq C \|\mu\| \|\nabla \rho^0\|_{L^d} \|\mu\| = C \|\mu\|^2 \|\nabla \rho^0\|_{L^d} \leq C_0(t) \|\mu\|^2 \end{aligned}$$

when  $d \geq 3$ . For the case  $d = 2$ , the Hardy-Littlewood-Sobolev inequality does not yield the desired estimate, but we have

$$-\langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle \leq \|\mathbf{w} \cdot \nabla \rho^0\| \|\mu\| \leq \|\mathbf{w}\|_{L^p} \|\nabla \rho^0\|_{L^q} \|\mu\| \quad (7.4)$$

where  $1/p + 1/q = 1/2$ . By Lemmas 7.7 and 7.8 we have

$$\|\mathbf{w}\|_{L^p} \leq C (\|\mathbf{w}\| + \|\mu\|). \quad (7.5)$$

Substituting this estimate into (7.4) gives, for any fixed  $q \in (2, \infty)$ ,

$$-\langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle \leq \|\mathbf{w} \cdot \nabla \rho^0\| \|\mu\| \leq C \|\nabla \rho^0\|_{L^q} (\|\mathbf{w}\| + \|\mu\|) \|\mu\| \leq C_0(t) (\|\mathbf{w}\| + \|\mu\|) \|\mu\|. \quad (7.6)$$

Applying the above estimates to (7.3), we see that for  $d \geq 3$ ,

$$\|\mu(t)\|^2 + \nu \int_0^t \|\nabla \mu(s)\|^2 ds \leq C_0(t) t \nu + C_0(t) \int_0^t \|\mu(s)\|^2 ds, \quad (7.7)$$

while for  $d = 2$ ,

$$\|\mu(t)\|^2 + \nu \int_0^t \|\nabla \mu(s)\|^2 ds \leq C_0(t)t\nu + \int_0^t C_0(s) \left( \|\mathbf{w}(s)\|^2 + \|\mu(s)\|^2 \right) ds. \quad (7.8)$$

Applying Lemma 1.2 (Grönwall's lemma) to (7.7) we conclude that  $\rho_\nu$  converges to  $\rho$  in  $L^\infty(0, T; L^2(\mathbb{R}^d))$  as  $\nu$  approaches zero. However, we must obtain a bound on  $\|\mathbf{w}\|$  for both  $d = 2$  and  $d \geq 3$  below in order to obtain the estimate in Theorem 7.3 on the difference of velocities in  $H^1$ . Therefore, we utilize (7.8) for both  $d = 2$  and  $d \geq 3$  in what follows.

We return to (7.2), but use (3) of Theorem 4.2 instead of (4) for an as-yet unspecified  $\varphi \in C([0, T]; L^2) \cap L^2(0, T; H^1)$ . In place of (7.2), then, we obtain

$$\begin{aligned} \int_0^t \sigma_1^{-1} \langle \partial_t \mathbf{w}, \nabla \varphi \rangle + \langle \rho^0, \mathbf{w} \cdot \nabla \varphi \rangle + \langle \mu, \mathbf{v}^\nu \cdot \nabla \varphi \rangle + (\sigma_1 + \sigma_2) \langle \mu \eta, \varphi \rangle \\ - \nu \langle \nabla \rho^\nu, \nabla \varphi \rangle = 0. \end{aligned} \quad (7.9)$$

Here we used

$$-(\partial_t \mu, \psi) = -\sigma_1^{-1} (\partial_t \operatorname{div} \mathbf{w}, \psi) = \sigma_1^{-1} (\partial_t \mathbf{w}, \nabla \psi) \quad (7.10)$$

for a.e  $t \in [0, T]$  for all  $\psi \in C_c^\infty(\mathbb{R}^d)$  and hence by density for  $\varphi$  in place of  $\psi$ .

Because  $\mu \eta$  belongs to all  $L^p$  spaces, the equality in (7.9) holds for all  $\varphi \in L^2(0, T; \dot{H}^1 \cap L^p)$  for any  $p \in [1, \infty]$ , any element of such a space being approximable by a sequence in  $Y$ . When  $\sigma_1 + \sigma_2 = 0$  only  $\nabla \varphi$  appears in (7.9), so equality holds for all  $\varphi \in L^2(0, T; \dot{H}^1)$ . Setting  $\varphi = \sigma_1 \Phi * \mu$  we note that for  $d \geq 2$  we have  $\psi \in L^2(0, T; \dot{H}^1)$  by Theorem 4.5 and Lemma 6.3. For  $d \geq 3$  we have  $\psi \in L^\infty(0, T; L^p)$  for any fixed  $p > d/(d-1)$ . Hence (7.9) holds for  $\varphi = \sigma_1 \Phi * \mu$  for both cases covered by this theorem.

Then  $\nabla \varphi = \mathbf{w}$  and (7.9) becomes

$$\int_0^t \sigma_1^{-1} \langle \partial_t \mathbf{w}, \mathbf{w} \rangle + \langle \rho^0, |\mathbf{w}|^2 \rangle + \langle \mu, \mathbf{v}^\nu \cdot \mathbf{w} \rangle + \sigma_1(\sigma_1 + \sigma_2) \langle \mu \eta, \Phi * \mu \rangle - \nu \langle \nabla \rho^\nu, \mathbf{w} \rangle = 0. \quad (7.11)$$

Again using Lemma 1.2 of Section III.1.4 of [24] (or Lemma 3.9) we have

$$\int_0^t \langle \partial_t \mathbf{w}, \mathbf{w} \rangle = \frac{1}{2} \int_0^t \partial_t \langle \mathbf{w}, \mathbf{w} \rangle = \frac{1}{2} \|\mathbf{w}(t)\|^2 - \frac{1}{2} \|\mathbf{w}(0)\|^2 = \frac{1}{2} \|\mathbf{w}(t)\|^2,$$

and we have the estimates

$$\begin{aligned} |\langle \rho^0, |\mathbf{w}|^2 \rangle| &\leq \|\rho^0\|_{L^\infty} \|\mathbf{w}\|^2 \leq C_0(t) \|\mathbf{w}\|^2, \\ |\langle \mu, \mathbf{v}^\nu \cdot \mathbf{w} \rangle| &\leq \|\mathbf{v}^\nu\|_{L^\infty} \|\mathbf{w}\| \|\mu\| \leq C_0(t) \|\mathbf{w}\|^2 + C_0(t) \|\mu\|^2, \\ \nu |\langle \nabla \rho^\nu, \mathbf{w} \rangle| &= \nu |\langle \nabla \mu, \mathbf{w} \rangle + \langle \nabla \rho^0, \mathbf{w} \rangle| \leq \frac{\nu |\sigma_1^{-1}|}{4} \|\nabla \mu\|^2 + \frac{2\nu + |\sigma_1^{-1}|}{2|\sigma_1^{-1}|} \|\mathbf{w}\|^2 + \frac{\nu^2}{2} \|\nabla \rho^0\|^2 \\ &\leq \frac{\nu |\sigma_1^{-1}|}{4} \|\nabla \mu\|^2 + \frac{2\nu + |\sigma_1^{-1}|}{2|\sigma_1^{-1}|} \|\mathbf{w}\|^2 + C_0(t) \nu^2. \end{aligned}$$

Now consider  $\langle \mu \eta, \Phi * \mu \rangle = \sigma_1 \langle \mu \eta, \operatorname{div}(\Phi * \mathbf{w}) \rangle$  for  $d \geq 3$ . Write

$$\begin{aligned} |\langle \mu \eta, \operatorname{div}(\Phi * \mathbf{w}) \rangle| &\leq \|\mu \eta\|_{L^{\frac{2d}{d+2}}} \|\nabla \Phi * \mathbf{w}\|_{L^{\frac{2d}{d-2}}} \\ &\leq \|\mu\|_{L^2} \|\eta\|_{L^d} \|\nabla \Phi * \mathbf{w}\|_{L^{\frac{2d}{d-2}}} \leq C_0(t) \|\mu\|_{L^2} \|\mathbf{w}\|_{L^2}, \end{aligned}$$

where we applied the Hardy-Littlewood-Sobolev inequality. The term  $\sigma_1(\sigma_1 + \sigma_2) \langle \mu \eta, \Phi * \mu \rangle$  disappears entirely when  $\sigma_1 + \sigma_2 = 0$ .

Substituting these bounds into (7.11) gives for all  $\nu \leq 1$

$$\|\mathbf{w}(t)\|^2 \leq \frac{\nu}{2} \int_0^t \|\nabla \mu(s)\|^2 ds + C_0(t)t\nu + \int_0^t C_0(s) \left( \|\mathbf{w}(s)\|^2 + \|\mu(s)\|^2 \right) ds.$$

Adding this inequality to that in (7.8) gives for all  $\nu \leq 1$

$$\|\mathbf{w}(t)\|^2 + \|\mu(t)\|^2 + \frac{\nu}{2} \int_0^t \|\nabla \mu(s)\|^2 ds \leq C_0(t)t\nu + \int_0^t C_0(s) \left( \|\mathbf{w}(s)\|^2 + \|\mu(s)\|^2 \right) ds.$$

Applying Lemma 1.2 (Grönwall's lemma) we conclude that

$$\|\mathbf{w}(t)\|^2 + \|\mu(t)\|^2 + \nu \int_0^t \|\nabla \mu(s)\|^2 ds \leq C_0(t)t\nu e^{C_0(t)}$$

for all  $\nu \leq 1$ . The proof is completed by observing that  $\|\mathbf{w}(t)\|_{H^1} \leq \|\mathbf{w}(t)\| + C\|\mu(t)\|$  by Lemma 7.8.  $\square$

**Remark 7.4.** *An examination of the proof of Theorem 7.3 shows that the conclusion holds as long as the solutions satisfy: (a)  $\rho^\nu, \rho^0 \in L^\infty(0, T; L^1 \cap L^\infty(\mathbb{R}^d))$  and (b)  $\nabla \rho^0 \in L^\infty(0, T; L^2 \cap L^d(\mathbb{R}^d))$  when  $d \geq 3$  or  $\nabla \rho^0 \in L^\infty(0, T; L^2 \cap L^q(\mathbb{R}^d))$  for some  $q > 2$  when  $d = 2$ . Our assumptions on the initial data imply that these conditions hold, though they are not minimal.*

Similar considerations yield the uniqueness result for solutions to the viscous equations in Theorem 7.5. Thus in Theorem 7.3 and later in Theorem 8.1,  $\rho^\nu$  can refer to the unique solution for the given value of  $\nu$ .

**Theorem 7.5.** *Fix  $\nu > 0$  and let  $d \geq 2$ . (1) Weak solutions to  $(GAG_\nu)$  as in Definition 4.1 are unique within the class of solutions for which (a)  $\rho^\nu \in L^\infty(0, T; L^1 \cap L^\infty(\mathbb{R}^d))$  and (b)  $\nabla \rho^\nu \in L^\infty(0, T; L^2 \cap L^d(\mathbb{R}^d))$  when  $d \geq 3$  or  $\nabla \rho^\nu \in L^\infty(0, T; L^2 \cap L^q(\mathbb{R}^d))$  for some  $q > 2$  when  $d = 2$ . (2) Let  $N > 1 + d/2$  and  $k \geq \max\{2, d/2\}$  be an integer. If  $\rho_0 \in H_N^k$  then there exists a unique weak solution to  $(GAG_\nu)$  lying in  $Y_N^k$ .*

*Proof.* (1) If instead of subtracting an inviscid from a viscous solution we had subtracted two solutions having the same viscosity then the proof of Theorem 7.3 would yield the uniqueness of solutions to  $(GAG_\nu)$  for  $\nu \geq 0$  in the class of solutions having the regularity of  $\rho^0$  stated in Remark 7.4. This covers the cases  $d \geq 3$  and  $d = 2$  with  $\sigma_1 + \sigma_2 = 0$ . The case  $d = 2$  with  $\sigma_1 + \sigma_2 \neq 0$  will be treated in Theorem 8.1.

(2) If  $\rho_0 \in H_N^k$  then  $\rho^\nu \in Y_N^k \subseteq C([0, T]; H^k)$  by Theorem 4.5. Then by the Galgardo-Nirenberg-Sobolev inequality,

$$\|\nabla \rho^\nu\|_{L^{2k}} \leq C \|D^k \rho^\nu\|_{L^\infty}^{\frac{1}{k}} \|\rho^\nu\|_{L^\infty}^{1-\frac{1}{k}}.$$

This shows that  $\rho^\nu$  has enough regularity to obtain uniqueness.  $\square$

**Remark 7.6.** *Since only one of the two weak solutions in the proof of Theorem 7.5 needs to have the required additional regularity over that of a weak solution, it also follows that a weak solution having the additional initial regularity given in Theorem 7.5 is a strong solution.*

We used the following two lemmas above.

**Lemma 7.7.** *Fix  $p_0 \in (2, \infty)$ . There exists  $C = C(p_0)$  such that for any  $p \in [p_0, \infty]$*

$$\|u\|_{L^p} \leq C (\|u\| + \|\nabla u\|)$$

*for any vector field  $u$  having components in  $H^1(\mathbb{R}^2)$ .*

*Proof.* Let  $u$  be a vector field having components in  $H^1(\mathbb{R}^2)$ . By the Gagliardo-Nirenberg-Sobolev inequality we have for any  $p \in (2, \infty)$

$$\|u\|_{L^p} \leq C \|u\|_{L^2}^{\frac{2}{p}} \|\nabla u\|^{1-\frac{2}{p}}.$$

But Young's inequality in the form,

$$A^r B^{1-r} \leq rA + (1-r)B$$

with

$$r = \frac{2}{p}, \quad A = \frac{p}{2} \|u\|, \quad B = \left(\frac{2}{p}\right)^{\frac{2}{p-2}} \|\nabla u\|$$

so that  $A^r B^{1-r} = \|u\|_{L^p}^{\frac{2}{p}} \|\nabla u\|^{1-\frac{2}{p}}$  gives

$$\|u\|_{L^p} \leq C \|u\| + C \frac{p-2}{p} \left(\frac{2}{p}\right)^{\frac{2}{2-p}} \|\nabla u\| \leq C (\|u\| + \|\nabla u\|).$$

The inequality also holds for  $p = \infty$  by the continuity of  $L^p$  norms.  $\square$

**Lemma 7.8.** *Assume that  $u = \nabla \Phi * \rho \in L^2(\mathbb{R}^d)$  with  $\rho \in L^2(\mathbb{R}^d)$ . Then  $\|\nabla u\| \leq C \|\rho\|$ .*

*Proof.* We have  $\nabla u = R\rho$  where  $R = \nabla \nabla \Delta^{-1}$ . But each component  $R^{ij}$  is a Calderon-Zygmund operator, which is bounded in  $L^2$ . Hence,  $\|\nabla u\| \leq C \|\rho\|$ . (In fact, knowing that  $\nabla u \in L^2$  one can justify an integration by parts to obtain  $\|\nabla u\| = \|\nabla \nabla \Phi * \rho\| = \|\Delta \Phi * \rho\| = \|\rho\|$ , but we will not need this.)  $\square$

## 8. THE VANISHING VISCOSITY LIMIT FOR $(GAG_\nu)$ FOR VELOCITIES NOT IN $L^2$

In this section, we consider the vanishing viscosity limit  $(VV)'$  (see Section 1) in the general 2D case. Throughout this section we make the assumptions in Remark 7.1.

**Theorem 8.1.** *Assume that  $d = 2$  and  $\rho_0 \in C^{1,\alpha}(\mathbb{R}^2)$  for some  $\alpha > 0$  is compactly supported. Define  $\mu, \mathbf{w}$  as in (7.1), and let*

$$\tilde{\mu} := \mu - m(\mu)g_0, \quad \tilde{\mathbf{w}} := \mathbf{w} - \boldsymbol{\theta}^\nu, \quad \boldsymbol{\theta}^\nu := \sigma_1 m(\mu) \boldsymbol{\tau}_0. \quad (8.1)$$

(For  $\tau_0, g_0$  see Definition 6.5; for the definition of  $m$  see (6.1).) Then for all  $t \in [0, T]$ ,  $\nu \leq 1$ ,

$$\|\tilde{\mathbf{w}}(t)\|_{H^1}^2 + \|\mu(t)\|^2 + \nu \int_0^t \|\nabla \mu\|^2 \leq C_0(t) \nu t e^{C_0(t)t}. \quad (8.2)$$

Moreover, for all  $k \geq 0$ ,

$$\|\mathbf{w}(t) - \tilde{\mathbf{w}}(t)\|_{C^k} = \|\boldsymbol{\theta}^\nu(t)\|_{C^k} \leq C_k \nu^{\frac{1}{2}} t^{\frac{3}{2}} e^{C_0(t)t}. \quad (8.3)$$

*Proof.* We have  $\operatorname{div} \tilde{\mathbf{w}} = \sigma_1 \tilde{\mu}$  and  $\tilde{\mathbf{w}} \in L^2(\mathbb{R}^d)$  for all  $t \in [0, T]$  by Lemma 6.3, since  $m(\tilde{\mu}) = 0$ .

We start off the same way as in the proof of Theorem 7.3. This leads to (7.3) and all the estimates following it that led to (7.7) and (7.8). We estimate the one term  $\langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle$  differently, however. First note that

$$|\langle \mathbf{w} \cdot \nabla \rho^0, \mu \rangle| \leq |\langle \tilde{\mathbf{w}} \cdot \nabla \rho^0, \mu \rangle| + |\langle (\sigma_1 m(\mu) \boldsymbol{\tau}_0) \cdot \nabla \rho^0, \mu \rangle|. \quad (8.4)$$

Following the proof of Theorem 7.3, we have

$$|\langle \tilde{\mathbf{w}} \cdot \nabla \rho^0, \mu \rangle| \leq \|\tilde{\mathbf{w}} \cdot \nabla \rho^0\| \|\mu\| \leq \|\tilde{\mathbf{w}}\|_{L^p} \|\nabla \rho^0\|_{L^q} \|\mu\| \quad (8.5)$$

where  $1/p + 1/q = 1/2$ . Let  $p \in (2, \infty)$  and observe that  $q = 2p/(p-2) \in (2, \infty)$  as well. From Lemmas 7.7 and 7.8 we have

$$\|\tilde{\mathbf{w}}\|_{L^p} \leq C(\|\tilde{\mathbf{w}}\| + \|\tilde{\mu}\|)$$

since  $\tilde{\mathbf{w}} = \nabla\Phi * \tilde{\mu}$ . Substituting this estimate into (8.5) gives, for any fixed  $q \in (2, \infty)$ ,

$$|\langle \tilde{\mathbf{w}} \cdot \nabla\rho^0, \mu \rangle| \leq C\|\nabla\rho^0\|_{L^q}(\|\tilde{\mathbf{w}}\| + \|\tilde{\mu}\|)\|\mu\| \leq C_0(t)(\|\tilde{\mathbf{w}}\| + \|\tilde{\mu}\|)\|\mu\|.$$

Now, by the definition of  $\tilde{\mu}$  and Propostion 7.2,

$$\begin{aligned} \|\tilde{\mu}(t)\| &\leq \|\mu(t)\| + |m(\mu(t))|\|g_0\| \leq \|\mu(t)\| + \|g_0\| |\sigma_1 + \sigma_2| \int_0^t \|\mu(s)\| \|\rho^\nu(s) + \rho^0(s)\| ds \\ &\leq \|\mu(t)\| + C_0(t) \int_0^t \|\mu(s)\| ds, \end{aligned}$$

so that

$$\begin{aligned} |\langle \tilde{\mathbf{w}} \cdot \nabla\rho^0, \mu \rangle| &\leq C_0(t)\|\tilde{\mathbf{w}}\|\|\mu\| + C_0(t)\|\mu\|^2 + C_0(t)\|\mu\| \int_0^t \|\mu(s)\| ds \\ &\leq C_0(t)\|\tilde{\mathbf{w}}\|^2 + C_0(t)\|\mu\|^2 + \left( \int_0^t \|\mu(s)\| ds \right)^2 \\ &\leq C_0(t)\|\tilde{\mathbf{w}}\|^2 + C_0(t)\|\mu\|^2 + C_0(t)t \int_0^t \|\mu(s)\|^2 ds, \end{aligned} \tag{8.6}$$

where we used Jensen's inequality (or Cauchy-Schwartz) in the last step. To estimate  $|\langle (\sigma_1 m(\mu)\tau_0) \cdot \nabla\rho^0, \mu \rangle|$ , we observe that, by Proposition 7.2,

$$|m(\mu(t))| \leq |\sigma_1 + \sigma_2| \int_0^t (\|\rho^\nu(s)\| + \|\rho^0(s)\|) \|\mu(s)\| ds \leq C_0(t) \int_0^t \|\mu(s)\| ds \tag{8.7}$$

so that

$$\begin{aligned} |\langle (\sigma_1 m(\mu)\tau_0) \cdot \nabla\rho^0, \mu \rangle| &\leq |\sigma_1| \|\nabla\rho^0\| \|\tau_0\|_{L^\infty} |m(\mu)| \|\mu\| \\ &\leq \frac{1}{2}\sigma_1^2 \|\nabla\rho^0\|^2 \|\tau_0\|_{L^\infty}^2 m(\mu)^2 + \frac{1}{2} \|\mu\|^2 \leq C_0(t) \left( \int_0^t \|\mu(s)\| ds \right)^2 + \frac{1}{2} \|\mu\|^2 \\ &\leq C_0(t)t \int_0^t \|\mu(s)\|^2 ds + \frac{1}{2} \|\mu\|^2, \end{aligned} \tag{8.8}$$

where we again used Jensen's inequality (or Cauchy-Schwartz) as well as Proposition 7.2. Hence, applying (8.6) and (8.8) to (8.4) we see that

$$|\langle \mathbf{w} \cdot \nabla\rho^0, \mu \rangle| \leq C_0(t) \|\tilde{\mathbf{w}}\|^2 + C_0(t) \|\mu\|^2 + C_0(t)t \int_0^t \|\mu(s)\|^2 ds.$$

Now,  $|\langle \mathbf{w} \cdot \nabla\rho^0, \mu \rangle|$  (as part of  $\langle \rho^0 \mathbf{w}, \nabla\mu \rangle$ ) appears in (7.3) integrated over time. Hence we need to remove the double time integral that would appear if we simply integrated the above estimate over time. We can do so by noting that

$$\int_0^t C_0(y)y \int_0^y \|\mu(s)\|^2 ds dy \leq C_0(t)t \int_0^t \int_0^t \|\mu(s)\|^2 ds dy \leq C_0(t)t^2 \int_0^t \|\mu(s)\|^2 ds. \tag{8.9}$$

Thus we see that in place of (7.8) we have

$$\|\mu(t)\|^2 + \nu \int_0^t \|\nabla\mu(s)\|^2 ds \leq C_0(t)t\nu + \int_0^t C_0(s) \left( \|\tilde{\mathbf{w}}(s)\|^2 + \|\mu(s)\|^2 \right) ds. \tag{8.10}$$

To bound  $\|\tilde{\mathbf{w}}\|$ , we derive the equivalent of (7.9) for  $\tilde{\mathbf{w}}$  for an arbitrary  $\varphi \in L^2(0, T; H^1)$ . The only change we need make is in (7.10), which, using  $\operatorname{div} \mathbf{w} = \operatorname{div} \tilde{\mathbf{w}} + \sigma_1 m(\mu) g_0$ , becomes

$$\begin{aligned} -(\partial_t \mu, \varphi) &= -\sigma_1^{-1} (\partial_t \operatorname{div} \mathbf{w}, \varphi) = -\sigma_1^{-1} (\partial_t \operatorname{div} \tilde{\mathbf{w}}, \varphi) - m'(\mu) (g_0, \varphi) \\ &= \sigma_1^{-1} \langle \partial_t \tilde{\mathbf{w}}, \nabla \varphi \rangle - (\sigma_1 + \sigma_2) m(\mu \eta) \langle g_0, \varphi \rangle \end{aligned}$$

with  $\eta$  as in (7.1) and using Proposition 7.2 to give  $m'(\nu) = (\sigma_1 + \sigma_2) \langle \mu, \eta \rangle = (\sigma_1 + \sigma_2) m(\mu \eta)$ . Thus, (7.9) becomes

$$\begin{aligned} \int_0^t \sigma_1^{-1} \langle \partial_t \tilde{\mathbf{w}}, \nabla \varphi \rangle - (\sigma_1 + \sigma_2) m(\mu \eta) \langle g_0, \varphi \rangle + \langle \rho^0, \mathbf{w} \cdot \nabla \varphi \rangle + \langle \mu, \mathbf{v}^\nu \cdot \nabla \varphi \rangle \\ + (\sigma_1 + \sigma_2) \langle \mu \eta, \varphi \rangle - \nu \langle \nabla \rho^\nu, \nabla \varphi \rangle = 0. \end{aligned} \quad (8.11)$$

Now set  $\varphi = \sigma_1 \Phi * \tilde{\mu} = \sigma_1 \Phi * (\mu - m(\mu) g_0)$  and fix  $p_0 \in (2, \infty]$ . Then  $\nabla \varphi \in L^\infty(0, T; L^2)$  with  $\varphi \in L^\infty(0, T; L^{p_0})$  by Lemma 6.3. Also,  $\mu \eta \in L^\infty(0, T; L^{q_0})$ , where  $q_0$  is Hölder conjugate to  $p_0$  because both  $\mu$  and  $\eta$  lie in  $L^\infty(0, T; L^p)$  for all  $p \in [1, \infty]$ . Hence, (8.11) holds for  $\varphi$  by a simple density argument. Using  $\nabla \varphi = \tilde{\mathbf{w}}$  and  $\langle \mu \eta, \varphi \rangle - m(\mu \eta) \langle g_0, \varphi \rangle = \langle \mu \eta - m(\mu \eta) g_0, \varphi \rangle$ , (8.11) becomes

$$\begin{aligned} \int_0^t \sigma_1^{-1} \langle \partial_t \tilde{\mathbf{w}}, \tilde{\mathbf{w}} \rangle + \langle \rho^0, \mathbf{w} \cdot \tilde{\mathbf{w}} \rangle + \langle \mu, \mathbf{v}^\nu \cdot \tilde{\mathbf{w}} \rangle + \sigma_1 (\sigma_1 + \sigma_2) \langle \mu \eta - m(\mu \eta) g_0, \Phi * \tilde{\mu} \rangle \\ - \nu \langle \nabla \rho^\nu, \tilde{\mathbf{w}} \rangle = 0. \end{aligned} \quad (8.12)$$

Now,

$$\begin{aligned} |\langle \rho^0, \tilde{\mathbf{w}} \cdot \mathbf{w} \rangle| &\leq \|\rho^0\|_{L^\infty} \|\tilde{\mathbf{w}}\|^2 + |\sigma_1| |m(\mu)| |\langle \tau_0 \cdot \tilde{\mathbf{w}}, \rho_0 \rangle| \leq C_0(t) \|\tilde{\mathbf{w}}\|^2 + C_0(t) |m(\mu)|^2, \\ |\langle \mu, \mathbf{v}^\nu \cdot \tilde{\mathbf{w}} \rangle| &\leq \|\mathbf{v}^\nu\|_{L^\infty} \|\tilde{\mathbf{w}}\| \|\mu\| \leq C_0(t) \|\tilde{\mathbf{w}}\|^2 + C_0(t) \|\mu\|^2, \\ \nu |\langle \nabla \rho^\nu, \tilde{\mathbf{w}} \rangle| &= \nu |\langle \nabla \mu, \tilde{\mathbf{w}} \rangle + \langle \nabla \rho^0, \tilde{\mathbf{w}} \rangle| \leq \frac{\nu |\sigma_1^{-1}|}{4} \|\nabla \mu\|^2 + \frac{2\nu + |\sigma_1^{-1}|}{2|\sigma_1^{-1}|} \|\tilde{\mathbf{w}}\|^2 + \frac{\nu^2}{2} \|\nabla \rho^0\|^2 \\ &\leq \frac{\nu |\sigma_1^{-1}|}{4} \|\nabla \mu\|^2 + \frac{2\nu + |\sigma_1^{-1}|}{2|\sigma_1^{-1}|} \|\tilde{\mathbf{w}}\|^2 + C_0(t) \nu^2. \end{aligned}$$

Substituting these bounds into (8.12), using (8.7) with Jensen's inequality, and applying Lemma 8.2 we obtain for all  $\nu \leq 1$

$$\|\tilde{\mathbf{w}}(t)\|^2 \leq \int_0^t \frac{\nu}{2} \|\nabla \mu(s)\|^2 ds + C_0(t) t \nu + \int_0^t C_0(s) \left( \|\tilde{\mathbf{w}}(s)\|^2 + \|\mu(s)\|^2 \right) ds.$$

Adding this inequality to that in (8.10) gives for all  $\nu \leq 1$

$$\|\tilde{\mathbf{w}}(t)\|^2 + \|\mu(t)\|^2 + \frac{\nu}{2} \int_0^t \|\nabla \mu(s)\|^2 ds \leq C_0(t) t \nu + \int_0^t C_0(s) \left( \|\tilde{\mathbf{w}}(s)\|^2 + \|\mu(s)\|^2 \right) ds.$$

Applying Lemma 1.2 (Grönwall's lemma), we conclude that

$$\|\tilde{\mathbf{w}}(t)\|^2 + \|\mu(t)\|^2 + \nu \int_0^t \|\nabla \mu(s)\|^2 ds \leq C_0(t) t \nu e^{C_0(t)} \quad (8.13)$$

for all  $\nu \leq 1$ . Also, by Lemma 6.3 and Proposition 7.2,

$$\begin{aligned}
\|\tilde{\mathbf{w}}(t)\|_{H^1} &\leq \|\tilde{\mathbf{w}}(t)\| + C \|\tilde{\mu}(t)\| \leq \|\tilde{\mathbf{w}}(t)\| + \|\mu(t)\| + |m(\mu(t))| \|g_0\| \\
&\leq \|\tilde{\mathbf{w}}(t)\| + \|\mu(t)\| + \|g_0\| |\sigma_1 + \sigma_2| \int_0^t \|\mu(s)\| \|\eta(s)\| ds \\
&\leq \|\tilde{\mathbf{w}}(t)\| + \|\mu(t)\| + C_0(t) \int_0^t \|\mu(s)\| ds \\
&\leq \|\tilde{\mathbf{w}}(t)\| + C_0(t) (t\nu)^{\frac{1}{2}} e^{C_0(t)} + C_0(t) \int_0^t C_0(s) s^{\frac{1}{2}} \nu^{\frac{1}{2}} e^{C_0(s)} ds \\
&\leq \|\tilde{\mathbf{w}}(t)\| + C_0(t) t^{\frac{1}{2}} e^{C_0(t)} \nu^{\frac{1}{2}}.
\end{aligned}$$

In the second-to-last inequality, we used (8.13). Combining this bound with (8.13) completes the proof of (8.2).

To prove (8.3), we simply observe that

$$\begin{aligned}
\|\mathbf{w}(t) - \tilde{\mathbf{w}}(t)\|_{L^\infty} &= \|\sigma_1 m(\mu) \boldsymbol{\tau}_0\|_{L^\infty} \leq |\sigma_1| |m(\mu)| \|\boldsymbol{\tau}_0\|_{L^\infty} \leq C |m(\mu)| \\
&\leq C_0(t) \int_0^t \|\mu(s)\| \|\eta(s)\| ds \leq C \nu^{\frac{1}{2}} t^{\frac{3}{2}} e^{C_0(t)t},
\end{aligned}$$

where we used Proposition 7.2 and (8.2). A similar bound holds for all spatial derivatives of  $\mathbf{w}(t) - \tilde{\mathbf{w}}(t)$ , yielding (8.3).  $\square$

We used the following lemma in the proof of Theorem 8.1, above.

**Lemma 8.2.** *Define  $\mu$ ,  $\eta$ ,  $\tilde{\mu}$ ,  $\tilde{\mathbf{w}}$  as in (7.1) and (8.1). When  $d = 2$ , we have,*

$$|\langle \mu\eta - m(\mu\eta)g_0, \Phi * \tilde{\mu} \rangle| \leq C_0(t) \|\tilde{\mathbf{w}}\| \|\mu\|.$$

*Proof.* First observe that  $\gamma := \mu\eta - m(\mu\eta)g_0$  lies in  $L_N^2(\mathbb{R}^2)$  because  $\mu$  and  $\eta$  both lie in  $L_N^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ . Also observe that  $\gamma$  has total mass zero. Thus  $\nabla\Phi * \gamma \in L^2(\mathbb{R}^2)$  by Lemma 6.3. Similarly,  $\tilde{\mu} \in L_N^2$  with total mass zero and  $\nabla\Phi * \tilde{\mu} = \tilde{\mathbf{w}} \in L^2(\mathbb{R}^2)$ . This allows us to integrate by parts, using  $\gamma = \text{div}(\nabla\Phi * \gamma)$ , to conclude that

$$|\langle \gamma, \Phi * \tilde{\mu} \rangle| = |\langle \nabla\Phi * \gamma, \tilde{\mathbf{w}} \rangle| \leq \|\nabla\Phi * \gamma\| \|\tilde{\mathbf{w}}\|.$$

By Lemma 6.4, with  $a$  as in Definition 1.1,

$$\begin{aligned}
\|\nabla\Phi * \gamma\|^2 &= -\langle \Phi * \gamma, \gamma \rangle = -\langle (a\Phi) * \gamma, \gamma \rangle - \langle ((1-a)\Phi) * \gamma, \gamma \rangle \\
&\leq \|a\Phi\|_{L^1} \|\gamma\|^2 + |\langle ((1-a)\Phi) * \gamma, \gamma \rangle| \\
&\leq C_0(t) \|\mu\|^2 + |\langle ((1-a)\Phi) * \gamma, \gamma \rangle|,
\end{aligned} \tag{8.14}$$

since

$$\begin{aligned}
\|\gamma\|^2 &= \|\mu\eta - m(\mu\eta)g_0\|^2 \leq (\|\mu\| \|\eta\|_{L^\infty} + |m(\mu\eta)| \|g_0\|)^2 \\
&\leq (\|\mu\| \|\eta\|_{L^\infty} + \|\mu\| \|\eta\| \|g_0\|)^2 \leq C_0(t) \|\mu\|^2.
\end{aligned}$$

It remains to estimate  $|\langle ((1-a)\Phi) * \gamma, \gamma \rangle|$ . Define  $g(x) := 1 + |x|^\varepsilon$  and write,

$$\begin{aligned}
|\langle ((1-a)\Phi) * \gamma, \gamma \rangle| &= |\langle (1/g) [((1-a)\Phi) * \gamma], g\gamma \rangle| \\
&\leq \|(1/g) [((1-a)\Phi) * \gamma]\|_{L^\infty} \|g\gamma\|_{L^1}.
\end{aligned}$$

Since  $\rho_0(t)$ ,  $\rho^\nu(t)$  are in  $L_N^2(\mathbb{R}^2)$ , Corollary 2.5 allows us to conclude that  $\|g\eta\| \leq C_0(t)$ . Therefore,

$$\begin{aligned} \|g\gamma\|_{L^1} &= \|\mu g\eta - m(\mu\eta)g g_0\|_{L^1} \leq \|\mu\| \|g\eta\| + |m(\mu\eta)| \|g g_0\|_{L^1} \\ &\leq C_0(t) \|\mu\| + \|\mu\| \|\eta\| \|g g_0\|_{L^1} \leq C_0(t) \|\mu\|, \end{aligned}$$

and we conclude that

$$|\langle ((1-a)\Phi) * \gamma, \gamma \rangle| \leq C_0(t) \|(1/g) [((1-a)\Phi) * \gamma]\|_{L^\infty} \|\mu\|. \quad (8.15)$$

We need to extract another factor of  $\|\mu\|$  from  $\|(1/g) [((1-a)\Phi) * \gamma]\|_{L^\infty}$ . We have,

$$\begin{aligned} \left| \frac{1}{g(x)} [((1-a)\Phi) * \gamma](x) \right| &= \left| \frac{1}{2\pi g(x)} \int_{\mathbb{R}^2} (1-a(x-y)) \log|x-y| \gamma(y) dy \right| \\ &\leq \frac{1}{2\pi g(x)} \int_{\mathbb{R}^2} \frac{(1-a(x-y)) \log|x-y|}{|x-y|^\varepsilon} |x-y|^\varepsilon |\gamma(y)| dy \\ &\leq \frac{1}{2\pi g(x)} \int_{\mathbb{R}^2} \frac{(1-a(x-y)) \log|x-y|}{|x-y|^\varepsilon} (|x|+|y|)^\varepsilon |\gamma(y)| dy \\ &\leq \frac{1}{2\pi g(x)} \int_{\mathbb{R}^2} \frac{(1-a(x-y)) \log|x-y|}{|x-y|^\varepsilon} |x|^\varepsilon |\gamma(y)| dy \\ &\quad + \frac{1}{2\pi g(x)} \int_{\mathbb{R}^2} \frac{(1-a(x-y)) \log|x-y|}{|x-y|^\varepsilon} |y|^\varepsilon |\gamma(y)| dy. \end{aligned}$$

So

$$\begin{aligned} \left\| \frac{1}{g} [((1-a)\Phi) * \gamma] \right\|_{L^\infty} &\leq \left\| \frac{|x|^\varepsilon}{2\pi g(x)} \int_{\mathbb{R}^2} \frac{(1-a(x-y)) \log|x-y|}{|x-y|^\varepsilon} |\gamma(y)| dy \right\|_{L_x^\infty} \\ &\quad + \left\| \frac{1}{2\pi g(x)} \int_{\mathbb{R}^2} \frac{(1-a(x-y)) \log|x-y|}{|x-y|^\varepsilon} |y|^\varepsilon |\gamma(y)| dy \right\|_{L_x^\infty} \\ &\leq C \int_{\mathbb{R}^2} |\gamma(y)| dy + C \int_{\mathbb{R}^2} |y|^\varepsilon |\gamma(y)| dy = C \|\gamma\|_{L^1} + C \| |x|^\varepsilon \gamma(x) \|_{L_x^1}. \end{aligned} \quad (8.16)$$

But,

$$\begin{aligned} \|\gamma\|_{L^1} &= \|\mu\eta - m(\mu\eta)g_0\|_{L^1} \leq \|\mu\| \|\eta\| + |m(\mu\eta)| \|g_0\|_{L^1} \\ &\leq C_0(t) \|\mu\| + \|\mu\| \|\eta\| \|g_0\|_{L^1} \leq C_0(t) \|\mu\| \end{aligned}$$

and

$$\begin{aligned} \| |x|^\varepsilon \gamma(x) \|_{L_x^1} &= \|\mu(x)(|x|^\varepsilon \eta(x)) - m(\mu\eta)(|x|^\varepsilon g_0(x))\|_{L_x^1} \\ &\leq \|\mu\| \| |x|^\varepsilon \eta(x) \|_{L_x^2} + |m(\mu\eta)| \| |x|^\varepsilon g_0(x) \|_{L_x^1} \\ &\leq C_0(t) \|\mu\| + \|\mu\| \|\eta\| \| |x|^\varepsilon g_0(x) \|_{L_x^1} \leq C_0(t) \|\mu\|. \end{aligned}$$

Substituting this estimate into (8.16), the resulting estimate into (8.15), and finally that estimate into (8.14), yields the desired bound.  $\square$

## 9. THE VANISHING VISCOSITY LIMIT IN THE $L^\infty$ -NORM

In this section, we use the results from Section 7 and Section 8 to prove in Theorem 9.1 that the vanishing viscosity limit (see (VV) in Section 1) holds in the  $L^\infty$ -norm of the density. Throughout this section we make the assumptions in Remark 7.1. Our main result in this section is Theorem 9.1:

**Theorem 9.1.** *Assume  $\rho_0 \in C^{1,\alpha}(\mathbb{R}^d) \cap L^2_N(\mathbb{R}^d)$  for some  $\alpha > 1$ ,  $N > 1 + d/2$ ,  $d \geq 2$  is compactly supported. Define  $\mu$  as in (7.1), and let  $T$  be as in Theorem 4.5. Then for  $t \in [0, T]$ ,  $\nu \leq 1$ , and  $\beta \in (0, 1)$ ,*

$$\|\mu(t)\|_{L^\infty} \leq C_0(t)(\nu t)^{\frac{2\beta}{2\beta+d}}.$$

Theorem 9.1 follows easily from interpolation once we establish a modulus of continuity on  $\rho^\nu$  that applies for all sufficiently small  $\nu$ . Working as we are in Hölder spaces, it is natural to obtain a bound on  $\rho^\nu$  in a Hölder space norm *uniformly* in  $\nu$ . We do this in Theorem 9.2, adapting the approach Hmidi and Keraani took in [12, 13] for the Navier-Stokes equations.

**Theorem 9.2.** *Assume  $\rho_0 \in C^\beta(\mathbb{R}^d)$  with  $\beta < 1$  and  $d \geq 2$  is compactly supported and let  $T$  be as in Theorem 4.5. Then  $\rho^\nu \in L^\infty(0, T; C^\beta(\mathbb{R}^d))$  and*

$$\|\rho^\nu(t)\|_{C^\beta} \leq C_0(T)e^{e^{C_0(T)}} \quad (9.1)$$

for all  $t \in [0, T]$ , where  $C_0(T)$  depends on  $\|\rho_0\|_{C^\beta}$  and  $\beta$ .

Before proving Theorem 9.2 and then Theorem 9.1 we must make a few definitions, most notably of the Littlewood-Paley operators.

Let  $\mathcal{S}(\mathbb{R}^d)$  denote the Schwartz space on  $\mathbb{R}^d$  and let  $\mathcal{S}'(\mathbb{R}^d)$  denote the space of all tempered distributions on  $\mathbb{R}^d$ . It is classical that there exists two functions  $\chi, \phi \in \mathcal{S}(\mathbb{R}^d)$  with  $\text{supp } \hat{\chi} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq \frac{5}{6}\}$  and  $\text{supp } \hat{\phi} \subset \{\xi \in \mathbb{R}^d : \frac{3}{5} \leq |\xi| \leq \frac{5}{3}\}$ , such that, if for every  $j \geq 0$  we set  $\phi_j(x) = 2^{jd}\phi(2^jx)$ , then

$$\hat{\chi} + \sum_{j \geq 0} \hat{\phi}_j = \hat{\chi} + \sum_{j \geq 0} \hat{\phi}(2^{-j}\cdot) \equiv 1.$$

For  $f \in \mathcal{S}'(\mathbb{R}^d)$  and  $j \geq -1$ , define the Littlewood-Paley operators  $\Delta_j$  by

$$\Delta_j f = \begin{cases} \chi * f, & j = -1 \\ \phi_j * f, & j \geq 0. \end{cases}$$

We will make use of the following classical lemma of Bernstein. A proof of the lemma can be found in Chapter 2 of [6]. Below,  $C_{a,b}(0)$  denotes the annulus with inner radius  $a$  and outer radius  $b$ .

**Lemma 9.3** (Bernstein's Lemma). *Let  $r_1$  and  $r_2$  satisfy  $0 < r_1 < r_2 < \infty$ , and let  $p$  and  $q$  satisfy  $1 \leq p \leq q \leq \infty$ . There exists a positive constant  $C$  such that for every integer  $k$ , if  $u$  belongs to  $L^p(\mathbb{R}^d)$ , and  $\text{supp } \hat{u} \subset B_{r_1\lambda}(0)$ , then*

$$\sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^k \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}. \quad (9.2)$$

Furthermore, if  $\text{supp } \hat{u} \subset C_{r_1\lambda, r_2\lambda}(0)$ , then

$$C^{-k} \lambda^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leq C^k \lambda^k \|u\|_{L^p}. \quad (9.3)$$

The following Littlewood-Paley definition of Hölder spaces is equivalent to the classical definition of Hölder spaces when  $\alpha$  is a positive non-integer (see Chapter 2 of [6]).

**Definition 9.4.** *For  $\alpha \in \mathbb{R}$ , the space  $C_*^\alpha$  is the set of functions  $f$  such that*

$$\sup_{j \geq -1} 2^{j\alpha} \|\Delta_j f\|_{L^\infty} < \infty.$$

We set

$$\|f\|_{C_*^\alpha} = \sup_{j \geq -1} 2^{j\alpha} \|\Delta_j f\|_{L^\infty}.$$

When  $\alpha$  is a positive non-integer, we will often write  $C^\alpha$  in place of  $C_*^\alpha$ , in view of the equivalence between the two spaces.

*Proof of Theorem 9.2.* This follows from Theorem 5.7 for  $\nu = 0$ . So assume that  $\nu > 0$ . That the solution  $\rho^\nu$  to  $(GAG_\nu)$  belongs in  $L^\infty(0, T; C^\beta(\mathbb{R}^d))$  for all  $\beta > 0$  follows from standard arguments; we show only the uniform control in viscosity of the  $C^\beta$ -norm when  $\beta < 1$ .

This theorem is essentially proved in [12, 13] for divergence-free vector fields  $\mathbf{v}^\nu$  in the more general setting of Besov spaces. We follow the proof from [13] below, with a slight modification to account for the assumption that  $\mathbf{v}^\nu$  is not divergence-free in our setting. Our modification relies on a commutator estimate established in Chapter 4 of [7].

As in the proof in [13], we apply the Littlewood-Paley operator  $\Delta_j$  to  $(GAG_\nu)$ , which gives

$$\partial_t \Delta_j \rho^\nu + \mathbf{v}^\nu \cdot \nabla \Delta_j \rho^\nu - \nu \Delta \Delta_j \rho^\nu = -[\Delta_j, \mathbf{v}^\nu \cdot \nabla] \rho^\nu + \sigma_2 \Delta_j (\rho^\nu)^2.$$

We can apply the maximum principle (see, for example, Lemma 5 of [12]) to write

$$\|\Delta_j \rho^\nu(t)\|_{L^\infty} \leq \|\Delta_j \rho_0\|_{L^\infty} + C \int_0^t (\|[\Delta_j, \mathbf{v}^\nu \cdot \nabla] \rho^\nu(s)\|_{L^\infty} + \|\Delta_j (\rho^\nu(s))^2\|_{L^\infty}) ds.$$

Multiplying through by  $2^{j\beta}$  and taking the supremum over  $j$  gives

$$\begin{aligned} \|\rho^\nu(t)\|_{C^\beta} &\leq \|\rho_0\|_{C^\beta} + C \int_0^t \left( \sup_j 2^{j\beta} \|[\Delta_j, \mathbf{v}^\nu \cdot \nabla] \rho^\nu(s)\|_{L^\infty} + \|(\rho^\nu(s))^2\|_{C^\beta} \right) ds \\ &\leq \|\rho_0\|_{C^\beta} + C \int_0^t \left( \sup_j 2^{j\beta} \|[\Delta_j, \mathbf{v}^\nu \cdot \nabla] \rho^\nu(s)\|_{L^\infty} + \|\rho^\nu(s)\|_{L^\infty} \|\rho^\nu(s)\|_{C^\beta} \right) ds, \end{aligned} \quad (9.4)$$

where we used the estimate  $\|(\rho^\nu)^2\|_{C^\beta} \leq C \|\rho^\nu\|_{L^\infty} \|\rho^\nu\|_{C^\beta}$  to obtain the last inequality. To bound the commutator on the right hand side in (9.4), we apply Lemma 9.6 below for the case  $r = \beta \in (0, 1)$ , which gives,

$$\|[\Delta_j, \mathbf{v}^\nu \cdot \nabla] \rho^\nu(s)\|_{L^\infty} \leq C 2^{-j\beta} \|\nabla \mathbf{v}^\nu(s)\|_{L^\infty} \|\rho^\nu(s)\|_{C^\beta}. \quad (9.5)$$

Substituting (9.5) into (9.4) and applying Lemma 1.2 (Grönwall's lemma) gives

$$\|\rho^\nu\|_{L^\infty(0, T; C^\beta)} \leq C e^{CV(t)} \|\rho_0\|_{C^\beta}, \quad (9.6)$$

where

$$V(t) = \int_0^t (\|\nabla \mathbf{v}^\nu(s)\|_{L^\infty} + \|\rho^\nu(s)\|_{L^\infty}) ds.$$

To complete the proof of Theorem 9.2, we apply Proposition 2.3.5 of [7] and write

$$\|\nabla \mathbf{v}^\nu(t)\|_{L^\infty} \leq \frac{C}{\beta} \|\nabla \mathbf{v}^\nu(t)\|_{C_*^0} \log \left( e + \frac{\|\nabla \mathbf{v}^\nu(t)\|_{C^\beta}}{\|\nabla \mathbf{v}^\nu(t)\|_{C_*^0}} \right).$$

Since  $x \mapsto x \log(e + \frac{C}{x})$  is increasing in  $x$  when  $C > 0$ , it follows from Lemma 9.5, the embedding  $L^\infty \hookrightarrow C_*^0$ , and the equivalence between  $C^\beta$  and  $C_*^\beta$  when  $\beta \in (0, 1)$  that

$$\begin{aligned} \|\nabla \mathbf{v}^\nu(t)\|_{L^\infty} &\leq \frac{C}{\beta} \|\rho^\nu\|_{L^1 \cap L^\infty} \log \left( e + \frac{\|\nabla \mathbf{v}^\nu(t)\|_{C^\beta}}{\|\rho^\nu\|_{L^1 \cap L^\infty}} \right) \leq \frac{C}{\beta} \|\rho^\nu\|_{L^1 \cap L^\infty} \log \left( e + \frac{\|\rho^\nu(t)\|_{L^1 \cap C^\beta}}{\|\rho^\nu\|_{L^1 \cap L^\infty}} \right) \\ &\leq \frac{C}{\beta} C_0(T) \log \left( e + \frac{\|\rho^\nu(t)\|_{L^1 \cap C^\beta}}{C_0(T)} \right), \end{aligned}$$

where we used Theorem 4.5 in the third inequality above. An application of 9.6 gives

$$\begin{aligned} \|\nabla \mathbf{v}^\nu(t)\|_{L^\infty} &\leq C_0(T) \log \left( e + Ce^{C_V(t)} \|\rho_0\|_{C^\beta} \right) \\ &\leq C_0(T) \log \left( e^{C_V(t)} (e + \|\rho_0\|_{C^\beta}) \right) = C_0(T) (V(t) + \log(e + \|\rho_0\|_{C^\beta})) \\ &= C_0(T) \log(e + \|\rho_0\|_{C^\beta}) + C_0(T) \int_0^t (\|\nabla \mathbf{v}^\nu(s)\|_{L^\infty} + \|\rho^\nu(s)\|_{L^\infty}) ds. \end{aligned}$$

Lemma 1.2 (Grönwall's lemma) and Theorem 4.5 imply that

$$\|\nabla \mathbf{v}^\nu(t)\|_{L^\infty} \leq C_0(T) e^{C_0(T)}.$$

Substituting this estimate into (9.6) gives (9.1).  $\square$

*Proof of Theorem 9.1.* Fix  $t < T$ ,  $p > d$ , and  $\beta \in (0, 1)$ . For fixed  $N \geq 0$  (to be chosen later), we use Bernstein's Lemma and the definition of the Holder space  $C^\beta(\mathbb{R}^d)$  as given in Definition 9.4 to write

$$\begin{aligned} \|(\rho^\nu - \rho^0)(t)\|_{L^\infty} &\leq \sum_{j=-1}^N \|\Delta_j(\rho^\nu - \rho^0)(t)\|_{L^\infty} + \sum_{j=N+1}^{\infty} \|\Delta_j(\rho^\nu - \rho^0)(t)\|_{L^\infty} \\ &\leq C \sum_{j=-1}^N 2^{j\frac{d}{2}} \|\Delta_j(\rho^\nu - \rho^0)(t)\|_{L^2} + \sum_{j=N+1}^{\infty} 2^{-j\beta} 2^{j\beta} \|\Delta_j(\rho^\nu - \rho^0)(t)\|_{L^\infty} \\ &\leq C \sum_{j=-1}^N 2^{j\frac{d}{2}} \|\Delta_j(\rho^\nu - \rho^0)(t)\|_{L^2} + C \sum_{j=N+1}^{\infty} 2^{-j\beta} (\|\rho^\nu(t)\|_{C^\beta} + \|\rho^0(t)\|_{C^\beta}) \\ &\leq C 2^{N\frac{d}{2}} \|(\rho^\nu - \rho^0)(t)\|_{L^2} + C_0(t) 2^{-N\beta}, \end{aligned} \tag{9.7}$$

where we applied Theorems 5.7 and 9.2 above to get the last inequality. By Theorems 7.3 and 8.1, for  $\nu \leq 1$ ,

$$\|(\rho^\nu - \rho^0)(t)\|_{L^2} \leq C_0(t) t \nu e^{C_0(t)}.$$

Substituting this estimate into (9.7) gives

$$\|(\rho^\nu - \rho^0)(t)\|_{L^\infty} \leq C_0(t) t e^{C_0(t)} \nu 2^{N\frac{d}{2}} + C_0(t) 2^{-N\beta}.$$

Now set  $N = -\frac{2}{2\beta+d} \log_2(\nu t)$ . We conclude that

$$\|(\rho^\nu - \rho^0)(t)\|_{L^\infty} \leq C_0(t) t e^{C_0(t)} (\nu t)^{1-\frac{d}{2\beta+d}} + C_0(t) (\nu t)^{\frac{2\beta}{2\alpha+d}} \leq C_0(t) e^{C_0(t)} (\nu t)^{\frac{2\beta}{2\beta+d}}.$$

This completes the proof of Theorem 9.1.  $\square$

Above, we used the following lemmas.

**Lemma 9.5.** For all  $r \in \mathbb{R}$ ,

$$\|\nabla \nabla \Phi * \rho\|_{C_*^r} \leq C(\|\rho\|_{L^1} + \|\rho\|_{C_*^r}).$$

*Proof.* Let  $\mathbf{v} = \nabla \Phi * \rho$ . Then using the definition of  $C_*^r$  as given in Definition 9.4, we have

$$\begin{aligned} \|\nabla \mathbf{v}\|_{C_*^r} &= \sup_{q \geq -1} 2^{qr} \|\Delta_q \nabla \mathbf{v}\|_{L^\infty} \leq 2^{-r} \|\Delta_{-1} \nabla \mathbf{v}\|_{L^\infty} + \sup_{q \geq 0} 2^{qr} \|\Delta_q \nabla \mathbf{v}\|_{L^\infty} \\ &\leq C \|\Delta_{-1} \rho\|_{L^1} + C \sup_{q \geq 0} 2^{qr} \|\Delta_q \rho\|_{L^\infty} \leq C \|\rho\|_{L^1} + C \|\rho\|_{C_*^r}. \end{aligned}$$

To obtain the second inequality above, we argued as in (3.6) of [25] when estimating the low frequency term and applied a classical lemma to bound the high frequencies (see, for example, Lemma 4.2 of [10]). This completes the proof.  $\square$

**Lemma 9.6.** *Assume  $\mathbf{u}$  is a vector field belonging to  $L^\infty(0, T; C^\alpha(\mathbb{R}^d))$  for every  $\alpha > 0$ . Let  $r > 0$ , and assume  $f$  belongs to  $L^\infty(0, T; C^r(\mathbb{R}^d))$ . For any function  $W(t)$  satisfying*

$$W(t) \geq \max \left\{ \|\nabla \mathbf{u}(t)\|_{L^\infty}, \frac{\|\nabla \mathbf{u}(t)\|_{C^{r-1}}}{r-1} \right\},$$

the following estimate holds:

$$\|[\mathbf{u} \cdot \nabla, \Delta_j]f\|_{L^\infty} \leq C(r)2^{-jr}W(t)\|f(t)\|_{C^r}. \quad (9.8)$$

*Proof.* The estimate (9.8) corresponds to (4.7) in [7], and is established in [7] as part of the proof of Lemma 4.1.1. We refer the reader to [7] for details.  $\square$

## 10. CONCLUDING REMARKS

It is possible to obtain a velocity formulation of  $(GAG_\nu)$ , in analogy with the Navier-Stokes and Euler equations. For any  $\nu \geq 0$ , we can write this in the form

$$\begin{cases} \partial_t \mathbf{v}^\nu + \mathbf{v}^\nu \cdot \nabla \mathbf{v}^\nu + \nabla q^\nu = \nu \Delta \mathbf{v}^\nu, \\ \operatorname{curl} \mathbf{v}^\nu = 0, \\ \mathbf{v}^\nu(0) = \mathbf{v}_0, \end{cases}$$

where the ‘‘pressure’’  $q^\nu$  satisfies  $\Delta q^\nu = \sigma_1(\rho^\nu)^2 - \nabla \mathbf{v}^\nu \cdot (\nabla \mathbf{v}^\nu)^T$ . This velocity formulation can be used to obtain the bounds on  $\|\mathbf{w}(t)\|$  in Section 7 and  $\|\tilde{\mathbf{w}}(t)\|$  in Section 8. Because  $\operatorname{div} \mathbf{w} \neq 0$ , however, the pressure does not disappear in these bounds. This requires a great deal of effort to properly bound the pressure so we took the shorter approach in Sections 7 and 8 leaving the elaboration of the velocity formulation to future work.

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## REFERENCES

- [1] David M. Ambrose, James P. Kelliher, Milton C. Lopes Filho, and Helena J. Nussenzveig Lopes. Serfati solutions to the 2D Euler equations on exterior domains. *Journal of Differential Equations*, 259(9):4509–4560, 2015. [30](#), [32](#)
- [2] A. Bertozzi, J. Garnett, T. Laurent, and J. Verdera. The regularity of the boundary of a multidimensional aggregation patch. *arXiv:1507.07831v1*, 2015. [2](#)
- [3] Andrea L. Bertozzi, Thomas Laurent, and Flavien Léger. Aggregation and spreading via the Newtonian potential: the dynamics of patch solutions. *Math. Models Methods Appl. Sci.*, 22(suppl. 1):1140005, 39, 2012. [2](#), [20](#)
- [4] Adrien Blanchet, José A. Carrillo, and Nader Masmoudi. Infinite time aggregation for the critical Patlak-Keller-Segel model in  $\mathbb{R}^2$ . *Comm. Pure Appl. Math.*, 61(10):1449–1481, 2008. [2](#)
- [5] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011. [10](#)
- [6] Jean-Yves Chemin. A remark on the inviscid limit for two-dimensional incompressible fluids. *Comm. Partial Differential Equations*, 21(11-12):1771–1779, 1996. [44](#)
- [7] Jean-Yves Chemin. *Perfect incompressible fluids*, volume 14 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie. [22](#), [24](#), [32](#), [35](#), [45](#), [47](#)

- [8] Peter Constantin. Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations. *Comm. Math. Phys.*, 104(2):311–326, 1986. [21](#)
- [9] Peter Constantin and Ciprian Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988. [19](#)
- [10] Elaine Cozzi and James P. Kelliher. Vanishing viscosity in the plane for vorticity in borderline spaces of Besov type. *Journal of Differential Equations*, 235(2):647–657, 2007. [47](#)
- [11] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010. [8](#)
- [12] Taoufik Hmidi. Régularité höldérienne des poches de tourbillon visqueuses. *J. Math. Pures Appl. (9)*, 84(11):1455–1495, 2005. [44](#), [45](#)
- [13] Taoufik Hmidi and Sahbi Keraani. Incompressible viscous flows in borderline Besov spaces. *Arch. Ration. Mech. Anal.*, 189(2):283–300, 2008. [44](#), [45](#)
- [14] James P. Kelliher. Expanding domain limit for incompressible fluids in the plane. *Comm. Math. Phys.*, 278(3):753–773, 2008. [27](#)
- [15] A. Kiselev and X. Xu. Suppression of chemotactic explosion by mixing. *arXiv:1508.05333v2*, 2013. [2](#)
- [16] Thomas Laurent. Local and global existence for an aggregation equation. *Comm. Partial Differential Equations*, 32(10-12):1941–1964, 2007. [2](#)
- [17] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I*. Springer-Verlag, New York-Heidelberg, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181. [10](#)
- [18] Pierre-Louis Lions. *Mathematical topics in fluid mechanics. Vol. 1*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1996. [20](#)
- [19] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002. [3](#), [6](#), [20](#), [24](#), [26](#), [35](#)
- [20] C. Marchioro and M. Pulvirenti. *Vortex methods in two-dimensional fluid dynamics*, volume 203 of *Lecture Notes in Physics*. Springer-Verlag, Berlin, 1984. [3](#), [20](#)
- [21] Carlo Marchioro and Mario Pulvirenti. *Mathematical theory of incompressible nonviscous fluids*, volume 96 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994. [3](#), [20](#), [23](#)
- [22] Nader Masmoudi and Ping Zhang. Global solutions to vortex density equations arising from superconductivity. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 22(4):441–458, 2005. [2](#), [3](#)
- [23] Benoit Perthame. *Transport equations in biology*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2007. [2](#), [3](#), [21](#)
- [24] Roger Temam. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition. [36](#), [37](#)
- [25] Misha Vishik. Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type. *Ann. Sci. École Norm. Sup. (4)*, 32(6):769–812, 1999. [6](#), [47](#)

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