WELL-POSEDNESS OF THE 2D EULER EQUATIONS WHEN VELOCITY GROWS AT INFINITY

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Abstract. We prove the uniqueness and finite-time existence of bounded-vorticity solutions to the 2D Euler equations having velocity growing slower than the square root of the distance from the origin, obtaining global existence for more slowly growing velocity fields. We also establish continuous dependence on initial data.

Compiled on Saturday 13 October 2018 at 11:18

Contents

1. Introduction

In [15], Ph. Serfati proved the existence and uniqueness of Lagrangian solutions to the 2D Euler equations having bounded vorticity and bounded velocity (for a rigorous proof, see [1]). Our goal here is to discover how rapidly the velocity at infinity can grow (keeping the vorticity bounded) and still obtain existence or uniqueness of solutions to the 2D Euler equations.

Serfati's approach in [15] centered around a novel identity he showed held for bounded vorticity, bounded velocity solutions. We write this identity, which we call *Serfati's identity* or the *Serfati identity*, in the form

$$u^j(t) - (u^0)^j = (a_\lambda K^j) * (\omega(t) - \omega^0) - \int_0^t \left(\nabla\nabla^\perp \left[(1 - a_\lambda)K^j\right]\right) *\cdot (u \otimes u)(s)\, ds, \qquad (1.1)$$

$j = 1, 2$. Here, u is a divergence-free velocity field, with $\omega = \operatorname{curl} u := \partial_1 u^2 - \partial_2 u^1$ its vorticity, and $K(x) := (2\pi)^{-1} x^\perp |x|^{-2}$ is the Biot-Savart kernel, $x^\perp := (-x_2, x_1)$. The function a_λ,

Date: Saturday 13$^{\text{th}}$ October, 2018.

2010 *Mathematics Subject Classification.* Primary 35Q35, 35Q92, 39A01.

Key words and phrases. Aggregation equations, Striated regularity, Transport equations.

$\lambda > 0$, is a scaled radial cutoff function (see Definition 1.3). Also, we have used the notation,

$$\begin{aligned} v \ast\cdot w &= v^i \ast w^i && \text{if } v \text{ and } w \text{ are vector fields,} \\ A \ast\cdot B &= A^{ij} \ast B^{ij} && \text{if } A,\, B \text{ are matrix-valued functions on } \mathbb{R}^2, \end{aligned}$$

where $\ast$ denotes convolution and where repeated indices imply summation.

The Biot-Savart law, $u = K \ast \omega$, recovers the unique divergence-free vector field u decaying at infinity from its vorticity (scalar curl) ω. It does not apply to bounded vorticity, bounded velocity solutions—indeed this is the greatest difficulty to overcome with such solutions—but because of the manner in which a_λ cuts off the Biot-Savart kernel in (1.1), we see that all the terms in Serfati's identity are finite. In fact, there is room for growth at infinity both in the vorticity and the velocity, though to avoid excessive complications, we only treat growth in the velocity. Because $\nabla\nabla^\perp\left[(1 - a_\lambda)K^j(x)\right]$ decays like $|x|^{-3}$, we see that as long as $|u(x)|$ grows more slowly than $|x|^{1/2}$, all the terms in (1.1) will at least be finite.

In brief, we will establish uniqueness of solutions having $o(|x|^{1/2})$ growth along with finite-time existence. We will obtain global existence only for much more slowly growing velocity fields.

To give a precise statement of our results, we must first describe the manner in which we prescribe the growth of the velocity field at infinity and give our formulation of a weak solution, including the function spaces in which the solution is to lie.

In what follows, an *increasing function* means nondecreasing; that is, not necessarily *strictly* increasing.

We define four types of increasingly restrictive bounds on the growth of the velocity, as follows:

Definition 1.1. *[Growth bounds]*

(i) A pre-growth bound is a function $h\colon [0,\infty) \to (0,\infty)$ that is concave, increasing, differentiable on $[0,\infty)$, and twice continuously differentiable on $(0,\infty)$.

(ii) A growth bound h is a pre-growth bound for which $\int_1^\infty h(s)s^{-2}\, ds < \infty$.

(iii) A well-posedness growth bound h is a growth bound for which h^2 is also a growth bound.

(iv) Let h be a well-posedness growth bound and define $H[h]\colon (0,\infty) \to (0,\infty)$ by

$$H[h](r) := \int_r^\infty \frac{h(s)}{s^2}\, ds, \tag{1.2}$$

noting that the condition in (ii) insures $H[h]$ and $H[h^2]$ are well-defined. We show in Lemma 2.3 that there always exists a continuous, convex function μ with $\mu(0) = 0$ for which

$$E(r) := \left(1 + r^{\frac12} H[h^2]\left(r^{\frac12}\right)\right)^2 r \le \mu(r).$$

We call h a global well-posedness growth bound if for some such μ,

$$\int_1^\infty \frac{dr}{\mu(r)} = \infty. \tag{1.3}$$

Definition 1.2. *Let h be a growth bound. We denote by S_h the Banach space of all divergence-free vector fields, u, on $\mathbb{R}^2$ for which u/h and $\omega(u)$ are bounded, where $\omega(u) := \partial_1 u^2 - \partial_2 u^1$ is the vorticity. We endow S_h with the norm*

$$\|u\|_{S_h} := \|u/h\|_{L^\infty} + \|\omega(u)\|_{L^\infty}.$$

Definition 1.3 (Radial cutoff function)**.** *Let a be a radially symmetric function in $C_c^\infty(\mathbb{R}^2)$ taking values in $[0,1]$, supported in $\overline{B_1(0)}$, with $a \equiv 1$ on $\overline{B_{1/2}(0)}$. (By $B_r(x)$ we mean the open ball of radius r centered at x.) We call any such function a radial cutoff function. For any $\lambda > 0$ define*

$$a_\lambda(x) := a\left(\frac{x}{\lambda}\right).$$

Definition 1.4 (Lagrangian solution)**.** *Fix $T > 0$ and let h be a growth bound. Assume that $u \in C(0,T;S_h)$, let $\omega = \operatorname{curl} u := \partial_1 u^2 - \partial_2 u^1$, and let X be the unique flow map for u. We say that u is a solution to the Euler equations in S_h without forcing and with initial velocity $u^0 = u|_{t=0}$ in S_h if the following conditions hold:*

(1) $\omega = \omega^0 \circ X^{-1}$ on $[0,T] \times \mathbb{R}^2$, where $\omega^0 := \operatorname{curl} u^0$;

(2) Serfati's identity (1.1) holds for all $\lambda > 0$.

The existence and uniqueness of the flow map X in Definition 1.4 is assured by $u \in C([0,T];S_h)$ (see Lemma 4.1). It also then follows easily that the vorticity equation, $\partial_t \omega + u \cdot \nabla \omega = 0$, holds in the sense of distributions—so u is also a weak Eulerian solution.

Our assumption that Serfati's identity holds in Definition 1.4 is needed in order to establish uniqueness of solutions. We discuss this necessity in more depth in the paragraphs following Corollary 1.8.

Our main results are Theorem 1.5 through Theorem 1.10.

Theorem 1.5. *Let $h_1(r) := (1+r)^\alpha$ for some $\alpha \in [0,1/2)$, $h_2(r) := \log^{\frac{1}{4}}(e+r)$. If $u^0 \in S_{h_j}$, $j = 1$ or 2, then there exists a unique solution to the Euler equations in S_{h_j} on $[0,T]$ as in Definition 1.4 for some $T > 0$. If $u^0 \in S_{h_2}$ then T can be made arbitrarily large.*

Loosely speaking, Theorem 1.5 says that solutions in S_h are unique and exist for finite time as long as h^2 is sublinear, while global-in-time solutions exist for velocities growing very slowly at infinity. These slowly growing velocities are somewhat analogous to the "slightly unbounded" vorticities of Yudovich [18], which extends the uniqueness result for bounded vorticities in [17]. We note that the existence of solutions for h_2 was shown in [5] using different techniques.

We will, in fact, prove more general existence and uniqueness theorems which include Theorem 1.5 as a special case. Specifically, in Theorem 1.6 and Theorem 1.7 below, we establish existence and uniqueness for solutions to the Euler equations in S_h, where h is any well-posedness growth bound. We show that Theorem 1.5 is a special case of Theorem 1.6 and Theorem 1.7 by demonstrating in Appendix A that h_1 and h_2 actually satisfy the pertinent conditions in Definition 1.1.

Theorem 1.6. *Let h be any well-posedness growth bound. For any $u^0 \in S_h$ there is a $T > 0$ such that there exists a solution to the Euler equations in S_h on $[0,T]$ as in Definition 1.4. If h is a global well-posedness growth bound then T can be chosen to be arbitrarily large.*

Theorem 1.7. *Let h be any well-posedness growth bound and let $\zeta \geq h$ be any growth bound for which ζ/h and ζh are also growth bounds. Let $u_1^0, u_2^0 \in S_h$ and let ω_1^0, ω_2^0 be the corresponding initial vorticities. Assume that there exist solutions, u_1, u_2, to the Euler equations in S_h on $[0,T]$ with initial velocities u_1^0, u_2^0, and let ω_1, ω_2 and X_1, X_2 be the corresponding vorticities and flow maps. Let*

$$a(T) := \|(u_1^0 - u_2^0)/\zeta\|_{L^\infty(\mathbb{R}^2)} + \|J/\zeta\|_{L^\infty((0,T)\times\mathbb{R}^2)}, \tag{1.4}$$

where

$$J(t,x) = \left((a_{h(x)}K) * (\omega_1^0 - \omega_2^0) \circ X_1^{-1}(t)\right)(x) - \left((a_{h(x)}K) * (\omega_1^0 - \omega_2^0)\right)(X_1(t,x)). \qquad (1.5)$$

Define

$$\begin{aligned}
\eta(t) &:= \left\| \frac{X_1(t,x) - X_2(t,x)}{\zeta(x)} \right\|_{L^\infty_x(\mathbb{R}^2)}, \\
L(t) &:= \left\| \frac{u_1(t, X_1(t,x)) - u_2(t, X_2(t,x))}{\zeta(x)} \right\|_{L^\infty_x(\mathbb{R}^2)}, \\
M(t) &:= \int_0^t L(s)\, ds, \\
Q(t) &:= \|(u_1(t) - u_2(t))/\zeta\|_{L^\infty(\mathbb{R}^2)}.
\end{aligned}$$

Then $\eta(t) \le M(t)$ *and*

$$\int_{ta(T)}^{M(t)} \frac{dr}{\overline{\mu}(C(T)r) + r} \le C(T)t \qquad (1.6)$$

for all $t \in [0,T]$, *where*

$$\overline{\mu}(r) := \begin{cases} -r\log r & \text{if } r \le e^{-1}, \\ e^{-1} & \text{if } r > e^{-1}. \end{cases} \qquad (1.7)$$

We have $M(T) \to 0$ *and* $\|Q\|_{L^\infty(0,T)} \to 0$ *as* $a(T) \to 0$. *Explicitly,*

$$M(t) \le (ta(T))^{e^{-C(T)t}}$$

holds until $M(t)$ *increases to* $C(T)e^{-1}$. *If* $ta(T) \ge C(T)e^{-1}$ *then*

$$M(t) \le C(T)ta(T).$$

For all $t \ge 0$, *we have*

$$Q(t) \le \left[a(T) + C(T)\overline{\mu}(C(T)M(t))\right] e^{C(T)t}.$$

Uniqueness is an immediate corollary of Theorem 1.7:

Corollary 1.8. *Let* h *be any well-posedness growth bound. Then solutions to the 2D Euler equations on* $[0,T]$ *in* S_h *are unique.*

Proof. Apply Theorem 1.7 with $u_1^0 = u_2^0 = u(0)$ so that $a(T) = 0$ and set $\zeta = h$, noting that $\zeta/h = 1$ and $\zeta h = h^2$ are both growth bounds. □

We remark that we must assume Serfati's identity holds in Definition 1.4 if we wish to have uniqueness of solutions. This is demonstrated in [11], where it is shown that for bounded vorticity, bounded velocity solutions, Serfati's identity must hold—up to the addition of a time-varying, constant-in-space vector field. This vector field, then, serves as the uniqueness criterion; its vanishing is equivalent to the sublinear growth of the pressure (used as the uniqueness criterion in [16]).

Bounded vorticity, bounded velocity solutions are a special case of the solutions we consider here, but the technology developed in [11] does not easily extend to velocities growing at infinity. Hence, we are unable to dispense with our assumption that the Serfati identity holds, as we need it as our uniqueness criterion.

Another closely related issue is that we are using Lagrangian solutions to the Euler equations, as we need to use the flow map in our uniqueness argument (with Serfati's identity). We note that it is not sufficient to simply know that the vorticity equation, $\partial_t \omega + u \cdot \nabla \omega = 0$, is satisfied. This tells us that the vorticity is transported, in a weak sense, by the unique flow map X, but we need that this weak transport equation has $\omega^0(X^{-1}(t, x))$ as its unique solution. Only then can we conclude that the curl of u truly is $\omega^0(X^{-1}(t, x))$. (For bounded velocity, bounded vorticity solutions, the transport estimate from [3] in Proposition 8.7 would be enough to obtain this uniqueness.)

The usual way to establish well-posedness of Eulerian solutions to the 2D Euler equations is to construct Lagrangian solutions (which are automatically Eulerian) and then prove uniqueness using the Eulerian formulation only. Such an approach works for bounded vorticity, bounded velocity solutions, as uniqueness using the Eulerian formulation was shown in [16] (see also [7]). Whether this can be extended to the solutions we study here is a subject for future work.

If $u_1^0 \neq u_2^0$ then $a(T)$ does not vanish. If $u_1^0, u_2^0 \in S_h$ for some well-posedness growth bound h, and $u_1^0 - u_2^0$ is small in S_h, we might expect $\|u_1(t) - u_2(t)\|_{S_h}$ to remain small at least for some time. The S_h norm, however, includes the L^∞ norm of $\omega_1(t) - \omega_2(t)$, with each vorticity being transported by different flow maps. Hence, we should expect $\|\omega_1(t) - \omega_2(t)\|_{L^\infty}$ to be of the same order as $\|\omega_j(t)\|_{L^\infty}$, $j = 1, 2$, immediately after time zero. Thus, it is too much to ask for continuous dependence on initial data in the S_h norm. In this regard, the situation is the same as for the classical bounded-vorticity solutions of Yudovich [17], and has nothing to do with lack of decay at infinity. The best we can hope to obtain is a bound on $(u_1(t) - u_2(t))/\zeta$ in L^∞—and so, by interpolation, in C^α for all $\alpha < 1$.

To obtain continuous dependence on initial data or control how changes at a distance from the origin affect the solution near the origin (*effect at a distance*, for short), we can employ the bound on Q in Theorem 1.7 to obtain a bound on how far apart the two solutions become, weighted by ζ. For continuous dependence on initial data, $\zeta = h$ is most immediately pertinent; for controlling effect at a distance, $\zeta \geq h$ is better.

The simplest form of continuous dependence on initial data, which follows from Theorem 1.9 applied to Theorem 1.7, shows that if the initial velocities are close in S_ζ then they remain close (in a weighted L^∞ space) for some time.

Theorem 1.9. *Make the assumptions in Theorem 1.7. Then*

$$a(T) \leq C \left\|u_1^0 - u_2^0\right\|_{S_\zeta}.$$

While Theorem 1.9 gives a theoretically meaningful measure of continuous dependence on initial data, the assumption that the initial velocities are close in S_ζ is overstrong. For instance, it would not apply to two vortex patches that do not quite coincide. One approach, motivated in part by this vortex patch example, is to make some assumption on the closeness of the initial vorticities locally uniformly in an L^p norm for $p < \infty$, as was done in [16, 1]. This assumption is, however, unnecessary (and in our setting somewhat artificial) as shown for the special case of bounded velocity ($h \equiv 1$) in [7].

In [7], the focus was not on continuous dependence on initial data, per se, but rather on understanding the effect at a distance. Hence, we used a function, $\zeta(r) = (1 + r)^\alpha$ for any $\alpha \in (0, 1)$, in place of h and obtained a bound on $(u_1(t) - u_2(t))/\zeta$ in terms of $(u_1^0 - u_2^0)/\zeta$, each in the L^∞ norm. In Theorem 1.10, we obtain a similar bound using very different techniques. Our need to assume bounded velocity ($h = 1$) arises from our inability to obtain usable transport estimates for non-Lipschitz vector fields growing at infinity. It is not clear whether this is only a technical issue or represents some fundamental new phenomenon causing unbounded velocities to have less stability, in the sense of effect at a distance, than bounded velocities. (See Remark 7.2.)

Theorem 1.10. *Let $u_1^0, u_2^0 \in S_1$ and let ζ be a growth bound. Let u_1, u_2 be the corresponding solutions in S_1 on $[0, T]$ with initial velocities u_1^0, u_2^0. Fix δ, α with $0 < \delta < \alpha < 1$ and choose any $T^* > 0$ such that*

$$T^* < \min \left\{ T, \frac{1+\delta}{CC_0} \right\}, \tag{1.8}$$

where

$$C_0 = \|u_1\|_{L^\infty(0,T;S_1)}. \tag{1.9}$$

Then

$$a(T^*) \le C_1 \Phi_\alpha \left(T, C \left\| \frac{u_1^0 - u_2^0}{\zeta} \right\|_{L^\infty}^\delta \right), \tag{1.10}$$

where,

$$\Phi_\alpha(t, x) := x + x^{\frac{e^{-C_0 t}}{\alpha + e^{-C_0 t}}}. \tag{1.11}$$

Remark 1.11. *To obtain a bound on $a(T)$, we iterate the bound in (1.10) N times, where $T = NT^*$ (decreasing T^* if necessary so as to obtain the minimum possible positive integer N), applying the bound on $Q(t)$ from Theorem 1.7 after each step. Since T^* depends only upon C_0, which does not change, we can always iterate this way. In principle, the resulting bound can be made explicit, at least for sufficiently small $a(T)$.*

The bound on $u_1 - u_2$ given by the combination of Theorems 1.7 and 1.10 is not optimal, primarily because we could use an L^1-in-time bound on $J(t, x)$ in place of the L^∞-in-time bound. (This would also be reflected in the bound on $a(T^*)$ in (1.10).) This would not improve the bounds sufficiently, however, to justify the considerable complications to the proofs.

The issue of well-posedness of solutions to the 2D Euler equations with bounded vorticity but velocities growing at infinity was taken up recently by Elgindi and Jeong in [8]. They prove existence and uniqueness of such solutions for velocity fields growing *linearly* at infinity under the assumption that the vorticity has m-fold symmetry for $m \ge 3$. We study here solutions with no preferred symmetry, and our approach is very different; nonetheless, aspects of our uniqueness argument were influenced by Elgindi's and Jeong's work. In particular, the manner in which they first obtain elementary but useful bounds on the flow map inspired Lemma 4.2, and the bound in Proposition 3.3 is the analog of Lemma 2.8 of [8] (obtained differently under different assumptions).

Finally, we remark that it would be natural to combine the approach in [8] with our approach here to address the case of two-fold symmetric vorticities ($m = 2$). The goal would be to obtain Elgindi's and Jeong's result, but for velocities growing infinitesimally less than linearly at infinity. A similar argument might also work for solutions to the 2D Euler equations in a half plane having sublinear growth at infinity.

This paper is organized as follows: Sections 2 to 4 contain preliminary material establishing useful properties of growth bounds, estimates on the Biot-Savart law and locally log-Lipschitz velocity fields, and bounds on flow maps for velocity fields growing at infinity. We establish the existence of solutions, Theorem 1.6, in Section 5. We prove Theorem 1.7 in Section 6, thereby establishing the uniqueness of solutions, Corollary 1.8. In Section 7, we prove Theorems 1.9 and 1.10, establishing continuous dependence on initial data and controlling the effect of changes at a distance.

In Section 8, we employ Littlewood-Paley theory to establish estimates in negative Hölder spaces for $u \in L^\infty(0, T; S_1)$. These estimates are used in the proof of Theorem 1.10 in Section 7.

Finally, in Appendix A we show that h_1 and h_2 of Theorem 1.5 are, in fact, well-posedness growth bounds, h_2 globally so, thereby establishing Theorem 1.5 as a corollary of Theorem 1.6 and Corollary 1.8.

2. Properties of growth bounds

We establish in this section a number of properties of growth bounds.

Definition 2.1. *We say that $f \colon [0, \infty) \to [0, \infty)$ is subadditive if*

$$f(r + s) \le f(r) + f(s) \text{ for all } r, s \ge 0. \tag{2.1}$$

Lemma 2.2. *Let h be a pre-growth bound. Then h is subadditive, as is h^2 if h is a well-posedness growth bound. Also, $h(r) \le cr + d$ with $c = h'(0)$, $d = h(0)$, and the analogous statement holds for h^2 when h is a well-posedness growth bound.*

Proof. Because h is concave with $h(0) \ge 0$, we have

$$ah(x) \le ah(x) + (1 - a)h(0) \le h(ax + (1 - a)0) = h(ax)$$

for all $x > 0$ and $a \in [0, 1]$. We apply this twice with $x = r + s > 0$ and $a = r/(r + s)$, giving

$$h(r + s) = ah(r + s) + (1 - a)h(r + s) \le h\left(a(r + s)\right) + h\left((1 - a)(r + s)\right) = h(r) + h(s).$$

Because $h'(0) < \infty$ and h is concave we also have $h(r) \le cr + d$. The facts regarding h^2 follow in the same way. □

Lemma 2.3 gives the existence of the function μ promised in Definition 1.1. A μ that yields a tighter bound on $E \le \mu$ will result in a longer existence time estimate for solutions, as we can see from the application of Lemma 5.1 in the proof of Theorem 1.6. The estimate we give in Lemma 2.3 is very loose; in specific cases, this bound can be much improved.

Lemma 2.3. *Let h be a well-posedness growth bound. There exists a continuous, convex function μ with $\mu(0) = 0$ for which $E \le \mu$, where E is as in Definition 1.1.*

Proof. Since $h(0) > 0$, $H(r) := H[h^2](r) \to \infty$ as $r \to 0^+$. Hence, L'Hospital's rule gives

$$\lim_{r \to 0^+} rH(r) = \lim_{r \to 0^+} \frac{H(r)}{r^{-1}} = -\lim_{r \to 0^+} \frac{H'(r)}{r^{-2}} = \lim_{r \to 0^+} \frac{h^2(r) r^{-2}}{r^{-2}} = h^2(0).$$

It follows that $r^{\frac{1}{2}} H(r^{\frac{1}{2}}) \le C$ for $r \in (0, 1)$. Then, since H decreases, we see that $(1 + r^{\frac{1}{2}} H(r^{\frac{1}{2}}))^2 \le (1 + Cr^{\frac{1}{2}})^2 \le 2(1 + Cr) \le C(1 + r)$. Hence, we can use $\mu(r) = Cr(1 + r)$. □

Remark 2.4. *Abusing notation, we will often write $h(x)$ for $h(|x|)$, treating h as a map from $\mathbb{R}^2$ to $\mathbb{R}$. Treated this way, h remains subadditive in the sense that*

$$h(y) = h(|y|) \le h(|x - y| + |x|) \le h(|x - y|) + h(|x|) = h(x - y) + h(x).$$

Here, we used the triangle inequality and that $h \colon [0, \infty) \to (0, \infty)$ is increasing and subadditive. Similarly, $|h(y) - h(x)| \le h(x - y)$.

Lemma 2.5. *Let h be a pre-growth bound as in Definition 1.1. Then for all $a \geq 1$ and $r \geq 0$,*

$$h(ar) \leq 2ah(r) \tag{2.2}$$

and

$$\begin{aligned} &h(ah(r)) \leq C(h)ah(r), \\ &h(h(r))/h(r) \leq C(h) \end{aligned} \tag{2.3}$$

where $C(h) = 2(h'(0) + h(h(0))/h(0))$.

Proof. To prove (2.2), we will show first that for any positive integer n and any $r \geq 0$,

$$h(nr) \leq nh(r). \tag{2.4}$$

For $n = 1$, (2.4) trivially holds, so assume that (2.4) holds for $n - 1 \geq 1$. Then because h is subadditive (Lemma 2.2),

$$h(nr) = h((n-1)r + r) \leq h((n-1)r) + h(r) \leq (n-1)h(r) + h(r) = nh(r).$$

Thus, (2.4) follows for all positive integers n by induction.

If $a = n + \alpha$ for some $\alpha \in [0, 1)$ then

$$\begin{aligned} h(ar) &= h(nr + \alpha r) \leq h(nr) + h(\alpha r) \leq nh(r) + h(r) = (n+1)h(r) \\ &= \frac{n+1}{n+\alpha}(n+\alpha)h(r) = \frac{n+1}{n+\alpha}ah(r) \leq \sup_{n \geq 1}\left[\frac{n+1}{n+\alpha}\right]ah(r) \leq 2ah(r). \end{aligned}$$

(Note that the supremum is over $n \geq 1$ since we assumed that $a \geq 1$.)

For $(2.3)_1$, let $c = h'(0)$, $d = h(0)$ as in Lemma 2.2, so that $h(r) \leq cr + d$. Then,

$$h(ah(r)) \leq h(acr + ad) \leq h(acr) + h(ad) \leq 2a(ch(r) + h(d)) \leq 2\left(c + \frac{h(d)}{h(0)}\right)ah(r),$$

which is $(2.3)_1$. From this, $(2.3)_2$ follows immediately. $\square$

In Section 6 we will employ the functions, $\Gamma_t, F_t \colon [0, \infty) \to (0, \infty)$, defined for any $t \in [0, T]$ in terms of an arbitrary growth bound h by

$$\int_a^{\Gamma_t(a)} \frac{dr}{h(r)} = Ct, \tag{2.5}$$

$$F_t(a) = F_t[h](a) := h(\Gamma_t(a)). \tag{2.6}$$

We know that Γ_t and so F_t are well-defined, because

$$\int_1^\infty \frac{dr}{h(r)} \geq \int_1^\infty \frac{1}{cr + d}\, dr = \infty, \tag{2.7}$$

recalling that $h(r) \leq cr + d$ by Lemma 2.2.

Remark 2.6. *If $h(0)$ were zero, then Γ_t would be the bound at time t on the spatial modulus of continuity of the flow map for a velocity field having h as its modulus of continuity. Much is known about properties of Γ_t (they are explored at length in [10]), and most of these properties are unaffected by $h(0)$ being positive. One key difference, however, is that $\Gamma_t(0) > 0$ and $\Gamma_t'(0) < \infty$. As we will see in Lemma 2.7, this implies that Γ_t is subadditive. This is in contrast to what happens when $h(0) = 0$, where $\Gamma_t(0) = 0$, $\Gamma_t'(0) = \infty$, and Γ satisfies the Osgood condition.*

Lemma 2.7 shows that F_t is a growth bound that is equivalent to h in that it is bounded above and below by constant multiples of h.

Lemma 2.7. *Assume that h is a (well-posedness, global well-posedness) growth bound and define F_t as in (2.6). For all $t \in [0,T]$, F_t is a (well-posedness, global well-posedness) growth bound as in Definition 1.1. Moreover, $F_t(r)$ is increasing in t and r with $h \le F_t \le C(t)h$, $C(t)$ increasing with time.*

Proof. First observe that $\Gamma_t'(r) = h(\Gamma_t(r))/h(r)$ follows from differentiating both sides of (2.5). Thus, Γ_t is increasing and continuously differentiable on $(0,\infty)$. Since $\Gamma_t'(0) = h(\Gamma_t(0))/h(0) \ge 1$ is finite, Γ_t is, in fact, differentiable on $[0,\infty)$. Also, $F_t'(0) = h'(\Gamma_t(0))\Gamma_t'(0)$ is finite and hence so is $(F_t^2)'(0)$, meaning that F_t and F_t^2 are differentiable on all of $[0,\infty)$.

We now show that F_t is increasing, concave, and twice differentiable on $(0,\infty)$, and that the same holds true for F_t^2 if h is a well-posedness growth bound. We do this explicitly for F_t^2, the proof for F_t being slightly simpler. Direct calculation gives

$$(F_t^2)'(r) = (h^2)'(\Gamma_t(r))\Gamma_t'(r), \quad (F_t^2)''(r) = (h^2)''(\Gamma_t(r))(\Gamma_t'(r))^2 + (h^2)'(\Gamma_t(r))\Gamma_t''(r).$$

But $(h^2)' \ge 0$ and $(h^2)'' \le 0$ because h^2 is increasing and concave. Also, h concave implies that Γ_t is concave: this is classical (see Lemma 8.3 of [10] for a proof). Hence, $\Gamma_t'' \le 0$, and we conclude that $(F_t^2)'' \le 0$, meaning that F_t^2 is concave.

We now show that $h \le F_t \le C(t)h$. We have $F_t(r) = h(\Gamma_t(r)) \ge h(r)$ since $\Gamma_t(r) \ge r$ and h is increasing. Because F_t is concave it is sublinear, so $F_t(r) \le c'r + d'$ for some c', d' increasing in time. Hence,

$$F_t(r) = h(\Gamma_t(r)) \le h(c'r + d') \le 2c'h(r) + h(d') \le (2c' + h(d')/h(0))h(r).$$

Here, we used Lemma 2.5 (we increase c' so that $c' \ge 1$ if necessary) and that h increasing gives $h(d') \le (h(d')/h(0))h(r)$. Hence, $h \le F_t \le C(t)h$, with $C(t)$ increasing with time.

Finally, if h is a global well-posedness growth bound then $C(t)\mu$ serves as a bound on the function E of Definition 1.1 for F_t. $\square$

Lemma 2.8. *Assume that h is a growth bound and let $g := 1/h$. Then g is a decreasing convex function; in particular, $|g'|$ is decreasing. Moreover*

$$|g'| \le c_0 g, \quad c_0 := h'(0)/h(0).$$

Proof. We have

$$g'(r) = -\frac{h'(r)}{h(r)^2} < 0$$

and

$$g''(r) = -\left(\frac{h'(r)}{h^2(r)}\right)' = \frac{2h(r)(h'(r))^2 - h^2(r)h''(x)}{h^4(r)} \ge 0,$$

since $h > 0$ and $h'' \le 0$. Thus, g is a decreasing convex function. Then, because g' is negative but increasing, $|g'|$ is decreasing.

Finally,

$$\frac{|g'(r)|}{g(r)} = \frac{h'(r)/h^2(r)}{1/h(r)} = \frac{h'(r)}{h(r)} = (\log h(r))'$$

is decreasing, since log is concave and h is concave so $\log h$ is concave. Therefore,

$$|g'(r)| \le (\log h)'(0)g(r).$$

$\square$

We will also need the properties of $\overline{\mu}$ (defined in (1.7)) given in Lemma 2.9.

Lemma 2.9. *For all $r \geq 0$ and $a \in [0,1]$, $a\overline{\mu}(r) \leq \overline{\mu}(ar)$.*

Proof. As in the proof of Lemma 2.2, because $\overline{\mu}$ is concave with $\overline{\mu}(0) = 0$, we have $a\overline{\mu}(r) = a\overline{\mu}(r) + (1-a)\overline{\mu}(0) \leq \overline{\mu}(ar + (1-a)0) = \overline{\mu}(ar)$ for all $r \geq 0$ and $a \in [0,1]$. □

Remark 2.10. *Similarly, μ of Definition 1.1 (iv) satisfies $\mu(ar) \leq a\mu(r)$ for all $r \geq 0$ and $a \in [0,1]$.*

3. Biot-Savart law and locally log-Lipschitz velocity fields

Proposition 3.1. *Let a_λ be as in Definition 1.3. There exists $C > 0$ such that, for all $x \in \mathbb{R}^2$ and all $\lambda > 0$ we have,*

$$\|a_\lambda(x-y)K(x-y)\|_{L^1_y(\mathbb{R}^2)} \leq C\lambda. \tag{3.1}$$

Let $U \subseteq \mathbb{R}^2$ have Lebesgue measure $|U|$. Then for any p in $[1,2)$,

$$\|K(x-\cdot)\|^p_{L^p(U)} \leq (2\pi(2-p))^{p-2}\,|U|^{1-\frac{p}{2}}\,. \tag{3.2}$$

Proof. See Propositions 3.1 and 3.2 of [1]. □

Proposition 3.2 is a refinement of Proposition 3.3 of [1] that better accounts for the effect of the measure of U.

Proposition 3.2. *Let X_1 and X_2 be measure-preserving homeomorphisms of $\mathbb{R}^2$. Let $U \subset \mathbb{R}^2$ have finite measure and assume that $\delta := \|X_1 - X_2\|_{L^\infty(U)} < \infty$. Then for any $x \in \mathbb{R}^2$,*

$$\|K(x - X_1(z)) - K(x - X_2(z))\|_{L^1_z(U)} \leq CR\overline{\mu}(\delta/R) \tag{3.3}$$

where $R = (2\pi)^{-1/2}\,|U|^{1/2}$.

Proof. As in the proof of Proposition 3.3 of [1], we have

$$\|K(x - X_1(z)) - K(x - X_2(z))\|_{L^1_z(U)} \leq CpR^{\frac{1}{p}}\delta^{1-\frac{1}{p}}.$$

In [1], $R^{\frac{1}{p}}$ was bounded above by $\max\{1, R\}$, which gave $p = -\log\delta$ as the minimizer of the norm as long as $\delta < e^{-1}$. Keeping the factor of $R^{\frac{1}{p}}$ we see that the minimum occurs when $p = -\log(\delta/R)$ as long as $\delta \leq e^{-1}R$, the minimum value being

$$\begin{aligned} -C\delta\log(\delta/R)R^{-\frac{1}{\log(\delta/R)}}\delta^{\frac{1}{\log(\delta/R)}} &= -C\delta\log(\delta/R)e^{-\frac{\log R}{\log(\delta/R)}}e^{\frac{\log\delta}{\log(\delta/R)}} = -Ce^{-1}\delta\log(\delta/R) \\ &= CR\overline{\mu}(\delta/R). \end{aligned}$$

This gives the bound for $\delta \leq e^{-1}R$; the $\delta > e^{-1}R$ bound follows immediately from (3.2) with $p = 1$. □

In Proposition 3.3, we establish a bound on the modulus of continuity of $u \in S_h$. In Lemma 2.8 of [8], the authors obtain the same bound as in Proposition 3.3 for $h(x) = 1 + |x|$, but under the assumption that the velocity field can be obtained from the vorticity via a symmetrized Biot-Savart law (which they show applies to m-fold symmetric vorticities for $m \geq 3$, but which does not apply for our unbounded velocities).

Proposition 3.3. *Let h be a pre-growth bound. Then for all $x, y \in \mathbb{R}^2$ such that $|y| \le C(1+|x|)$ for some constant $C > 0$, we have, for all $u \in S_h$,*

$$|u(x+y) - u(x)| \le C \|u\|_{S_h} h(x) \overline{\mu}\left(\frac{|y|}{h(x)}\right).$$

If $h \equiv C$, we need no restriction on $|y|$.

Proof. Fix $x \in \mathbb{R}^2$ and let ψ be the stream function for u on $\mathbb{R}^2$ chosen so that $\psi(x) = 0$. Let $\phi = a_2$, so that $\operatorname{supp} \phi \subseteq \overline{B_2(0)}$ with $\phi \equiv 1$ on $\overline{B_1(0)}$ and let $\phi_{x,R}(y) := \phi(R^{-1}(y-x))$ for any $R > 0$. Let $\overline{u} = \nabla^\perp(\phi_{x,R}\psi)$ and let $\overline{\omega} = \operatorname{curl} \overline{u}$.

Applying Morrey's inequality gives, for any y with $|y| \le R$, and any $p > 2$,

$$|u(x+y) - u(x)| = |\overline{u}(x+y) - \overline{u}(x)| \le C \|\nabla \overline{u}\|_{L^p(\mathbb{R}^2)} |y|^{1-\frac{2}{p}}. \tag{3.4}$$

Because $\overline{\omega}$ is compactly supported, $\overline{u} = K * \overline{\omega}$. Thus, we can apply the Calderon-Zygmund inequality to obtain

$$\begin{aligned}
|u(x+y) - u(x)| &\le C \inf_{p>2} \{p \|\overline{\omega}\|_{L^p(\mathbb{R}^2)} |y|^{1-\frac{2}{p}}\} \\
&= C \inf_{p>2} \{p \|\overline{\omega}\|_{L^p(B_{2R}(x))} |y|^{1-\frac{2}{p}}\} \le C \|\overline{\omega}\|_{L^\infty(\mathbb{R}^2)} \inf_{p>2} \{R^{2/p} p |y|^{1-\frac{2}{p}}\} \\
&= C |y| \|\overline{\omega}\|_{L^\infty(\mathbb{R}^2)} \inf_{p>2} \{p \left|R^{-1} y\right|^{-\frac{2}{p}}\}.
\end{aligned}$$

When $R^{-1}|y| \le e^{-1}$ (meaning also that $|y| \le R$, as required), the infimum occurs at $p = -2\log(\left|R^{-1}y\right|)$ and we have

$$|u(x+y) - u(x)| \le -C \|\overline{\omega}\|_{L^\infty(\mathbb{R}^2)} |y| \log \left|R^{-1}y\right|.$$

Having minimized over p for a fixed R, we must now choose R. First observe that $\overline{\omega} = \Delta(\phi_{x,R}\psi) = \phi_{x,R}\Delta\psi + \Delta\phi_{x,R}\psi + 2\nabla\phi_{x,R} \cdot \nabla\psi = \phi_{x,R}\omega + \Delta\phi_{x,R}\psi + 2\nabla^\perp\phi_{x,R} \cdot u$. Also, for all $z \in B_{2R}(x)$

$$|\psi(z)| \le \int_{|x|}^{|x|+2R} \|gu\|_{L^\infty} h(r)\, dr \le \|gu\|_{L^\infty} R h(|x| + 2R),$$

where $g := 1/h$. Hence,

$$\begin{aligned}
\|\overline{\omega}\|_{L^\infty(\mathbb{R}^2)} &\le \|\phi_{x,R}\|_{L^\infty} \|\omega\|_{L^\infty(B_{2R}(x))} + 2\|\nabla^\perp\phi_{x,R}\|_{L^\infty} \|u\|_{L^\infty(B_{2R}(x))} \\
&\qquad + \|gu\|_{L^\infty} h(|x| + 2R) R \|\Delta\phi_{x,R}\|_{L^\infty(B_{2R}(x))} \\
&\le \|\omega\|_{L^\infty(\mathbb{R}^2)} + C R^{-1} h(|x| + 2R) \|gu\|_{L^\infty(\mathbb{R}^2)}
\end{aligned} \tag{3.5}$$

so

$$|u(x+y) - u(x)| \le -C \left[\|\omega\|_{L^\infty(\mathbb{R}^2)} + R^{-1} h(|x| + 2R) \|gu\|_{L^\infty(\mathbb{R}^2)}\right] |y| \log \left|R^{-1}y\right|.$$

Now choose $R = h(x)$. Then

$$\begin{aligned}
R^{-1} h(|x| + 2R) &= g(x) h(|x| + 2h(x)) \le C g(x) h(|x|) + h(2h(x)) \\
&\le C(1 + C g(x) h(h(x))) \le C,
\end{aligned}$$

where we used the subadditivity of h (see Lemma 2.2) and Lemma 2.5. Hence, we have, for $|y| \le e^{-1}h(x)$,

$$\begin{aligned} |u(x+y) - u(x)| &\le -C \|u\|_{S_h} |y| (\log |y| + \log g(x)) \\ &= C \|u\|_{S_h} |y| \log \left(\frac{h(x)}{|y|} \right) = C \|u\|_{S_h} h(x) \overline{\mu} \left(\frac{|y|}{h(x)} \right), \end{aligned}$$

the last equality holding as long as $|y| \le h(x)e^{-1} = e^{-1}R \le R$.

If $|y| \ge e^{-1}R$, then $\overline{\mu}\left(\frac{|y|}{h(x)}\right) = e^{-1}$, and

$$\begin{aligned} |u(x+y) - u(x)| &\le C \|u\|_{S_h} (h(x) + h(x+y)) \le C \|u\|_{S_h} h(x) \\ &= C \|u\|_{S_h} h(x) \overline{\mu} \left(\frac{|y|}{h(x)} \right), \end{aligned}$$

where we applied Lemma 2.5, using $|y| \le C(1 + |x|)$. Note that if $h \equiv C$, however, we need no restriction on $|y|$ to reach this conclusion, since $h(x) = h(x+y) = C$. □

In proving uniqueness in Section 6, we will need to bound the term in the Serfati identity (1.1) coming from a convolution of the difference between two vorticities. Since the vorticities have no assumed regularity, we will need to rearrange the convolution so as to use an estimate on the Biot-Savart kernel that involves the difference of the flow maps, as in Proposition 3.4. This proposition is a refinement of Proposition 6.2 of [1] that better accounts for the effect of the parameter λ in the cutoff function a_λ. Note that although we assume the solutions lie in some S_h space, h does not appear directly in the estimates, rather it appears indirectly via the value of $\delta(t)$, as one can see in the application of the proposition.

Proposition 3.4. *Let X_1 and X_2 be measure-preserving homeomorphisms of $\mathbb{R}^2$ and let $\omega^0 \in L^\infty(\mathbb{R}^2)$. Fix $x \in \mathbb{R}^2$ and $\lambda > 0$. Let $V = \operatorname{supp} a_\lambda(X_1(s,x) - X_1(s,\cdot)) \cup \operatorname{supp} a_\lambda(X_1(s,x) - X_2(s,\cdot))$ and assume that*

$$\delta(t) := \|X_1(t,\cdot) - X_2(t,\cdot)\|_{L^\infty(V)} < \infty. \tag{3.6}$$

Then we have

$$\left| \int (a_\lambda K(X_1(s,x) - X_1(s,y)) - a_\lambda K(X_1(s,x) - X_2(s,y)))\omega^0(y)\, dy \right| \le C\|\omega^0\|_{L^\infty} \lambda \overline{\mu}(\delta(t)/\lambda).$$

The constant, C, depends only on the Lipschitz constant of a.

Proof. We have,

$$\int (a_\lambda K(X_1(s,x) - X_1(s,y)) - a_\lambda K(X_1(s,x) - X_2(s,y)))\omega^0(y)\, dy = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &:= \int a_\lambda(X_1(s,x) - X_1(s,y)) \left(K(X_1(s,x) - X_1(s,y)) - K(X_1(s,x) - X_2(s,y))\right) \omega^0(y)\, dy, \\ I_2 &:= \int \left(a_\lambda(X_1(s,x) - X_1(s,y)) - a_\lambda(X_1(s,x) - X_2(s,y))\right) K(X_1(s,x) - X_2(s,y))\omega^0(y)\, dy. \end{aligned}$$

To bound I_1, let $U = \operatorname{supp} a_\lambda(X_1(s,x) - X_1(s,\cdot)) \subseteq V$, which we note has measure $4\pi\lambda^2$ independently of x. Then

$$\begin{aligned} |I_1| &\le \|K(X_1(s,x) - X_1(s,y)) - K(X_1(s,x) - X_2(s,y))\|_{L^1_y(U)} \|\omega^0\|_{L^\infty(\mathbb{R}^2)} \\ &\le C\lambda \|\omega^0\|_{L^\infty(\mathbb{R}^2)} \overline{\mu}(\delta(t)/\lambda). \end{aligned}$$

Here, we applied Proposition 3.2 at the point $X_1(s,x)$.

For I_2, we have,

$$
\begin{aligned}
|I_2| &\le \int \left| (a_\lambda(X_1(s,x)-X_1(s,y)) - a_\lambda(X_1(s,x)-X_2(s,y)))\, K(X_1(s,x)-X_2(s,y))\omega^0(y) \right| \, dy \\
&\le \frac{C}{\lambda} \int_V |X_1(s,y)) - X_2(s,y))| \, |K(X_1(s,x)-X_2(s,y))| \, |\omega^0(y)| \, dy \\
&\le \frac{C}{\lambda} \|\omega^0\|_{L^\infty} \delta(t) \int_V |K(X_1(s,x)-X_2(s,y))| \, dy \le C\|\omega^0\|_{L^\infty}\delta(t).
\end{aligned}
$$

Here, we used (3.2) with $p=1$ and that the Lipschitz constant of a_λ is $C\lambda^{-1}$. Also, though,

$$
|I_2| \le 2\|\omega^0\|_{L^\infty} \int_V |K(X_1(s,x)-X_2(s,y))| \, dy \le C\lambda \|\omega^0\|_{L^\infty},
$$

again using (3.2). The result then follows from observing that $\delta(t) \le \lambda \overline{\mu}(\delta(t)/\lambda)$ for $\delta(t) \le \lambda e^{-1}$ and $\overline{\mu}(\delta(t)/\lambda) = e^{-1}$ for $\delta(t) \ge \lambda e^{-1}$. □

4. Flow map bounds

In this section we develop bounds related to the flow map for solutions to the Euler equations in S_h on $[0,T]$. First, though, is the matter of existence and uniqueness:

Lemma 4.1. *Let h be a pre-growth bound (which we note includes $h(x) = C(1+|x|)$) and assume that $u \in L^\infty(0,T;S_h)$. Then there exists a unique flow map, X, for u; that is, a function $X\colon [0,T]\times\mathbb{R}^2 \to \mathbb{R}^2$ for which*

$$
X(t,x) = x + \int_0^t u(s, X(s,x)) \, ds
$$

for all $(t,x) \in [0,T]\times\mathbb{R}^2$.

Proof. Because u is locally log-Lipschitz by Proposition 3.3, this is (essentially) classical. □

Lemma 4.2. *Let h be a pre-growth bound. Assume that $u_1, u_2 \in L^\infty(0,T;S_h)$. Let F_t be the function defined in (2.6). We have,*

$$
\frac{|X_1(t,x)-X_2(t,x)|}{F_t(x)} \le C_0 t, \quad \frac{|X_j(t,x)-x|}{F_t(x)} \le C_0 t, \tag{4.1}
$$

where $C_0 = \|u_j\|_{L^\infty(0,T;S_h)}$.

Proof. For $j=1,2$, $|u_j(t,x)| \le \|u_j(t)\|_{S_h} |h(x)|$, so

$$
|X_j(t,x)| \le |x| + \int_0^t |u_j(s, X_j(s,x))| \, ds \le |x| + C_0 \int_0^t h(X_j(s,x)) \, ds.
$$

Hence by Osgood's inequality, $|X_j(t,x)| \le \Gamma_t(x)$, where Γ_t is defined in (2.5). We also have

$$
|X_j(t,x)-x| \le \int_0^t |u_j(s,X_j(s,x))| \, ds \le C_0 \int_0^t h(X_j(s,x)) \, ds \le C_0 t F_t(x).
$$

Similarly,

$$
|X_1(t,x)-X_2(t,x)| \le \int_0^t (|u_1(s,X_1(s,x))| + |u_2(s,X_2(s,x))|) \, ds \le C_0 t F_t(x).
$$

These bounds yield the result. □

Lemmas 2.7 and 4.2 together show that over time, the flow transports a "particle" of fluid at a distance r from the origin by no more than a constant times $h(r)$. This will allow us to control the growth at infinity of the velocity field over time so that it remains in S_h (for at least a finite time), as we shall see in the next section. As the fluid evolves over time, however, the flow can move two points farther and farther apart; that is, its spatial modulus of continuity can worsen, though in a controlled way, as we show in Lemma 4.3. (A similar bound to that in Lemma 4.3 holds for any growth bound, but we restrict ourselves to the special case of bounded vorticity, bounded velocity velocity fields, for that is all we will need.)

Lemma 4.3. *Let $u \in L^\infty(0, T; S_1)$ and let X be the unique flow map for u. Let $C_0 = \|u\|_{L^\infty(0,T;S_1)}$. For any $t \in [0, T]$ define the function,*

$$\chi_t(r) := \left\{ \begin{array}{ll} r^{e^{-C_0 t}} & \text{when } r \le 1 \\ r & \text{when } r > 1 \end{array} \right\} \le r + r^{e^{-C_0 t}}.$$

Then for all $x, y \in \mathbb{R}^2$,

$$|X(t, x) - X(t, y)| \le C(T)\chi_t(|x - y|).$$

The same bound holds for X^{-1}.

Proof. The bounds,

$$|X(t, x) - X(t, y)|, \left|X^{-1}(t, x) - X^{-1}(t, y)\right| \le C(T)\,|x - y|^{e^{-C_0 t}}$$

are established in Lemma 8.2 of [12]. We note, however, that that proof applies only for all sufficiently small $|x - y|$. A slight refinement of the proof produces the bounds as we have stated them. □

The following simple bound will be useful later in the proof of Proposition 8.3:

$$\chi_t(ar) \le a^{e^{-C_0 t}} \chi_t(r) \text{ for all } a \in [0, 1], r > 0. \tag{4.2}$$

5. Existence

Our proof of existence differs significantly from that in [1] only in the use of the Serfati identity to obtain a bound in $L^\infty(0, T; S_h)$ of a sequence of approximating solutions and to show that the sequence is Cauchy, which is more involved than in [1]. Although velocities in S_h are not log-Lipschitz in the whole plane (unless h is constant), they are log-Lipschitz in any compact subset of $\mathbb{R}^2$. Since the majority of the proof of existence involves obtaining convergence on compact subsets, this has little effect on the proof. Therefore, we give only the details of the bound on $L^\infty(0, T; S_h)$ using the Serfati identity, as this is the main modification of the existence proof. We refer the reader to [1] for the remainder of the argument.

Proof of existence in Theorem 1.6. Let $u^0 \in S_h$ and assume that u^0 does not vanish identically; otherwise, there is nothing to prove. Let $(u_n^0)_{n=1}^\infty$ and $(\omega_n^0)_{n=1}^\infty$ be compactly supported approximating sequences to the initial velocity, u^0, and initial vorticity, ω^0, obtained by cutting off the stream function and mollifying by a smooth, compactly supported mollifier. (This is as done in Proposition B.2 of [1], which simplifies tremendously when specializing to all of $\mathbb{R}^2$.) Let u_n be the classical, smooth solution to the Euler equations with initial velocity u_n^0, and note that its vorticity is compactly supported for all time. The existence and uniqueness of such solutions follows, for instance, from [14] and references therein. (See also Chapter 4 of [12] or Chapter 4 of [4].) Finally, let $\omega_n = \operatorname{curl} u_n$.

As we stated above, we give only the uniform $L^\infty([0,T]; S_h)$ bound for this sequence, the rest of the proof differing little from that in [1].

We have,

$$\left\|u_n^0\right\|_{S_h} \le C \left\|u^0\right\|_{S_h}. \tag{5.1}$$

It follows that the Serfati identity holds for the approximate solutions. Hence,

$$\begin{aligned}|u_n(t,x)| &\le \left|u_n^0(x)\right| + \left|(a_\lambda K) * (\omega_n(t) - \omega_n^0)(x)\right| \\ &\quad + \int_0^t \left|\Big(\nabla\nabla^\perp\left[(1-a_\lambda)K\right]\Big) *\cdot (u_n \otimes u_n)(s,x)\right| \, ds.\end{aligned}$$

We note here that λ can vary arbitrarily with x.

The first convolution we bound using (3.1) and (5.1) as

$$\left|(a_\lambda K) * (\omega_n(t) - \omega_n^0)(x)\right| \le C\left(\lambda\|\omega_n(t)\|_{L^\infty(B_\lambda(x))} + \lambda\|\omega_n^0\|_{L^\infty(B_\lambda(x))}\right) \le C\lambda.$$

For the second convolution, we have,

$$\begin{aligned}\left|\Big(\nabla\nabla^\perp\left[(1-a_\lambda)K\right]\Big) *\cdot (u_n \otimes u_n)(s)\right| &\le \int_{B_\lambda(x)^C} \frac{C}{|x-y|^3} \left|u_n(s,y)\right|^2 dy \\ &= C\int_{B_\lambda(x)^C} \frac{h(y)^2}{|x-y|^3}\left|\frac{u_n(s,y)}{h(y)}\right|^2 dy \\ &\le C\left\|\frac{u_n(s)}{h}\right\|_{L^\infty}^2 \left[\int_{B_\lambda(x)^C} \frac{h(x-y)^2}{|x-y|^3}\, dy + h(x)^2 \int_{B_\lambda(x)^C} \frac{1}{|x-y|^3}\, dy\right] \\ &= C\left\|\frac{u_n(s)}{h}\right\|_{L^\infty}^2 \left[H(\lambda(x)) + C\frac{h(x)^2}{\lambda(x)}\right],\end{aligned}$$

where $H = H[h^2]$ is defined in (1.2). The second inequality follows from the subadditivity of h^2 (as in Remark 2.4). Hence,

$$|u_n(t,x)| \le \left|u_n^0(x)\right| + C\lambda(x) + C\int_0^t \left\|\frac{u_n(s)}{h}\right\|_{L^\infty}^2 \left[H(\lambda(x)) + C\frac{h(x)^2}{\lambda(x)}\right] ds.$$

Dividing both sides by $h(x)$ gives

$$\left|\frac{u_n(t,x)}{h(x)}\right| \le \left|\frac{u_n^0(x)}{h(x)}\right| + C\frac{\lambda(x)}{h(x)} + C\int_0^t \left\|\frac{u_n(s)}{h}\right\|_{L^\infty}^2 \left[\frac{H(\lambda(x))}{h(x)} + C\frac{h(x)}{\lambda(x)}\right] ds. \tag{5.2}$$

Now, for any fixed t, we can set

$$\lambda = \lambda(t,x) = h(x)\left(\int_0^t \left\|\frac{u_n(s)}{h}\right\|_{L^\infty}^2 ds\right)^{\frac{1}{2}}. \tag{5.3}$$

Defining

$$\Lambda(s) := \left\|\frac{u_n(s)}{h}\right\|_{L^\infty}^2$$

this leads to

$$
\begin{aligned}
\left|\frac{u_n(t,x)}{h(x)}\right| &\le \left|\frac{u_n^0(x)}{h(x)}\right| + C\|\omega^0\|_{L^\infty}\left(\int_0^t \Lambda(s)\,ds\right)^{\frac12} \\
&\qquad + C\int_0^t \Lambda(s) H\left(h(x)\left(\int_0^t \Lambda(r)\,dr\right)^{\frac12}\right) g(x)\,ds + C\left(\int_0^t \Lambda(s)\,ds\right)^{\frac12} \\
&\le C + Cg(x)H\left(h(x)\left(\int_0^t \Lambda(s)\,ds\right)^{\frac12}\right)\int_0^t \Lambda(s)\,ds + C\left(\int_0^t \Lambda(s)\,ds\right)^{\frac12} \\
&\le C + CH\left(\left(\int_0^t \Lambda(s)\,ds\right)^{\frac12}\right)\int_0^t \Lambda(s)\,ds + C\left(\int_0^t \Lambda(s)\,ds\right)^{\frac12} \\
&\le C + C\left(\int_0^t \Lambda(s)\,ds\right)^{\frac12} + Cf\left(\left(\int_0^t \Lambda(s)\,ds\right)^{\frac12}\right)\left(\int_0^t \Lambda(s)\,ds\right)^{\frac12} \\
&\le C + C\left[1 + f\left(\left(\int_0^t \Lambda(s)\,ds\right)^{\frac12}\right)\right]\left(\int_0^t \Lambda(s)\,ds\right)^{\frac12},
\end{aligned}
$$

where $f(z) := zH(z)$. In the third-to-last inequality we used that H is decreasing and $h(x) \ge h(0) > 0$.

Observe that although this inequality was obtained by choosing $\lambda = \lambda(t,x)$ for one fixed t, the inequality itself holds for all $t \in [0,T]$.

Taking the supremum over $x \in \mathbb{R}^2$ and squaring both sides, we have

$$\Lambda(t) \le C + E\left(\int_0^t \Lambda(s)\,ds\right) \le C + \mu\left(\int_0^t \Lambda(s)\,ds\right), \tag{5.4}$$

where E, μ are as in Definition 1.1. Now we can apply Lemmas 2.3 and 5.1 to conclude that $u_n \in L^\infty(0,T;S_h)$ with a norm bounded uniformly over n. Lemma 5.1 also gives global-in-time existence (T arbitrarily large) when (1.3) holds. □

Lemma 5.1. *Assume that $\Lambda\colon [0,\infty) \to [0,\infty)$ is continuous with*

$$\Lambda(t) \le \Lambda_0 + \mu\left(\int_0^t \Lambda(s)\,ds\right) \tag{5.5}$$

for some $\Lambda_0 \ge 0$, where $\mu\colon [0,\infty) \to [0,\infty)$ is convex. Then for all $t \le 1$,

$$\int_{\Lambda_0}^{\Lambda(t)} \frac{ds}{\mu(s)} \le t \tag{5.6}$$

and for all $t \in [0,T]$ for any fixed $T \ge 1$,

$$\int_{\Lambda_0}^{\Lambda(t)} \frac{ds}{\mu(Ts)} \le \frac{t}{T}. \tag{5.7}$$

Moreover, if

$$\int_1^\infty \frac{ds}{\mu(s)} = \infty \tag{5.8}$$

then $\Lambda \in L^\infty_{loc}([0,\infty))$.

Proof. Because μ is convex, we can apply Jensen's inequality to conclude that

$$\Lambda(t) \le \Lambda_0 + \mu\left(\int_0^t t\Lambda(s)\,\frac{ds}{t}\right) \le \Lambda_0 + \int_0^t \mu(t\Lambda(s))\,\frac{ds}{t}.$$

As long as $t \le 1$, Remark 2.10 allows us to write

$$\Lambda(t) \le \Lambda_0 + \int_0^t \mu(\Lambda(s))\,ds,$$

and Osgood's lemma gives (5.6). Now suppose that $T > 1$. Then Remark 2.10 gives the weaker bound,

$$\mu(t\Lambda(s)) = \mu\left(\frac{t}{T}T\Lambda(s)\right) \le \frac{t}{T}\mu\left(T\Lambda(s)\right)$$

so that

$$\Lambda(t) \le \Lambda_0 + \frac{1}{T}\int_0^t \mu(T\Lambda(s))\,ds,$$

leading to (5.7).

Finally, if (5.8) holds then applying Osgood's lemma to (5.7) shows that Λ is bounded on any interval $[0, T]$, so that $\Lambda \in L^\infty_{loc}([0,\infty))$. □

We make a few remarks on our proof of Theorem 1.6.

Lemma 5.1 allows us to obtain finite-time or global-in-time existence of solutions, but unless we have a stronger condition on μ, neither the finite time of existence nor the bound on the growth of the L^∞ norm that results will be optimal. For both of our example growth bounds in Theorem 1.5 there are stronger conditions; namely, if μ_1, μ_2 are the function in Definition 1.1 corresponding to h_1, h_2 then for all $a, r \ge 0$,

$$\mu_1(ar) \le C_0 a^{1+\alpha}\mu_1(r), \quad \mu_2(ar) \le C_0 a\mu_2(r).$$

It is easy to see that the condition on μ_2 in fact, implies (5.8), though the condition on μ_1 is too weak to do so. Both conditions improve the bound on the L^∞ norm resulting from Lemma 5.1 and for h_1, the time of existence.

Moreover, Lemma 2.3 shows that, up to a constant factor, $\mu(r) := Cr(1 + r)$ works for all well-posedness growth bounds (and gives $\mu_2(ar) \le C_0 a^2 \mu_2(r)$ for all $a, r \ge 0$). This *suggests* that a slight weakening of the condition we placed on growth bounds in (ii) of Definition 1.1 could be made that would still allow finite-time existence to be obtained.

6. Uniqueness

In this section we prove Theorem 1.7, from which uniqueness immediately follows. Our argument is a an adaptation of the approach of Serfati as it appears in [1]. It starts, however, by exploiting the flow map estimates in Lemma 4.2, inspired by the proof of Lemma 2.13 of [8], which is itself an adaptation of Marchioro's and Pulvirenti's elegant uniqueness proof for 2D Euler in [13], in which a weight is introduced.

Proof of Theorem 1.7. We will use the bound on X_1 and X_2 given by Lemma 4.2 with the growth bound, $F_T[\zeta]$, defined in (2.6). This is valid since $\zeta \ge h$. By Lemma 2.7, $F_T[\zeta]$ is a growth bound that is equivalent to ζ, up to a factor of $C(T)$; hence, we will use ζ in place of $F_T[\zeta]$, which will simply introduce a factor $C(T)$ into our bounds.

By the expression for the flow maps in Lemma 4.1,

$$\eta(t) \le \left\| \int_0^t \frac{|u_1(s, X_1(s,x)) - u_2(s, X_2(s,x))|}{\zeta(x)} \, ds \right\| \le \int_0^t L(s) \, ds = M(t).$$

We also have,

$$L(s) \le \|A_1(s,x)\|_{L^\infty_x(\mathbb{R}^2)} + \|A_2(s,x)\|_{L^\infty_x(\mathbb{R}^2)},$$

where

$$\begin{aligned} A_1(s,x) &:= \frac{u_2(s, X_1(s,x)) - u_2(s, X_2(s,x))}{\zeta(x)}, \\ A_2(s,x) &:= \frac{u_1(s, X_1(s,x)) - u_2(s, X_1(s,x))}{\zeta(x)}. \end{aligned}$$

For A_1, first observe that Lemma 4.2 shows that $|X_1(s,x) - X_2(s,x)| \le Ct\zeta(x) \le Ct(1 + |x|)$ and $|X_1(s,x) - x| \le Ct\zeta(x) \le Ct(1 + |x|)$. Hence, we can apply Proposition 3.3 with ζ in place of h to give

$$\begin{aligned} &|u_2(s, X_1(s,x)) - u_2(s, X_2(s,x))| \le C \|u_2\|_{S_\zeta} \zeta(x) \overline{\mu} \left(|X_1(s,x) - X_2(s,x)| / \zeta(x) \right) \\ &\qquad \le C\zeta(x)\overline{\mu}(\eta(s)). \end{aligned}$$

It follows that

$$|A_1(s,x)| \le C \frac{\zeta(x)}{\zeta(x)} \overline{\mu}(\eta(s)) \le C\overline{\mu}(\eta(s)). \tag{6.1}$$

To bound A_2, we use the Serfati identity, choosing $\lambda(x) = h(x)$, to write

$$|A_2(s,x)| \le \frac{\left| u_1^0(x) - u_2^0(x) \right|}{\zeta(x)} + A_2^1(s,x) + A_2^2(s,x),$$

where

$$\begin{aligned} A_2^1(s,x) &= \frac{1}{\zeta(x)} \left| (a_{h(x)} K) * (\omega_1(s) - \omega_2(s))(X_1(s,x)) - (a_{h(x)} K) * (\omega_1^0 - \omega_2^0)(X_1(s,x)) \right|, \\ A_2^2(s,x) &= \frac{1}{\zeta(x)} \left| \int_0^s \left(\nabla \nabla^\perp [(1 - a_{h(x)}) K] \ast\cdot (u_1 \otimes u_1 - u_2 \otimes u_2) \right) (r, X_1(r,x)) \, dr \right|. \end{aligned}$$

We write,

$$\begin{aligned} &(a_{h(x)} K) * (\omega_1(s) - \omega_2(s))(X_1(s,x)) \\ &\qquad = \int (a_{h(x)} K(X_1(s,x) - z))(\omega_1^0(X_1^{-1}(s,z)) - \omega_2^0(X_2^{-1}(s,z))) \, dz \\ &\qquad = \int (a_{h(x)} K(X_1(s,x) - z))(\omega_2^0(X_1^{-1}(s,z)) - \omega_2^0(X_2^{-1}(s,z))) \, dz \\ &\qquad\quad + \int (a_{h(x)} K(X_1(s,x) - z))(\omega_1^0(X_1^{-1}(s,z)) - \omega_2^0(X_1^{-1}(s,z))) \, dz. \end{aligned}$$

Making the two changes of variables, $z = X_1(s,y)$ and $z = X_2(s,y)$, we can write

$$\begin{aligned} A_2^1(s,x) &\le \frac{1}{\zeta(x)} \left| \int (a_{h(x)} K(X_1(s,x) - X_1(s,y)) - a_{h(x)} K(X_1(s,x) - X_2(s,y))) \omega_2^0(y) \, dy \right| \\ &\qquad + \frac{|J(s,x)|}{\zeta(x)}. \end{aligned}$$

We are thus in a position to apply Proposition 3.4 to bound A_2^1. To do so, we set

$$U_j := \{y \in \mathbb{R}^2 \colon\ |X_1(s,x) - X_j(s,y)| \leq h(x)\}$$

so that $V := U_1 \cup U_2$ is as in Proposition 3.4. Then with δ as in (3.6), we have

$$\begin{aligned} \delta(s) &\leq \eta(s) \sup_{y \in V} \zeta(y) \leq \eta(s) \zeta(|x| + h(x) + Ct\zeta(|x| + h(x))) \\ &= C\eta(s)\zeta\left(|x| + \zeta(|x|) + CT\zeta(|x| + \zeta(|x|))\right) \leq C_1 \eta(s) \zeta(x), \end{aligned} \tag{6.2}$$

where $C_1 = C(T)$. Above, we applied Lemma 4.2 in the second inequality and the last inequality follows from repeated applications of Lemma 2.5 to ζ. Hence, Proposition 3.4 gives

$$A_2^1(s,x) \leq C\|\omega^0\|_{L^\infty} \frac{h(x)}{\zeta(x)} \overline{\mu}\left(\frac{\delta(s)}{h(x)}\right) + \frac{|J(s,x)|}{\zeta(x)}.$$

But by Lemma 2.9 (noting that $h(x)/\zeta(x) \leq 1$) and (6.2),

$$\frac{h(x)}{\zeta(x)} \overline{\mu}\left(\frac{\delta(s)}{h(x)}\right) \leq \overline{\mu}\left(\frac{h(x)}{\zeta(x)} \frac{\delta(s)}{h(x)}\right) = \overline{\mu}\left(\frac{\delta(s)}{\zeta(x)}\right) \leq \overline{\mu}(C_1\eta(s)).$$

Hence,

$$A_2^1(s,x) \leq C\overline{\mu}\left(C_1\eta(s)\right) + \frac{|J(s,x)|}{\zeta(x)}. \tag{6.3}$$

We now bound $A_2^2(x)$. We have,

$$\begin{aligned} A_2^2(s,x) &\leq \frac{C}{\zeta(x)} \int_0^s \max\left\{ \left\| \frac{u_1}{h} \right\|_{L^\infty(0,T)\times\mathbb{R}^2}, \left\| \frac{u_2}{h} \right\|_{L^\infty(0,T)\times\mathbb{R}^2} \right\} \\ &\qquad \int_{B_{\frac{h(x)}{2}}(X_1(r,x))^C} \frac{(\zeta h)(y)}{|X_1(r,x) - y|^3} \frac{|u_1(r,y) - u_2(r,y)|}{\zeta(y)}\, dy\, dr \\ &\leq \frac{C}{\zeta(x)} \int_0^s \left(\sup_{z \in \mathbb{R}^2} \frac{|u_1(r,z) - u_2(r,z)|}{\zeta(z)} \int_{B_{\frac{h(x)}{2}}(X_1(r,x))^C} \frac{(\zeta h)(y)}{|X_1(r,x) - y|^3}\, dy \right) dr \\ &= \frac{C}{\zeta(x)} \int_0^s \left(Q(r) \int_{B_{\frac{h(x)}{2}}(X_1(r,x))^C} \frac{(\zeta h)(y)}{|X_1(r,x) - y|^3}\, dy \right)\, dr. \end{aligned}$$

Because ζh is subadditive (being a pre-growth bound), letting $w = X_1(r,x)$, we have

$$\begin{aligned} \int_{B_{\frac{h(x)}{2}}(w)^C} \frac{(\zeta h)(y)}{|w - y|^3}\, dy &\leq \int_{B_{\frac{h(x)}{2}}(w)^C} \frac{(\zeta h)(w - y)}{|w - y|^3}\, dy + (\zeta h)(w) \int_{B_{\frac{h(x)}{2}}(w)^C} \frac{1}{|w - y|^3}\, dy \\ &= 2\pi H[\zeta h](h(x)/2) + C\frac{(\zeta h)(w)}{h(x)/2} \leq C + C\zeta(w) \leq C(1 + \zeta(x)). \end{aligned} \tag{6.4}$$

Here we used that ζh is a growth bound and Lemma 4.2. It follows that

$$A_2^2(s,x) \leq C\frac{1 + \zeta(x)}{\zeta(x)} \int_0^s Q(r)\, dr \leq C \int_0^s Q(r)\, dr. \tag{6.5}$$

But,

$$\begin{aligned}
Q(r) &= \sup_{z\in\mathbb{R}^2} \frac{|u_1(r,z)-u_2(r,z)|}{\zeta(z)} = \sup_{z\in\mathbb{R}^2} \frac{|u_1(r,X_1(r,z))-u_2(r,X_1(r,z))|}{\zeta(X_1(r,z))} \\
&\le C \sup_{z\in\mathbb{R}^2} \frac{|u_2(r,X_1(r,z))-u_2(r,X_2(r,z))|}{\zeta(z)} + C \sup_{z\in\mathbb{R}^2} \frac{|u_2(r,X_2(r,z))-u_1(r,X_1(r,z))|}{\zeta(z)} \\
&\le C\left(\overline{\mu}(\eta(r)) + L(r)\right),
\end{aligned}$$

where we used Lemma 4.2 in the first inequality and (6.1) in the last inequality. Hence,

$$A_2^2(s,x) \le C \int_0^s (\overline{\mu}(\eta(r)) + L(r))\,dr.$$

It follows from all of these estimates that

$$\begin{aligned}
\eta(t) &\le \int_0^t L(s)\,ds = M(t), \\
L(s) &\le \left\| \frac{u_1^0 - u_2^0}{\zeta} \right\|_{L^\infty} + C\overline{\mu}(C_1\eta(s)) + \frac{|J(s,x)|}{\zeta(x)} + C\int_0^s (\overline{\mu}(\eta(r)) + L(r))\,dr \qquad (6.6)\\
&= a(T) + C\overline{\mu}(C_1\eta(s)) + C\int_0^s (\overline{\mu}(C_1\eta(r)) + L(r))\,dr.
\end{aligned}$$

We therefore have

$$\begin{aligned}
M(t) &\le ta(T) + \int_0^t \left(C\overline{\mu}(C_1\eta(s)) + C\int_0^s (\overline{\mu}(C_1\eta(r)) + L(r))\,dr \right) ds \\
&\le ta(T) + C\int_0^t \left(\overline{\mu}(C_1 M(s)) + \int_0^s (\overline{\mu}(C_1 M(r)) + L(r))\,dr \right) ds \\
&= ta(T) + C\int_0^t \left(\overline{\mu}(C_1 M(s)) + M(s) + \int_0^s \overline{\mu}(C_1 M(r))\,dr \right) ds \qquad (6.7)\\
&\le ta(T) + C\int_0^t \left((1+s)\overline{\mu}(C_1 M(s)) + M(s) \right) ds \\
&\le ta(T) + C\int_0^t \left(\overline{\mu}(C_1 M(s)) + M(s) \right) ds,
\end{aligned}$$

where we note that the final $C = C(T)$ increases with T. In the second inequality we used $\overline{\mu}$ increasing and $\eta(s) \le M(s)$, while in the third inequality we used that $\overline{\mu}$ and M are both increasing. The bound in (1.6) follows from Osgood's lemma.

We now obtain the bounds on $M(t)$ and $Q(t)$.

The bound on Q we made earlier shows that

$$Q(t) \le C(T)(\overline{\mu}(\eta(t)) + L(t)) \le C(T)(\overline{\mu}(C_1 M(t)) + L(t)), \qquad (6.8)$$

since $\eta(t) \le M(t)$. Then by (6.6),

$$\begin{aligned}
L(t) &\le a(T) + C(T)\overline{\mu}(C_1 M(t)) + C(T)\int_0^t (\overline{\mu}(C_1 M(s)) + L(s))\,ds \\
&\le a(T) + C(T)\overline{\mu}(C_1 M(t))) + C(T)\int_0^t L(s)\,ds.
\end{aligned}$$

Applying Gronwall's inequality,

$$L(t) \le (a(T) + C\overline{\mu}(C_1 M(t)))e^{C(T)t}.$$

Since we can absorb a constant, this same bound holds for $Q(t)$:

$$Q(t) \le (a(T) + C\overline{\mu}(C_1 M(t)))e^{C(T)t}.$$

Hence, we can easily translate a bound on M to a bound on Q.

Returning, then, to (1.6), we have, for $a(T)$ sufficiently small,

$$\int_{ta(T)}^{M(t)} \frac{ds}{\overline{\mu}(C_1 s)}\, ds \le C(T)t.$$

Integrating gives

$$-\log\log s\Big|_{C_1 ta(T)}^{C_1 M(t)} \le C(T)t$$

from which we conclude that

$$M(t) \le (ta(T))^{e^{-C(T)t}}.$$

This holds as long as $M(t) \le C_1^{-1}e^{-1}$, which gives a bound on the time t.

On the other hand, if $s > C_1^{-1}e^{-1}$ then $\overline{\mu}(C_1 s) = e^{-1}$, so for $ta(T) > C_1^{-1}e^{-1}$,

$$\int_{ta(T)}^{M(t)} \frac{ds}{\overline{\mu}(C_1 s)}\, ds = \int_{ta(T)}^{M(t)} e\, ds \le C(T)t.$$

Thus,

$$e(M(t) - ta(T)) \le C(T)t \le C(T)ta(T),$$

giving

$$M(t) \le C(T)ta(T).$$

(For intermediate values of $a(T)$ we still obtain a usable bound, it is just more difficult to be explicit.) □

We were able to use a growth bound ζ larger than h and obtain a result for an arbitrary T because, unlike the proof of existence in Section 5, we are assuming that we already know that u_1, u_2 lies in S_h. Hence, the quadratic term in the Serfati identity can in effect be made linear.

Theorem 1.7 gives a bound on the difference in velocities over time. It remains, however, to characterize $a(T)$ in a useful way in terms of u_1^0, u_2^0, and $u_1^0 - u_2^0$ and so obtain Theorem 1.10. This, the subject of the next section, is not as simple as it may seem.

7. Continuous dependence on initial data

In this section, we prove Theorems 1.9 and 1.10, bounding $a(T)$ of (1.4). The difficulty in bounding $a(T)$ lies wholly in bounding J/ζ, with $J = J(t, x)$ as in (1.5). We can write $J = J_2 - J_1$, where

$$\begin{aligned} J_1(t, x) &= (a_{h(x)}K) * (\omega_1^0 - \omega_2^0)(X_1(t, x)), \\ J_2(t, x) &= (a_{h(x)}K) * ((\omega_1^0 - \omega_2^0) \circ (X_1^{-1}(t))(x). \end{aligned}$$

Both J_1/ζ and J_2/ζ are easy to bound, as we do in Theorem 1.9, if we assume that $\omega_1^0 - \omega_2^0$ is close in L^∞, an assumption that is physically unreasonable, however, as discussed in Section 1.

Proof of Theorem 1.9. We have

$$\begin{aligned}\left|\frac{J_1(s,x)}{\zeta(x)}\right| &\le \left\|(a_{h(x)}K) * (\omega_1^0 - \omega_2^0)(X_1(s,z))\right\|_{L^\infty_z} / \zeta(x) \\ &= \left\|(a_{h(x)}K) * (\omega_1^0 - \omega_2^0)\right\|_{L^\infty} / \zeta(x) \\ &\le \left\|a_{h(x)}K\right\|_{L^1} \left\|\omega_1^0 - \omega_2^0\right\|_{L^\infty} / \zeta(x) \le C(h(x)/\zeta(x)) \left\|\omega_1^0 - \omega_2^0\right\|_{L^\infty} \\ &\le C \left\|\omega_1^0 - \omega_2^0\right\|_{L^\infty}.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\frac{J_2(s,x)}{\zeta(x)}\right| &\le \left\|(a_{h(x)}K) * ((\omega_1^0 - \omega_2^0) \circ X_1^{-1}(s))(x)\right\|_{L^\infty_z} / \zeta(x) \\ &\le \left\|a_{h(x)}K\right\|_{L^1} \left\|(\omega_1^0 - \omega_2^0)(X_1^{-1}(s,z))\right\|_{L^\infty_z} / \zeta(x) \le C(h(x)/\zeta(x)) \left\|\omega_1^0 - \omega_2^0\right\|_{L^\infty} \\ &\le C \left\|\omega_1^0 - \omega_2^0\right\|_{L^\infty}.\end{aligned}$$

Combined, these two bounds easily yield the bound on $a(T)$. □

More interesting is a measure of $a(T)$ in terms of $u_1^0 - u_2^0$ without involving $\omega_1^0 - \omega_2^0$. Now, J_1 is fairly easily bounded in terms of $u_1^0 - u_2^0$ (using Lemma 7.5) since $X_1(s,x)$ has no effect on the L^∞ norm. But in J_2, the composition of the initial vorticity with the flow map complicates matters considerably. What we seek is a bound on $a(T)$ of (1.4) in terms of $\|(u_1^0 - u_2^0)/\zeta\|_{L^\infty}$ and constants that depend upon $\|u_1^0\|_{S_1}$, $\|u_2^0\|_{S_1}$. That is the primary purpose of Theorem 1.10, which we now prove, making forward references to a number of results that appear following its proof. These include Propositions 8.3 and 8.7 and Lemma 8.9, which employ Littlewood-Paley theory and Hölder spaces of negative index, and which we defer to Section 8, where we introduce the necessary technology.

Remark 7.1. *In the proof of Theorem 1.10, we make use for the first time in this paper of Hölder spaces, with negative and fractional indices. We are not using the classical definition of these spaces, but rather one based upon Littlewood-Paley theory. For non-integer indices, they are equivalent, but the constant of equivalency (in one direction) blows up as the index approaches an integer (see Remark 8.2). Because we will be comparing norms with different indices, it is important that we use a consistent definition of these spaces. In this section, the only fact we use regarding Hölder spaces (in the proof of Lemma 7.6) is that $\|\operatorname{div} v\|_{C^{r-1}} \le C\|v\|_{C^r}$ for any $r \in (0,1)$ for a constant C independent of r. For that reason, we defer our definition of Hölder spaces to Section 8.*

Remark 7.2. *With the exception of Proposition 8.7, versions of all of the various lemmas and propositions that we use in the proof of Theorem 1.10 can be obtained for solutions in S_h for any well-posedness growth bound h. Should a way be found to also extend Proposition 8.7 to S_h then a version of Theorem 1.10 would hold for S_h as well.*

Proof of Theorem 1.10. Let ω_1^0, ω_2^0 be the initial vorticities, and let ω_1, ω_2 and X_1, X_2 be the vorticities and flow maps of u_1, u_2.

To bound $a(T)$, let $\overline{\omega}_0 = \omega_1^0 - \omega_2^0 = \operatorname{curl}(u_1^0 - u_2^0)$. Then, since $h \equiv 1$, we can write,

$$\begin{aligned}\frac{J_1(s,x)}{\zeta(x)} &= \frac{\zeta(X_1(s,x))}{\zeta(x)} \frac{\left[(aK) * \operatorname{curl}(u_1^0 - u_2^0)\right](X_1(s,x))}{\zeta(X_1(s,x))}, \\ \frac{J_2(s,x)}{\zeta(x)} &= \frac{(aK) * (\overline{\omega}_0 \circ X_1^{-1}(s))(x)}{\zeta(x)}.\end{aligned}$$

Applying Lemma 2.5, Lemma 4.2, and Lemma 7.5, we see that

$$\frac{J_1(s,x)}{\zeta(x)} \le C \left\| \frac{u_1^0 - u_2^0}{\zeta} \right\|_{L^\infty}, \tag{7.1}$$

the bound holding uniformly over $s \in [0, T]$.

Bounding J_2 is much more difficult, because the flow map appears inside the convolution, which prevents us from writing it as the curl of a divergence-free vector field. Instead, we apply a sequence of bounds, starting with

$$\begin{aligned} \left| \frac{J_2(s,x)}{\zeta(x)} \right| &= \left| \frac{(\zeta \circ X_1^{-1}(s))(x)}{\zeta(x)} \right| \left| \frac{(aK) * (\overline{\omega}_0 \circ X_1^{-1}(s))(x)}{(\zeta \circ X_1^{-1}(s))(x)} \right| \\ &\le C \left| \frac{(aK) * (\overline{\omega}_0 \circ X_1^{-1}(s))(x)}{(\zeta \circ X_1^{-1}(s))(x)} \right| \\ &\le C \Phi_\alpha \left(s, \left\| \frac{\overline{\omega}_0}{\zeta} \circ X_1^{-1}(s, \cdot) \right\|_{C^{-\alpha}} \right) \end{aligned}$$

for all $s \in [0, T]$. Here we used Lemma 4.2 and applied Proposition 8.3 with $f = \overline{\omega}_0 \circ X_1^{-1}(s)$.

Applying Proposition 8.7 with $\alpha = \delta_t$ and $\beta = 2C(\delta)$, followed by Lemma 7.6 gives

$$\left| \frac{J_2(s,x)}{\zeta(x)} \right| \le C \Phi_\alpha \left(s, 2 \left\| \frac{\overline{\omega}_0}{\zeta} \right\|_{C^{-\delta}} \right) \le C \Phi_\alpha \left(s, 2 \left\| \frac{u_1^0 - u_2^0}{\zeta} \right\|_{C^{1-\delta}} \right)$$

for all $s \in [0, T^*]$. Note that the condition on δ_{T^*} in Proposition 8.7 is satisfied because of our definition of T^* and because $\|u\|_{LL} \le C \|u\|_{S_1}$, which follows from Proposition 3.3. We also used, and use again below, that Φ_α is increasing in its second argument.

We apply Lemma 8.9 with $r = 1 - \delta$, obtaining

$$\left\| \frac{u_1^0 - u_2^0}{\zeta} \right\|_{C^{1-\delta}} C \le \left\| \frac{u_1^0 - u_2^0}{\zeta} \right\|_{L^\infty}^{\delta} \left[\left\| \frac{u_1^0 - u_2^0}{\zeta} \right\|_{L^\infty}^{1-\delta} + \left\| \frac{u_1^0 - u_2^0}{\zeta} \right\|_{S_1}^{1-\delta} \right].$$

But,

$$\begin{aligned} \left\| \frac{u_1^0 - u_2^0}{\zeta} \right\|_{S_1} &\le \left\| \frac{u_1^0 - u_2^0}{\zeta} \right\|_{L^\infty} + \left\| \frac{1}{\zeta} \right\|_{L^\infty} \left\| \omega_1^0 - \omega_2^0 \right\|_{L^\infty} + \left\| \nabla \left(\frac{1}{\zeta} \right) \right\|_{L^\infty} \left\| u_1^0 - u_2^0 \right\|_{L^\infty} \\ &\le C \left\| u_1^0 - u_2^0 \right\|_{S_1} \le C, \end{aligned}$$

where we used the identity, $\operatorname{curl}(fu) = f \operatorname{curl} u - \nabla f \cdot u^\perp$, and that $1/\zeta$ is Lipschitz (though $1/\zeta \notin C^1(\mathbb{R}^2)$ unless ζ is constant, because $\nabla \zeta$ is not defined at the origin). Therefore,

$$\left\| \frac{u_1^0 - u_2^0}{\zeta} \right\|_{C^{1-\delta}} \le C \left\| \frac{u_1^0 - u_2^0}{\zeta} \right\|_{L^\infty}^{\delta}.$$

We conclude that

$$\left| \frac{J_2(s,x)}{\zeta(x)} \right| \le C_1 \Phi_\alpha \left(T, C \left\| \frac{u_1^0 - u_2^0}{\zeta} \right\|_{L^\infty}^{\delta} \right)$$

for all $0 \le s \le T^*$. Since the bound on J_1/ζ in (7.1) is better than that on J_2/ζ, this completes the proof. □

Remark 7.3. *In the application of Proposition 8.7 and Lemma 8.4 (which was used in the proof of Proposition 8.3) the value of $C_0 = \|u_1\|_{L^\infty(0,T;S_1)}$ enters into the constants. A bound on $\|u_1\|_{L^\infty(0,T;S_1)}$ comes from the proof of existence in Section 5. While we did not explicitly* *calculate it, for S_1 it yields an exponential-in-time bound, as in [15, 1]. Hence, our bound on $a(T)$ is doubly exponential (it would be worse for unbounded velocities). It is shown in [9] (extending [19]) for bounded velocity, however, that $\|u_1\|_{L^\infty(0,T;S_1)}$ can be bounded linearly in time, which means that C_0 actually only increases singly exponentially in time.*

Whether an improved bound can be obtained for a more general h is an open question: If it could, it would extend the time of existence of solutions, possibly expanding the class of growth bounds for which global-in-time existence holds.

Lemma 7.4. *Let $Z \in S_1$. For any $\lambda > 0$,*

$$(a_\lambda K) * \operatorname{curl} Z = \operatorname{curl}(a_\lambda K) * Z. \tag{7.2}$$

Proof. Note that $((a_\lambda K) * \operatorname{curl} Z)^i = (a_\lambda K^i) * (\partial_1 Z^2 - \partial_2 Z^1) = (\partial_1(a_\lambda K^i)) * Z^2 - (\partial_2(a_\lambda K^i)) * Z^1$. Thus, (7.2) is not just a matter of moving the curl from one side of the convolution to the other. Using both that Z is divergence-free and that a is radially symmetric, however, (7.2) is proved in Lemma 4.4 of [11]. □

The following is a twist on Proposition 4.6 of [2].

Lemma 7.5. *Let ζ be a pre-growth bound and suppose that $Z \in S_1$. For any $\lambda > 0$,*

$$\begin{aligned} &\|(a_\lambda K) * \operatorname{curl} Z\|_{L^\infty(\mathbb{R}^2)} \le 2 \|Z\|_{L^\infty(\mathbb{R}^2)}, \\ &\left\| \frac{(a_\lambda K) * \operatorname{curl} Z}{\zeta} \right\|_{L^\infty(\mathbb{R}^2)} \le \left(1 + 6 \frac{\zeta(\lambda)}{\zeta(0)}\right) \left\| \frac{Z}{\zeta} \right\|_{L^\infty(\mathbb{R}^2)}. \end{aligned} \tag{7.3}$$

Proof. Lemma 7.4 gives $(a_\lambda K) * \operatorname{curl} Z = \operatorname{curl}(a_\lambda K) * Z$. But $\operatorname{curl}(a_\lambda K) = -\operatorname{div}(a_\lambda K^\perp) = -a_\lambda \operatorname{div} K^\perp - \nabla a_\lambda \cdot K^\perp = \delta - \nabla a_\lambda \cdot K^\perp$, where δ is the Dirac delta function, since $a_\lambda(0) = 1$. Hence,

$$(a_\lambda K) * \operatorname{curl} Z = Z - (\nabla a_\lambda \cdot K^\perp) * Z = Z - \varphi_\lambda * Z, \tag{7.4}$$

where $\varphi_\lambda := \nabla a_\lambda \cdot K^\perp \in C_c^\infty(\mathbb{R}^2)$. Then $(7.3)_1$ follows from

$$\|(a_\lambda K) * \operatorname{curl} Z\|_{L^\infty(\mathbb{R}^2)} \le C\left(1 + \|\varphi_\lambda\|_{L^1}\right) \|Z\|_{L^\infty(\mathbb{R}^2)} = 2 \|Z\|_{L^\infty(\mathbb{R}^2)}.$$

Here, we have used that $\|\varphi_\lambda\|_{L^1} = 1$, as can easily be verified by integrating by parts. (In fact, $\varphi_\lambda *$ is a mollifier, though we will not need that.)

Using (7.4), we have

$$\left\| \frac{(a_\lambda K) * \operatorname{curl} Z}{\zeta} \right\|_{L^\infty} \le \left\| \frac{Z}{\zeta} \right\|_{L^\infty(\mathbb{R}^2)} + \left\| \frac{\varphi_\lambda * Z}{\zeta} \right\|_{L^\infty(\mathbb{R}^2)}.$$

But $\varphi_\lambda(x - \cdot)$ is supported in $B_\lambda(x)$, so

$$\begin{aligned} \left| \frac{\varphi_\lambda * Z(x)}{\zeta(x)} \right| &= \left| \frac{\varphi_\lambda * (\mathbb{1}_{B_\lambda(x)} Z)(x)}{\zeta(x)} \right| \le \frac{\|\varphi_\lambda\|_{L^1} \left\|\mathbb{1}_{B_\lambda(x)} Z\right\|_{L^\infty}}{\zeta(x)} = \frac{\left\|\mathbb{1}_{B_\lambda(x)} Z\right\|_{L^\infty}}{\zeta(x)} \\ &\le \left\| \frac{Z}{\zeta} \right\|_{L^\infty} \frac{\zeta(|x| + \lambda)}{\zeta\left(\max\{|x| - \lambda, 0\}\right)}. \end{aligned}$$

Taking the supremum over $x \in \mathbb{R}^2$, we have

$$\left\| \frac{(a_\lambda K) * \operatorname{curl} Z}{\zeta} \right\|_{L^\infty(\mathbb{R}^2)} \le \left\| \frac{Z}{\zeta} \right\|_{L^\infty} \left(1 + \sup_{x \in \mathbb{R}^2} \frac{\zeta(|x| + \lambda)}{\zeta\left(\max\{|x| - \lambda, 0\}\right)}\right).$$

If $|x| > 2\lambda$ then using Lemma 2.5,

$$\frac{\zeta(|x|+\lambda)}{\zeta\left(\max\{|x|-\lambda,0\}\right)} \le \frac{\zeta(3\,|x|/2)}{\zeta(|x|/2)} \le 6\frac{\zeta(|x|/2)}{\zeta(|x|/2)} = 6,$$

while if $|x| \le 2\lambda$ then

$$\frac{\zeta(|x|+\lambda)}{\zeta\left(\max\{|x|-\lambda,0\}\right)} \le \frac{\zeta(3\lambda)}{\zeta(0)} \le 6\frac{\zeta(\lambda)}{\zeta(0)}.$$

Hence, we obtain $(7.3)_2$. $\square$

Lemma 7.6. *Let ζ be a pre-growth bound, $\alpha \in (0,1)$, and $\overline{\omega}_0 = \omega_1^0 - \omega_2^0 = \operatorname{curl}(u_1^0 - u_2^0)$ with u_1^0, $u_2^0 \in S_\zeta$. Then*

$$\left\|\frac{\overline{\omega}_0}{\zeta}\right\|_{C^{\alpha-1}} \le C\left\|\frac{u_1^0-u_2^0}{\zeta}\right\|_{C^\alpha}.$$

Proof. Let $g := 1/\zeta$ and $v = -(u_1^0 - u_2^0)$ so that $\overline{\omega}_0 = \operatorname{div} v^\perp$. Then

$$g\overline{\omega}_0 = g \operatorname{div} v^\perp = \operatorname{div}(gv^\perp) - \nabla g\cdot v^\perp.$$

Thus (see Remark 7.1),

$$\|g\overline{\omega}_0\|_{C^{\alpha-1}} \le C\left(\|gv^\perp\|_{C^\alpha} + \|\nabla g\cdot v^\perp\|_{C^{\alpha-1}}\right) = C\left(\|gv\|_{C^\alpha} + \|\nabla g\cdot v^\perp\|_{C^{\alpha-1}}\right).$$

But by virtue of Lemma 2.8, we also have

$$\|\nabla g\cdot v^\perp\|_{C^{\alpha-1}} \le \|\nabla g\cdot v^\perp\|_{L^\infty} \le c_0\|gv\|_{L^\infty} \le c_0\|gv\|_{C^\alpha}.$$

$\square$

8. Hölder space estimates

In this section, we make use of the Littlewood-Paley operators Δ_j, $j \ge -1$. A detailed definition of these operators and their properties can be found in chapter 2 of [4]. We note here, only that $\Delta_j f = \varphi_j * f$, where $\varphi_j(\cdot) = 2^{2j}\varphi(2^j\cdot)$ for $j \ge 0$, φ is a Schwartz function, and the Fourier transform of φ is supported in an annulus. We can write $\Delta_{-1}f$ as a convolution with a Schwartz function χ whose Fourier transform is supported in a ball.

We will also make use of the following Littlewood-Paley definition of Holder spaces.

Definition 8.1. *Let $r \in \mathbb{R}$. The space $C_*^r(\mathbb{R}^2)$ is defined to be the set of all tempered distributions on $\mathbb{R}^2$ for which*

$$\|f\|_{C_*^r} := 2^{rj}\sup_{j\ge -1}\|\Delta_j f\|_{L^\infty} < \infty.$$

Remark 8.2. *It follows from Propositions 2.3.1, 2.3.2 of [4] that the C_*^r norm is equivalent to the classical Hölder space C^r norm when r is a positive non-integer: $\|f\|_{C^r} \le a_r\|f\|_{C_*^r}$, $\|f\|_{C_*^r} \le b_r\|f\|_{C^r}$ for constants, a_r, $b_r > 0$, though $a_r \to \infty$ as r approaches an integer. See Remark 7.1.*

Proposition 8.3. *Let ζ be a pre-growth bound and let $u \in L^\infty(0,T;S_1)$ with X its associated flow map. Let $t \in [0,T]$ and set $\eta = \zeta\circ X^{-1}(t)$. For any $\alpha > 0$, $\lambda > 0$, and $f \in L^\infty(\mathbb{R}^2)$,*

$$\left\|\frac{(a_\lambda K)*f}{\eta}\right\|_{L^\infty} \le C(1+\|f\|_{L^\infty})\Phi_\alpha\left(t, \left\|\frac{f}{\eta}\right\|_{C^{-\alpha}}\right), \tag{8.1}$$

where $C = C(T,\zeta,\lambda)$, and Φ_α is defined in (1.11) (using $C_0 = \|u\|_{L^\infty(0,T;S_1)}$).

Proof. Define $g = 1/\eta$. For fixed $N \geq -1$ (to be chosen later), write

$$\begin{aligned}
\left|\frac{(a_\lambda K) * f(x)}{\eta(x)}\right| &= \left|g(x)\int_{\mathbb{R}^2} a_\lambda(y)K(y)\eta(x-y)(f/\eta)(x-y)\,dy\right| \\
&= \left|g(x)\int_{\mathbb{R}^2} a_\lambda(y)K(y)\eta(x-y)\sum_{j\geq -1}(\Delta_j(f/\eta))(x-y)\,dy\right| \\
&\leq \sum_{-1\leq j\leq N}\left|g(x)\int_{\mathbb{R}^2} a_\lambda(y)K(y)\eta(x-y)(\Delta_j(f/\eta))(x-y)\,dy\right| \\
&\quad + \sum_{j>N}\left|g(x)\int_{\mathbb{R}^2} a_\lambda(y)K(y)\eta(x-y)(\Delta_j(f/\eta))(x-y)\,dy\right| =: I + II.
\end{aligned} \tag{8.2}$$

We first estimate I. Exploiting Definition 8.1,

$$\begin{aligned}
I &= \sum_{-1\leq j\leq N} 2^{-j\alpha}2^{j\alpha}\left|g(x)\int_{\mathbb{R}^2} a_\lambda(y)K(y)\eta(x-y)(\Delta_j(f/\eta))(x-y)\,dy\right| \\
&\leq \sup_j 2^{-j\alpha}\|\Delta_j(f/\eta)\|_{L^\infty}\sum_{-1\leq j\leq N} 2^{j\alpha}g(x)\int_{\mathbb{R}^2}|a_\lambda(y)K(y)\eta(x-y)|\,dy \\
&\leq C2^{\alpha N}\|f/\eta\|_{C^{-\alpha}}g(x)\int_{\mathbb{R}^2}|a_\lambda(y)K(y)\eta(x-y)|\,dy.
\end{aligned} \tag{8.3}$$

Since $\eta = \zeta \circ X^{-1}(t)$, where ζ is a pre-growth bound, Lemma 8.4 implies that

$$\begin{aligned}
g(x)&\int_{\mathbb{R}^2}|a_\lambda(y)K(y)\eta(x-y)|\,dy \leq Cg(x)\int_{\mathbb{R}^2}|a_\lambda(y)K(y)\zeta(x-y)|\,dy \\
&\leq C\int_{\mathbb{R}^2}|a_\lambda(y)K(y)g(x)(\zeta(x)+\zeta(y))|\,dy \\
&\leq C\int_{\mathbb{R}^2}|a_\lambda(y)K(y)|(1+g(x)\zeta(y))\,dy \leq C\int_0^\lambda(1+g(x)\zeta(r))\,dr \\
&\leq C\lambda\,(1+g(x)\zeta(\lambda)) \leq C\lambda(1+\zeta(\lambda)) = C(\lambda),
\end{aligned}$$

where we used boundedness of g. Substituting this estimate into (8.3), we conclude that

$$I \leq C2^{\alpha N}\|f/\eta\|_{C^{-\alpha}}. \tag{8.4}$$

We now estimate II by introducing a commutator and utilizing the Holder continuity of η, writing

$$\begin{aligned}
II &\leq \sum_{j>N}\left|g(x)\int_{\mathbb{R}^2} a_\lambda(y)K(y)\Delta_j f(x-y)\,dy\right| \\
&\quad + \sum_{j>N}\left|g(x)\int_{\mathbb{R}^2} a_\lambda(y)K(y)[\Delta_j f(x-y) - \eta(x-y)(\Delta_j(f/\eta))(x-y)]\,dy\right| \\
&=: III + IV.
\end{aligned} \tag{8.5}$$

We rewrite III as a convolution, noting that the Littlewood-Paley operators commute with convolutions, and apply Bernstein's Lemma and Lemma 8.5 to give

$$
\begin{aligned}
III &= \sum_{j>N} |g(x)((a_\lambda K) * \Delta_j f(x))| = g(x) \sum_{j>N} |\Delta_j (a_\lambda K * f)(x)| \\
&\le g(x) \sum_{j>N} 2^{-j} \left\| \nabla \Delta_j (a_\lambda K * f) \right\|_{L^\infty} \le C g(x) \left\| f \right\|_{L^\infty} \sum_{j>N} 2^{-j} \le C g(x) 2^{-N} \left\| f \right\|_{L^\infty}.
\end{aligned}
$$

To estimate IV, we apply Lemma 8.6 and (3.1) to obtain

$$
\begin{aligned}
IV &\le \sum_{j>N} g(x) \int_{\mathbb{R}^2} |a_\lambda(y) K(y)| \left| \Delta_j f(x-y) - \eta(x-y)(\Delta_j(f/\eta))(x-y) \right| dy \\
&\le \sum_{j>N} C \left\| f \right\|_{L^\infty} 2^{-je^{-C_0 t}} g(x) \int_{\mathbb{R}^2} |a_\lambda(y) K(y)| \, dy \le C \left\| f \right\|_{L^\infty} 2^{-Ne^{-C_0 t}} g(x) \lambda.
\end{aligned}
$$

Substituting the estimates for III and IV into (8.5) gives

$$
II \le C(1+\lambda) \left\| f \right\|_{L^\infty} 2^{-Ne^{-C_0 t}} g(x).
$$

Finally, substituting the estimates for I and II into (8.2) yields

$$
\left| \frac{(a_\lambda K) * f(x)}{\eta(x)} \right| \le C \left(1 + \left\| f \right\|_{L^\infty}\right) \left(2^{\alpha N} \left\| f/\eta \right\|_{C^{-\alpha}} + 2^{-Ne^{-C_0 t}} g(x) \right), \tag{8.6}
$$

where C depends on λ. Now, we are free to choose the integer $N \ge -1$ any way we wish, but if we chose N as close to

$$
N^* := \frac{1}{\alpha + e^{-C_0 t}} \log_2 \left(\frac{1}{\left\| f/\eta \right\|_{C^{-\alpha}}} \right) = -\frac{1}{\alpha + e^{-C_0 t}} \log_2 \left\| f/\eta \right\|_{C^{-\alpha}}
$$

as possible, we will be near the minimizer of the bound in (8.6), as long as $N^* \ge -1$ (because such an N^* balances the two terms). Calculating with $N = N^*$ gives

$$
\begin{aligned}
2^{\alpha N} \left\| f/\eta \right\|_{C^{-\alpha}} &= 2^{\frac{\alpha}{\alpha + e^{-C_0 t}} \log_2 \left(\frac{1}{\| f/\eta \|_{C^{-\alpha}}} \right)} \left\| f/\eta \right\|_{C^{-\alpha}} = \left\| f/\eta \right\|_{C^{-\alpha}}^{\frac{e^{-C_0 t}}{\alpha + e^{-C_0 t}}}, \\
2^{-Ne^{-C_0 t}} g(x) &= 2^{\frac{e^{-C_0 t}}{\alpha + e^{-C_0 t}} \log_2 \| f/\eta \|_{C^{-\alpha}}} g(x) \le C \left\| f/\eta \right\|_{C^{-\alpha}}^{\frac{e^{-C_0 t}}{\alpha + e^{-C_0 t}}},
\end{aligned}
$$

since g is bounded. Rounding N^* up or down to the nearest integer will only introduce a multiplicative constant no larger than $C2^\alpha$, so this yields

$$
\left| \frac{(a_\lambda K) * f(x)}{\eta(x)} \right| \le C \left(1 + \left\| f \right\|_{L^\infty}\right) \left\| f/\eta \right\|_{C^{-\alpha}}^{\frac{e^{-C_0 t}}{\alpha + e^{-C_0 t}}} \tag{8.7}
$$

as long an $N^* \ge -1$.

If $N^* < -1$, then it must be that

$$
\left\| f/\eta \right\|_{C^{-\alpha}}^{\frac{1}{\alpha + e^{-C_0 t}}} \ge 2
$$

so

$$
\left\| f/\eta \right\|_{C^{-\alpha}}^{\frac{e^{-C_0 t}}{\alpha + e^{-C_0 t}}} \ge 1 \ge C g(x)
$$

for some constant $C > 0$. Then, from (8.6),

$$\left|\frac{(a_\lambda K) * f(x)}{\eta(x)}\right| \le C\left(1 + \|f\|_{L^\infty}\right)\left(2^{\alpha N}\|f/\eta\|_{C^{-\alpha}} + 2^{-Ne^{-C_0 t}}\|f/\eta\|_{C^{-\alpha}}^{\frac{e^{-C_0 t}}{\alpha + e^{-C_0 t}}}\right).$$

Choosing $N = -1$, yields

$$\begin{aligned}\left|\frac{(a_{h(x)} K) * f(x)}{\eta(x)}\right| &\le C\left(1 + \|f\|_{L^\infty}\right)\left(\|f/\eta\|_{C^{-\alpha}} + \|f/\eta\|_{C^{-\alpha}}^{\frac{e^{-C_0 t}}{\alpha + e^{-C_0 t}}}\right)\\ &= C\left(1 + \|f\|_{L^\infty}\right)\Phi_\alpha\left(t, \left\|\frac{f}{\eta}\right\|_{C^{-\alpha}}\right).\end{aligned} \tag{8.8}$$

Adding the bounds in (8.7) and (8.8), we conclude (8.1) holds for all values of $\|f/\eta\|_{C^{-\alpha}}$. $\square$

Lemma 8.4. *Let ζ be a pre-growth bound and let $u \in L^\infty(0, T; S_1)$ with X its associated flow map. Then for all $t \in [0, T]$, $x \in \mathbb{R}^2$,*

$$C(T)\zeta(X^{-1}(t, x)) \le \zeta(x) \le C'(T)\zeta(X^{-1}(t, x)) \tag{8.9}$$

for constants, $0 < C(T) < C'(T)$.

Proof. We rewrite (8.9) as

$$C(T)\zeta(x) \le \zeta(X(t, x)) \le C'(T)\zeta(x)$$

for all $x \in \mathbb{R}^2$, a bound that we see easily follows from Lemmas 2.5 and 4.2. $\square$

Lemma 8.5. *For all $j \ge -1$ and all $\lambda > 0$,*

$$\|\nabla\Delta_j((a_\lambda K) * f)\|_{L^\infty} \le C\|f\|_{L^\infty}.$$

Proof. Write

$$(a_\lambda K) * f(z) = \left(K * (a_\lambda(\cdot - z)f)\right)(z).$$

For fixed z, define the divergence-free vector field,

$$b_z(w) := \left(K * (a_\lambda(\cdot - z)f)\right)(w).$$

We know (for instance, from Lemma 4.2 of [6]) that $\|\nabla\Delta_j b_z\|_{L^\infty} \le C\|\Delta_j \operatorname{curl} b_z\|_{L^\infty}$. Thus,

$$\begin{aligned}\|\nabla\Delta_j((a_\lambda K) * f)\|_{L^\infty} &= \|\nabla\Delta_j b_z\|_{L^\infty} \le C\|\Delta_j \operatorname{curl} b_z\|_{L^\infty} = C\|\Delta_j(a_\lambda(\cdot - z)f)\|_{L^\infty}\\ &\le C\|a_\lambda(\cdot - z)f\|_{L^\infty} \le C\|f\|_{L^\infty}.\end{aligned}$$

$\square$

Lemma 8.6. *With the assumptions as in Proposition 8.3, for all $x, y \in \mathbb{R}^2$, we have*

$$|\Delta_j f(x - y) - \eta(x - y)(\Delta_j(f/\eta))(x - y)| \le C\|f\|_{L^\infty}\|\nabla\zeta\|_{L^\infty} 2^{-je^{-C_0 t}}.$$

Proof. First observe that for any x, $y \in \mathbb{R}^2$

$$|\eta(x) - \eta(y)| = \frac{\left|\zeta(X_1^{-1}(t, x)) - \zeta(X_1^{-1}(t, y))\right|}{\left|X_1^{-1}(t, x) - X_1^{-1}(t, y)\right|}\left|X_1^{-1}(t, x) - X_1^{-1}(t, y)\right| \le \|\nabla\zeta\|_{L^\infty}\chi_t(|x - y|)$$

by Lemma 4.3. We then have,

$$
\begin{aligned}
&|\Delta_j f(x-y) - \eta(x-y)(\Delta_j(f/\eta))(x-y)| \\
&\qquad = \left| \int_{\mathbb{R}^2} (\varphi_j(z)(\eta f/\eta)(x-y-z) - \eta(x-y)\varphi_j(z)(f/\eta)(x-y-z))\, dz \right| \\
&\qquad \le \int_{\mathbb{R}^2} |\varphi_j(z)(f/\eta)(x-y-z)|\, |\eta(x-y-z) - \eta(x-y)|\, dz \\
&\qquad \le \|\nabla\zeta\|_{L^\infty} \int_{\mathbb{R}^2} \chi_t(|z|)\, |\varphi_j(z)|\, |(f/\eta)(x-y-z)|\, dz \le C\, \|\nabla\zeta\|_{L^\infty}\, \|f\|_{L^\infty}\, 2^{-je^{-C_0 t}}.
\end{aligned}
$$

We used here that, because φ is Schwartz-class, by virtue of (4.2), we have

$$
\begin{aligned}
\int_{\mathbb{R}^2} |\varphi_j(z)|\, \chi_t(|z|)\, dz &= 2^{2j} \int_{\mathbb{R}^2} \left|\varphi(2^j z)\right| \chi_t(|z|)\, dz = \int_{\mathbb{R}^2} |\varphi(w)|\, \chi_t(2^{-j}|w|)\, dw \\
&\le 2^{-je^{-C_0 t}} \int_{\mathbb{R}^2} |\varphi(w)|\, \chi_t(|w|)\, dw \le C 2^{-je^{-C_0 t}}.
\end{aligned}
$$

□

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The following proposition is a simplified version of Theorem 3.28 of [3].

Proposition 8.7. *Let $u \in L^\infty(0,T;S_1)$. Assume $f \in C([0,T];L^\infty)$ solves the transport equation*

$$
\begin{aligned}
&\partial_t f + u \cdot \nabla f = 0 \\
&f|_{t=0} = f^0.
\end{aligned}
$$

For fixed $\delta \in (-1,0)$, there exists a constant $C = C(\delta)$ such that for any $\beta > C$, if

$$
\delta_t = \delta - \beta \int_0^t \|u(s)\|_{LL}\, ds, \tag{8.10}
$$

and if T^* satisfies $\delta_{T^*} \ge -1$, then

$$
\sup_{t\in[0,T^*]} \|f(t)\|_{C^{\delta_t}} \le \frac{\beta}{\beta - C} \|f_0\|_{C^\delta}.
$$

Remark 8.8. *Proposition 8.7 is proved in greater generality in [3]. The authors assume, for example, that f belongs to the appropriate negative Hölder space. Here, we apply Proposition 8.7 with $f = \frac{\bar{\omega}^0}{\zeta} \circ X_1^{-1}$, which clearly belongs to $C([0,T];L^\infty)$ and therefore to all negative Hölder spaces. In addition, the authors integrate a quantity in (8.10) which differs from $\|u\|_{LL}$ but is bounded above and below by $C\|u\|_{LL}$ for a constant $C > 0$.*

Lemma 8.9. *For any $u \in S_1(\mathbb{R}^2)$ and $r \in (0,1)$,*

$$
\|u\|_{C^r} \le C \left(\|u\|_{L^\infty} + \|u\|_{L^\infty}^{1-r} \|u\|_{S_1}^r \right).
$$

Proof. We apply Definition 8.1 and write

$$
\begin{aligned}
\|u\|_{C_r} &= \sup_{j\ge -1} 2^{jr} \|\Delta_j u\|_{L^\infty} \\
&\le C\, \|u\|_{L^\infty} + \sup_{j\ge 0} 2^{jr} \|\Delta_j u\|_{L^\infty}^{1-r} \|\Delta_j u\|_{L^\infty}^r \\
&\le C\, \|u\|_{L^\infty} + C\, \|u\|_{L^\infty}^{1-r} \sup_{j\ge 0} 2^{jr} 2^{-jr} \|\Delta_j \nabla u\|_{L^\infty}^r,
\end{aligned}
$$

where we used Bernstein's Lemma to get the second inequality. But, using Lemma 4.2 of [6],

$$\|\Delta_j \nabla u\|_{L^\infty} \le C \|\Delta_j \operatorname{curl} u\|_{L^\infty} \le C \|\operatorname{curl} u\|_{L^\infty} \le C \|u\|_{S^1},$$

which yields the result. □

Appendix A. Examples of growth bounds

Proof of Theorem 1.5. We will show that h_1 and h_2 are well-posedness growth bounds, h_2 globally. Once this is established, Theorem 1.5 follows as an immediate corollary of Theorem 1.6 and Corollary 1.8.

First consider $h(r) = h_1(r) = (1+r)^\alpha$ for some $\alpha \in [0, 1/2)$. Clearly, h and so h^2 are increasing and both h and h^2 are concave and infinitely differentiable on $(0, \infty)$, and $h'(0) = \alpha < \infty$. This gives (i) of Definition 1.1 for h and h^2. Then for $n = 1, 2$,

$$\int_1^\infty \frac{h^n(s)}{s^2}\, ds = \int_1^\infty \frac{(1+s)^{n\alpha}}{s^2}\, ds \le 2^{n\alpha} \int_1^\infty s^{n\alpha - 2}\, ds = \frac{2^{n\alpha}}{1 - n\alpha} < \infty,$$

giving (ii) of Definition 1.1 for h and h^2. It follows that h is a well-posedness growth bound. (An additional calculation, which we suppress since we do not strictly need it, shows that $E(r) \le \mu(r) := C(\alpha)(1 + r^\alpha)r$, improving the coarse bound of Lemma 2.3.)

Now assume that $h(r) = h_2(r) = \log^{\frac{1}{4}}(e+r)$. Then $h^2(r) = \log^{\frac{1}{2}}(e+r)$. Then h^2 is infinitely differentiable on $(0,1)$, increasing, and concave and hence so also is $h = \sqrt{h^2}$, being a composition of increasing concave functions infinitely differentiable on $(0, 1)$. Also,

$$(h^2)'(0) = \frac{1}{2}(x+e)^{-1} \log^{-\frac{1}{2}}(x+e)|_{x=0} = \frac{1}{2e} < \infty.$$

This gives (i) of Definition 1.1.

Noting that $h_2(r) \le h_1(r) = (1+r)^\alpha$ for any $\alpha > 0$, (ii) and (iii) of Definition 1.1 follows for h and h^2 from our result for h_1. Hence, $h = h_2$ is a well-posedness growth bound.

To obtain (1.3), we make the change of variables, $w = 1/s$, giving

$$H(r) = \int_0^{\frac{1}{r}} h^2(1/w)\, dw := \int_0^{\frac{1}{r}} \log^{\frac{1}{2}}\left(e + \frac{1}{w}\right) dw.$$

Now, if $w \le 1/e$ then

$$\log^{\frac{1}{2}}\left(e + \frac{1}{w}\right) \le \log^{\frac{1}{2}}\left(\frac{2}{w}\right)$$

so for $r \ge e$,

$$\begin{aligned} H(r) &\le \int_0^{\frac{1}{r}} \log^{\frac{1}{2}}\left(\frac{2}{w}\right) dw = \lim_{a \to 0^+} \left[w\sqrt{\log\left(\frac{2}{w}\right)} - \sqrt{\pi}\operatorname{erf}\sqrt{\log\left(\frac{2}{w}\right)} \right]_a^{\frac{1}{r}} \\ &= \frac{1}{r}\sqrt{\log(2r)} - \sqrt{\pi}\operatorname{erf}\sqrt{\log(2r)} + \sqrt{\pi} = \frac{1}{r}\sqrt{\log(2r)} + \sqrt{\pi}\operatorname{erfc}\sqrt{\log(2r)}, \end{aligned}$$

where

$$\operatorname{erf}(r) := \frac{2}{\sqrt{\pi}} \int_0^r e^{-s^2}\, ds, \qquad \operatorname{erfc}(r) = 1 - \operatorname{erf}(r).$$

From the well-known inequality, $\operatorname{erfc}(r) \le e^{-r^2}$, it follows that for $r \ge e$,

$$H(r) \le \frac{1}{r}\sqrt{\log(2r)} + \frac{\sqrt{\pi}}{2r}.$$

If $w > 1/e$ then

$$\log^{\frac{1}{2}}\left(e + \frac{1}{w}\right) \le \log^{\frac{1}{2}}(2e)$$

so that for $r < e$,

$$\begin{aligned} H(r) &= \int_0^{\frac{1}{e}} h^2(1/w)\,dw + \int_{\frac{1}{e}}^{\frac{1}{r}} h^2(1/w)\,dw \le H(e) + \int_{\frac{1}{e}}^{\frac{1}{r}} \log^{\frac{1}{2}}(2e)\,dw \\ &\le \frac{1}{e}\sqrt{\log(2e)} + \frac{\sqrt{\pi}}{2e} + \log^{\frac{1}{2}}(2e)\,\frac{e-r}{er} = \frac{\sqrt{\pi}}{2e} + \frac{\log^{\frac{1}{2}}(2e)}{r} \le 2\frac{\log^{\frac{1}{2}}(2e)}{r}. \end{aligned}$$

In the final inequality, we used that

$$\frac{\sqrt{\pi}}{2e} < 1 < \frac{\log^{\frac{1}{2}}(2e)}{r}.$$

We see, then, that an inequality that works for all $r > 0$ is

$$H(r) \le 2\frac{\sqrt{\log(2e + 2r)}}{r}.$$

This bound on H gives

$$\begin{aligned} E(r) &\le 2(1 + rH(r^{\frac{1}{2}})^2)r \le 2(1 + 4\log(2e + 2r^{\frac{1}{2}}))r \\ &\le C\,(1 + \log(e + r))\,r =: \mu(r). \end{aligned}$$

Then $\mu(0) = 0$, μ is continuous, and μ is convex, since $\mu''(r) = C(r + 2e)/(e + r)^2 > 0$.

Finally,

$$\begin{aligned} \int_1^\infty \frac{dr}{(1 + \log(e + r))\,r} &\ge \frac{1}{2}\int_1^\infty \frac{dr}{(\log(e + r))\,r} \ge \frac{1}{2}\int_{e+1}^\infty \frac{dr}{r\log r} \\ &= \frac{1}{2}\int_{\log(e+1)}^\infty \frac{dx}{x} = \infty, \end{aligned}$$

where we made the change of variables, $x = \log r$. This shows that (1.3) holds. □

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1439786 while the second author was in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the Spring 2017 semester. The first author was supported by Simons Foundation Grant No. 429578.

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