Boundary layers for the Navier-Stokes equations linearized around a stationary Euler flow

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Abstract. We study the viscous boundary layer that forms at small viscosity near a rigid wall for the solution to the Navier-Stokes equations linearized around a smooth and stationary Euler flow (LNSE for short) in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ under no-slip boundary conditions. LNSE is supplemented with smooth initial data and smooth external forcing, assumed ill-prepared, that is, not compatible with the no-slip boundary condition. We construct an approximate solution to LNSE on the time interval $[0, T]$, $0 < T < \infty$, obtained via an asymptotic expansion in the viscosity parameter, such that the difference between the linearized Navier-Stokes solution and the proposed expansion vanishes as the viscosity tends to zero in $L^2(\Omega)$ uniformly in time, and remains bounded independently of viscosity in the space $L^2([0, T]; H^1(\Omega))$. We make this construction both for a 3D channel domain and a smooth domain with a curved boundary. The zero-viscosity limit for LNSE, that is, the convergence of the LNSE solution to the solution of the linearized Euler equations around the same profile when viscosity vanishes, then naturally follows from the validity of this asymptotic expansion. This article generalizes and improves earlier works, such as Temam and Wang [20], Xin and Yanagisawa [23], and Gie [4].

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1. Introduction

We study the boundary layer formed near a rigid wall by a low-viscosity incompressible fluid that solves the Navier-Stokes equations (NSE) linearized around a smooth and stationary Euler flow. In exterior domains, such equations model the flow around an obstacle moving at constant velocity, the classical Oseen system, where the steady profile is also spatially homogeneous. In a smooth bounded domain, they model the behavior of nearly inviscid flows in bodies with cavities, as in simplified models of the earth's mantle.

For simplicity, we assume that the fluid occupies a bounded, connected region Ω in $\mathbb{R}^3$ with a C^∞ boundary Γ. We can then write NSE as the system of PDEs,

$$\begin{cases} \dfrac{\partial \boldsymbol{u}^\varepsilon}{\partial t} + \boldsymbol{u}^\varepsilon \cdot \nabla \boldsymbol{u}^\varepsilon + \nabla p^\varepsilon = \boldsymbol{f} + \varepsilon \Delta \boldsymbol{u}^\varepsilon & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}^\varepsilon = 0 & \text{in } \Omega. \end{cases}$$

Above, $\varepsilon > 0$ is the given, constant viscosity coefficient, $\boldsymbol{u}^\varepsilon$ is the velocity field, p^ε is the pressure field, and $\boldsymbol{f}$ is a given time-dependent external force (independent of ε). Together, $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ is the solution to NSE. We impose *no-slip* boundary conditions,

$$\boldsymbol{u}^\varepsilon = 0,$$

and we give an initial condition $\boldsymbol{u}_0$ on the velocity alone. The no-slip condition is the most appropriate at rigid, smooth walls.

As customary in fluid mechanics, we denote by H the function space

$$H = \{\boldsymbol{u} \in L^2(\Omega) | \operatorname{div} \boldsymbol{u} = 0 \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}, \tag{1.1}$$

endowed with the L^2 norm, where $\boldsymbol{n}$ represents the unit outer normal to Γ.

In the domain Ω, we consider a smooth vector field $\boldsymbol{U}$ of class $H \cap C^\infty(\overline{\Omega})$, which is a solution to the stationary Euler equations (EE),

$$\begin{cases} \boldsymbol{U} \cdot \nabla \boldsymbol{U} + \nabla \pi = \boldsymbol{F} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{U} = 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\boldsymbol{F} \in C^\infty(\Omega)$. We impose on $\boldsymbol{U}$ the *no-penetration* condition,

$$\boldsymbol{U} \cdot \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma.$$

One can always construct solutions to the equation above provided $\boldsymbol{F}$ is appropriately given. For example, when $\boldsymbol{F} = \boldsymbol{0}$, one can obtain steady Euler solutions, called *Beltrami flows*, from eigenfunctions of the curl operator.

We linearize NSE about $\boldsymbol{U}$ in the usual manner. We let $\boldsymbol{v}^\varepsilon = \boldsymbol{u}^\varepsilon - \boldsymbol{U}$ so that $\boldsymbol{u}^\varepsilon = \boldsymbol{v}^\varepsilon + \boldsymbol{U}$. Using (1.2) gives an equation for $\boldsymbol{v}^\varepsilon$ from NSE, in which the pressure can be identified (up to a constant) with $p^\varepsilon - \pi$. LNSE is then obtained by retaining all linear terms in $\boldsymbol{v}^\varepsilon$. The pressure that ensures the divergence-free condition on $\boldsymbol{v}^\varepsilon$ can still be identified with $p^\varepsilon - \pi$. Given $\boldsymbol{U}$ and the initial conditions on $\boldsymbol{u}^\varepsilon$, it follows that $\boldsymbol{v}^\varepsilon$ satisfies a non-homogeneous boundary condition on Γ, namely, $\boldsymbol{v}^\varepsilon = -\boldsymbol{U}$. However, by performing a lift of the boundary value that is divergence free (e.g. via an harmonic vector potential), we can WLOG assume that $\boldsymbol{v}^\varepsilon = 0$, provided the right-hand-side of the equation is changed accordingly. Finally, with abuse of notation, we relabel $\boldsymbol{v}^\varepsilon$ by $\boldsymbol{u}^\varepsilon$ and $p^\varepsilon - \pi$ by p^ε. Then, the initial-boundary-value problem (IBVP for short) for LNSE is the system,

$$\begin{cases} \dfrac{\partial \boldsymbol{u}^\varepsilon}{\partial t} - \varepsilon\Delta \boldsymbol{u}^\varepsilon + \boldsymbol{U}\cdot\nabla\boldsymbol{u}^\varepsilon + \boldsymbol{u}^\varepsilon\cdot\nabla\boldsymbol{U} + \nabla p^\varepsilon = \boldsymbol{f} - \boldsymbol{F} + \varepsilon\Delta\boldsymbol{U} \quad \text{in } \Omega\times(0,T), \\ \operatorname{div} \boldsymbol{u}^\varepsilon = 0 \quad \text{in } \Omega\times(0,T), \\ \boldsymbol{u}^\varepsilon = 0 \quad \text{on } \Gamma\times(0,T), \\ \boldsymbol{u}^\varepsilon|_{t=0} = \boldsymbol{u}_0 \quad \text{in } \Omega, \end{cases} \tag{1.3}$$

for any fixed $T > 0$.

It should be noted that often in the literature, the Stokes equation, obtained by dropping the non-linear terms directly in NSE, is called the LNSE. Here, instead we refer to the LNSE as an Oseen-type equation, that is NSE linearized around a non-trivial profile $\boldsymbol{U}$. The vanishing viscosity limit and associated boundary layer for the Stokes system can be analyzed adapting techniques used for the heat equation and do not require the use of correctors.

The goal of this work is to analyze the boundary layer that arise in the system (1.3) at small viscosity due to the mismatch in its boundary conditions and those of the corresponding limit problem (1.6) below, which is obtained by formally setting $\varepsilon = 0$. Our main task is to build an incompressible boundary layer corrector (and hence an asymptotic expansion of $\boldsymbol{u}^\varepsilon$ in ε) assuming sufficient regularity of the data. Since the minimal regularity requirement for the data is not our focus, we assume that

$$\boldsymbol{f} \in C^1(0,T; C^\infty(\overline{\Omega})), \quad \boldsymbol{u}_0 \in H \cap C^\infty(\overline{\Omega}). \tag{1.4}$$

However, this smooth data may not necessarily vanish on the boundary and, in this sense, the initial data is ill-prepared; that is, the boundary and initial conditions in (1.3) are not compatible.

As is the case for the unsteady Stokes system (see e.g [18, 19]), under the assumption (1.4), for any $0 < T < \infty$ there exists a unique, strong solution $\boldsymbol{v}^\varepsilon$ to the IBVP for LNSE (1.3) at fixed ε. Moreover, the solution $\boldsymbol{u}^\varepsilon$ satisfies *uniformly in* ε,

$$\boldsymbol{u}^\varepsilon \in L^2(0,T;V)\cap C([0,T];H), \quad \frac{\partial \boldsymbol{u}^\varepsilon}{\partial t} \in L^2(0,T;H), \tag{1.5}$$

by standard energy estimates. (The limit problem satisfies these same estimates.)

Formally setting $\varepsilon = 0$ in (1.3), we obtain the corresponding limit problem,

$$\begin{cases} \dfrac{\partial \boldsymbol{u}^0}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{u}^0 + \boldsymbol{u}^0 \cdot \nabla \boldsymbol{U} + \nabla p^0 = \boldsymbol{f} - \boldsymbol{F} & \text{in } \Omega \times (0,T), \\ \operatorname{div} \boldsymbol{u}^0 = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{u}^0 \cdot \boldsymbol{n} = 0 & \text{on } \Gamma \times (0,T), \\ \boldsymbol{u}^0|_{t=0} = \boldsymbol{u}_0 & \text{in } \Omega. \end{cases} \tag{1.6}$$

By analogy with (1.3), we call the above system the linearized Euler equations or LEE. Under the assumptions (1.4), for any $0 < T < \infty$ the system (1.6) possesses a unique strong solution $\boldsymbol{u}^0$ (with pressure p^0, unique up to an additive constant) such that

$$\boldsymbol{u}^0 \in C^1(0,T; H \cap C^\infty(\overline{\Omega})). \tag{1.7}$$

(We refer to the results in [13] for a proof.)

In this work, we systematically employ the method of correctors as proposed by J. L. Lions [14] to analyse the boundary layer for LNSE. The corrector that we construct accounts for the difference between the solution to LNSE and LEE due to the discrepancies between the boundary values of the viscous and inviscid solutions, and it accounts for the rapid variation of the functions and their normal derivatives in the boundary layer.

We assume the same regular asymptotic expansions in powers of $\sqrt{\varepsilon}$ and scaling as in Prandtl theory [17]. In particular, the thickness of the layer where the effect of the corrector is not negligible is of order $\sqrt{\varepsilon}$ as in Prandtl theory. At the same time, as verified by rigorous analysis below, the corrector shares the major estimates and properties of the corrector introduced by Kato in [11] to study the vanishing viscosity limit. This fact is not unexpected given that the zero-viscosity limit holds in this case. However, in Kato's work the effects of viscosity, in terms of viscous energy dissipation rate, must be controlled in a layer of order ε to pass to the zero-viscosity limit.

The main idea behind the corrector method is to propose a form for an approximate solution to LNSE, which is the given solution to the limit problem plus the corrector. Formal matched asymptotic analysis and physical considerations are used to derive the form of the corrector and the effective equations it satisfies. Then the validity of this asymptotic expansions is established by energy estimates on the difference of the viscous solution and the proposed expansion, performed on the whole domain Ω. To enforce the incompressibility condition on the corrector, we follow the original approach in [4], where the viscous boundary layer for the Stokes equations is investigated. In this article, we generalize the analysis in [4] by studying the asymptotic behavior of the Navier-Stokes equations linearized around a stationary Euler flow. The asymptotic expansion proposed in this article provides complete structural information of the boundary layers for LNSE.

Establishing the zero-viscosity limit is nontrivial even in the absence of boundaries, due to the singular nature of the limit. When boundaries are present, the analysis of flows at small viscosity is significantly more challenging, given that rigid walls generate vorticity. When no-slip boundary conditions are impose on the viscous solutions, the lack of control of growth normal to the boundary of the tangential velocity components keeps the problem still essentially open, unless strong conditions such as analyticity or symmetry are imposed on the data or the solution (we refer to [6, 16] and references therein for a survey of recent results) or the equations are linearized. In this work, we consider another simplified situation where the vanishing viscosity limit and the associated boundary layer can be rigorously studied. The analysis of the boundary layer for LNSE can be the first step in elucidating the role of (a potentially strong) tangential advection on the stability of the layer. Indeed, recent results on the ill-posedness of Prandtl equations have as their starting point linear instabilities around shear flows [3, 8, 9].

While Oseen-type equations were studied in the context of the vanishing viscosity limit and Prandtl-type approximations in [15, 20, 21, 22, 23] in domains with flat boundaries such as a channel, in this article we consider the LNSE in a general smooth domain with a curved boundary and hence extend those earlier results. As a matter of fact, when the boundary is curved, the expansion of the viscous solutions in powers of the viscosity, assumed small, that is obtained on domains with flat boundaries does not give a suitable approximation. As exemplified in Equation (1.8) and Theorem 1.1 below, an additional corrector for the pressure is required in order to account the lower-order error caused by curvature.

In addition, prior works consider only the case of well-prepared or compatible initial data, that is, data that vanish on the boundary. This work, instead, is the first that analyzes the boundary and initial layers for Oseen-type equations (or LNSE around a stationary Euler flow) when the initial data is ill prepared. In the case of incompatible data, an initial layer forms in the viscous equations that needs to be accounted for in the analysis. Indeed, when the limit solution is steady, the contribution from the initial layer may persist in the limit of vanishing viscosity.

For a curved boundary, as in the case treated here, it is necessary to introduce a pressure corrector at zero order. We therefore write the approximate expansion of the LNSE solution as

$$\begin{aligned} &\boldsymbol{u}^\varepsilon \approx \boldsymbol{u}^0 + \boldsymbol{\Theta}, \\ &p^\varepsilon \approx \begin{cases} p^0, & \text{for the case of a 3D channel domain,} \\ p^0 + q, & \text{for the case of a 3D smooth domain.} \end{cases} \end{aligned} \tag{1.8}$$

To isolate and so clarify these technical difficulties, we treat the case of LNSE in a channel first in Section 2 before tackling the the more technically involved case of a curved boundary in Section 3. The corrector, $\boldsymbol{\Theta}$, is given explicity in (2.7) and (2.8) for the channel and in (3.16) and (3.17) for a curved domain.

Our main result is the following error estimate with sharp rates of convergence in viscosity:

Theorem 1.1. *Make the assumptions in (1.4) and let* $\boldsymbol{w}^\varepsilon := \boldsymbol{u}^\varepsilon - (\boldsymbol{u}^0 + \boldsymbol{\Theta})$*, the difference between the linearized Navier-Stokes solution and its asymptotic expansion, as given in (1.8). Then* $\boldsymbol{w}^\varepsilon$ *vanishes with the viscosity parameter in the sense that*

$$\|\boldsymbol{w}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}} \|\nabla \boldsymbol{w}^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{2}}, \tag{1.9}$$

for a constant κ *depending on the data, but independent of* ε*. Moreover, as* ε *tends to zero,* $\boldsymbol{u}^\varepsilon$ *converges to the Euler solution* $\boldsymbol{u}^0$ *in the sense that*

$$\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}^0\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{4}}. \tag{1.10}$$

This paper is organized as follows. We introduce and study the velocity corrector $\boldsymbol{\Theta}$ in Sections 2 and 3 for a channel geometry. We introduce the pressure corrector q in Section 3, where we discuss the case of a curved boundary and prove Theorem 1.1. There, we also introduce a suitable coordinate system in a collar neighborhood of the boundary Γ, used in the analysis of the viscous layer.

Throughout, we will use the fairly standard notation in which a subscript on an equation number signifies the ordering of the equation in that reference. For example, $(2.3)_3$ means the third equation in system (2.3).

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2. Boundary layers for LNSE in a 3D channel domain

In this section, we consider the problems (1.3) and (1.6) when the domain is a 3D periodic channel, identified with

$$\Omega =: (0, L)^2 \times (0, h) \subset \mathbb{R}^3,$$

under periodic conditions in the x_1 and x_2 directions. Periodicity makes the domain bounded and ensures uniqueness of solutions to the fluid equations.

2.1. Asymptotic expansion of solutions to LNSE

We start with the ansatz that to the first-order, only the velocity needs to be corrected to obtain the LNSE solution from the LEE solution:

$$\boldsymbol{u}^\varepsilon \approx \boldsymbol{u}^0 + \boldsymbol{\Theta}, \qquad p^\varepsilon \approx p^0. \tag{2.1}$$

The hypothesis that the approximate pressure is the Euler pressure is justified as in Prandtl theory, and will be rigorously verified.

To obtain an equation for the corrector, we start by supposing that (2.1) holds exactly, so that $\boldsymbol{\Theta} = \boldsymbol{u}^\varepsilon - \boldsymbol{u}^0$ and $p^\varepsilon - p^0 = 0$. Subtracting $(1.6)_1$ from $(1.3)_2$ gives

$$\partial_t \boldsymbol{\Theta} - \varepsilon \Delta \boldsymbol{u}^\varepsilon + \boldsymbol{U} \cdot \nabla \boldsymbol{\Theta} + \boldsymbol{\Theta} \cdot \nabla \boldsymbol{U} = \varepsilon \Delta \boldsymbol{U}. \tag{2.2}$$

We refine (2.2) by making an ansatz like that of Prandtl: we assume that (2.2) holds exactly only outside of a boundary layer of width proportional to $\sqrt{\varepsilon}$. As in the Prandtl theory, this gives that in the boundary layer, $\partial/\partial x_i \sim \sqrt{\varepsilon}(\partial/\partial x_3)$ and $\Theta_i \sim \varepsilon^{-1/2}\, \Theta_3$, $i = 1, 2$. We see that only the x_3 derivatives contribute at leading order to $\varepsilon \Delta \boldsymbol{u}^\varepsilon$ and that $\varepsilon \Delta \boldsymbol{U}$ is of lower order in ε. This yields

$$\frac{\partial \Theta_i}{\partial t} - \varepsilon \frac{\partial^2 u_i^\varepsilon}{\partial x_3^2} + \sum_{j=1}^{3} U_j \frac{\partial \Theta_i}{\partial x_j} + \sum_{j=1}^{2} \Theta_j \frac{\partial U_i}{\partial x_j} = 0.$$

Also, because $\varepsilon \partial_{x_3}^2 \boldsymbol{u}^0$ is of lower order in ε it can be added to the equation, allowing us to replace $\varepsilon \partial_{x_3}^2 u_i^\varepsilon$ by $\varepsilon \partial_{x_3}^2 \Theta_i$. Supplemented with initial and boundary conditions, we have the following formal asymptotic expansion for the corrector:

$$\begin{cases} \dfrac{\partial \Theta_i}{\partial t} - \varepsilon \dfrac{\partial^2 \Theta_i}{\partial x_3^2} + \displaystyle\sum_{j=1}^{3} U_j \frac{\partial \Theta_i}{\partial x_j} + \sum_{j=1}^{2} \Theta_j \frac{\partial U_i}{\partial x_j} = 0 & \text{in } \Omega \times (0, T), \quad i = 1, 2, \\ \operatorname{div} \boldsymbol{\Theta} = 0 & \text{in } \Omega \times (0, T), \\ \boldsymbol{\Theta} = -\boldsymbol{u}^0 & \text{on } \Gamma \times (0, T), \\ \boldsymbol{\Theta}|_{t=0} = 0 & \text{in } \Omega. \end{cases} \tag{2.3}$$

Because there are two boundary components, we will construct the corrector from two boundary layer functions, θ_L and θ_R, each defined on a half-space. (We use the subscript L and R for left and right layer functions assuming the channel is vertically oriented.) The layer functions satisfy drift-diffusion equations on half spaces with boundary, respectively, given by the planes $x_3 = 0$ and $x_3 = h$:

$$\begin{cases} \dfrac{\partial \theta_{i,\,L}}{\partial t} - \varepsilon \Delta \theta_{i,\,L} + \displaystyle\sum_{j=1}^{2} U_j\big|_{x_3=0} \dfrac{\partial \theta_{i,\,L}}{\partial x_j} + x_3 \sigma_L \dfrac{\partial U_3}{\partial x_3}\Big|_{x_3=0} \dfrac{\partial \theta_{i,\,L}}{\partial x_3} \\ \qquad + \displaystyle\sum_{j=1}^{2} \theta_{j,\,L} \dfrac{\partial U_i}{\partial x_j}\Big|_{x_3=0} = 0, \quad (0,L)^2 \times (0,\infty) \times (0,T), \quad i = 1,2, \\ \theta_{i,\,L} = -u_i^0, \quad \text{at } x_3 = 0, \\ \theta_{i,\,L} = 0, \quad \text{at } t = 0, \end{cases} \tag{2.4}$$

and

$$\begin{cases} \dfrac{\partial \theta_{i,\,R}}{\partial t} - \varepsilon \Delta \theta_{i,\,R} + \displaystyle\sum_{j=1}^{2} U_j\big|_{x_3=h} \dfrac{\partial \theta_{i,\,R}}{\partial x_j} - \widehat{x}_3 \sigma_R \dfrac{\partial U_3}{\partial x_3}\Big|_{x_3=h} \dfrac{\partial \theta_{i,\,R}}{\partial x_3} \\ + \displaystyle\sum_{j=1}^{2} \theta_{j,\,R} \dfrac{\partial U_i}{\partial x_j}\Big|_{x_3=h} = 0, \quad (0,L)^2 \times (-\infty,h) \times (0,T), \quad i = 1,2, \\ \theta_{i,\,R} = -u_i^0, \quad \text{at } x_3 = h, \\ \theta_{i,\,R} = 0, \quad \text{at } t = 0, \end{cases} \tag{2.5}$$

where $\widehat{x}_3 := h - x_3$ and the cut-offs σ_L and σ_R are given by

$$\sigma_L(x_3) = \begin{cases} 1, \ 0 \le x_3 \le h/4, \\ 0, \ x_3 \ge h/2, \end{cases} \qquad \sigma_R(x_3) = \sigma_L(h - x_3). \tag{2.6}$$

Informally, the parabolic layer function $\theta_{i,\,L}$ $\theta_{i,\,R}$) represents the tangential component Θ_i, $i = 1, 2$, of the corrector $\boldsymbol{\Theta}$ near the boundary $x_3 = 0$ ($x_3 = h$). However, we want the corrector $\boldsymbol{\Theta}$ to belong to the space H to avoid dealing directly with the pressure in the convergence analysis of the error $\boldsymbol{w}^\varepsilon$. To this end, we first introduce, as is customary in boundary layer analysis, appropriate cut-off functions σ_L and σ_R so that the domain of the truncated layer functions is Ω and the approximate corrector satisfies the boundary conditions on Γ. Then, to enforce the divergence-free condition, we define the tangential components of the corrector Θ_i, $i = 1, 2$, as follows:

$$\begin{aligned} \Theta_i(\boldsymbol{x}, t) &= \sigma_L\, \theta_{i,\,L} + \sigma_L' \int_0^{x_3} \theta_{i,\,L}\, dx_3' + \sigma_R\, \theta_{i,\,R} + \sigma_R' \int_h^{x_3} \theta_{i,\,R}\, dx_3' \\ &= \frac{\partial}{\partial x_3} \Big\{ \sigma_L \int_0^{x_3} \theta_{i,\,L}\, dx_3' + \sigma_R \int_h^{x_3} \theta_{i,\,R}\, dx_3' \Big\}, \quad i = 1, 2 \end{aligned} \tag{2.7}$$

Finally, we use the divergence-free condition on $\boldsymbol{\Theta}$ to obtain the normal component of the corrector Θ_3 from its tangential components:

$$\Theta_3(\boldsymbol{x}, t) = -\sum_{i=1}^{2} \Big\{ \sigma_L \int_0^{x_3} \frac{\partial \theta_{i,\,L}}{\partial x_i}\, dx_3' + \sigma_R \int_h^{x_3} \frac{\partial \theta_{i,\,R}}{\partial x_i}\, dx_3' \Big\}. \tag{2.8}$$

Hence $\mathbf{\Theta}$ belongs to the space H;

$$\text{div } \mathbf{\Theta} = 0 \text{ in } \Omega \quad \text{and} \quad \Theta_3 = 0 \text{ on } \Gamma.$$

We can write the corrector $\mathbf{\Theta}$ as a sum of three vector fields in the form,

$$\mathbf{\Theta} = \boldsymbol{\theta} + \boldsymbol{\varphi} + \boldsymbol{\psi}, \tag{2.9}$$

where, for $i = 1, 2$,

$$\begin{cases} \theta_i = \sigma_L \, \theta_{i,\,L} + \sigma_R \, \theta_{i,\,R}, \\ \varphi_i = \sigma_L' \displaystyle\int_0^{\infty} \theta_{i,\,L} \, dx_3' + \sigma_R' \int_h^{-\infty} \theta_{i,\,R} \, dx_3', \\ \psi_i = \sigma_L' \displaystyle\int_{\infty}^{x_3} \theta_{i,\,L} \, dx_3' + \sigma_R' \int_{-\infty}^{x_3} \theta_{i,\,R} \, dx_3', \end{cases} \tag{2.10}$$

and

$$\begin{cases} \theta_3 = 0, \\ \varphi_3 = -\sigma_L \displaystyle\int_0^{\infty} \sum_{i=1}^{2} \frac{\partial \theta_{i,\,L}}{\partial x_i} \, dx_3' - \sigma_R \int_h^{-\infty} \sum_{i=1}^{2} \frac{\partial \theta_{i,\,R}}{\partial x_i} \, dx_3', \\ \psi_3 = -\sigma_L \displaystyle\int_{\infty}^{x_3} \sum_{i=1}^{2} \frac{\partial \theta_{i,\,L}}{\partial x_i} \, dx_3' - \sigma_R \int_{-\infty}^{x_3} \sum_{i=1}^{2} \frac{\partial \theta_{i,\,R}}{\partial x_i} \, dx_3'. \end{cases} \tag{2.11}$$

As we will verify in the following subsection, the main part $\boldsymbol{\theta}$ of $\mathbf{\Theta}$ is a fast decaying *boundary layer function*, which agrees with the classical theory of boundary layers, while the remaining parts $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are *supplementary vector fields* (which are small when ε is small) to maintain the corrector $\mathbf{\Theta}$ in the space H.

2.2. Estimates on the corrector

In this section, we will derive estimates on the correctors in various norms. These estimates are needed to establish error bounds on the approximate LNSE solution. We begin by estimating the layer functions $\theta_{i,\,L}$, $\theta_{i,\,R}$, $i = 1, 2$, in L^p. The main contribution in ε to the L^p norm comes from the Laplacian, the zero-order term giving possibly an exponential growth in time only. Therefore, we utilize well-known estimates for solutions to the heat equation as an intermediate step.

We let $\theta_{\text{heat},\,i}$, $i = 1, 2$, be the solution of the following IBVP for the heat equation in a half space:

$$\begin{cases} \dfrac{\partial \theta_{\text{heat},\,i}}{\partial t} - \varepsilon \dfrac{\partial^2 \theta_{\text{heat},\,i}}{\partial x_3^2} = 0, \quad x_3 > 0, \\ \theta_{\text{heat},\,i} = -u_i^0, \quad \text{at } x_3 = 0, \\ \theta_{\text{heat},\,i} = 0, \quad \text{at } t = 0. \end{cases} \tag{2.12}$$

We recall the following L^p estimates holds on $\theta_{\text{heat},\, i}$ (see e.g. [2]; see [5, 4] for an application to viscous boundary layers):

$$\Big\| \Big(\frac{x_3}{\sqrt{\varepsilon}} \Big)^{\frac{\ell}{p}} \frac{\partial^{k+m} \theta_{\text{heat},\, i}}{\partial x_j^k \partial x_3^m} \Big\|_{L^p((0,L)^2 \times (0,\infty))} \leq \kappa \big(1 + t^{\frac{1}{2p} - \frac{2}{m}} \big) \, \varepsilon^{\frac{1}{2p} - \frac{2}{m}}, \quad t > 0, \tag{2.13}$$

for $0 \leq p \leq \infty$, $k, \ell \geq 0$, $1 \leq m \leq 3$, and $i, j = 1, 2$.

Denoting $\widetilde{\theta}_i = \theta_{i,\, L} - \theta_{\text{heat},\, i}$, one finds that $\widetilde{\theta}_i$, $i = 1, 2$, satisifes

$$\begin{cases} \dfrac{\partial \widetilde{\theta}_i}{\partial t} - \varepsilon \Delta \widetilde{\theta}_i + \displaystyle\sum_{j=1}^{2} U_j \big|_{x_3 = 0} \dfrac{\partial \widetilde{\theta}_i}{\partial x_j} + x_3 \sigma_L \dfrac{\partial U_3}{\partial x_3} \Big|_{x_3=0} \dfrac{\partial \widetilde{\theta}_i}{\partial x_3} + \displaystyle\sum_{j=1}^{2} \widetilde{\theta}_j \, \dfrac{\partial U_i}{\partial x_j} \Big|_{x_3=0} \\ \quad = \widetilde{E}_i, \quad \text{in } (0,L)^2 \times (0, \infty), \\ \widetilde{\theta}_i = 0, \quad \text{at } x_3 = 0 \text{ or } t = 0, \end{cases} \tag{2.14}$$

where

$$\begin{aligned} \widetilde{E}_i = \ & \varepsilon \sum_{j=1}^{2} \frac{\partial^2 \theta_{\text{heat},\, i}}{\partial x_j^2} - \sum_{j=1}^{2} U_j \big|_{x_3=0} \frac{\partial \theta_{\text{heat},\, i}}{\partial x_j} - x_3 \sigma_L \frac{\partial U_3}{\partial x_3} \Big|_{x_3=0} \frac{\partial \theta_{\text{heat},\, i}}{\partial x_3} \\ & - \sum_{j=1}^{2} \theta_{\text{heat},\, j} \, \frac{\partial U_i}{\partial x_j} \Big|_{x_3=0}. \end{aligned}$$

A standard energy estimate gives bounds similar to (2.13) on $\widetilde{\theta}_i$, hence $\theta_{i,\, L}$, and by symmetry $\theta_{i,\, R}$, satisfies the following estimates, which we record in a lemma for convenience.

Lemma 2.1. *For $i, j = 1, 2$ and $k, \ell \geq 0$, and $1 \leq p \leq \infty$, we have*

$$\begin{aligned} \Big\| \Big(\frac{x_3}{\sqrt{\varepsilon}} \Big)^{\frac{\ell}{p}} \frac{\partial^k \theta_{i,\, L}}{\partial x_j^k} \Big\|_{L^\infty(0,T; L^p((0,L)^2 \times (0,\infty)))} & + \\ \varepsilon^{\frac{1}{2p} + \frac{1}{4}} \Big\| \Big(\frac{x_3}{\sqrt{\varepsilon}} \Big)^{\frac{\ell}{2}} \nabla \frac{\partial^k \theta_{i,\, L}}{\partial x_j^k} \Big\|_{L^2(0,T; L^2((0,L)^2 \times (0,\infty)))} & \leq \kappa_T \, \varepsilon^{\frac{1}{2p}}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} \Big\| \frac{\partial^{k+1} \theta_{i,\, L}}{\partial x_j^k \partial x_3} \Big\|_{L^\infty(0,T; L^2((0,L)^2 \times (0,\infty)))} + \varepsilon^{\frac{1}{2}} \Big\| \nabla \frac{\partial^{k+1} \theta_{i,\, L}}{\partial x_j^k \partial x_3} \Big\|_{L^2(0,T; L^2((0,L)^2 \times (0,\infty)))} \\ \leq \kappa_T \, \varepsilon^{-\frac{1}{4}}, \end{aligned} \tag{2.16}$$

for a constant κ_T depending on T and other data, but independent of ε.

Similar estimates hold for $\theta_{i,\, R}$ if $\theta_{i,\, L}$, $x_3/\sqrt{\varepsilon}$, and $(0,L)^2 \times (0,\infty)$ are replaced by $\theta_{i,\, R}$, $(h - x_3)/\sqrt{\varepsilon}$, and $(0,L)^2 \times (-\infty, 0)$ respectively.

Even if the proof of this lemma follows by standard arguments, we include a proof for the reader's sake.

Proof. We prove (2.15) by induction on ℓ.

Using the bounds on $\theta_{\mathrm{heat},\,i}$ and the definition of $\widetilde{E}_i$, $i=1,2$, we have

$$\Big\| \Big(\frac{x_3}{\sqrt{\varepsilon}}\Big)^{\ell} \frac{\partial^{k+m}\widetilde{E}_i}{\partial x_j^k \partial x_3^m} \Big\|_{L^p((0,L)^2\times(0,\infty))} \le \kappa\big(1+t^{\frac{1}{2p}-\frac{2}{m}}\big)\,\varepsilon^{\frac{1}{2p}-\frac{2}{m}}, \quad t>0, \tag{2.17}$$

for $0\le p\le\infty$, $k,\ell\ge 0$, $1\le m\le 3$, and $i,j=1,2$. In addition,

$$\|\widetilde{E}_i|_{x_3=0}\|_{L^\infty((0,T)\times(0,L)^2)} \le \kappa_T. \tag{2.18}$$

To prove (2.15) for $\ell=0$, we multiply $(2.14)_1$ by $\widetilde{\theta}_i^{p-1}$ where $p>1$ is a simple fraction q/r with q an even integer. Integrating over $(0,L)^2\times(0,\infty)$ gives

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\widetilde{\theta}_i\|^p_{L^p((0,L)^2\times(0,\infty))} + \varepsilon(p-1)\int_{(0,L)^2}\int_0^\infty |\nabla\widetilde{\theta}_i|^2\widetilde{\theta}_i^{p-2}\,dx_3dx_1dx_2\\
&\quad\le \Big|\int_{(0,L)^2}\int_0^\infty \Big(\sum_{j=1}^2 U_j\big|_{x_3=0}\frac{\partial\widetilde{\theta}_i}{\partial x_j} + x_3\sigma_L\frac{\partial U_3}{\partial x_3}\Big|_{x_3=0}\frac{\partial\widetilde{\theta}_i}{\partial x_3}\Big)\widetilde{\theta}_i^{p-1}\,dx_3dx_1dx_2\Big|\\
&\qquad+\Big|\int_{(0,L)^2}\int_0^\infty \Big(-\sum_{j=1}^2\widetilde{\theta}_j\frac{\partial U_i}{\partial x_j}\Big|_{x_3=0} + \widetilde{E}_i\Big)\widetilde{\theta}_i^{p-1}\,dx_3dx_1dx_2\Big|.
\end{aligned} \tag{2.19}$$

We bound each term on the right-hand side separately, starting with the first:

$$\begin{aligned}
&\Big|\int_{(0,L)^2}\int_0^\infty \sum_{j=1}^2 U_j\big|_{x_3=0}\frac{\partial\widetilde{\theta}_i}{\partial x_j}\widetilde{\theta}_i^{p-1}\,dx_3dx_1dx_2\Big|\\
&\quad=\Big|\frac{1}{p}\sum_{j=1}^2\int_0^\infty\int_{(0,L)^2}\frac{\partial U_j}{\partial x_j}\Big|_{x_3=0}\widetilde{\theta}_i^p\,dx_1dx_2dx_3\Big| \le \frac{\kappa_T}{p}\|\widetilde{\theta}_i\|^p_{L^p((0,L)^2\times(0,\infty))},
\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}
&\Big|\int_{(0,L)^2}\int_0^\infty x_3\sigma_L\frac{\partial U_3}{\partial x_3}\Big|_{x_3=0}\frac{\partial\widetilde{\theta}_i}{\partial x_3}\widetilde{\theta}_i^{p-1}\,dx_3dx_1dx_2\Big|\\
&=\Big|\frac{1}{p}\int_{(0,L)^2}\int_0^\infty\frac{\partial(x_3\sigma_L)}{\partial x_3}\frac{\partial U_3}{\partial x_3}\Big|_{x_3=0}\widetilde{\theta}_i^p\,dx_3dx_1dx_2\Big| \le \frac{\kappa_T}{p}\|\widetilde{\theta}_i\|^p_{L^p((0,L)^2\times(0,\infty))}.
\end{aligned} \tag{2.21}$$

For the second term on the right-hand side of (2.19), we apply Hölder's and Young's inequalities with $1/p$ and $(p-1)/p$ and write

$$\begin{aligned}
&\Big| \int_{(0,L)^2} \int_0^\infty \Big(- \sum_{j=1}^2 \widetilde{\theta}_j \, \frac{\partial U_i}{\partial x_j}\Big|_{x_3=0} + \widetilde{E}_i \Big) \widetilde{\theta}_i^{p-1} \, dx_3 dx_1 dx_2 \Big| \\
&\leq \kappa_T \int_{(0,L)^2} \int_0^\infty \Big(\sum_{j=1}^2 |\widetilde{\theta}_j| + |\widetilde{E}_i| \Big) |\widetilde{\theta}_i|^{p-1} \, dx_3 dx_1 dx_2 \\
&\leq \kappa_T \Big(\sum_{j=1}^2 \|\widetilde{\theta}_j\|_{L^p((0,L)^2\times(0,\infty)))} + \|\widetilde{E}_i\|_{L^p((0,L)^2\times(0,\infty)))} \Big) \|\widetilde{\theta}_i\|^{p-1}_{L^p((0,L)^2\times(0,\infty)))} \\
&\leq \frac{\kappa_T}{p} \sum_{j=1}^2 \|\widetilde{\theta}_j\|^p_{L^p((0,L)^2\times(0,\infty)))} + \frac{\kappa_T}{p} \|\widetilde{E}_i\|^p_{L^p((0,L)^2\times(0,\infty)))} \\
&\quad + \kappa_T \|\widetilde{\theta}_i\|^p_{L^p((0,L)^2\times(0,\infty)))}.
\end{aligned} \tag{2.22}$$

Now, it follows from (2.17) and (2.19) – (2.22) that

$$\begin{aligned}
&\frac{d}{dt} \Big(\sum_{i=1}^2 \|\widetilde{\theta}_i\|^p_{L^p((0,L)^2\times(0,\infty))} \Big) + \varepsilon p(p-1) \int_{(0,L)^2} \int_0^\infty \sum_{i=1}^2 |\nabla \widetilde{\theta}_i|^2 \widetilde{\theta}_i^{p-2} \\
&\qquad \leq \kappa_T \varepsilon^{\frac{1}{2}} + \kappa_T p \sum_{i=1}^2 \|\widetilde{\theta}_i\|^p_{L^p((0,L)^2\times(0,\infty))}.
\end{aligned} \tag{2.23}$$

Then, by applying Grönwall's inequality with an integrating factor $\exp(-\kappa_T \, p)$ and by using the continuity of L^p norm in p, we deduce that, $i = 1, 2$,

$$\|\widetilde{\theta}_i\|_{L^\infty(0,T;L^p((0,L)^2\times(0,\infty)))} + \varepsilon^{\frac{1}{2p}+\frac{1}{4}} \|\nabla \widetilde{\theta}_i\|_{L^2(0,T;L^2((0,L)^2\times(0,\infty)))} \leq \kappa_T \, \varepsilon^{\frac{1}{2p}}, \tag{2.24}$$

for any $1 \leq p \leq \infty$. Again, we can replace $\widetilde{\theta}_i$ by $\theta_{i,\,L}$ in (2.24), owing to estimate (2.13) for $\theta_{i,\,\mathrm{heat}}$.

Next, we observe that any tangential derivative of $\widetilde{\theta}_i$ in x_j, $j = 1, 2$, satisfies an equation similar to (2.14) up to lower-order derivatives in x_j, in which the source term is replaced by a tangential derivative of $\widetilde{E}_i$. Then, thanks to (2.17), we can verify (2.15) with $\ell = 0$ for all $k \geq 0$ by an argument similar to the one above.

Now we assume that (2.15) holds true for $0 \leq \ell \leq \ell' - 1$ as our induction hypothesis, and establish (2.15) with $\ell = \ell'$.

We multiply $(2.14)_1$ by $(x_3/\sqrt{\varepsilon})^{\ell'} \, \widetilde{\theta}_i^{p-1}$, where again $p > 1$ is a simple fraction q/r with an even integer q. Integrating over $(0, L)^2 \times (0, \infty)$ and integrating by parts gives

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\left\|\Big(\frac{x_3}{\sqrt{\varepsilon}}\Big)^{\frac{\ell'}{p}}\widetilde{\theta}_i\right\|^p_{L^p((0,L)^2\times(0,\infty))}+\varepsilon(p-1)\int_{(0,L)^2}\int_0^\infty\Big(\frac{x_3}{\sqrt{\varepsilon}}\Big)^{\ell'}|\nabla\widetilde{\theta}_i|^2\widetilde{\theta}_i^{p-2}\\
&\quad\leq\left|\int_{(0,L)^2}\int_0^\infty\Big(\sum_{j=1}^2 U_j\big|_{x_3=0}\frac{\partial\widetilde{\theta}_i}{\partial x_j}+x_3\sigma_L\frac{\partial U_3}{\partial x_3}\Big|_{x_3=0}\frac{\partial\widetilde{\theta}_i}{\partial x_3}\Big)\Big(\frac{x_3}{\sqrt{\varepsilon}}\Big)^{\ell'}\widetilde{\theta}_i^{p-1}\right|\\
&\qquad+\left|\int_{(0,L)^2}\int_0^\infty\Big(-\sum_{j=1}^2\widetilde{\theta}_j\frac{\partial U_i}{\partial x_j}\Big|_{x_3=0}+\widetilde{E}_i\Big)\Big(\frac{x_3}{\sqrt{\varepsilon}}\Big)^{\ell'}\widetilde{\theta}_i^{p-1}\right|\\
&\qquad+\kappa\left\|\Big(\frac{x_3}{\sqrt{\varepsilon}}\Big)^{\frac{\ell'}{p}}\widetilde{\theta}_i\right\|^p_{L^p((0,L)^2\times(0,\infty))}+\kappa\left\|\Big(\frac{x_3}{\sqrt{\varepsilon}}\Big)^{\frac{\ell'-2}{p}}\widetilde{\theta}_i\right\|^p_{L^p((0,L)^2\times(0,\infty))}.
\end{aligned}
\tag{2.25}
$$

Thanks to the induction hypothesis and estimate (2.13), the same computations that led to (2.24) give (2.15) for $k=0$. Again, because any tangential derivative of $\widetilde{\theta}_i$ in x_j, $j=1,2$, satisfies the equation similar to (2.14) (up to lower order derivatives in x_j, we deduce that (2.15) holds for all $k\geq 0$ as well.

To prove (2.16), we derive the IBVP for $\partial\widetilde{\theta}_i/\partial x_3$. First, we differentiate $(2.14)_{1,3}$ in x_3:

$$
\begin{cases}
\dfrac{\partial}{\partial t}\dfrac{\partial\widetilde{\theta}_i}{\partial x_3}-\varepsilon\Delta\dfrac{\partial\widetilde{\theta}_i}{\partial x_3}+\displaystyle\sum_{j=1}^2 U_j\big|_{x_3=0}\dfrac{\partial}{\partial x_j}\dfrac{\partial\widetilde{\theta}_i}{\partial x_3}+(\sigma_L+x_3\sigma')\dfrac{\partial U_3}{\partial x_3}\Big|_{x_3=0}\dfrac{\partial\widetilde{\theta}_i}{\partial x_3}\\
\quad+x_3\dfrac{\partial U_3}{\partial x_3}\Big|_{x_3=0}\dfrac{\partial^2\widetilde{\theta}_i}{\partial x_3^2}+\displaystyle\sum_{j=1}^2\dfrac{\partial\widetilde{\theta}_j}{\partial x_3}\dfrac{\partial U_i}{\partial x_j}\Big|_{x_3=0}=\dfrac{\partial\widetilde{E}_i}{\partial x_3}\quad\text{in } (0,L)^2\times(0,\infty),\\
\dfrac{\partial\widetilde{\theta}_i}{\partial x_3}=0\quad\text{at } t=0.
\end{cases}
\tag{2.26}
$$

Second, to obtain a boundary condition for $\partial\widetilde{\theta}_i/\partial x_3$, given the regularity of the data, we simply restrict $(2.14)_2$ to $x_3=0$:

$$
\frac{\partial^2\widetilde{\theta}_i}{\partial x_3^2}=-\frac{1}{\varepsilon}\widetilde{E}_i,\quad\text{at } x_3=0,
\tag{2.27}
$$

which is of order ε^{-1} by (2.18).

A standard energy estimate gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{2}\left\|\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|^2_{L^2((0,L)^2\times(0,\infty))}\right)+\varepsilon\sum_{i=1}^{2}\left\|\nabla\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|^2_{L^2((0,L)^2\times(0,\infty))}\\
&\quad\le\varepsilon\sum_{i=1}^{2}\left|\int_{(0,L)^2\times\{x_3=0\}}\frac{\partial^2\widetilde{\theta}_i}{\partial x_3^2}\frac{\partial\widetilde{\theta}_i}{\partial x_3}\,dx_1dx_2\right|+\kappa\sum_{i=1}^{2}\left\|\frac{\partial\widetilde{E}_i}{\partial x_3}\right\|^2_{L^2((0,L)^2\times(0,\infty))}\\
&\qquad+\kappa\sum_{i=1}^{2}\left\|\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|^2_{L^2((0,L)^2\times(0,\infty))}.
\end{aligned}\tag{2.28}$$

To estimate the boundary integral in the right-hand side, we use(2.27) and the trace theorem:

$$\begin{aligned}
&\varepsilon\left|\int_{(0,L)^2\times\{x_3=0\}}\frac{\partial^2\widetilde{\theta}_i}{\partial x_3^2}\frac{\partial\widetilde{\theta}_i}{\partial x_3}\,dx_1dx_2\right|\le\kappa\left\|\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|_{L^2((0,L)^2\times\{x_3=0\})}\\
&\le\kappa\left\|\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|^{\frac{1}{2}}_{L^2((0,L)^2\times(0,\infty))}\left\|\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|^{\frac{1}{2}}_{H^1((0,L)^2\times(0,\infty))}\\
&\le\kappa\left\|\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|_{L^2((0,L)^2\times(0,\infty))}+\kappa\left\|\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|^{\frac{1}{2}}_{L^2((0,L)^2\times(0,\infty))}\left\|\nabla\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|^{\frac{1}{2}}_{L^2((0,L)^2\times(0,\infty))}\\
&\le\kappa\varepsilon^{-\frac{1}{2}}+\left\|\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|^2_{L^2((0,L)^2\times(0,\infty))}\\
&+\kappa\varepsilon^{\frac{1}{2}}\left\|\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|_{L^2((0,L)^2\times(0,\infty))}\left\|\nabla\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|_{L^2((0,L)^2\times(0,\infty))}\\
&\le\kappa\varepsilon^{-\frac{1}{2}}+\left\|\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|^2_{L^2((0,L)^2\times(0,\infty))}+\frac{1}{2}\varepsilon\left\|\nabla\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|^2_{L^2((0,L)^2\times(0,\infty))},
\end{aligned}\tag{2.29}$$

so that

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{i=1}^{2}\left\|\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|^2_{L^2((0,L)^2\times(0,\infty))}\right)+\varepsilon\sum_{i=1}^{2}\left\|\nabla\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|^2_{L^2((0,L)^2\times(0,\infty))}\\
&\qquad\le\kappa_T(1+t^{-\frac{1}{2}})\varepsilon^{\frac{1}{2}}+\kappa t^{-\frac{1}{2}}\sum_{i=1}^{2}\left\|\frac{\partial\widetilde{\theta}_i}{\partial x_3}\right\|^2_{L^2((0,L)^2\times(0,\infty))}.
\end{aligned}\tag{2.30}$$

Finally, an application of Grönwall's inequality with $\exp(-2\kappa t^{1/2})$ as integrating factor gives (2.16), employing again (2.13) to replace $\widetilde{\theta}_i$ with $\theta_{i,\,L}$.

□

Our goal in the rest of this section is to derive the equations satisfied by the three components of the corrector, and estimate the data in terms of ε. We begin with $\boldsymbol{\theta}$, the main component of the corrector. Thanks to its definition, we can write,

$$
\begin{aligned}
&\frac{\partial \theta_i}{\partial t} - \varepsilon \Delta \theta_i + \sum_{j=1}^{2} U_j\big|_{x_3=0} \frac{\partial(\sigma_L\, \theta_{i,\,L})}{\partial x_j} + \sum_{j=1}^{2} U_j\big|_{x_3=h} \frac{\partial(\sigma_R\, \theta_{i,\,R})}{\partial x_j} \\
&+ x_3\, \sigma_L\, \frac{\partial U_3}{\partial x_3}\bigg|_{x_3=0} \frac{\partial(\sigma_L\, \theta_{i,\,L})}{\partial x_3} - (h - x_3)\, \sigma_R\, \frac{\partial U_3}{\partial x_3}\bigg|_{x_3=h} \frac{\partial(\sigma_R\, \theta_{i,\,R})}{\partial x_3} \\
&+ \sum_{j=1}^{2} \sigma_L\, \theta_{j,\,L}\, \frac{\partial U_i}{\partial x_j}\bigg|_{x_3=0} + \sum_{j=1}^{2} \sigma_R\, \theta_{j,\,R}\, \frac{\partial U_i}{\partial x_j}\bigg|_{x_3=h} = E_{\mathrm{temp},i}(\theta_i), \quad i = 1, 2,
\end{aligned}
\tag{2.31}
$$

where

$$
\begin{aligned}
E_{\mathrm{temp},i}(\theta_i) \quad &= -\varepsilon \Big\{ 2\sigma_L'\, \frac{\partial \theta_{i,\,L}}{\partial x_3} + \sigma_L''\, \theta_{i,\,L} - 2\sigma_R'\, \frac{\partial \theta_{i,\,R}}{\partial x_3} + \sigma_R''\, \theta_{i,\,R} \Big\} \\
&\quad + x_3\, \sigma_L\, \frac{\partial U_3}{\partial x_3}\bigg|_{x_3=0} \sigma_L' \theta_{j,\,L} - (h - x_3)\, \sigma_R\, \frac{\partial U_3}{\partial x_3}\bigg|_{x_3=h} \sigma_R' \theta_{j,\,R}.
\end{aligned}
\tag{2.32}
$$

By the estimates in Lemma 2.1,

$$
\|E_{\mathrm{temp},i}(\theta_i)\|_{L^\infty(0,T;L^2(\Omega))} \le \kappa_T \varepsilon^{\frac{3}{4}}. \tag{2.33}
$$

Also, Taylor's theorem applied to $\boldsymbol{U}$ in x_3 at $x_3 = 0$ gives, for $i, j = 1, 2$,

$$
\begin{aligned}
\bigg\| \Big(U_j - U_j\big|_{x_3=0} \Big) \frac{\partial(\sigma_L \theta_{i,\,L})}{\partial x_j} \bigg\|_{L^\infty(0,T;L^2(\Omega))} \quad &\le \kappa \bigg\| x_3 \frac{\partial(\sigma_L \theta_{i,\,L})}{\partial x_j} \bigg\|_{L^\infty(0,T;L^2(\Omega))} \\
&\le \kappa_T \varepsilon^{\frac{3}{4}},
\end{aligned}
\tag{2.34}
$$

$$
\begin{aligned}
&\bigg\| \bigg(U_3 - x_3\, \sigma_L\, \frac{\partial U_3}{\partial x_3}\bigg|_{x_3=0} \bigg) \frac{\partial(\sigma_L \theta_{i,\,L})}{\partial x_3} \bigg\|_{L^2(0,T;L^2(\Omega))} \\
&\qquad \le \kappa \bigg\| x_3^2\, \sigma_L \frac{\partial \theta_{i,\,L}}{\partial x_3} \bigg\|_{L^2(0,T;L^2(\Omega))} \le \kappa_T \varepsilon^{\frac{3}{4}},
\end{aligned}
\tag{2.35}
$$

and

$$
\begin{aligned}
\bigg\| \sigma_L \theta_{j,\,L} \bigg(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_i}{\partial x_j}\bigg|_{x_3=0} \bigg) \bigg\|_{L^\infty(0,T;L^2(\Omega))} \quad &\le \kappa \Big\| \sigma_L \theta_{j,\,L}\, x_3 \Big\|_{L^\infty(0,T;L^2(\Omega))} \\
&\le \kappa_T \varepsilon^{\frac{3}{4}}.
\end{aligned}
\tag{2.36}
$$

Consequently, we can write the (vectorial) equation for $\boldsymbol{\theta}$ as

$$
\left\{
\begin{aligned}
\frac{\partial \boldsymbol{\theta}}{\partial t} - \varepsilon \Delta \boldsymbol{\theta} + \boldsymbol{U} \cdot \nabla \boldsymbol{\theta} + \boldsymbol{\theta} \cdot \nabla \boldsymbol{U} = \boldsymbol{E}(\boldsymbol{\theta}) \quad & \text{in } \Omega \times (0, T), \\
\boldsymbol{\theta} = -\boldsymbol{u}^0 \quad & \text{on } \Gamma \times (0, T), \\
\boldsymbol{\theta}\big|_{t=0} = 0 \quad & \text{in } \Omega,
\end{aligned}
\right.
\tag{2.37}
$$

where

$$
\boldsymbol{E}(\boldsymbol{\theta}) := \big(E_1(\theta_1),\, E_2(\theta_2),\, 0 \big) \tag{2.38}
$$

satisfies the estimate,

$$
\|E_i(\theta_i)\|_{L^2(0,T;L^2(\Omega))} \le \kappa_T \varepsilon^{\frac{3}{4}}, \quad i = 1, 2. \tag{2.39}
$$

We now turn to estimating the supplementary layer functions for the corrector $\boldsymbol{\Theta}$, starting with $\boldsymbol{\varphi}$. First, using (2.10), (2.11), and (2.15) we see that

$$\begin{aligned}\Big\|\frac{\partial^{k+m}\boldsymbol{\varphi}}{\partial x_j^k \partial x_3^m}\Big\|_{L^\infty(0,T;L^\infty(\Omega))} &\leq \kappa \sum_{i,\ell=1}^{2} \Big\|\frac{\partial^{k+1}\theta_{i,\,L}}{\partial x_j^k \partial x_\ell}\Big\|_{L^\infty(0,T;L^1((0,L)^2\times(0,\infty)))} \\ &\qquad + \Big\|\frac{\partial^{k+1}\theta_{i,\,R}}{\partial x_j^k \partial x_\ell}\Big\|_{L^\infty(0,T;L^1((0,L)^2\times(-\infty,h)))} \\ &\leq \kappa_T \varepsilon^{\frac{1}{2}}, \quad j=1,2,\ k,m\geq 0. \end{aligned} \tag{2.40}$$

Next, we can write

$$\frac{\partial \varphi_i}{\partial t} = T_{i,\,1} + T_{i,\,2}, \quad i=1,2, \tag{2.41}$$

thanks to (2.4) and (2.10), where

$$\begin{aligned} T_{i,\,1} &:= \varepsilon\sigma_L' \int_0^\infty \frac{\partial^2 \theta_{i,\,L}}{\partial x_3^2}\, dx_3' + \varepsilon\sigma_R' \int_h^{-\infty} \frac{\partial^2\theta_{i,\,R}}{\partial x_3^2}\, dx_3' \\ &= -\varepsilon\sigma_L' \frac{\partial\theta_{i,\,L}}{\partial x_3}\Big|_{x_3=0} - \varepsilon\sigma_R' \frac{\partial\theta_{i,\,R}}{\partial x_3}\Big|_{x_3=h}, \\ T_{i,\,2} &:= \sigma_L' \int_0^\infty \varepsilon\Delta_{\boldsymbol{\tau}}\theta_{i,\,L} - \sum_{j=1}^{2} U_j\big|_{x_3=0} \frac{\partial \theta_{i,\,L}}{\partial x_j}\, dx_3' \\ &\quad -\sigma_L' \int_0^\infty x_3'\sigma_L \frac{\partial U_3}{\partial x_3}\Big|_{x_3=0} \frac{\partial\theta_{i,\,L}}{\partial x_3} + \sum_{j=1}^{2} \theta_{j,\,L} \frac{\partial U_i}{\partial x_j}\Big|_{x_3=0}\, dx_3' \\ &\quad +\sigma_R' \int_h^{-\infty} \varepsilon\Delta_{\boldsymbol{\tau}}\theta_{i,\,R} - \sum_{j=1}^{2} U_j\big|_{x_3=h} \frac{\partial\theta_{i,\,R}}{\partial x_j}\, dx_3' \\ &\quad -\sigma_R' \int_h^{-\infty} \widehat{x}_3'\sigma_R \frac{\partial U_3}{\partial x_3}\Big|_{x_3=h} \frac{\partial\theta_{i,\,R}}{\partial x_3} + \sum_{j=1}^{2} \theta_{j,\,R} \frac{\partial U_i}{\partial x_j}\Big|_{x_3=h}\, dx_3', \end{aligned} \tag{2.42}$$

with $\Delta_{\boldsymbol{\tau}} = \sum_{i=1,2} \partial^2/\partial^2 x_i^2$ and $\widehat{x}_3' = h - x_3'$. By the standard trace theorem,

$$\|T_{i,\,1}\|_{L^2(\Omega)} \leq \kappa_T\varepsilon \Big\|\frac{\partial\theta_i}{\partial x_3}\Big\|_{L^2(\Gamma)} \leq \kappa_T\varepsilon \Big\|\frac{\partial\theta_i}{\partial x_3}\Big\|_{L^2(\Omega)}^{\frac{1}{2}} \Big\|\frac{\partial\theta_i}{\partial x_3}\Big\|_{H^1(\Omega)}^{\frac{1}{2}}, \quad i=1,2. \tag{2.43}$$

Then, it follows again from the estimates in Lemma 2.1 that

$$\|T_{i,\,1}\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T\varepsilon^{\frac{1}{2}}, \quad i=1,2. \tag{2.44}$$

$$\|T_{i,\,2}\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa_T\varepsilon^{\frac{1}{2}}, \quad i=1,2. \tag{2.45}$$

Combining (2.44) and (2.45) for $\partial\varphi_i/\partial t$, $i=1,2$, and observing that $\partial\varphi_3/\partial t$ enjoys the same estimates, we finally see that

$$\Big\|\frac{\partial\boldsymbol{\varphi}}{\partial t}\Big\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T\varepsilon^{\frac{1}{2}}. \tag{2.46}$$

We deduce from the estimates above that

$$\Big\| \boldsymbol{E}(\boldsymbol{\varphi}) := \frac{\partial \boldsymbol{\varphi}}{\partial t} - \varepsilon \Delta \boldsymbol{\varphi} + \boldsymbol{U} \cdot \nabla \boldsymbol{\varphi} + \boldsymbol{\varphi} \cdot \nabla \boldsymbol{U} \Big\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{2}}. \tag{2.47}$$

In addition,

$$\boldsymbol{\varphi}|_\Gamma = \boldsymbol{\varphi}|_{t=0} = 0. \tag{2.48}$$

We tackle $\boldsymbol{\psi}$, which is defined in (2.10). We temporarily set

$$\psi_{i,\,L} := \sigma_L' \int_0^\infty \theta_{i,\,L} \, dx_3', \quad i = 1, 2, \tag{2.49}$$

which, by (2.4), satisfies

$$\begin{aligned} &\frac{\partial \psi_{i,\,L}}{\partial t} - \varepsilon \Delta \psi_{i,\,L} + \sum_{j=1}^{2} U_j\big|_{x_3=0} \frac{\partial \psi_{i,\,L}}{\partial x_j} + x_3 \sigma_L \frac{\partial U_3}{\partial x_3}\Big|_{x_3=0} \frac{\partial \psi_{i,\,L}}{\partial x_3} \\ &\qquad + \sum_{j=1}^{2} \psi_{j,\,L} \frac{\partial U_i}{\partial x_j}\Big|_{x_3=0} = \widehat{E}_{\mathrm{temp},i}(\theta_{i,\,L}), \end{aligned} \tag{2.50}$$

where

$$\widehat{E}_{\mathrm{temp},i}(\theta_{i,\,L}) = -\varepsilon \sigma_L''' \int_\infty^{x_3} \theta_{i,\,L} \, dx_3' - 2\varepsilon \sigma_L'' \theta_{i,\,L} + x_3 \sigma_L'' \int_\infty^{x_3} \theta_{i,\,L} \, dx_3', \quad i = 1, 2. \tag{2.51}$$

Using Lemma 2.1 once again, we can estimate this source term by

$$\Big\| \Big(\frac{x_3}{\sqrt{\varepsilon}} \Big)^{\frac{\ell}{p}} \widehat{E}_{\mathrm{temp},i}(\theta_{i,\,L}) \Big\|_{L^\infty(0,T;L^p(\Omega))} \leq \kappa_T \, \varepsilon^{\frac{1}{2}}, \quad \ell \geq 0, \ 1 \leq p \leq \infty, \tag{2.52}$$

noticing that σ_L and all its derivatives vanish for $x_3 \geq 1/2$, so that

$$\begin{aligned} \Big| \Big(\frac{x_3}{\sqrt{\varepsilon}} \Big)^{\ell} \sigma_L \textstyle\int_{x_3}^\infty \theta_{i,\,L} \, dx_3' \Big| &\leq \sigma_L \textstyle\int_{x_3}^\infty \Big(\frac{1/2}{\sqrt{\varepsilon}} \Big)^{\ell} |\theta_{i,\,L}| \, dx_3' \\ &\leq \kappa \Big\| \Big(\frac{x_3}{\sqrt{\varepsilon}} \Big)^{\ell} \theta_{i,\,L} \Big\|_{L^1((0,L)^2 \times (0,\infty))}, \end{aligned} \tag{2.53}$$

for $\ell \geq 0$ and $0 < t < T$. An energy estimate then gives

$$\begin{aligned} &\Big\| \Big(\frac{x_3}{\sqrt{\varepsilon}} \Big)^{\frac{\ell}{2p}} \frac{\partial^k \psi_{i,\,L}}{\partial x_j^k} \Big\|_{L^\infty(0,T;L^p(\Omega))} + \varepsilon^{\frac{1}{2p} + \frac{1}{4}} \Big\| \Big(\frac{x_3}{\sqrt{\varepsilon}} \Big)^{\frac{\ell}{2}} \nabla \frac{\partial^k \psi_{i,\,L}}{\partial x_j^k} \Big\|_{L^2(0,T;L^2(\Omega))} \\ &\leq \kappa_T \, \varepsilon^{\frac{1}{2p}}, \end{aligned} \tag{2.54}$$

for $i, j = 1, 2$ and $k, \ell \geq 0$, and $1 \leq p \leq \infty$.

From (2.10) and (2.11), it follows that $(\psi_i - \psi_{i,\,L})$ (hence ψ_i) enjoys the same estimate as in (2.54) with x_3 replaced by $\widehat{x}_3 = h - x_3$. In addition, given its definition, the normal component ψ_3 satisfies the same type of estimates as well. We conclude from (2.34), (2.35), and (2.36) that $\boldsymbol{\psi}$ satisfies

$$\begin{cases} \dfrac{\partial \boldsymbol{\psi}}{\partial t} - \varepsilon \Delta \boldsymbol{\psi} + \boldsymbol{U} \cdot \nabla \boldsymbol{\psi} + \boldsymbol{\psi} \cdot \nabla \boldsymbol{U} = \boldsymbol{E}(\boldsymbol{\psi}) & \text{in } \Omega \times (0,T), \\ \boldsymbol{\psi} = 0 & \text{on } \Gamma \times (0,T), \\ \boldsymbol{\psi}|_{t=0} = 0 & \text{in } \Omega, \end{cases} \tag{2.55}$$

where

$$\|\boldsymbol{E}(\boldsymbol{\psi})\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{2}}, \quad i = 1, 2. \tag{2.56}$$

We are now in a position to prove our main result when the geometry is that of a (periodized) channel.

2.3. Proof of Theorem 1.1: The case of a 3D channel domain

Setting

$$\boldsymbol{w}^\varepsilon := \boldsymbol{u}^\varepsilon - (\boldsymbol{u}^0 + \boldsymbol{\Theta}), \qquad \pi^\varepsilon := p^\varepsilon - p^0,$$

we employ the equations satisfied by the corrector $\boldsymbol{\Theta}$ and by u^ε and u^0 (equations (1.3), (1.6), (2.1), (2.37), (2.47), (2.48), (2.55)) along with the divergence-free condition to write the IBVP for the error $(\boldsymbol{w}^\varepsilon, \pi^\varepsilon)$ as

$$\begin{cases} \dfrac{\partial \boldsymbol{w}^\varepsilon}{\partial t} - \varepsilon \Delta \boldsymbol{w}^\varepsilon + \boldsymbol{U} \cdot \nabla \boldsymbol{w}^\varepsilon + \boldsymbol{w}^\varepsilon \cdot \nabla \boldsymbol{U} + \nabla \pi^\varepsilon = \boldsymbol{E}(\boldsymbol{\Theta}) & \text{in } \Omega \times (0,T), \\ \operatorname{div} \boldsymbol{w}^\varepsilon = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{w}^\varepsilon = 0 & \text{on } \Gamma \times (0,T), \\ \boldsymbol{w}^\varepsilon|_{t=0} = 0 & \text{in } \Omega, \end{cases} \tag{2.57}$$

where

$$\boldsymbol{E}(\boldsymbol{\Theta}) = \boldsymbol{E}(\boldsymbol{\theta}) + \boldsymbol{E}(\boldsymbol{\varphi}) + \boldsymbol{E}(\boldsymbol{\psi}). \tag{2.58}$$

A simple energy estimate gives (1.9), thanks to the bounds (2.39), (2.47), and (2.56).

Finally, the vanishing viscosity limit (1.10) follows from (1.9) and the smallness of the corrector in $L^\infty(0,T;L^2(\Omega))$.

3. The case of a 3D smooth domain

We now turn to the study of the boundary layer of LNSE (1.3) in the more general and difficult case when Ω is a bounded domain in $\mathbb{R}^3$ with curved boundary Γ. Following the analysis of the Stokes problem, we will utilize a curvilinear system adapted to the boundary.

3.1. Elements of differential geometry

We assume that a bounded domain Ω in $\mathbb{R}^3$ has boundary Γ given by a compact, orientable 2D manifold of class C^∞. We choose a small $\delta > 0$ and define a tubular (collar) neighborhood $\Omega_{3\delta}$ of Ω as the set of all point in Ω within distance 3δ of Γ. We will place coordinates on $\Omega_{3\delta}$ following the procedure described in detail [7], which we now summarize.

Because Γ is compact, we can cover it with a finite number of overlapping charts. We will develop the corrector in a single chart, the resulting estimates applying to the whole manifold because the number of charts is finite. Let us focus, then, on a single chart on Γ, on which have a curvilinear coordinate system which we label $\boldsymbol{\xi}' = (\xi_1, \xi_2) \in \omega$, where ω is an open subset of $\mathbb{R}^2$. This means that there exists a smooth function, $\widetilde{\boldsymbol{x}} \colon \omega \to \Gamma$ mapping points in ω to points on Γ. (There is a condition on the transition maps between charts, but such conditions do not concern us here.)

Letting

$$\widetilde{\boldsymbol{g}}_i(\xi') := \frac{\partial \widetilde{\boldsymbol{x}}}{\partial \xi_i}, \quad i = 1, 2$$

gives a covariant basis, $(\widetilde{\boldsymbol{g}}_1, \widetilde{\boldsymbol{g}}_2)$, locally on Γ. We do not assume orthogonality of this frame. We extend our coordinate system to a chart on $\Omega_{3\delta}$ by setting ξ_3 to be the negative of the distance from a point in $\Omega_{3\delta}$ to the boundary. We label the point, $\boldsymbol{\xi} = (\boldsymbol{\xi}', \xi_3) = (\xi_1, \xi_2, \xi_3)$, and we have a covariant basis, $(\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3)$, locally of $\Omega_{3\delta}$, where

$$\boldsymbol{g}_i(\boldsymbol{\xi}) = \frac{\partial \boldsymbol{x}}{\partial \xi_i}(\boldsymbol{\xi}) = \widetilde{\boldsymbol{g}}_i(\xi') - \xi_3 \frac{\partial \boldsymbol{n}}{\partial \xi_i}(\xi'),\ i = 1, 2, \quad \boldsymbol{g}_3(\boldsymbol{\xi}) = -\boldsymbol{n}(\xi').$$

Let $g_{ij} := \boldsymbol{g}_i \cdot \boldsymbol{g}_j$ and $g = \det(g_{ij})_{1 \le i,j \le 3}$.

As shown in [7], we can write the metric tensor in covariant form as

$$(g_{ij}) = \begin{pmatrix} g_{22} & -g_{12} & 0 \\ -g_{21} & g_{11} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $g := \det(g_{ij})_{1 \le i,j \le 3} > 0$ locally in $\Omega_{3\delta}$. The function, $h := g^{1/2} > 0$, is the magnitude of the Jacobian determinant for the transformation from $\boldsymbol{x}$ to $\boldsymbol{\xi}$.

From the covariant basis, $(\boldsymbol{g}_i)$, we introduce

$$h_i := |\boldsymbol{g}_i| = h_i(\boldsymbol{\xi}),\ i = 1, 2, \quad h_3 = 1. \tag{3.1}$$

Defining the normalized covariant basis, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, where

$$\boldsymbol{e}_i := \frac{\boldsymbol{g}_i}{|\boldsymbol{g}_i|},$$

we represent a vector-valued function, $\boldsymbol{F}$, in the form

$$\boldsymbol{F} = \sum_{i=1}^{3} F^i(\boldsymbol{\xi}) \boldsymbol{e}_i. \tag{3.2}$$

We now have the tools we need to represent covariant differential operators for smooth functions on $\Omega_{3\delta}$ in an effective manner. The divergence operator acting on $\boldsymbol{F}$ can then be written in the $\boldsymbol{\xi}$ coordinates as

$$\operatorname{div} \boldsymbol{F} = \frac{1}{h}\sum_{i=1}^{2} \frac{\partial}{\partial \xi_i}\Big(\frac{h}{h_i}F^i\Big) + \frac{1}{h}\frac{\partial (hF^3)}{\partial \xi_3}, \tag{3.3}$$

while the Laplacian of $\boldsymbol{F}$ takes the form,

$$\Delta \boldsymbol{F} = \sum_{i=1}^{3}\Big(\mathcal{S}^i\boldsymbol{F} + \mathcal{L}F^i + \frac{\partial^2 F^i}{\partial \xi_3^2}\Big)\boldsymbol{e}_i, \tag{3.4}$$

where

$$\begin{cases} \mathcal{S}^i\boldsymbol{F} = \begin{pmatrix} \text{linear combination of tangential derivatives} \\ \text{of } F^j,\ 1 \le j \le 3, \text{ in } \xi', \text{ up to order 2} \end{pmatrix}, \\ \mathcal{L}F^i = \big(\text{proportional to } \partial F^i/\partial \xi_3\big). \end{cases} \tag{3.5}$$

The coefficients of $\mathcal{S}^i$ and $\mathcal{L}^i$, $1 \le i \le 3$ in (3.5), are multiples of h , $1/h$, h_i , $1/h_i$, $i = 1, 2$, g_{12}, g_{21}, and their derivatives.

Finally, we compute the covariant derivative $\nabla_{\boldsymbol{F}}\boldsymbol{G}$, of $\boldsymbol{G}$ in the direction $\boldsymbol{F}$ for smooth vector fields $\boldsymbol{F}$, $\boldsymbol{G} : \Omega_{3\delta,\boldsymbol{\xi}} \to \mathbb{R}^3$ in the $\boldsymbol{\xi}$ coordinates:

$$\nabla_{\boldsymbol{F}}\boldsymbol{G} = \sum_{i=1}^{3}\Big\{\mathcal{P}^i(\boldsymbol{F},\boldsymbol{G}) + F^3\frac{\partial G^i}{\partial \xi_3} + \mathcal{Q}^i(\boldsymbol{F},\boldsymbol{G}) + \mathcal{R}^i(\boldsymbol{F},\boldsymbol{G})\Big\}\boldsymbol{e}_i, \tag{3.6}$$

where

$$\mathcal{P}^i(\boldsymbol{F},\boldsymbol{G}) = \begin{pmatrix} \text{linear combination of the products of} \\ \text{the tangential component } F^1 \text{ or } F^2 \text{ and} \\ \text{the tangential derivative of } G_i \text{ in } \xi_j,\ j = 1, 2 \end{pmatrix}, \tag{3.7}$$

$$\mathcal{Q}^i(\boldsymbol{F},\boldsymbol{G}) = \big(\text{linear combination of } F^jG^k,\ 1 \le j,k \le 2\big), \tag{3.8}$$

$$\mathcal{R}^i(\boldsymbol{F},\boldsymbol{G}) = (\text{linear combination of } F^jG^k,\ j = 3 \text{ or } k = 3). \tag{3.9}$$

The $\mathcal{Q}^i(\boldsymbol{F},\boldsymbol{G})$ and $\mathcal{R}^i(\boldsymbol{F},\boldsymbol{G})$ are related to the Christoffel symbols of the second kind, which comes from the twisting effects of the curvilinear system $\boldsymbol{\xi}$. For the case of an orthogonal system, the explicit expressions are given in Appendix 2 of [1].

The formula above for the covariant derivative will be used to compute the convective term in the curvilinear coordinate system.

3.2. Asymptotic expansion of solutions to LNSE

As in the case of the 3D channel, we postulate an expansion for the approximate LNSE in the form

$$\boldsymbol{u}^\varepsilon \approx \boldsymbol{u}^0 + \boldsymbol{\Theta}, \qquad p^\varepsilon \approx p^0 + q, \tag{3.10}$$

where $\boldsymbol{\Theta}$ is again the velocity corrector, and we have now also a pressure corrector q.

We construct the correctors using the coordinates $\boldsymbol{\xi}$, that is, in the collar neighborhood $\Omega_{3\delta}$). On this collar neighborhood, we implictly assume the representation (3.2) for all vector fields. We remark that δ is chosen independently of ε, and hence the collar neighborhood contains the viscous boundary layer for all sufficiently small ε.

We can again formally derive the equations for the correctors from (1.3) and (1.6), though we no longer assume the pressures are identical. This gives

$$\begin{cases} \dfrac{\partial \boldsymbol{\Theta}}{\partial t} - \varepsilon\Delta\boldsymbol{\Theta} + \boldsymbol{U}\cdot\nabla\boldsymbol{\Theta} + \boldsymbol{\Theta}\cdot\nabla\boldsymbol{U} + \nabla(p^\varepsilon - p^0) \approx \varepsilon\Delta(\boldsymbol{U}+\boldsymbol{u}^0) \quad \text{in } \Omega, \\ \qquad\qquad\qquad \operatorname{div}\,\boldsymbol{\Theta} = 0 \quad \text{in } \Omega, \\ \qquad\qquad\qquad \boldsymbol{\Theta} = -\boldsymbol{u}^0 \quad \text{on } \Gamma, \\ \qquad\qquad\qquad \boldsymbol{\Theta}|_{t=0} = 0 \quad \text{in } \Omega. \end{cases} \tag{3.11}$$

The formal asymptotic expansion is performed along the same lines as that for the channel. Using the coordinate ξ_3 allows us to make scaling arguments similar to those in the Prandtl theory, which lead to a viscous boundary layer of thickness $\varepsilon^{1/2}$, and to the assumption that $\partial/\partial\xi_i \simeq \sqrt{\varepsilon}(\partial/\partial\xi_3)$, $i=1,2$. Then, from (3.3) and $(3.11)_2$, it follows that $\Theta^i \simeq \varepsilon^{-1/2}\,\Theta^3$, $i=1,2$. Using these observations as well as the differential geometric foumulas of Section 3.1, we write the equations (3.11) in the $\boldsymbol{\xi}$ variables, and collect the leading order terms in ε, yielding

$$\begin{cases} \dfrac{\partial \Theta^i}{\partial t} - \varepsilon\dfrac{\partial^2\Theta^i}{\partial\xi_3^2} + \mathcal{P}^i(\boldsymbol{U},\boldsymbol{\Theta}) + U^3\dfrac{\partial\Theta^i}{\partial\xi_3} + \mathcal{Q}^i(\boldsymbol{U},\boldsymbol{\Theta}) + \mathcal{P}^i(\boldsymbol{\Theta},\boldsymbol{U}) + \mathcal{Q}^i(\boldsymbol{\Theta},\boldsymbol{U}) \\ \qquad\qquad \approx 0 \quad \text{in } \Omega_{3\delta}\times(0,T) \text{ (at least)}, \quad i=1,2, \\ \mathcal{Q}^3(\boldsymbol{\Theta},\boldsymbol{U}) + \mathcal{Q}^3(\boldsymbol{U},\boldsymbol{\Theta}) + \dfrac{\partial q}{\partial\xi_3} = 0 \quad \text{in } \Omega_{3\delta}\times(0,T) \text{ (at least)}, \\ \qquad \operatorname{div}\,\boldsymbol{\Theta} = 0 \quad \text{in } \Omega\times(0,T), \\ \qquad \Theta^i = -\widetilde{u}^i \quad \text{on } \Gamma\times(0,T), \quad i=1,2, \\ \qquad \Theta^3 = 0 \quad \text{on } \Gamma\times(0,T), \\ \qquad \boldsymbol{\Theta}|_{t=0} = 0 \quad \text{in } \Omega. \end{cases} \tag{3.12}$$

Here and below, for any function f expressed in the $\boldsymbol{\xi}$ variables, we denote by $\widetilde{f}$ the restriction to the plane $\xi_3=0$ in $\mathbb{R}^3_{\boldsymbol{\xi}}$. So, for instance, $\widetilde{u}^i := \big(\boldsymbol{u}^0\cdot\boldsymbol{e}_i\big)|_{\xi_3=0}$.

In contrast to the 3D channel of Section 2, the curvature of the domain induces a small effect on the tangential components Θ^i, $i=1,2$, in the normal direction, which requires a pressure corrector q to cancel. Our task now is to build a corrector $\boldsymbol{\Theta}$ as an approximating solution to this system.

We exploit the insight gained from the construction of the incompressible corrector in Section 2.1, performing the matching asymptotics in the equations $(3.12)_1$ and collecting the leading order terms with respect to a small parameter ε. To construct the approximate solution to the corrector equations, we again solve a drift-diffusion equation in the half space $\xi_3>0$

in $\mathbb{R}^3_{\boldsymbol{\xi}}$, and use the solution in the tangential components of $\boldsymbol{\Theta}$:

$$\begin{cases} \dfrac{\partial \theta^i}{\partial t} - \varepsilon \Delta_{\boldsymbol{\xi}} \theta^i + \widetilde{\mathcal{P}}^i(\widetilde{\boldsymbol{U}}, \boldsymbol{\theta}) + \xi_3\, \sigma\, \dfrac{\partial \widetilde{U}^3}{\partial \xi_3} \dfrac{\partial \theta^i}{\partial \xi_3} + \widetilde{\mathcal{Q}}^i(\widetilde{\boldsymbol{U}}, \boldsymbol{\theta}) + \widetilde{\mathcal{P}}^i(\boldsymbol{\theta}, \widetilde{\boldsymbol{U}}) + \widetilde{\mathcal{Q}}^i(\boldsymbol{\theta}, \widetilde{\boldsymbol{U}}) \\ \qquad = 0, \quad \text{in } \omega \times (0,\infty) \times (0,T), \\ \theta^i = -\widetilde{u}^i, \quad \text{at } \xi_3 = 0, \\ \boldsymbol{\theta} = 0, \quad \text{at } t = 0, \end{cases} \tag{3.13}$$

where $\boldsymbol{\theta} = \sum_{i=1}^{2} \theta^i\, \boldsymbol{e}_i$ and σ is a smooth cut-off function near the boundary, such that

$$\sigma(\xi_3) = \begin{cases} 1, \ 0 \le \xi_3 \le \delta, \\ 0, \ \xi_3 \ge 2\delta. \end{cases} \tag{3.14}$$

The $\widetilde{\mathcal{P}}^i(\widetilde{\boldsymbol{U}}, \boldsymbol{\theta})$ is the value of $\mathcal{P}^i(\boldsymbol{U}, \boldsymbol{\theta})$ with $\boldsymbol{U}$ and the other geometric functions, e.g., g and h, evaluated at $\xi_3 = 0$. The other terms, $\widetilde{\mathcal{Q}}^i(\widetilde{\boldsymbol{U}}, \boldsymbol{\theta})$, $\widetilde{\mathcal{P}}^i(\boldsymbol{\theta}, \widetilde{\boldsymbol{U}})$, and $\widetilde{\mathcal{Q}}^i(\boldsymbol{\theta}, \widetilde{\boldsymbol{U}})$, are defined in a similar way.

In addition, for convenience, we have set, with a slight abuse of notation,

$$\Delta_{\boldsymbol{\xi}} v = \sum_{i=1}^{3} \frac{\partial^2 v}{\partial \xi_i^2} \quad \text{for any scalar function } v \text{ defined in } \omega \times (0,\infty), \tag{3.15}$$

(which is not the Laplacian expressed in the $\boldsymbol{\xi}$ variables.) Hence, equation (3.13) depends on ξ_3 only through $\boldsymbol{\theta}$ and the terms containing ξ_3.

As we did for a channel domain, we define the tangential components Θ_i, $i = 1, 2$, of the corrector $\boldsymbol{\Theta}$ to be

$$\Theta^i(\boldsymbol{\xi}, t) = \frac{h_i}{h}(\boldsymbol{\xi})\, \frac{\widetilde{h}}{\widetilde{h}_i}(\boldsymbol{\xi}', 0)\, \frac{\partial}{\partial \xi_3}\Big\{ \sigma(\xi_3) \int_0^{\xi_3} \theta^i(\boldsymbol{\xi}', \eta, t)\, d\eta \Big\}, \quad i = 1, 2. \tag{3.16}$$

Then, using (3.3), we define the normal component Θ_3 by enforcing the divergence-free constraint on $\boldsymbol{\Theta}$; that is,

$$\Theta^3(\boldsymbol{\xi}, t) = -\frac{1}{h}(\boldsymbol{\xi})\, \sigma(\xi_3) \sum_{i=1}^{2} \frac{\partial}{\partial \xi_i} \Big\{ \frac{\widetilde{h}}{\widetilde{h}_i}(\boldsymbol{\xi}', 0) \int_0^{\xi_3} \theta^i(\boldsymbol{\xi}', \eta, t)\, d\eta \Big\}. \tag{3.17}$$

As a consequence of this construction, $\boldsymbol{\Theta}$ belongs to the space H, since

$$\operatorname{div} \boldsymbol{\Theta} = 0 \text{ in } \Omega \quad \text{and} \quad \Theta_3 = 0 \text{ on } \Gamma \text{ at } \xi_3 = 0.$$

To elucidate the structure of the corrector further, we write $\boldsymbol{\Theta}$ as a sum of three vector fields in the form,

$$\boldsymbol{\Theta} = \boldsymbol{\theta} + \boldsymbol{\varphi} + \boldsymbol{\psi}, \tag{3.18}$$

where, for $i = 1, 2$,

$$\begin{cases} \theta^i = \dfrac{h_i}{h}\,\dfrac{\widetilde{h}}{\widetilde{h}_i}\,\sigma\,\theta^i, \\[2ex] \varphi^i = \dfrac{h_i}{h}\,\dfrac{\widetilde{h}}{\widetilde{h}_i}\,\sigma' \displaystyle\int_0^\infty \theta^i\,d\eta, \\[2ex] \psi^i = \dfrac{h_i}{h}\,\dfrac{\widetilde{h}}{\widetilde{h}_i}\,\sigma' \displaystyle\int_\infty^{\xi_3} \theta^i\,d\eta, \end{cases} \tag{3.19}$$

and

$$\begin{cases} \theta^3 = 0, \\[2ex] \varphi^3 = -\dfrac{1}{h}\,\sigma \displaystyle\sum_{i=1}^{2} \dfrac{\partial}{\partial \xi_i}\Big\{ \dfrac{\widetilde{h}}{\widetilde{h}_i} \displaystyle\int_0^\infty \theta^i\,d\eta \Big\}, \\[2ex] \psi^3 = -\dfrac{1}{h}\,\sigma \displaystyle\sum_{i=1}^{2} \dfrac{\partial}{\partial \xi_i}\Big\{ \dfrac{\widetilde{h}}{\widetilde{h}_i} \displaystyle\int_\infty^{\xi_3} \theta^i\,d\eta \Big\}. \end{cases} \tag{3.20}$$

As was the cases for a channel domain, $\boldsymbol{\theta}$, the main part of $\boldsymbol{\Theta}$, is a fast decaying *boundary layer function* which agrees with one in the classical theory of boundary layers, while the remaining parts $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are small *supplementary vector fields* (with respect to a small ε) to ensure that $\boldsymbol{\Theta}$ belongs to the space H.

We define the pressure corrector q in the form,

$$q = \sigma \int_0^{\xi_3} \sigma^{-1} \Big(\widetilde{\mathcal{Q}}^3(\boldsymbol{\theta}, \widetilde{\boldsymbol{U}}) + \widetilde{\mathcal{Q}}^3(\widetilde{\boldsymbol{U}}, \boldsymbol{\theta}) \Big)\,d\eta. \tag{3.21}$$

With the choice made above for velocity corrector $\boldsymbol{\Theta}$, which naturally follows from a Prandtl-type analysis, there is small error (of order $\varepsilon^{1/4}$ in L^2) in $(3.12)_2$ (see (3.32) below as well).Then, $-\partial q/\partial \xi_3 = -\mathcal{Q}^3(\boldsymbol{\Theta}, \boldsymbol{U}) - \mathcal{Q}^3(\boldsymbol{U}, \boldsymbol{\Theta})$ up to a small error, as discussed in more detail later.

3.3. Estimates on the corrector

As for the channel, our goal in this section is to derive estimates for the corrector, by deriving the equations that its three parts satisfy and estimating the data in terms of ε.

First, we prove the estimates on θ^i, $i = 1, 2$, below exactly in the same fashion as in the proof of Lemma 2.1. For $i, j = 1, 2$ and $k, \ell \geq 0$, and $1 \leq p \leq \infty$, we have

$$\begin{aligned} &\Big\| \Big(\frac{\xi_3}{\sqrt{\varepsilon}} \Big)^{\frac{\ell}{p}} \frac{\partial^k \theta^i}{\partial \xi_j^k} \Big\|_{L^\infty(0,T;L^p(\omega \times (0,\infty)))} \\ &+ \varepsilon^{\frac{1}{2p}+\frac{1}{4}} \Big\| \Big(\frac{\xi_3}{\sqrt{\varepsilon}} \Big)^{\frac{\ell}{2}} \nabla \frac{\partial^k \theta^i}{\partial \xi_j^k} \Big\|_{L^2(0,T;L^2(\omega \times (0,\infty)))} \leq \kappa_T\, \varepsilon^{\frac{1}{2p}}, \quad \text{and} \end{aligned} \tag{3.22}$$

$$\Big\| \frac{\partial^{k+1}\theta^i}{\partial \xi_j^k \partial \xi_3} \Big\|_{L^\infty(0,T;L^2(\omega\times(0,\infty)))} + \varepsilon^{\frac{1}{2}} \Big\| \nabla \frac{\partial^{k+1}\theta^i}{\partial \xi_j^k \partial \xi_3} \Big\|_{L^2(0,T;L^2(\omega\times(0,\infty)))} \leq \kappa_T\, \varepsilon^{-\frac{1}{4}}, \tag{3.23}$$

for a constant κ_T depending on T and other data, but independent of ε.

We derive the equation for $\boldsymbol{\theta}$ from its definition (3.18)-(3.20) and equation (3.13) for θ^i:

$$\begin{aligned} &\frac{\partial \theta^i}{\partial t} - \varepsilon \Delta \boldsymbol{\theta} \cdot \boldsymbol{e}_i + \widetilde{\mathcal{P}}^i(\widetilde{\boldsymbol{U}}, \boldsymbol{\theta}) + \xi_3\, \sigma\, \widetilde{\frac{\partial U^3}{\partial \xi_3}} \frac{\partial \theta^i}{\partial \xi_3} + \widetilde{\mathcal{Q}}^i(\widetilde{\boldsymbol{U}}, \boldsymbol{\theta}) + \widetilde{\mathcal{P}}^i(\boldsymbol{\theta}, \widetilde{\boldsymbol{U}}) + \widetilde{\mathcal{Q}}^i(\boldsymbol{\theta}, \widetilde{\boldsymbol{U}}) \\ &= E^i_{\text{temp}}(\boldsymbol{\theta}), \quad i = 1, 2, \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} E^i_{\text{temp}}(\boldsymbol{\theta}) &= -\varepsilon(\Delta \boldsymbol{\theta} \cdot \boldsymbol{e}_i - \Delta_{\boldsymbol{\xi}} \theta^i) \\ &\quad -\varepsilon \frac{\partial^2}{\partial \xi_3^2}\Big(\sigma \frac{h_i}{h}\Big) \frac{\widetilde{h}}{\widetilde{h}_i}\, \theta^i - 2\varepsilon \frac{\partial}{\partial \xi_3}\Big(\sigma \frac{h_i}{h}\Big) \frac{\widetilde{h}}{\widetilde{h}_i} \frac{\partial \theta^i}{\partial \xi_3} \\ &\quad + \sum_{j=1}^{2} \frac{1}{\widetilde{h}_j} \widetilde{U}^j \frac{\partial}{\partial \xi_j}\Big(\frac{h_i}{h} \frac{\widetilde{h}}{\widetilde{h}_i}\Big) \sigma\, \theta^i + \xi_3\, \sigma\, \widetilde{\frac{\partial U^3}{\partial \xi_3}} \frac{\partial}{\partial \xi_3}\Big(\sigma \frac{h_i}{h}\Big) \widetilde{\frac{h}{h_i}}\, \theta^i. \end{aligned} \tag{3.25}$$

Using the differential geometric formulae for the differential operators as well as the estimates (3.22), we notice that

$$\| E^i_{\text{temp}}(\boldsymbol{\theta}) \|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{3}{4}}. \tag{3.26}$$

To estimate the convective terms, we apply Taylor's theorem at $\xi_3 = 0$ to $\boldsymbol{U}$ and use the estimates in (3.22) to obtain,

$$\big\| \mathcal{P}^i(\boldsymbol{U}, \boldsymbol{\theta}) - \widetilde{\mathcal{P}}^i(\widetilde{\boldsymbol{U}}, \boldsymbol{\theta}) \big\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa \sum_{j=1}^{2} \Big\| \xi_3 \sigma \frac{\partial \theta^i}{\partial x_j} \Big\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{3}{4}}, \tag{3.27}$$

$$\Big\| \Big(U^3 - \xi_3 \sigma \frac{\partial \widetilde{U}^3}{\partial \xi_3} \Big) \frac{\partial \theta^i}{\partial x_3} \Big\|_{L^2(0,T;L^2(\Omega))} \leq \kappa \Big\| \xi_3^2\, \sigma \frac{\partial \theta^i}{\partial \xi_3} \Big\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{3}{4}}, \tag{3.28}$$

and

$$\begin{aligned} &\big\| \mathcal{Q}^i(\boldsymbol{U}, \boldsymbol{\theta}) - \widetilde{\mathcal{Q}}^i(\widetilde{\boldsymbol{U}}, \boldsymbol{\theta}) \big\|_{L^\infty(0,T;L^2(\Omega))} + \big\| \mathcal{P}^i(\boldsymbol{\theta}, \boldsymbol{U}) - \widetilde{\mathcal{P}}^i(\boldsymbol{\theta}, \widetilde{\boldsymbol{U}}) \big\|_{L^\infty(0,T;L^2(\Omega))} \\ &+ \big\| \mathcal{Q}^i(\boldsymbol{\theta}, \boldsymbol{U}) - \widetilde{\mathcal{Q}}^i(\boldsymbol{\theta}, \widetilde{\boldsymbol{U}}) \big\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa \sum_{j=1}^{2} \| \xi_3 \theta^j \|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{3}{4}}. \end{aligned} \tag{3.29}$$

From all the estimates above, we find that the tangential part of the equation for $\boldsymbol{\theta}$ can be written as, for $i = 1, 2$,

$$\frac{\partial \theta^i}{\partial t} - \varepsilon \Delta \boldsymbol{\theta} \cdot \boldsymbol{e}_i + (\boldsymbol{U} \cdot \nabla \boldsymbol{\theta}) \cdot \boldsymbol{e}_i + (\boldsymbol{\theta} \cdot \nabla \boldsymbol{U}) \cdot \boldsymbol{e}_i = E^i(\boldsymbol{\theta}), \tag{3.30}$$

where

$$\|E^i(\boldsymbol{\theta})\|_{L^2(0,T;L^2(\Omega))} \le \kappa_T \varepsilon^{\frac{3}{4}}, \quad i = 1,2. \tag{3.31}$$

We proceed in a similar fashion for the normal component of the $\boldsymbol{\theta}$-equation. First, the differential geometric formulae (3.4) and (3.7)-(3.9) give

$$\left(\frac{\partial \boldsymbol{\theta}}{\partial t} - \varepsilon\Delta\boldsymbol{\theta} + \boldsymbol{U}\cdot\nabla\boldsymbol{\theta} + \boldsymbol{\theta}\cdot\nabla\boldsymbol{U}\right)\cdot\boldsymbol{e}_3 = -\varepsilon\mathcal{S}^3\boldsymbol{\theta} + \mathcal{Q}^3(\boldsymbol{U},\boldsymbol{\theta}) + \mathcal{P}^3(\boldsymbol{\theta},\boldsymbol{U}) + \mathcal{Q}^3(\boldsymbol{\theta},\boldsymbol{U}). \tag{3.32}$$

Noticing that the leading order term on the right-hand side of (3.32) is $\widetilde{\mathcal{Q}}^3(\boldsymbol{\theta},\widetilde{\boldsymbol{U}}) + \widetilde{\mathcal{Q}}^3(\widetilde{\boldsymbol{U}},\boldsymbol{\theta})$, we are led to define a pressure corrector q as in (3.21). Then,

$$\left(\frac{\partial \boldsymbol{\theta}}{\partial t} - \varepsilon\Delta\boldsymbol{\theta} + \boldsymbol{U}\cdot\nabla\boldsymbol{\theta} + \boldsymbol{\theta}\cdot\nabla\boldsymbol{U}\right)\cdot\boldsymbol{e}_3 - \nabla q = \boldsymbol{E}(q), \qquad \boldsymbol{E}(q) = \sum_{i=1}^{3} E^i(q)\boldsymbol{e}_i, \tag{3.33}$$

where

$$\begin{cases} E^i(q) = (\text{linear combination of the tangential derivatives of } q),\ i = 1,2, \\ E^3(q) = (\text{RHS of (3.32)}) - \dfrac{\partial}{\partial \xi_3}\left(\sigma \displaystyle\int_0^{\xi_3} \sigma^{-1}\left(\widetilde{\mathcal{Q}}^3(\boldsymbol{\theta},\widetilde{\boldsymbol{U}}) + \widetilde{\mathcal{Q}}^3(\widetilde{\boldsymbol{U}},\boldsymbol{\theta})\right) d\eta\right). \end{cases} \tag{3.34}$$

Thanks to (3.22) and estimates similar to those in (3.29), one can verify that

$$\|\boldsymbol{E}(q)\|_{L^\infty(0,T;L^2(\Omega))} \le \kappa_T \varepsilon^{\frac{1}{2}}. \tag{3.35}$$

Combining (3.30) and (3.33), gives finally the system satisfied by $(\boldsymbol{\theta}, q)$:

$$\begin{cases} \dfrac{\partial \boldsymbol{\theta}}{\partial t} - \varepsilon\Delta\boldsymbol{\theta} + \boldsymbol{U}\cdot\nabla\boldsymbol{\theta} + \boldsymbol{\theta}\cdot\nabla\boldsymbol{U} + \nabla q = \boldsymbol{E}(\boldsymbol{\theta}) + \boldsymbol{E}(q) \quad \text{in } \Omega\times(0,T), \\ \boldsymbol{\theta} = -\boldsymbol{u}^0 \quad \text{on } \Gamma\times(0,T), \\ \boldsymbol{\theta}|_{t=0} = 0 \quad \text{in } \Omega. \end{cases} \tag{3.36}$$

We next turn to the supplementary part $\boldsymbol{\varphi}$ of the corrector $\boldsymbol{\Theta}$. Its definition in (3.18)-(3.20) gives

$$\boldsymbol{\varphi}|_\Gamma = \boldsymbol{\varphi}|_{t=0} = \boldsymbol{0}. \tag{3.37}$$

We define the contribution of $\boldsymbol{\varphi}$ to the error as

$$\boldsymbol{E}(\boldsymbol{\varphi}) := \frac{\partial \boldsymbol{\varphi}}{\partial t} - \varepsilon\Delta\boldsymbol{\varphi} + \boldsymbol{U}\cdot\nabla\boldsymbol{\varphi} + \boldsymbol{\varphi}\cdot\nabla\boldsymbol{U}.$$

To bound $\boldsymbol{E}(\boldsymbol{\varphi})$, we first observe that, by (3.18)-(3.20), and (3.22), for $j = 1,2$, and $k, m \ge 0$,

$$\left\|\frac{\partial^{k+m}\boldsymbol{\varphi}}{\partial\xi_j^k\partial\xi_3^m}\right\|_{L^\infty(0,T;L^\infty(\Omega))} \le \kappa \sum_{i,\ell=1}^{2}\left\|\frac{\partial^{k+1}\theta^i}{\partial\xi_j^k\partial\xi_\ell}\right\|_{L^\infty(0,T;L^1(\omega\times(0,\infty)))} \le \kappa_T\varepsilon^{\frac{1}{2}}. \tag{3.38}$$

From calculations similar to those in (2.41)-(2.45), we also have

$$\Big\| \frac{\partial \boldsymbol{\varphi}}{\partial t} \Big\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{2}}. \tag{3.39}$$

These estimates imply that

$$\Big\| \boldsymbol{E}(\boldsymbol{\varphi}) \Big\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{2}}. \tag{3.40}$$

Finally, we derive the equation for $\boldsymbol{\psi}$ (defined also in (3.18)-(3.20). We use (3.13) and write the equation of $\boldsymbol{\psi}$ in the ξ_i direction, $i = 1, 2$, as

$$\begin{aligned} &\frac{\partial \psi^i}{\partial t} - \varepsilon \Delta \boldsymbol{\psi} \cdot \boldsymbol{e}_i + \widetilde{\mathcal{P}}^i(\widetilde{\boldsymbol{U}}, \boldsymbol{\psi}) + \xi_3\, \sigma\, \widetilde{\frac{\partial U^3}{\partial \xi_3}} \frac{\partial \psi^i}{\partial \xi_3} + \widetilde{\mathcal{Q}}^i(\widetilde{\boldsymbol{U}}, \boldsymbol{\psi}) + \widetilde{\mathcal{P}}^i(\boldsymbol{\psi}, \widetilde{\boldsymbol{U}}) \\ &+ \widetilde{\mathcal{Q}}^i(\boldsymbol{\psi}, \widetilde{\boldsymbol{U}}) = \widehat{E}^i_{\text{temp}}(\boldsymbol{\psi}), \quad i = 1, 2, \end{aligned} \tag{3.41}$$

where

$$\begin{aligned} E^i_{\text{temp}}(\boldsymbol{\psi}) \;&= -\varepsilon(\Delta \boldsymbol{\psi} \cdot \boldsymbol{e}_i - \Delta_{\boldsymbol{\xi}} \psi^i) \\ &\quad -\varepsilon \sum_{j=1}^{2} \frac{\partial^2}{\partial \xi_j^2} \Big(\frac{h_i}{h} \frac{\widetilde{h}}{\widetilde{h}_i} \Big) \sigma' \int_\infty^{\xi_3} \theta^i \, d\eta - 2\varepsilon \sum_{j=1}^{2} \frac{\partial}{\partial \xi_j} \Big(\frac{h_i}{h} \widetilde{\frac{h}{h_i}} \Big) \sigma' \int_\infty^{\xi_3} \frac{\partial \theta^i}{\partial \xi_j} \, d\eta \\ &\quad -\varepsilon \frac{\partial^2}{\partial \xi_3^2} \Big(\sigma' \, \frac{h_i}{h} \Big) \widetilde{\frac{h}{h_i}} \int_\infty^{\xi_3} \theta^i \, d\eta - 2\varepsilon \frac{\partial}{\partial \xi_3} \Big(\sigma' \, \frac{h_i}{h} \Big) \widetilde{\frac{h}{h_i}} \, \theta^i \\ &\quad + \sum_{j=1}^{2} \frac{1}{\widetilde{h}_j} \widetilde{U}^j \, \frac{\partial}{\partial \xi_j} \Big(\frac{h_i}{h} \widetilde{\frac{h}{h_i}} \Big) \sigma' \int_\infty^{\xi_3} \theta^i \, d\eta \\ &\quad + \xi_3 \, \sigma \, \widetilde{\frac{\partial U^3}{\partial \xi_3}} \frac{\partial}{\partial \xi_3} \Big(\sigma' \, \frac{h_i}{h} \Big) \widetilde{\frac{h}{h_i}} \int_\infty^{\xi_3} \theta^i \, d\eta, \quad i = 1, 2. \end{aligned} \tag{3.42}$$

Since σ and all its derivatives vanish for $\xi_3 \geq 2\delta$, we use (3.22) and find that

$$\Big\| \Big(\frac{\xi_3}{\sqrt{\varepsilon}} \Big)^{\frac{\ell}{p}} \widehat{E}^i_{\text{temp}}(\boldsymbol{\psi}) \Big\|_{L^\infty(0,T;L^p(\Omega))} \leq \kappa_T \, \varepsilon^{\frac{1}{2}}, \quad \ell \geq 0, \; 1 \leq p \leq \infty. \tag{3.43}$$

By the energy estimate for (3.41) (which is identical to one in (2.19)), we then obtain

$$\Big\| \Big(\frac{\xi_3}{\sqrt{\varepsilon}} \Big)^{\frac{\ell}{p}} \frac{\partial^k \psi^i}{\partial \xi_j^k} \Big\|_{L^\infty(0,T;L^p(\Omega))} + \varepsilon^{\frac{3}{4}} \Big\| \Big(\frac{\xi_3}{\sqrt{\varepsilon}} \Big)^{\frac{\ell}{2}} \nabla \frac{\partial^k \psi^i}{\partial \xi_j^k} \Big\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \, \varepsilon^{\frac{1}{2}}, \tag{3.44}$$

for $i, j = 1, 2$ and $k, \ell \geq 0$, and $1 \leq p \leq \infty$.

The normal component ψ^3 satisfies similar bounds given that the two expressions for ψ^i, $i = 1, 2$, and for ψ^3 are similar. Using these estimates as well as (3.27), (3.28), and (3.29), we conclude that the second supplementary part $\boldsymbol{\psi}$ of the corrector satisfies

$$\begin{cases} \dfrac{\partial \boldsymbol{\psi}}{\partial t} - \varepsilon \Delta \boldsymbol{\psi} + \boldsymbol{U} \cdot \nabla \boldsymbol{\psi} + \boldsymbol{\psi} \cdot \nabla \boldsymbol{U} = \boldsymbol{E}(\boldsymbol{\psi}) & \text{in } \Omega \times (0,T), \\ \boldsymbol{\psi} = 0 & \text{on } \Gamma \times (0,T), \\ \boldsymbol{\psi}|_{t=0} = 0 & \text{in } \Omega, \end{cases} \tag{3.45}$$

where

$$\|\boldsymbol{E}(\boldsymbol{\psi})\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{2}}, \quad i = 1, 2. \tag{3.46}$$

3.4. Proof of Theorem 1.1: The case of a 3D smooth domain

Recall that the error is given by $\boldsymbol{w}^\varepsilon := \boldsymbol{u}^\varepsilon - (\boldsymbol{u}^0 + \boldsymbol{\Theta})$, $\pi^\varepsilon = p^\varepsilon - (p^0 + q)$.

Then, thanks to the equations satisfied by $\boldsymbol{u}^\varepsilon$, $\boldsymbol{u}^0$, and the corrector $\boldsymbol{\Theta}$, along with the divergence-free condition on $\boldsymbol{\Theta}$, the equation for $(\boldsymbol{w}^\varepsilon, \pi^\varepsilon)$ can be written as

$$\begin{cases} \dfrac{\partial \boldsymbol{w}^\varepsilon}{\partial t} - \varepsilon \Delta \boldsymbol{w}^\varepsilon + \boldsymbol{U} \cdot \nabla \boldsymbol{w}^\varepsilon + \boldsymbol{w}^\varepsilon \cdot \nabla \boldsymbol{U} + \nabla \pi^\varepsilon = \boldsymbol{E}(\boldsymbol{\Theta}) + \boldsymbol{E}(q) & \text{in } \Omega \times (0,T), \\ \text{div } \boldsymbol{w}^\varepsilon = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{w}^\varepsilon = 0 & \text{on } \Gamma \times (0,T), \\ \boldsymbol{w}^\varepsilon|_{t=0} = 0 & \text{in } \Omega. \end{cases} \tag{3.47}$$

where

$$\boldsymbol{E}(\boldsymbol{\Theta}) = \boldsymbol{E}(\boldsymbol{\theta}) + \boldsymbol{E}(\boldsymbol{\varphi}) + \boldsymbol{E}(\boldsymbol{\psi}). \tag{3.48}$$

for $(\boldsymbol{w}^\varepsilon, \pi^\varepsilon)$

Using the bounds derived in the previous sections (specifically (3.31), (3.35), (3.40), and (3.46)), the error estimate in (1.9) follows from a simple energy estimate.

The vanishing viscosity limit (1.10) is a consequence of (1.9) and the smallness of the corrector in $L^\infty(0,T;L^2(\Omega))$. The proof of Theorem 1.1 is complete.

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