

THE VANISHING VISCOSITY LIMIT FOR SOME SYMMETRIC FLOWS

GUNG-MIN GIE¹, JAMES P. KELLIHER², MILTON C. LOPES FILHO³, ANNA L. MAZZUCATO⁴, AND HELENA J. NUSSENZVEIG LOPES³

ABSTRACT. The focus of this paper is on the analysis of the boundary layer and the associated vanishing viscosity limit for two classes of flows with symmetry, namely, Plane-Parallel Channel Flows and Parallel Pipe Flows. We construct explicit boundary layer correctors, which approximate the difference between the Navier-Stokes and the Euler solutions. Using properties of these correctors, we establish convergence of the Navier-Stokes solution to the Euler solution as viscosity vanishes with optimal rates of convergence. In addition, we investigate vorticity production on the boundary in the limit of vanishing viscosity. Our work significantly extends prior work in the literature.

CONTENTS

1. Introduction	2
2. Symmetric flows: an overview	6
3. Lighthill principle for PCF	7
4. Plane-parallel channel flows	18
4.1. Viscous approximation and convergence result	19
4.2. Estimates of the corrector Θ	22
4.3. Proof of Theorem 4.1	23
5. Parallel pipe flows	27
5.1. Viscous approximation and convergence result	29
5.2. Estimates of the corrector Θ	33
5.3. Proof of Theorem 5.1	34
Appendix A. One-dimensional heat equations	40
A.1. On the one-dimensional heat equation with small diffusivity	40
A.2. On a drift-diffusion equation with small diffusivity	42
Appendix B. A few auxiliary results	47
Appendix C. Vorticity accumulation on the boundary	48
Acknowledgments	48
References	49

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1. INTRODUCTION

This article concerns the behavior of incompressible, viscous fluids at small viscosity in the presence of boundaries under the classical “no-slip” boundary conditions. We let Ω be a bounded domain in two or three space dimensions with boundary Γ of class C^∞ . Viscous, incompressible (Newtonian) fluid flow is modeled by solutions of the Navier-Stokes equations (NSE for short). We consider the following initial-value problem:

$$\left\{ \begin{array}{ll} \frac{\partial \mathbf{u}^\varepsilon}{\partial t} + (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{u}^\varepsilon = -\nabla p^\varepsilon + \varepsilon \Delta \mathbf{u}^\varepsilon + \mathbf{f}, & \text{in } \Omega \times (0, T), \\ \operatorname{div} \mathbf{u}^\varepsilon = 0, & \text{in } \Omega \times (0, T), \\ \mathbf{u}^\varepsilon = 0, & \text{on } \Gamma \times (0, T), \\ \mathbf{u}^\varepsilon|_{t=0} = \mathbf{u}_0, & \text{in } \Omega. \end{array} \right. \quad (1.1)$$

Where \mathbf{u}^ε is the Eulerian fluid velocity, p^ε is the pressure, \mathbf{f} are given external forces, and \mathbf{u}_0 is the given initial velocity. Here ε is a small, strictly positive parameter, representing the kinematic viscosity of the fluid, assumed homogeneous, $T > 0$ is a fixed, positive time, \mathbf{f} and \mathbf{u}_0 are smooth, divergence-free vector fields. The boundary condition in (1.1) is referred to as the *no-slip* condition or *no-slip, no-penetration* condition.

By formally setting $\varepsilon = 0$ in NSE we obtain the Euler equations (EE for short), which model the flow of inviscid, incompressible fluids. The initial-value problem for EE is given by:

$$\left\{ \begin{array}{ll} \frac{\partial \mathbf{u}^0}{\partial t} + (\mathbf{u}^0 \cdot \nabla) \mathbf{u}^0 = -\nabla p^0 + \mathbf{f}, & \text{in } \Omega \times (0, T), \\ \operatorname{div} \mathbf{u}^0 = 0, & \text{in } \Omega \times (0, T), \\ \mathbf{u}^0 \cdot \mathbf{n} = 0, & \text{on } \Gamma \times (0, T), \\ \mathbf{u}^0|_{t=0} = \mathbf{u}_0, & \text{in } \Omega. \end{array} \right. \quad (1.2)$$

where \mathbf{n} is the unit outer normal vector on $\partial\Omega$. The boundary condition in (1.2) is referred to simply as *no-penetration*, and reflects the assumption that the fluid is in a container with rigid walls. For the types of flows considered in this paper, it is convenient to take the initial velocity for NSE to be independent of ε and equal to the initial velocity for EE, although this assumption can be weakened. The assumption that the data and the boundary of the domain are smooth can also be weakened, but we will not seek optimal regularity conditions, as our focus is on a detailed analysis of the fluid boundary behavior at small viscosity. By passing to a moving frame, it is possible to consider the case in which the boundary is allowed to move rigidly along itself, as in the classical case of the Taylor-Couette flow. Then, the no-slip boundary condition reads $\mathbf{u}^\varepsilon \equiv \mathbf{U}$ on $\Gamma \times (0, T)$, where \mathbf{U} is a given vector field tangent to the boundary.

A main question in fluid mechanics is whether viscous fluids at low viscosity are well approximated by inviscid fluids. Near the boundary, this approximation cannot hold uniformly in ε as there must be a discrepancy in the tangential components of \mathbf{u}^ε and \mathbf{u}^0 at the boundary, unless \mathbf{u}^0 happens to vanish on the boundary identically over time. This discrepancy leads to the potential creation of large gradients of velocity in a layer near the boundary, called a *viscous boundary layer*, where the fluid is hence neither well modeled by solutions of NSE nor by solutions of EE. (We refer to [79] and references therein for an introduction to the theory of boundary layers.) Understanding the behavior of a fluid in the viscous layer is one of the most challenging problems in fluid mechanics, and mathematically it is far from understood, even though progress has been

made recently. A related mathematical problem is whether solutions of NSE converge in a suitable norm to solution of EE as ε goes to zero. We will say that the (classical) *vanishing viscosity limit* or *inviscid limit* holds if solutions of (1.1) converge to solutions of (1.2) in the energy norm, that is, strongly in $L^\infty((0, T); L^2(\Omega))$. Whether the classical vanishing viscosity limit holds generically, at least for short time, is an open question even for C^∞ initial data and in simple geometries, such as a disk in the plane, under the classical no-slip boundary conditions. Except in special situations, one does not expect the vanishing viscosity limit to hold over long intervals of time (assuming the Euler solution exists over such intervals) because of the observed phenomenon of *boundary layer separation*. Although separation is a well-documented phenomenon experimentally, and even numerically, it is still not well understood from a mathematical point of view. It is, however, intimately related to the issue of stability of layer profiles. We discuss further this issue later in the Introduction. Instabilities in the viscous layer and subsequent detachment are thought to be one of the main mechanism for onset of turbulence. In fact, the vanishing viscosity limit is at the core of mathematical theories of turbulence based on irregular, dissipative solutions to the inviscid equations and their uniqueness, as in *Onsager conjecture*. While Onsager conjecture was recently settled by use of convex integration techniques, even in bounded domains [21, 4, 27, 48], the question of whether dissipative solutions can actually be achieved in the vanishing viscosity limit remains essentially unanswered (see [5] for the case of shear flows).

There is an extensive literature on the vanishing viscosity limit when the boundary layer is absent or very weak. For solutions in the whole space or in a periodic domain, the vanishing viscosity limit has been rigorously proved in various norms ([80, 50, 51, 26, 68]). The limit also holds if some slip is allowed at the boundary for viscous flows or if the production of vorticity at the boundary is prescribed, such as under so-called Navier-friction boundary conditions [14, 15, 88, 19, 18, 16, 17, 49, 37]. In this context, the vanishing viscosity limit has also been used initially as a mean to establish existence of 2D Euler solutions (see ([89], [60, pp. 87–98], [2], and [62, pp. 129–131]). The boundary layer is studied for Navier conditions in 2D in [24, 65, 53] and in 3D in [46, 47, 69, 38]. Lastly, the limit can be shown to hold for non-characteristic boundary conditions [81, 82, 44, 36], such as with injection and suction at the boundary.

For the classical no-slip boundary conditions considered here, there is no general convergence result, though there are physically-motivated criteria for convergence, dating back to the work of Kato [52] and subsequent generalizations (see in particular [8, 6] and references therein). A formal asymptotic analysis as $\varepsilon \rightarrow 0$ leads to the Prandtl equations for the velocity in the boundary layer, which exhibit both ill-posedness and instabilities [28, 39, 30, 32, 43, 31], unless the boundary and the data have some degree of analyticity [1, 78, 63, 67, 23, 58, 33] or the data is monotonic in the normal direction to the boundary [75, 76, 57]. Another situation in which the Prandtl equations are well behaved and the boundary layer can be analyzed is when the initial data and the geometry of the domain have special symmetries. In this paper, we discussed several examples of this last situation. Specifically, we investigate *plane-parallel channel* and *parallel pipe flows* in three space dimensions. These are well-known examples of exact solutions of the fluid equations that can be viewed as generalizations of plane Couette and Poiseuille flows, and have been investigated before in the context of boundary layers and the vanishing viscosity limit. A special case of parallel pipe flows is that of

planar flows, which reduce to two-dimensional, *circularly-symmetric flows*. These flows are naturally of interest for the study of boundary layers, as the inviscid limit holds because Kato's criterion, specifically, the generalization due to Temam and Wang [83, 87], applies. In fact, they represent interesting, physically motivated, test cases, since the Prandtl approximation can be rigorously established. In addition, an analysis of the vorticity production by the boundary, in the vanishing viscosity limit, can be carried out.

In this article, we extend significantly prior work on these classes of flows, some of which was done by the same authors of the present manuscript, giving a unified treatment of different classes of flows, focusing in particular on vorticity production at the boundary and ill-prepared, or non-compatible, data. By *ill-prepared* initial velocity we mean that the tangential component of \mathbf{u}^0 does not vanish at the boundary, so that the no-slip boundary condition in (1.1) is not satisfied at time $t = 0$, and the forcing need not be compatible with the initial data at $t = 0$. The smooth initial data is assumed to be only in the space

$$H = \{\mathbf{v} \in L^2(\Omega) \mid \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\},$$

but not in the space

$$V = \{\mathbf{v} \in H_0^1(\Omega) \mid \operatorname{div} \mathbf{v} = 0\}.$$

The case of ill-prepared data is mathematically more difficult to treat and physically more interesting when the inviscid solution is steady, which is the case for circularly-symmetric data, as recalled below. In this case, there is constant production of vorticity at the boundary in the limit [64]. Production of vorticity at the boundary was already discussed for plane Couette and Poiseuille flows in [74], using physical arguments.

In Section 2, we introduce the special symmetric flows we will be concerned with, and we make some general remarks about the vanishing viscosity limit. The simplest case of symmetric flows is that of circularly symmetric flows, which are 2D solutions of the fluid equations for which the streamlines are circles centered at the origin. Such solutions can be obtained from any radial stream function or, equivalently, any radial vorticity function, via the Biot-Savart law. They are a special case of parallel pipe flows, discussed in Section 5. It is immediate to verify that any circularly symmetric, sufficiently regular Euler flow is steady, that is, $\mathbf{u}^0(t) = \mathbf{u}^0(0) = \mathbf{u}_0$, and that the solution to NSE with the same data actually solves a two-dimensional heat equation with no pressure. Since the dynamics is completely linear, this example is more pedagogical in nature. It arises also in the context of stability of boundary layers around steady profiles, a challenging and fundamental problem, which we do not tackle in this paper. However, it suffices to say that recently progress has been made in characterizing spectral instability of layer profiles around shear flows, primarily in the context of Prandtl equation and in the context of so-called *inviscid damping* [13, 12, 10, 11]. A spectral instability occurs in the tangential direction at frequencies that depends on the stability of the underlying shear flow (see [39, 42] for the case of an unstable shear flow, and [40, 41] for the case of stable shear flows). This instability can be overcome if the data is analytic [77] or in a suitable Gevrey class [33]. These instabilities arise when considering non-symmetric flows. The symmetry assumption does not allow the instability to manifest. Circularly symmetric flows and their instability play also a role in regards to uniqueness and non-uniqueness of Euler flows [7, 84, 85].

A first proof of the vanishing viscosity limit in this class can be found in [70] (see also [20]). A more general convergence result, allowing for a rough boundary velocity \mathbf{U} , which precludes the use of Kato's criterion, appears in [64]. A simple argument to show that the vanishing viscosity limit holds is given in [55, Theorem 6.1], though without a rate of convergence.

In Section 3 we discuss the Lighthill principle for viscous flows between two parallel planes and we use it to deduce an L^1 estimate for the vorticity of plane parallel channel flows, uniform with respect to viscosity. We focus on the argument which leads to the Lighthill principle and on the role of the Lighthill principle in quantifying vorticity production at the boundary. This section can be read independently from the remainder of our work.

In Section 4, we discuss plane-parallel flows in a periodized channel. These are flows for which the streamlines lie on parallel planes, and the velocity is independent of one of the horizontal variables, but depends on the vertical variable, making the flow three dimensional. For plane-parallel flows, the Euler solution \mathbf{u}^0 will not be steady, even for zero forcing, and both EE and NSE retain their non-linear nature, albeit only as a weakly non-linear system with zero pressure, making this a substantially more difficult problem to study. A proof of the validity of the vanishing viscosity limit for ill-prepared data and the analysis of the boundary layer corrector were carried out in [73], using a parametrix construction for a diffusion-drift equation and layer potential techniques. Convergence of the corrected velocity was obtained only in L^∞ . A Prandtl-type expansion was used in [72] to obtain convergence in H^1 uniformly in time, but only for well-prepared data. In this article, we extend these results to obtain strong convergence of the corrected velocity in $L^\infty((0, T); H^1(\Omega))$ for ill-prepared data and study vorticity production at the boundary in the limit.

Parallel pipe flows, the subject of Section 5, combine the features of both circularly symmetric flows and plane-parallel flows. The domain is a straight, infinite, circular pipe that is periodized along the direction of the axis. As with the channel geometry, symmetry and periodicity ensure uniqueness of solutions to NSE and EE, excluding in particular non-trivial pressure-driven flows. The velocity is independent of the variable along the pipe axis and, in any circular cross section of the pipe, it is the sum of a circularly symmetric, planar velocity field and a velocity field pointing in the direction of the axis. Again, NSE and EE reduce to a weakly non-linear system. A substantial complication over plane-parallel flows is that the non-vanishing curvature now becomes an important factor in the analysis. Also, much as in the case of axisymmetric flows in the whole space, the behavior of the solution near the axis cannot be controlled as well as it can be away from the axis in cylindrical coordinates. To deal with this difficulty, one can adapt techniques from [71] and [45], which entails the use of a two-step localization, close to the boundary and near the pipe axis, or employ suitable weighted inequalities. Since our focus in this work is on the behavior of the flow near the boundary, we restrict ourselves to considering pipes with annular cross-section.

We close this Introduction with some notational conventions.

Notation 1.1. We introduce generic constants,

$$\kappa := \kappa(\mathbf{u}_0, \mathbf{f}, \Omega), \quad \kappa_T := \kappa_T(\mathbf{u}_0, \mathbf{f}, \Omega, T),$$

depending on the indicated data, but independent of ε or t .

Notation 1.2. By the appellation *e.s.t.* associated to a function t we mean that the function or constant has exponentially small norm in all Sobolev spaces H^s (and thus in all Hölder's spaces C^s) with a bound on the norm of the form $c_{1,s} e^{-c_{2,s}/\varepsilon^{\gamma_s}}$, $c_{1,s}, c_{2,s}, \gamma_s > 0$, for each s . We will say that a constant is *e.s.t.* if it satisfies a similar bound.

2. SYMMETRIC FLOWS: AN OVERVIEW

The focus of this work is the analysis of the boundary layer and vanishing viscosity limit for two classes of flows with symmetry.

Below and throughout the paper, we employ the following standard notation: if (ζ, η, ξ) represents an orthogonal system of coordinates in \mathbb{R}^3 , then $\{\mathbf{e}_\zeta, \mathbf{e}_\eta, \mathbf{e}_\xi\}$ represents the associated orthonormal frame, and similarly for coordinates in the plane. We will denote by (x, y, z) the Cartesian coordinates in \mathbb{R}^3 , by (r, ϕ) the polar coordinates in \mathbb{R}^2 , and by (x, r, ϕ) the cylindrical coordinates in \mathbb{R}^3 .

In this work we will be concerned with the following symmetric flows.

(CSF) Circularly symmetric flows: these are planar flows in a disk centered at the origin $\Omega = \{x^2 + y^2 < R^2\}$. The velocity is of the form:

$$\mathbf{u} = V(r, t)\mathbf{e}_\phi.$$

The vorticity, which can be identified with a scalar for planar flows, is also radially symmetric.

(PCF) Plane-parallel channel flows: these are 3D flows in an infinite channel, with periodicity imposed in the x and y -directions. The velocity takes the form:

$$\mathbf{u} = (u_1(z, t), u_2(x, z, t), 0),$$

and is defined on the domain

$$\Omega := (0, L)^2 \times (0, h).$$

Here h is the width of the channel and u_1, u_2 satisfy periodic boundary conditions in x and y with period L . The boundary is identified with the set $\Gamma := \partial\Omega = [0, L]^2 \times \{0, h\}$

(PPF) Parallel pipe flows: these are 3D flows in an infinite straight, circular pipe, with periodicity imposed along the pipe axis. The velocity is of the form

$$\mathbf{u} = u_\phi(r, t)\mathbf{e}_\phi + u_x(\phi, r, t)\mathbf{e}_x,$$

in cylindrical coordinates on the domain

$$\Omega := \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 < R^2, 0 < x < L\}.$$

Here R is the radius of the circular cross-section of the pipe and u_ϕ, u_x satisfy periodic boundary conditions in x . The boundary is identified with the set $\Gamma = [0, L] \times \{(y, z) \in \mathbb{R}^2 \mid y^2 + z^2 = R^2\}$.

CSF is a special case of PPF when the component of the velocity along the axis is zero, that is, the flow can be identified with a two-dimensional flow. In fact, the cross-sectional components of any PPF can be identified with a CSF in the cross-section of the pipe. In all three cases, the symmetry of the initial data is preserved in time for both \mathbf{u}^ε and \mathbf{u}^0 as long as the forcing has the same spatial symmetry as the initial velocity.

Uniqueness holds not only in the class of strong solutions, but also in the class of weak solutions (see [3] and references therein).

For any initial velocity $\mathbf{u}_0 \in H$, due to the energy inequality for solutions of NSE, weak sequential compactness implies the existence of $\mathbf{v} \in L^\infty(0, T; H)$ and some subsequence of $(\mathbf{u}^\varepsilon)_{\varepsilon>0}$ converging weakly to \mathbf{v} in $L^\infty(0, T; H)$ (see [56]). Additional information is required to conclude that \mathbf{v} is a weak solution of EE.

Let $\omega^\varepsilon = \text{curl } \mathbf{u}^\varepsilon$ be the vorticity. We will find that, in each of these examples,

$$(\omega^\varepsilon) \text{ is bounded in } L^\infty(0, T; L^1(\Omega)) \text{ uniformly in } \varepsilon. \quad (2.1)$$

Except in the very special case when \mathbf{u}^0 vanishes on the boundary for all $t \in (0, T)$, it is not possible to have (ω^ε) uniformly bounded in $L^\infty(0, T; L^p(\Omega))$ for any $p > 1$ (see [56]). Hence, (2.1) is the strongest possible condition (in the class of Lebesgue spaces) one could expect on (ω^ε) .

However, not even (2.1) is enough to ensure that the classical vanishing viscosity limit,

$$\mathbf{u}^\varepsilon \rightarrow \mathbf{u}^0 \text{ in } L^\infty(0, T; H), \quad (2.2)$$

holds true. In fact, a slightly stronger condition would be sufficient, namely that $\{\omega^\varepsilon\}$ be bounded in $L^\infty(0, T; X)$, for some Banach space X which is compactly imbedded in H^{-1} . This follows from an easy adaptation of Theorem 1.1 in [66]. Within the Lebesgue hierarchy, L^1 is critical for this imbedding. In fact, L^p is compactly imbedded in H^{-1} , for any $p > 1$.

It should be noted that, even for CSF, (2.1) is not straightforward to establish (in fact, the lack of an L^p vorticity bound, for $p > 1$, uniform in viscosity, is a diffusive effect, present even when inertial terms vanish). Here, it is a byproduct of establishing convergence in stronger norms than the energy norm for the corrected velocity. (For recent, related criteria on the validity of the vanishing viscosity limit, see [25].)

Another common property among all types of flows under study is the fact that $\Delta \mathbf{u}^\varepsilon \cdot \mathbf{n} = 0$ on the boundary. Though we will not use this property directly, we will exploit some related implications, in particular that the Laplace and Stokes operators agree when applied to \mathbf{u}^ε and that $\nabla p_\varepsilon \cdot \mathbf{n} = 0$, providing a boundary condition for the pressure. (The pressure will vanish entirely for CSF and PCF.) By comparison, in [17] (see also [16]) the authors study NSE under boundary conditions of the form $\text{curl}^k \mathbf{u}^\varepsilon \cdot \mathbf{n} = 0$, $k = 0, 1, 2$, which can also be written as

$$\mathbf{u}^\varepsilon \cdot \mathbf{n} = \text{curl } \mathbf{u}^\varepsilon \cdot \mathbf{n} = \Delta \mathbf{u}^\varepsilon \cdot \mathbf{n} = 0.$$

For the 3D examples of PCF and PPF, the first and third of these boundary conditions are satisfied. The second boundary condition is not satisfied, though $\text{curl } \mathbf{u}^\varepsilon \cdot \mathbf{n}$ is of an especially simple form, containing only a tangential derivative of one of the components of the velocity.

3. Lighthill Principle for PCF

In this section we introduce the Lighthill principle, which we prove for the case of flow between two parallel planes, and we use it to derive an L^1 estimate on vorticity for PCF, independent of viscosity. The Lighthill principle is a property of viscous incompressible flow in a domain with a rigid boundary. Roughly speaking it is a way of expressing the flux, through the rigid boundary, of the vorticity components tangent to the boundary,

in terms of tangential derivatives of pressure at the boundary. We will see that, for flow between two parallel planes, the vorticity vector is actually tangent to the boundary, so that Lighthill principle provides a complete set of boundary conditions for the viscous vorticity equation, provided that the pressure is known at the boundary. For a discussion of the Lighthill principle see [74], and for the original source, see [59].

The results covered in this section are not used in the remainder of the text. Our purpose in including this material is twofold. First, to present the argument which leads to the Lighthill principle. This is what will actually be used in the remainder of the article. Our second objective is to illustrate the use of Lighthill's idea in estimating vorticity production by the viscous friction between the fluid and the boundary in a rigorous form.

We are interested in solutions of the 3D Navier-Stokes equations (1.1) between two parallel planes, say $\{z = 0\}$ and $\{z = h\}$. We will also assume the flow is periodic in the other two directions. Let $L > 0$ and set $Q_L = [0, L] \times [0, L]$ to be the periodic box of sides L ; in this section $\Omega = Q_L \times (0, h)$. In this section we will assume that $\mathbf{f} \equiv 0$.

We fix \mathbf{u}_0 a smooth, divergence-free vector field in Ω , tangent to $\Gamma = \partial\Omega \equiv Q_L \times \{0, h\}$, horizontally periodic, and consider \mathbf{u}^ε the (smooth) solution of problem (1.1) with $\mathbf{f} = 0$.

Proposition 3.1. *Let $\boldsymbol{\omega}^\varepsilon = \text{curl } \mathbf{u}^\varepsilon = (\omega_1^\varepsilon, \omega_2^\varepsilon, \omega_3^\varepsilon)$. We have that, at any point on $\Gamma = \partial\Omega$,*

$$\omega_3^\varepsilon = 0.$$

We also have, at the boundary, that:

$$\begin{aligned} \frac{\partial \omega_1^\varepsilon}{\partial z} &= -\frac{1}{\varepsilon} \frac{\partial p^\varepsilon}{\partial y} \\ \frac{\partial \omega_2^\varepsilon}{\partial z} &= \frac{1}{\varepsilon} \frac{\partial p^\varepsilon}{\partial x}. \end{aligned} \tag{3.1}$$

Proof. Set $\mathbf{n}_\pm = (0, 0, \pm 1)$, so that \mathbf{n}_+ is the unit exterior normal to $Q_L \times \{z = h\}$ and \mathbf{n}_- is the unit exterior normal to $Q_L \times \{z = 0\}$. We claim that

$$\pm \omega_3^\varepsilon \equiv \boldsymbol{\omega}^\varepsilon \cdot \mathbf{n}_\pm = 0 \text{ on } \partial\Omega \times (0, T). \tag{3.2}$$

Indeed, it is immediate that $\pm \omega_3^\varepsilon \equiv \boldsymbol{\omega}^\varepsilon \cdot \mathbf{n}_\pm$. We write

$$\boldsymbol{\omega}^\varepsilon = (\partial_y u_3^\varepsilon - \partial_z u_2^\varepsilon, \partial_z u_1^\varepsilon - \partial_x u_3^\varepsilon, \partial_x u_2^\varepsilon - \partial_y u_1^\varepsilon).$$

Hence, at Γ , $\omega_3^\varepsilon = \partial_x u_2^\varepsilon - \partial_y u_1^\varepsilon = 0$, because $\mathbf{u}^\varepsilon = 0$ at Γ and both ∂_x and ∂_y are tangential derivatives along the boundary. This establishes (3.2).

Next, we observe that, from the vector calculus identity below:

$$\text{curl curl } \mathbf{u} = \nabla \text{div } \mathbf{u} - \Delta \mathbf{u}$$

together with the fact that \mathbf{u}^ε is divergence-free, it follows that

$$\Delta \mathbf{u}^\varepsilon = -\text{curl } \boldsymbol{\omega}^\varepsilon. \tag{3.3}$$

Assume that the Navier-Stokes equations (1.1) remain valid up to the boundary. Then, since $\mathbf{u}^\varepsilon = 0$ on $\partial\Omega \times (0, T)$, we find, using (3.3),

$$\text{curl } \boldsymbol{\omega}^\varepsilon = -\frac{1}{\varepsilon} \nabla p^\varepsilon \text{ on } \partial\Omega \times (0, T). \tag{3.4}$$

We will take the cross-product of (3.4) with \mathbf{n}_\pm .

We first compute $\text{curl } \boldsymbol{\omega}^\varepsilon \times \mathbf{n}_\pm$ and we find:

$$\text{curl } \boldsymbol{\omega}^\varepsilon \times \mathbf{n}_\pm = \pm(-\partial_x \omega_3^\varepsilon + \partial_z \omega_1^\varepsilon, -\partial_y \omega_3^\varepsilon + \partial_z \omega_2^\varepsilon, 0).$$

However, on Γ we now know that $\omega_3^\varepsilon = 0$. Hence, since ∂_x and ∂_y are tangential derivatives, we find, on Γ , that

$$\text{curl } \boldsymbol{\omega}^\varepsilon \times \mathbf{n}_\pm = \pm(\partial_z \omega_1^\varepsilon, \partial_z \omega_2^\varepsilon, 0). \quad (3.5)$$

Next we compute $\nabla p^\varepsilon \times \mathbf{n}_\pm$. We obtain:

$$\nabla p^\varepsilon \times \mathbf{n}_\pm = \pm(\partial_y p^\varepsilon, -\partial_x p^\varepsilon, 0). \quad (3.6)$$

We easily deduce, from (3.4), (3.5) and (3.6), the desired system of equations in the statement, (3.1). □

Lighthill principle, as expressed above, provides a complete set of boundary conditions for the vorticity form of the Navier-Stokes equations. The two tangential components of vorticity satisfy a non-homogeneous Neumann condition and the normal component satisfies a homogeneous Dirichlet condition.

Next we will focus on the special case of plane-parallel channel flows in Ω (PCF). As discussed in the previous section, PCF have the form

$$\mathbf{u}^\varepsilon = \mathbf{u}^\varepsilon(x, y, z, t) \equiv (u_1^\varepsilon(z, t), u_2^\varepsilon(x, z, t), 0). \quad (3.7)$$

This symmetry is preserved by both the Euler and Navier-Stokes evolution. Note that the divergence-free condition for velocity is automatically satisfied.

We will use the following notation for the initial velocity:

$$\mathbf{u}_0 = \mathbf{u}_0(x, y, z) = (g_1(z), g_2(x, z), 0).$$

Under this symmetry the Navier-Stokes equations reduce to:

$$\left\{ \begin{array}{ll} \frac{\partial u_1^\varepsilon}{\partial t} = -\frac{\partial p^\varepsilon}{\partial x} + \varepsilon \frac{\partial^2 u_1^\varepsilon}{\partial z^2}, & \text{in } \Omega \times (0, T), \\ \frac{\partial u_2^\varepsilon}{\partial t} + u_1^\varepsilon \frac{\partial u_2^\varepsilon}{\partial x} = -\frac{\partial p^\varepsilon}{\partial y} + \varepsilon \Delta_{x,z} u_2^\varepsilon, & \text{in } \Omega \times (0, T), \\ 0 = -\frac{\partial p^\varepsilon}{\partial z}, & \text{in } \Omega \times (0, T), \\ u_1^\varepsilon = u_2^\varepsilon = 0, & \text{on } Q_L \times \{0, h\} \times (0, T), \\ u_1^\varepsilon, u_2^\varepsilon \text{ } L\text{-periodic} & \text{in } x, y, \text{ for each } z \in (0, h), t \in (0, T), \\ u_1^\varepsilon|_{t=0} = g_1(z), & \text{in } \Omega, \\ u_2^\varepsilon|_{t=0} = g_2(x, z), & \text{in } \Omega. \end{array} \right. \quad (3.8)$$

Above, the pressure p^ε may be chosen to vanish identically. Indeed, we deduce, from the evolution equations for u_1^ε and u_2^ε , that $\partial^2 p^\varepsilon / \partial x^2 = \partial^2 p^\varepsilon / \partial y^2 = 0$. As we are assuming periodic boundary conditions on all the unknowns we find that $p^\varepsilon = p^\varepsilon(z, t)$ and, since $\partial p^\varepsilon / \partial z = 0$, p^ε is constant in z ; we choose $p^\varepsilon = 0$.

In what follows we are interested primarily in the behavior of vorticity when the data is not compatible, i.e., when g_1 and g_2 are not necessarily vanishing at $z = 0$ and $z = h$. The compatible case is much simpler to treat. We will begin our analysis with the observation that, taking the curl of the velocity equation (3.8) and using Theorem 3.1 and $p^\varepsilon = 0$, we obtain the system of equations below, for $\boldsymbol{\omega}^\varepsilon = (\omega_1^\varepsilon, \omega_2^\varepsilon, \omega_3^\varepsilon) \equiv \left(-\frac{\partial u_2^\varepsilon}{\partial z}, \frac{\partial u_1^\varepsilon}{\partial z}, \frac{\partial u_2^\varepsilon}{\partial x}\right)$.

$$\left\{ \begin{array}{ll} \frac{\partial \omega_1^\varepsilon}{\partial t} + u_1^\varepsilon \frac{\partial \omega_1^\varepsilon}{\partial x} - \omega_2^\varepsilon \omega_3^\varepsilon = \varepsilon \Delta_{x,z} \omega_1^\varepsilon, & \text{in } \Omega \times (0, T), \\ \frac{\partial \omega_2^\varepsilon}{\partial t} = \varepsilon \frac{\partial^2 \omega_2^\varepsilon}{\partial z^2}, & \text{in } \Omega \times (0, T), \\ \frac{\partial \omega_3^\varepsilon}{\partial t} + u_1^\varepsilon \frac{\partial \omega_3^\varepsilon}{\partial x} = \varepsilon \Delta_{x,z} \omega_3^\varepsilon, & \text{in } \Omega \times (0, T), \\ \frac{\partial \omega_1^\varepsilon}{\partial z} = \frac{\partial \omega_2^\varepsilon}{\partial z} = \omega_3^\varepsilon = 0, & \text{on } \partial\Omega \times (0, T), \\ \boldsymbol{\omega}^\varepsilon \text{ } L\text{-periodic,} & \text{in } x, y, \text{ for each } z \in (0, h), t \in (0, T), \\ \omega_1^\varepsilon|_{t=0} = -\frac{\partial g_2}{\partial z}, & \text{in } \Omega, \\ \omega_2^\varepsilon|_{t=0} = \frac{dg_1}{dz}, & \text{in } \Omega. \\ \omega_3^\varepsilon|_{t=0} = \frac{\partial g_2}{\partial x}, & \text{in } \Omega. \end{array} \right. \quad (3.9)$$

If the initial data g_i , $i = 1, 2$, were compatible then even the spatial derivatives of the solution $u_1^\varepsilon, u_2^\varepsilon$ to (3.8) would be continuous in time, see [29] Chapter 7, §7.1, Theorem 5, allowing for energy methods to produce bounds on vorticity. As our main interest is non-compatible data, we will use a different approach.

We will obtain bounds for the vorticity $\boldsymbol{\omega}^\varepsilon$ in $L^\infty((0, T); L^1(\Omega))$, uniform with respect to ε , by approximating the non-compatible problem for (3.8) by a sequence of compatible problems. We will argue that the sequence of velocities converge, in the sense of distributions, to the solution of the non-compatible problem and we will derive estimates for the curl of the approximate velocities, uniform along the sequence, in $L^\infty((0, T); L^1(\Omega))$. It follows by the weak lower semicontinuity of the L^1 -norm that these estimates remain true for the limit problem.

As stated in the beginning of this section, this material is independent from the remainder of the article and serves mostly a pedagogical purpose. Hence, in the theorem below, we choose to be rather loose regarding the precise regularity of the solutions involved. We point out that solutions of the heat equation are certainly as smooth as needed in the calculations performed in the proof.

Theorem 3.2. *Fix $h, L > 0$ and $T > 0$. Let $Q_L = [0, L]^2$ be the periodic box of sides L and set $\Omega = Q_L \times (0, h)$. Let $g_1 = g_1(z) \in C^\infty([0, h])$, $g_2 = g_2(x, z) \in C^\infty([0, L] \times [0, h])$, and suppose $g_2(0, z) = g_2(L, z)$. Assume that neither g_1 nor g_2 vanish for $z \in \{0, h\}$. Consider plane-parallel channel flow $\mathbf{u}^\varepsilon = \mathbf{u}^\varepsilon(x, y, z, t) \equiv (u_1^\varepsilon(z, t), u_2^\varepsilon(x, z, t), 0)$. Then*

$u_1^\varepsilon, u_2^\varepsilon$ is the solution of

$$\left\{ \begin{array}{ll} \frac{\partial u_1^\varepsilon}{\partial t} = \varepsilon \frac{\partial^2 u_1^\varepsilon}{\partial z^2}, & \text{in } (0, h) \times (0, T), \\ u_1^\varepsilon = 0, & \text{at } \{z = 0, h\} \times (0, T), \\ u_1^\varepsilon|_{t=0} = g_1(z), & \text{in } (0, h), \\ \frac{\partial u_2^\varepsilon}{\partial t} + u_1^\varepsilon \frac{\partial u_2^\varepsilon}{\partial x} = \varepsilon \Delta_{x,z} u_2^\varepsilon, & \text{in } [0, L] \times (0, h) \times (0, T), \\ u_2^\varepsilon = 0, & \text{on } [0, L] \times \{z = 0, h\} \times (0, T), \\ u_2^\varepsilon \text{ } L\text{-periodic} & \text{in } x, \text{ for each } z \in (0, h), t \in (0, T), \\ u_2^\varepsilon|_{t=0} = g_2(x, z), & \text{in } [0, L] \times (0, h). \end{array} \right. \quad (3.10)$$

In addition, if $\text{curl } \mathbf{u}^\varepsilon = \boldsymbol{\omega}^\varepsilon = (\omega_1^\varepsilon, \omega_2^\varepsilon, \omega_3^\varepsilon)$ then

$$\omega_1^\varepsilon = -\frac{\partial u_2^\varepsilon}{\partial z}, \quad \omega_2^\varepsilon = \frac{\partial u_1^\varepsilon}{\partial z}, \quad \omega_3^\varepsilon = \frac{\partial u_2^\varepsilon}{\partial x}$$

and

$$\begin{aligned} \|\omega_1^\varepsilon(\cdot, t)\|_{L^1(\Omega)} &\leq \left\| \frac{\partial g_2}{\partial z} \right\|_{L^1(\Omega)} + 2L \left(\|g_2\|_{L^\infty(\Omega)} + \left\| \frac{\partial^2 g_2}{\partial x^2} \right\|_{L^\infty(\Omega)} \right) \\ &+ T \left(\left\| \frac{dg_1}{dz} \right\|_{L^1(\Omega)} + \|g_1\|_{L^\infty(\Omega)} \right) \left\| \frac{\partial g_2}{\partial x} \right\|_{L^\infty(\Omega)}; \end{aligned} \quad (3.11)$$

$$\|\omega_2^\varepsilon(\cdot, t)\|_{L^1(\Omega)} \leq \left\| \frac{dg_1}{dz} \right\|_{L^1(\Omega)} + 2\|g_1\|_{L^\infty(\Omega)}; \quad (3.12)$$

$$\|\omega_3^\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \left\| \frac{\partial g_2}{\partial x} \right\|_{L^\infty(\Omega)}, \quad (3.13)$$

for all $0 \leq t < T$.

Proof. We begin by noticing that, as u_1^ε is independent of x, y , u_2^ε is independent of y , and $p^\varepsilon \equiv 0$, it follows from (3.8) that $u_1^\varepsilon, u_2^\varepsilon$ satisfy (3.10).

Next, let us introduce $\alpha_n = \alpha_n(t) \in C^\infty([0, +\infty))$ as below:

$$\alpha_n = \alpha_n(t) \equiv 1 \text{ if } t > \frac{1}{n}, \quad \alpha_n = \alpha_n(t) \equiv 0 \text{ if } 0 \leq t < \frac{1}{2n}, \quad 0 \leq \alpha_n \leq 1, \quad \alpha_n'(t) \geq 0. \quad (3.14)$$

We will start with an approximation to u_1^ε , from which we will derive the bound (3.12) for $\omega_2^\varepsilon = \partial u_1^\varepsilon / \partial z$.

We introduce w_1^ε , the solution of

$$\left\{ \begin{array}{ll} \frac{\partial w_1^\varepsilon}{\partial t} = \varepsilon \frac{\partial^2 w_1^\varepsilon}{\partial z^2}, & \text{in } (0, h) \times (0, T), \\ w_1^\varepsilon = -g_1, & \text{at } \{z = 0, h\} \times (0, T), \\ w_1^\varepsilon|_{t=0} = 0, & \text{in } (0, h). \end{array} \right. \quad (3.15)$$

Observe that the data is not compatible.

Let us also introduce v_1^ε such that

$$u_1^\varepsilon = v_1^\varepsilon + g_1 + w_1^\varepsilon.$$

Now, v_1^ε satisfies a compatible problem – both initial and boundary data vanish identically – for a heat equation with smooth forcing, given by $\varepsilon \partial_z^2 g_1$. We will use an approximation for w_1^ε .

Set $w_1^{\varepsilon,n}$ to be the solution of

$$\begin{cases} \frac{\partial w_1^{\varepsilon,n}}{\partial t} = \varepsilon \frac{\partial^2 w_1^{\varepsilon,n}}{\partial z^2}, & \text{in } (0, h) \times (0, T), \\ w_1^{\varepsilon,n} = -\alpha_n(t)g_1, & \text{at } \{z = 0, h\} \times (0, T), \\ w_1^{\varepsilon,n}|_{t=0} = 0, & \text{in } (0, h). \end{cases} \quad (3.16)$$

Since $\alpha_n(0) = 0$ this problem is compatible.

Let $u_1^{\varepsilon,n} \equiv v_1^\varepsilon + g_1 + w_1^{\varepsilon,n}$.

Claim 3.3. *We have, passing to subsequences as needed,*

$$u_1^{\varepsilon,n} \rightharpoonup u_1^\varepsilon,$$

in $\mathcal{D}'([0, T] \times (0, h))$, as $n \rightarrow \infty$.

Proof of Claim: Clearly, it is enough to show that

$$w_1^{\varepsilon,n} \rightharpoonup w_1^\varepsilon.$$

Next, we lift the boundary data as a forcing term in the equation. Set $\overline{w_1^{\varepsilon,n}} = w_1^{\varepsilon,n} + \alpha_n(t)g_1$. Then $\overline{w_1^{\varepsilon,n}}$ satisfies

$$\begin{cases} \frac{\partial \overline{w_1^{\varepsilon,n}}}{\partial t} = \varepsilon \frac{\partial^2 \overline{w_1^{\varepsilon,n}}}{\partial z^2} - \varepsilon \alpha_n(t) \frac{d^2 g_1}{dz^2} + \alpha_n'(t)g_1, & \text{in } (0, h) \times (0, T), \\ \overline{w_1^{\varepsilon,n}} = 0, & \text{at } \{z = 0, h\} \times (0, T), \\ \overline{w_1^{\varepsilon,n}}|_{t=0} = 0, & \text{in } (0, h). \end{cases} \quad (3.17)$$

We begin by observing that $\overline{w_1^{\varepsilon,n}}$ is bounded, uniformly in n , in $L^\infty((0, T; L^2(0, h)))$. Indeed, multiply the equation by $\overline{w_1^{\varepsilon,n}}$, integrate over $(0, h)$, and divide by $\|\overline{w_1^{\varepsilon,n}}\|_{L^2}$ to find

$$\frac{d}{dt} \|\overline{w_1^{\varepsilon,n}}\|_{L^2} \leq 2\varepsilon \alpha_n(t) \left\| \frac{d^2 g_1}{dz^2} \right\|_{L^2} + 2\alpha_n'(t) \|g_1\|_{L^2},$$

where we used that $\alpha_n, \alpha_n' \geq 0$. We obtain the uniform estimate upon integrating in time, using that $\|\overline{w_1^{\varepsilon,n}}(0)\|_{L^2} = 0$ and that $\alpha_n \leq 1$, $\int_0^T \alpha_n'(s) ds = 1$.

It follows from the Banach-Alaoglu theorem that, passing to subsequences as needed, there exists $R \in L^\infty((0, T; L^2(0, h)))$ such that $\overline{w_1^{\varepsilon,n}} \rightharpoonup R$ weak- $*$ $L^\infty((0, T; L^2(0, h)))$. Hence, we also have $\overline{w_1^{\varepsilon,n}} \rightharpoonup R$ in $\mathcal{D}'([0, T] \times (0, h))$.

Let $\varphi \in C^\infty([0, T] \times [0, h])$. Assume $\varphi(\cdot, z) \in C_c^\infty([0, T])$ for every $z \in [0, h]$ and, additionally, $\varphi(t, 0) = \varphi(t, h) = 0$ for each $t \geq 0$. Multiply the equation for $\overline{w_1^{\varepsilon,n}}$ by φ

and integrate in time and space, transferring all derivatives to φ , including the time-derivative of α_n , to obtain a weak formulation for (3.17). Since the equation is linear it follows, from weak convergence of $\overline{w_1^{\varepsilon,n}}$ to R and because $\alpha_n \rightarrow \chi_{(0,+\infty)}$ strongly in L^1 , that

$$-\int_0^T \int_0^h \partial_t \varphi R = \varepsilon \int_0^T \int_0^h \partial_z^2 \varphi R - \varepsilon \int_0^T \int_0^h \varphi \frac{d^2 g_1}{dz^2} + \int_0^h \varphi(0, z) g_1.$$

Let us now introduce $S \equiv R - g_1$. Clearly, it holds that:

$$-\int_0^T \int_0^h \partial_t \varphi S = \varepsilon \int_0^T \int_0^h \partial_z^2 \varphi S + \varepsilon \int_0^T \int_0^h \partial_z^2 \varphi g_1 - \varphi \frac{d^2 g_1}{dz^2}.$$

Taking $\varphi \in C_c^\infty((0, T) \times (0, h))$ we obtain that S is a distributional solution of the heat equation in $(0, T) \times (0, h)$. Taking now $\varphi \in C_c^\infty((0, T) \times [0, h])$, with $\varphi(t, 0) = \varphi(t, h) = 0$, we deduce that $S = -g_1$ at $z = 0, h$. Finally, taking $\varphi \in C_c^\infty([0, T] \times (0, h))$ we deduce that $S = 0$ at $t = 0$. Hence, by uniqueness for (3.15), it follows that $S = w_1^\varepsilon$. \square

Having established the claim, we now prove uniform estimates for $\omega_2^n \equiv \partial_z u_1^{\varepsilon,n}$. Then (3.12) will follow from these estimates, together with the weak convergence $u_1^{\varepsilon,n} \rightharpoonup u_1^\varepsilon$.

Start by observing that $u_1^{\varepsilon,n}$ satisfies

$$\begin{cases} \frac{\partial u_1^{\varepsilon,n}}{\partial t} = \varepsilon \frac{\partial^2 u_1^{\varepsilon,n}}{\partial z^2}, & \text{in } (0, h) \times (0, T), \\ u_1^{\varepsilon,n} = (1 - \alpha_n(t))g_1, & \text{at } \{z = 0, h\} \times (0, T), \\ u_1^{\varepsilon,n}|_{t=0} = g_1(z), & \text{in } (0, h). \end{cases} \quad (3.18)$$

Differentiate the equation for $u_1^{\varepsilon,n}$ with respect to z to find, easily,

$$\frac{\partial \omega_2^n}{\partial t} = \varepsilon \frac{\partial^2 \omega_2^n}{\partial z^2}.$$

We proceed in the spirit of (3.1): evaluate the evolution equation for $u_1^{\varepsilon,n}$, (3.18) at the boundary $z = 0, z = h$, to find

$$\frac{\partial \omega_2^n}{\partial z} \Big|_{z=0,h} = -\frac{1}{\varepsilon} \alpha_n'(t) g_1 \Big|_{z=0,h}. \quad (3.19)$$

The initial condition for ω_2^n is clearly $\omega_2^n(z, t = 0) = dg_1/dz$. Putting together the equation for ω_2^n , the boundary condition (3.19), and the initial data yields the Cauchy problem below:

$$\begin{cases} \frac{\partial \omega_2^n}{\partial t} = \varepsilon \frac{\partial^2 \omega_2^n}{\partial z^2}, & \text{in } (0, h) \times (0, T), \\ \frac{\partial \omega_2^n}{\partial z} = -\frac{1}{\varepsilon} \alpha_n' g_1, & \text{at } \{z = 0, h\} \times (0, T), \\ \omega_2^n|_{t=0} = \frac{dg_1}{dz}, & \text{in } (0, h). \end{cases} \quad (3.20)$$

Fix $\delta > 0$ and set $\varphi_\delta = \varphi_\delta(s) \equiv \sqrt{\delta^2 + s^2}$. Of course, $\varphi_\delta(s) \rightarrow |s|$ pointwise, as $\delta \rightarrow 0$. In addition,

$$|\varphi'_\delta(s)| \leq 1; \quad \varphi''_\delta(s) \geq 0.$$

Multiply the equation for ω_2^n by $\varphi'_\delta(\omega_2^n)$ and integrate on $(0, h)$ to find, upon integration by parts and using the Neumann boundary condition (3.19):

$$\begin{aligned} \frac{d}{dt} \int_0^h \varphi_\delta(\omega_2^n) dz &= \varepsilon \int_0^h \varphi'_\delta(\omega_2^n) \frac{\partial^2 \omega_2^n}{\partial z^2} dz \\ &= -\varepsilon \int_0^h \varphi''_\delta(\omega_2^n) \left(\frac{\partial \omega_2^n}{\partial z} \right)^2 dz + \varepsilon \int_0^h \partial_z [\varphi'_\delta(\omega_2^n) \partial_z \omega_2^n] dz \\ &\leq -\alpha'_n(t) [\varphi'_\delta(\omega_2^n) g_1] \Big|_{z=0}^{z=h} \\ &\leq |\alpha'_n(t)| (|g_1(h)| + |g_1(0)|). \end{aligned} \tag{3.21}$$

Integrating (3.21) in time from 0 to t and taking the limit $\delta \rightarrow 0$ we obtain

$$\|\omega_2^n(t)\|_{L^1(0,h)} \leq \|\omega_2^n(t=0)\|_{L^1(0,h)} + |g_1(h)| + |g_1(0)| \leq \|dg_1/dz\|_{L^1(0,h)} + 2\|g_1\|_{L^\infty(0,h)}.$$

Estimate (3.12) follows by taking $n \rightarrow \infty$, using the weak lower semicontinuity of $\|\cdot\|_{L^1}$, in view of the convergence $u_1^{\varepsilon,n} \rightarrow u_1^\varepsilon$ in the sense of distributions.

Next we treat u_2^ε . The approximation is quite similar. We introduce w_2^ε , the solution of

$$\begin{cases} \frac{\partial w_2^\varepsilon}{\partial t} + u_1^\varepsilon \frac{\partial w_2^\varepsilon}{\partial x} = \varepsilon \Delta_{x,z} w_2^\varepsilon, & \text{in } (0, L) \times (0, h) \times (0, T), \\ w_2^\varepsilon = -g_2, & \text{at } (0, L) \times \{0, h\} \times (0, T), \\ w_2^\varepsilon|_{t=0} = 0, & \text{in } (0, L) \times (0, h). \end{cases} \tag{3.22}$$

We also require w_2^ε to be periodic in x with period L . We note, as before, that the data is not compatible.

We introduce v_2^ε so that

$$u_2^\varepsilon = v_2^\varepsilon + g_2 + w_2^\varepsilon.$$

As before, v_2^ε satisfies a compatible problem – both initial and boundary data vanish identically – for a drift-diffusion equation, with drift u_1^ε , and with smooth (in the interior of $(0, L) \times (0, h) \times (0, T)$) forcing, given by $\varepsilon \Delta_{x,z} g_2 - u_1^\varepsilon \partial_x g_2$. Similarly to what we did for u_1^ε , we will make use of an approximation for w_2^ε .

Set $w_2^{\varepsilon,n}$ to be the solution of

$$\begin{cases} \frac{\partial w_2^{\varepsilon,n}}{\partial t} + u_1^\varepsilon \frac{\partial w_2^{\varepsilon,n}}{\partial x} = \varepsilon \Delta_{x,z} w_2^{\varepsilon,n}, & \text{in } (0, L) \times (0, h) \times (0, T), \\ w_2^{\varepsilon,n} = -\alpha_n(t) g_2, & \text{at } (0, L) \times \{0, h\} \times (0, T), \\ w_2^{\varepsilon,n}|_{t=0} = 0, & \text{in } (0, L) \times (0, h). \end{cases} \tag{3.23}$$

Impose periodic boundary conditions at $x = 0$, $x = L$. Since $\alpha_n(0) = 0$ this problem is compatible.

Let $u_2^{\varepsilon,n} \equiv v_2^\varepsilon + g_2 + w_2^{\varepsilon,n}$.

Claim 3.4. *We have, passing to subsequences as needed,*

$$u_2^{\varepsilon,n} \rightharpoonup u_2^\varepsilon,$$

in $\mathcal{D}'([0, T] \times (0, L) \times (0, h))$, periodic in x , as $n \rightarrow \infty$.

Proof of Claim: As before, clearly, it is enough to show that

$$w_2^{\varepsilon,n} \rightharpoonup w_2^\varepsilon.$$

We lift the boundary data as a forcing term in the equation. Set $\overline{w_2^{\varepsilon,n}} = w_2^{\varepsilon,n} + \alpha_n(t)g_2$. Then $\overline{w_2^{\varepsilon,n}}$ satisfies

$$\left\{ \begin{array}{ll} \frac{\partial \overline{w_2^{\varepsilon,n}}}{\partial t} + u_1^\varepsilon \frac{\partial \overline{w_2^{\varepsilon,n}}}{\partial x} = \varepsilon \Delta_{x,z} \overline{w_2^{\varepsilon,n}} - \varepsilon \alpha_n(t) \Delta_{x,z} g_2 \\ \quad + u_1^\varepsilon \alpha_n(t) \frac{\partial g_2}{\partial x} + \alpha_n'(t) g_2, & \text{in } (0, L) \times (0, h) \times (0, T), \\ \overline{w_2^{\varepsilon,n}} = 0, & \text{at } (0, L) \times \{0, h\} \times (0, T), \\ \overline{w_2^{\varepsilon,n}}|_{t=0} = 0, & \text{in } (0, L) \times (0, h). \end{array} \right. \quad (3.24)$$

Additionally, $\overline{w_2^{\varepsilon,n}}$ is periodic in x .

We note that $w_2^{\varepsilon,n}$ is bounded, uniformly in n , in $L^\infty((0, T; L^2((0, L) \times (0, h))))$. Indeed, we have, easily,

$$\frac{d}{dt} \|\overline{w_2^{\varepsilon,n}}\|_{L^2} \leq 2\varepsilon \alpha_n(t) \|\Delta_{x,z} g_2\|_{L^2} + 2\alpha_n(t) \|u_1^\varepsilon \partial_x g_2\|_{L^2} + 2\alpha_n'(t) \|g_2\|_{L^2},$$

where we used, once again, that $\alpha_n, \alpha_n' \geq 0$. We obtain the uniform estimate upon integrating in time, using that $\|\overline{w_2^{\varepsilon,n}}(0)\|_{L^2} = 0$, that $\sup_{(0,T)} \|u_1^\varepsilon g_2\|_{L^2} < \infty$, and that

$$\alpha_n \leq 1, \quad \int_0^T \alpha_n'(s) ds = 1.$$

The remainder of the argument used to establish Claim 3.3 can now be used, with the appropriate modifications, to conclude the proof of the present claim. \square

Next, we use Claim 3.4 to establish (3.13).

Note that $u_2^{\varepsilon,n}$ satisfies

$$\left\{ \begin{array}{ll} \frac{\partial u_2^{\varepsilon,n}}{\partial t} + u_1^\varepsilon \frac{\partial u_2^{\varepsilon,n}}{\partial x} = \varepsilon \Delta_{x,z} u_2^{\varepsilon,n}, & \text{in } (0, L) \times (0, h) \times (0, T), \\ u_2^{\varepsilon,n} = (1 - \alpha_n(t)) g_2, & \text{at } (0, L) \times \{0, h\} \times (0, T), \\ u_2^{\varepsilon,n}|_{t=0} = g_2(z), & \text{in } (0, L) \times (0, h). \end{array} \right. \quad (3.25)$$

Moreover, $u_2^{\varepsilon,n}$ is periodic in x .

Let $\omega_3^n \equiv \partial u_2^{\varepsilon,n} / \partial x$. Differentiating the equation for $u_2^{\varepsilon,n}$ with respect to x yields, easily,

$$\frac{\partial \omega_3^n}{\partial t} + u_1^\varepsilon \frac{\partial \omega_3^n}{\partial x} = \varepsilon \Delta_{x,z} \omega_3^n.$$

We now evaluate the evolution equation for u_2^n , (3.25), at the boundary $[0, L] \times \{0, h\}$, to obtain

$$\omega_3^n(x, \cdot, t) \Big|_{z=0,h} = (1 - \alpha_n(t)) \partial_x g_2(x, \cdot) \Big|_{z=0,h}. \quad (3.26)$$

In addition we have $\omega_3^n(x, z, t=0) = \partial_x g_2(x, z)$. Putting together the equation for ω_3^n , the boundary condition (3.26), and the initial data yields the Cauchy problem below:

$$\left\{ \begin{array}{ll} \frac{\partial \omega_3^n}{\partial t} + u_1^\varepsilon \frac{\partial \omega_3^n}{\partial x} = \varepsilon \Delta_{x,z} \omega_3^n, & \text{in } [0, L] \times (0, h) \times (0, T), \\ \omega_3^n(\cdot, t) = (1 - \alpha_n(t)) \partial_x g_2(\cdot), & \text{on } [0, L] \times \{z = 0, h\} \times (0, T), \\ \omega_3^n \Big|_{t=0} = \partial_x g_2(x, z), & \text{in } [0, L] \times (0, h). \end{array} \right. \quad (3.27)$$

Since all the coefficients and data are smooth (u_1^ε is smooth for $t > 0$), we will have a smooth solution to which we can apply the maximum principle for the operator $\partial_t + u_1^\varepsilon \partial_x - \varepsilon \Delta_{x,z}$. We deduce that

$$\max_{[0,L] \times [0,h] \times [0,T]} |\omega_3^n(x, z, t)| = \max \left\{ \max_{[0,L] \times \{0,h\} \times [0,T]} |\omega_3^n|, \max_{[0,L] \times [0,h]} |\omega_3^n(\cdot, \cdot, 0)| \right\},$$

i.e.,

$$\|\omega_3^n\|_{L^\infty([0,L] \times [0,h] \times [0,T])} \leq \|\partial_x g_2\|_{L^\infty([0,L] \times [0,h])}.$$

Estimate (3.13) follows by taking $n \rightarrow \infty$, given that $u_2^n \rightarrow u_2^\varepsilon$ in the sense of distributions.

Finally, we analyze $\omega_1^\varepsilon = -\partial_z u_2^\varepsilon$. We note that the equation for ω_1^ε is the most complicated because it is the only equation with a vorticity stretching term, namely, $-\omega_2^\varepsilon \omega_3^\varepsilon$. This will impact the analysis for the approximations as well.

Let $\omega_1^n \equiv -\partial u_2^{\varepsilon,n} / \partial z$. Differentiate the equation for $u_2^{\varepsilon,n}$ with respect to z to obtain

$$\frac{\partial \omega_1^n}{\partial t} + u_1^\varepsilon \frac{\partial \omega_1^n}{\partial x} - \omega_2^n \omega_3^n = \varepsilon \Delta_{x,z} \omega_1^n.$$

As in (3.1), assume that the evolution equation for $u_2^{\varepsilon,n}$, (3.25), remains valid up to the boundary $[0, L] \times \{0, h\}$, so that, since u_1^ε vanishes at this boundary, for all $t > 0$, we have

$$\frac{\partial u_2^n}{\partial t} \Big|_{z=0,h} = \varepsilon (1 - \alpha_n(t)) \partial_x^2 g_2 - \varepsilon \frac{\partial \omega_1^n}{\partial z} \Big|_{z=0,h},$$

hence

$$\frac{\partial \omega_1^n}{\partial z} \Big|_{z=0,h} = \frac{1}{\varepsilon} \alpha_n' g_2 \Big|_{z=0,h} + (1 - \alpha_n(t)) \partial_x^2 g_2. \quad (3.28)$$

In addition we have $\omega_1^n(z, t = 0) = -\partial_z g_2$. Putting together the equation for ω_1^n , the boundary conditions (3.28), and the initial data yields the Cauchy problem below:

$$\left\{ \begin{array}{ll} \frac{\partial \omega_1^n}{\partial t} + u_1^\varepsilon \frac{\partial \omega_1^n}{\partial x} - \omega_2^n \omega_3^n = \varepsilon \Delta_{x,z} \omega_1^n, & \text{in } [0, L] \times (0, h) \times (0, T), \\ \frac{\partial \omega_1^n}{\partial z} = \frac{1}{\varepsilon} \alpha_n' g_2 + (1 - \alpha_n(t)) \partial_x^2 g_2, & \text{at } [0, L] \times \{z = 0, h\} \times (0, T), \\ \omega_1^n|_{t=0} = -\partial_z g_2(x, z), & \text{in } [0, L] \times (0, h). \end{array} \right. \quad (3.29)$$

Fix $\delta > 0$ and consider $\varphi_\delta = \varphi_\delta(s)$. As we did for ω_2^n , multiply the equation for ω_1^n by $\varphi_\delta'(\omega_1^n)$ and integrate on $[0, L] \times (0, h)$ to find, upon integration by parts and using the Neumann boundary condition (3.28):

$$\begin{aligned} \frac{d}{dt} \int_0^L \int_0^h \varphi_\delta(\omega_1^n) dx dz &= \varepsilon \int_0^L \int_0^h \varphi_\delta'(\omega_1^n) \Delta_{x,z} \omega_1^n dx dz + \int_0^L \int_0^h \varphi_\delta'(\omega_1^n) \omega_2^n \omega_3^n dx dz \\ &= -\varepsilon \int_0^L \int_0^h \varphi_\delta''(\omega_1^n) |\nabla_{x,z} \omega_1^n|^2 dx dz + \varepsilon \int_0^L \int_0^h \partial_z [\varphi_\delta'(\omega_1^n) \partial_z \omega_1^n] dx dz \\ &\quad + \int_0^L \int_0^h \varphi_\delta'(\omega_1^n) \omega_2^n \omega_3^n dx dz \\ &\leq \alpha_n'(t) \int_0^L [\varphi_\delta'(\omega_1^n) g_2] \Big|_{z=0}^{z=h} dx + \varepsilon (1 - \alpha_n(t)) \int_0^L [\varphi_\delta'(\omega_1^n) \partial_x^2 g_2] \Big|_{z=0}^{z=h} dx \\ &\quad + \|\omega_2^n(\cdot, t)\|_{L^1(0,h)} \|\omega_3^n(\cdot, t)\|_{L^\infty([0,L] \times [0,h])}. \end{aligned} \quad (3.30)$$

Integrating (3.30) in time from 0 to t and taking the limit $\delta \rightarrow 0$ we obtain

$$\begin{aligned} \|\omega_2^n(t)\|_{L^1([0,L] \times (0,h))} &\leq \|\omega_1^n|_{t=0}\|_{L^1([0,L] \times (0,h))} + 2L(\|g_2\|_{L^\infty([0,L] \times (0,h))} + \|\partial_x^2 g_2\|_{L^\infty([0,L] \times (0,h))}) \\ &\quad + T \|\omega_2^n\|_{L^\infty((0,T); L^1(0,h))} \|\omega_3^n\|_{L^\infty([0,L] \times [0,h] \times (0,T))} \\ &= \|\partial_z g_2\|_{L^1([0,L] \times (0,h))} + 2L(\|g_2\|_{L^\infty([0,L] \times (0,h))} + \|\partial_x^2 g_2\|_{L^\infty([0,L] \times (0,h))}) \\ &\quad + T(\|dg_1/dz\|_{L^1(0,h)} + 2\|g_1\|_{L^\infty([0,h])}) \|\partial_x g_2\|_{L^\infty([0,L] \times [0,h])}. \end{aligned}$$

Estimate (3.11) follows by taking $n \rightarrow \infty$, given that $u_2^n \rightharpoonup u_2^\varepsilon$ in the sense of distributions. This concludes the proof. \square

Analogous results hold for flow in the pipe. More precisely, it is possible to obtain a version of Proposition 3.1 for flow in a pipe and a version of Theorem 3.2 for PPF.

4. PLANE-PARALLEL CHANNEL FLOWS

In this section, we will present the asymptotic description of the vanishing viscosity limit for PCF, significantly extending the analysis in [72, 73].

We consider NSE and EE in an infinite channel, but impose periodic boundary conditions in the streamwise direction. The fluid domain is

$$\Omega := (0, L)^2 \times (0, h) \text{ with } \Gamma := \partial\Omega = (0, L)^2 \times \{0, h\},$$

for a fixed $h > 0$, and we consider flows which are periodic in both the x and y directions, with period $L > 0$.

We study plane-parallel solutions of the fluid equations of the form:

$$\mathbf{u} = (u_1(z, t), u_2(x, z, t), 0). \quad (4.1)$$

The initial data and the forcing will be taken to satisfy the same symmetry, that is:

$$\mathbf{f} = (f_1(z, t), f_2(x, z, t), 0), \quad \mathbf{u}_0 = (u_{0,1}(z), u_{0,2}(x, z), 0).$$

Under this symmetry, all vector fields are divergence free and automatically satisfy the no-penetration condition. In addition, it is easy to see that the pressure can be taken to be zero in both the NSE and EE and, therefore, will not enter the ensuing calculations.

It can be shown that the forward evolution under the NSE and EE preserves the symmetry, at least for strong solutions (cf. e.g. [73]). We hence consider the symmetry-reduced NSE (1.1), which become the following weakly non-linear system:

$$\left\{ \begin{array}{l} \frac{\partial u_1^\varepsilon}{\partial t} - \varepsilon \frac{\partial^2 u_1^\varepsilon}{\partial z^2} = f_1 \text{ in } \Omega \times (0, T), \\ \frac{\partial u_2^\varepsilon}{\partial t} - \varepsilon \frac{\partial^2 u_2^\varepsilon}{\partial x^2} - \varepsilon \frac{\partial^2 u_2^\varepsilon}{\partial z^2} + u_1^\varepsilon \frac{\partial u_2^\varepsilon}{\partial x} = f_2 \text{ in } \Omega \times (0, T), \\ u_2^\varepsilon \text{ is periodic in } x \text{ with period } L, \\ u_i^\varepsilon = 0, \quad i = 1, 2, \text{ on } \Gamma, \\ u_i^\varepsilon|_{t=0} = u_{0,i}, \quad i = 1, 2, \text{ in } \Omega. \end{array} \right. \quad (4.2)$$

Similarly, we consider the symmetry-reduced EE (1.2):

$$\left\{ \begin{array}{l} \frac{\partial u_1^0}{\partial t} = f_1 \text{ in } \Omega \times (0, T), \\ \frac{\partial u_2^0}{\partial t} + u_1^0 \frac{\partial u_2^0}{\partial x} = f_2 \text{ in } \Omega \times (0, T), \\ u_2^0 \text{ is periodic in } x \text{ direction with period } L, \\ u_i^0|_{t=0} = u_{0,i}, \quad i = 1, 2, \text{ in } \Omega. \end{array} \right. \quad (4.3)$$

We assume that the data, \mathbf{u}_0 and \mathbf{f} , are sufficiently regular, but ill-prepared in the sense that

$$\mathbf{u}_0 \in H \cap H^k(\Omega), \quad \mathbf{f} \in C(0, T; H \cap H^k(\Omega)), \quad \text{for a sufficiently large } k \geq 0. \quad (4.4)$$

Note that we do not assume that \mathbf{u}_0 vanishes at the boundary, nor does \mathbf{f} have to be compatible with \mathbf{u}_0 at $t = 0$. Under these regularity assumptions, the NSE and EE have, both, global-in-time strong solutions (see e.g. [73]).

Under the plane-parallel symmetry, the tangential components of the EE velocity \mathbf{u}^0 need not vanish. Therefore, a viscous boundary layer is expected to form to account for the mismatch in the tangential components of the NSE and EE velocities at the boundary. The fact that the data is ill prepared leads to an initial layer for NSE, which also affects the zero-viscosity limit.

In the following subsections, we construct correctors for the Euler flow that lead to an asymptotic expansion of \mathbf{u}^ε at small viscosity ε . This expansion will be used to study the boundary layer and the vanishing viscosity limit. This expansion is not a Prandtl-type expansion as that used in [72] and lends itself somewhat naturally to study accumulation of vorticity at the boundary in the limit.

4.1. Viscous approximation and convergence result. We postulate an approximation of the viscous solution of the form

$$\mathbf{u}^\varepsilon \cong \mathbf{u}^0 + \Theta, \quad (4.5)$$

where Θ is a corrector to the inviscid solution \mathbf{u}^0 . The corrector depends on ε , but for sake of notation, we will not explicitly denote it. The corrector will be assumed to satisfy the same symmetry as the fluid velocities, that is,

$$\Theta = (\Theta_1(z, t), \Theta_2(x, z, t), 0). \quad (4.6)$$

This assumption is justified by the fact that the flow remains laminar and there is no boundary layer separation in this case (see e.g. [73]).

In recent related work, see [72], a viscous approximation to the NS solution similar to (4.5) was introduced, where the corrector, which we call $\Upsilon = (\Upsilon_1(z, t), \Upsilon_2(x, z, t), 0)$ here, is defined as the solution of the *weakly coupled parabolic system*,

$$\left\{ \begin{array}{l} \frac{\partial \Upsilon_1}{\partial t} - \varepsilon \frac{\partial^2 \Upsilon_1}{\partial z^2} = 0 \quad \text{in } \Omega \times (0, T), \\ \frac{\partial \Upsilon_2}{\partial t} - \varepsilon \frac{\partial^2 \Upsilon_2}{\partial z^2} + \Upsilon_1 \frac{\partial \Upsilon_2}{\partial x} + u_1^0 \frac{\partial \Upsilon_2}{\partial x} + \Upsilon_1 \frac{\partial u_2^0}{\partial x} = 0 \quad \text{in } \Omega \times (0, T), \\ \Upsilon_i = -u_i^0 \quad \text{on } \Gamma \times (0, T), \quad i = 1, 2, \\ \Upsilon|_{t=0} = 0. \end{array} \right. \quad (4.7)$$

In fact, (4.7) is a reduced form of Prandtl's equations under the plane-parallel symmetry given in (4.1). Assuming well-prepared data, bounds on Υ with an explicit dependency on ε in the Sobolev space H^k , for some $k \geq 0$, were derived from energy estimates. Using these bounds, the authors verify the validity of the vanishing viscosity limit. Certain higher-order expansions are discussed as well.

In this article, we tackle the case of ill-prepared data. Since the second component of the velocity is advected by the first component in (4.2), we first construct Θ_1 ; then, using Θ_1 , we construct the second component of the corrector, Θ_2 .

The ingredients needed to construct both Θ_1 and Θ_2 are an explicit solution of the heat equation on the half-line and a solution of a drift-diffusion equation in a periodic channel.

Let $\Phi[g] = \Phi[g](\eta, t)$, $\eta > 0$, $t > 0$, denote the solution of (A.1) with boundary data $g = g(t)$. Set

$$\begin{cases} \Theta_{1,L}(z, t) = \Phi[g_L](z, t) \\ \Theta_{1,U}(z, t) = \Phi[g_U](h - z, t), \end{cases} \quad (4.8)$$

where $g_L = g_L(t) \equiv u_1^0(0, t)$ and $g_U = g_U(t) \equiv u_1^0(h, t)$ are the boundary data at $z = 0$ and $z = h$, respectively, for the first component of the EE solution.

Next, we introduce a smooth cut-off function, σ such that

$$\sigma \in C^\infty, \quad 0 \leq \sigma \leq 1, \quad \sigma(z) = \begin{cases} 1, & 0 \leq z \leq h/4, \\ 0, & z \geq h/2. \end{cases} \quad (4.9)$$

The role of this cut-off is to localize the correctors near each of the channel walls.

Using (4.8) and (4.9), we define the first component of the corrector as:

$$\Theta_1(z, t) = \sigma(z) \Theta_{1,L}(z, t) + \sigma(h - z) \Theta_{1,U}(z, t). \quad (4.10)$$

It is easy to see that Θ_1 satisfies

$$\begin{cases} \frac{\partial \Theta_1}{\partial t} - \varepsilon \frac{\partial^2 \Theta_1}{\partial z^2} = -\varepsilon \left\{ 2\sigma'(z) \frac{\partial \Theta_{1,L}}{\partial z} - 2\sigma'(h - z) \frac{\partial \Theta_{1,U}}{\partial z} \right. \\ \quad \left. + \sigma''(z) \Theta_{1,L} + \sigma''(h - z) \Theta_{1,U} \right\} \quad \text{in } \Omega \times (0, T), \\ \Theta_1 = -u_1^0 \quad \text{on } \Gamma \times (0, T), \\ \Theta_1|_{t=0} = 0. \end{cases} \quad (4.11)$$

The right-hand side of (4.11)₁ is an *e.s.t.*, a fact that will be verified later.

Next, we turn to the construction of the second component of the corrector. Let $\Psi[\mathcal{U}, G, g] = \Psi[\mathcal{U}, G, g](\tau, \eta, t)$, $0 < \tau < L$, $\eta > 0$, $t > 0$, periodic in τ , denote the solution of the drift-diffusion equation (A.6) with drift velocity $\mathcal{U} = \mathcal{U}(\eta, t)$, forcing $G = G(\tau, \eta, t)$ and boundary data $g = g(\tau, t)$, at $\eta = 0$. We introduce the lower and upper drift velocity, forcing and boundary data, as follows:

$$\begin{aligned} \mathcal{U}_L &= \mathcal{U}_L(\eta, t) \equiv u_1^0(\eta, t) + \Theta_1(\eta, t) \\ G_L &= G_L(\tau, \eta, t) \equiv -\Theta_1(\eta, t) \partial_\tau u_2^0(\tau, \eta, t) \\ g_L &= g_L(\tau, t) = -u_2^0(\tau, 0, t) \end{aligned} \quad (4.12)$$

and

$$\begin{aligned} \mathcal{U}_U &= \mathcal{U}_U(\eta, t) \equiv u_1^0(h - \eta, t) + \Theta_1(h - \eta, t) \\ G_U &= G_U(\tau, \eta, t) \equiv -\Theta_1(h - \eta, t) \partial_\tau u_2^0(\tau, h - \eta, t) \\ g_U &= g_U(\tau, t) = -u_2^0(\tau, h, t). \end{aligned} \quad (4.13)$$

Set

$$\begin{cases} \Theta_{2,L}(x, z, t) = \Psi[\mathcal{U}_L, G_L, g_L](x, z, t) \\ \Theta_{2,U}(x, z, t) = \Psi[\mathcal{U}_U, G_U, g_U](x, h - z, t), \end{cases} \quad (4.14)$$

and note that $\Theta_{2,L}$ is defined on $x \in (0, L)$, $z > 0$, $t > 0$, while $\Theta_{2,U}$ is defined on $x \in (0, L)$, $z < h$, $t > 0$.

We will use once more the cut-off σ (4.9) to define the second component of our corrector:

$$\Theta_2(x, z, t) = \sigma(z) \Theta_{2,L}(x, z, t) + \sigma(h - z) \Theta_{2,U}(x, z, t). \quad (4.15)$$

It is a simple calculation to verify that Θ_2 satisfies:

$$\left\{ \begin{array}{l} \frac{\partial \Theta_2}{\partial t} - \varepsilon \left(\frac{\partial^2 \Theta_2}{\partial z^2} + \frac{\partial^2 \Theta_2}{\partial x^2} \right) + (u_1^0 + \Theta_1) \frac{\partial \Theta_2}{\partial x} \\ \quad = -\Theta_1 \frac{\partial u_2^0}{\partial x} - \varepsilon \left\{ 2\sigma'(z) \frac{\partial \Theta_{2,L}}{\partial z} - 2\sigma'(h - z) \frac{\partial \Theta_{2,U}}{\partial z} \right. \\ \quad \quad \left. + \sigma''(z) \Theta_{2,L} + \sigma''(h - z) \Theta_{2,U} \right\} \quad \text{in } \Omega \times (0, T), \\ \Theta_2 = -u_2^0 \quad \text{on } \Gamma \times (0, T), \\ \Theta_2|_{t=0} = 0, \end{array} \right. \quad (4.16)$$

where the second term on the right hand side of (4.16) is again an *e.s.t.*.

The needed bounds in L^p on the corrector Θ with an explicit dependence on ε are stated and proved separately in Section 4.2 below.

Following a well-known approach [61, 86], we define an approximation to the viscous solution combining the corrector with the inviscid solution, and estimate the approximation error explicitly in terms of ε in various norms. We then set:

$$\mathbf{v}^\varepsilon = (v_1^\varepsilon(z, t), v_2^\varepsilon(x, z, t), 0) := \mathbf{u}^\varepsilon - \mathbf{u}^0 - \Theta, \quad (4.17)$$

and let ω^ε denote the associated vorticity,

$$\omega^\varepsilon = (\omega_1^\varepsilon(x, z, t), \omega_2^\varepsilon(z, t), \omega_3^\varepsilon(x, z, t)) := \text{curl } \mathbf{v}^\varepsilon = \left(-\frac{\partial v_2^\varepsilon}{\partial z}, \frac{\partial v_1^\varepsilon}{\partial z}, \frac{\partial v_2^\varepsilon}{\partial x} \right). \quad (4.18)$$

With these definitions, we are ready to state and prove our main convergence result for PCF.

Theorem 4.1. *Under the assumptions (4.4), we have:*

$$\begin{aligned} \|\mathbf{v}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}} \|\nabla \mathbf{v}^\varepsilon\|_{L^2(0,T;L^2(\Omega))} &\leq \kappa_T \varepsilon, \\ \|\omega_1^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}} \|\nabla \omega_1^\varepsilon\|_{L^2(0,T;L^2(\Omega))} &\leq \kappa_T \varepsilon^{\frac{1}{4}}, \\ \|\omega_2^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}} \|\nabla \omega_2^\varepsilon\|_{L^2(0,T;L^2(\Omega))} &\leq \kappa_T \varepsilon^{\frac{3}{4}}, \\ \|\omega_3^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}} \|\nabla \omega_3^\varepsilon\|_{L^2(0,T;L^2(\Omega))} &\leq \kappa_T \varepsilon, \\ \|\omega^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}} \|\nabla \omega^\varepsilon\|_{L^2(0,T;L^2(\Omega))} &\leq \kappa_T \varepsilon^{\frac{1}{4}}, \\ \|\mathbf{v}^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} + \varepsilon^{\frac{1}{2}} \|\mathbf{v}^\varepsilon\|_{L^2(0,T;H^2(\Omega))} &\leq \kappa_T \varepsilon^{\frac{1}{4}}. \end{aligned} \quad (4.19)$$

In particular, the vanishing viscosity limit holds with convergence rate:

$$\|\mathbf{u}^\varepsilon - \mathbf{u}^0\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{4}}. \quad (4.20)$$

In addition,

$$\text{curl } \mathbf{u}^\varepsilon \rightarrow \text{curl } \mathbf{u}^0 + (\mathbf{u}_0 \times \mathbf{n})\mu \quad \text{weak}^* \text{ in } L^\infty(0, T; \mathcal{M}(\overline{\Omega})), \quad (4.21)$$

where $\mathcal{M}(\overline{\Omega})$ is the space of Radon measures on $\overline{\Omega}$ and μ is a measure supported on Γ , on which $\mu|_{\Gamma}$ agrees with normalized surface area.

Since the bounds on the corrector in 4.2 allow to estimate its size, by the triangle inequality it follows that the rates of convergence in Theorem 4.1 are optimal in viscosity. We postpone the proof of the theorem until Section 4.3.

4.2. Estimates of the corrector Θ . In this subsection, we derive bounds in Lebesgue and Sobolev spaces for the corrector, $\Theta = (\Theta_1(z, t), \Theta_2(x, z, t), 0)$, using the estimates obtained in Appendix A. These bounds, in turn, will be crucial in establishing the convergence rates of Theorem 4.1.

We begin with bounds for the first component of the corrector Θ_1 , defined in (4.10), which satisfies (4.11). Then, exploiting the pointwise estimates in Lemma A.1, satisfied by $\Theta_{1,L}$ and $\Theta_{1,R}$ with η replaced by z and $h - z$ respectively, gives:

$$(\text{right-hand side of (4.11)}) = e.s.t.. \quad (4.22)$$

Therefore, we can apply the L^p estimates in Lemma A.2 with η replaced by z and $h - z$ respectively, and obtain:

$$\left\| \frac{\partial^m \Theta_1}{\partial z^m} \right\|_{L^p(\Omega)} \leq \kappa_T (1 + t^{\frac{1}{2p} - \frac{m}{2}}) \varepsilon^{\frac{1}{2p} - \frac{m}{2}}, \quad 1 \leq p \leq \infty, \quad 0 \leq m \leq 2, \quad 0 < t < T. \quad (4.23)$$

Similarly, for the second component of the corrector Θ_2 , defined in (4.15), we can employ the estimates in Lemma A.4 on $\Theta_{2,L}$ or $\Theta_{2,R}$, as they both satisfy the parabolic system (A.6) with τ and η replaced by x and z (or $h - z$) in the domain $(0, L) \times (0, \infty)$ (or $(0, L) \times (-\infty, h)$) respectively, and with

$$\begin{cases} \mathcal{U} = \Theta_1 + u_1^0, \\ G = -\Theta_1 \frac{\partial u_2^0}{\partial x}, \\ g = -u_2^0|_{z=0 \text{ (or } h)}, \end{cases}$$

in the notation of the Appendix. Consequently, it holds:

$$\begin{cases} \left\| \frac{\partial^k \Theta_2}{\partial x^k} \right\|_{L^\infty(0,T;L^p(\Omega))} + \varepsilon^{\frac{1}{2p} + \frac{1}{4}} \left\| \nabla \frac{\partial^k \Theta_2}{\partial x^k} \right\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{2p}}, & 1 \leq p \leq \infty, \\ \left\| \frac{\partial^{k+1} \Theta_2}{\partial x^k \partial z} \right\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}} \left\| \nabla \frac{\partial^{k+1} \Theta_2}{\partial x^k \partial z} \right\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{-\frac{1}{4}}, \\ \left\| \frac{\partial^{k+1} \Theta_2}{\partial x^k \partial z} \right\|_{L^\infty(0,T;L^1(\Omega))} \leq \kappa_T, \end{cases} \quad (4.24)$$

with $k \geq 0$. Above, κ_T is a constant depending on T and the other data, but not on ε . Note that the continuity of L^p norm in a bounded interval is used for the L^1 bound of $\partial^k \Theta_2 / \partial x^k$. Moreover, one can verify that

$$(\text{the second term on the right hand side of (4.16)}) = e.s.t., \quad (4.25)$$

by performing an energy estimate on Ψ (or its derivative) in (A.6) multiplied by any derivative of the cut-off function in (4.9). We omit the proof of this last estimate for the sake of exposition.

In particular, it follows from (4.23) and (4.24) that

$$\|\operatorname{curl} \Theta\|_{L^\infty(0,T;L^1(\Omega))} \leq \kappa_T, \quad (4.26)$$

owing to the particular form of the curl of a vector field under plane-parallel symmetry (cf. (4.18)).

4.3. Proof of Theorem 4.1. In this subsection, we sketch the proof of Theorem 4.1, using the estimates for the corrector.

To prove (4.19)₁, we will perform energy estimates on the approximation remainder \mathbf{v}^ε . Throughout, we will use standard inequalities (e.g. Hölder, Cauchy-Schwarz, Young) and the fact that \mathbf{u}^0 is a strong solution of the Euler equation under the assumptions (4.4) on the data.

We first derive the initial-boundary problem for \mathbf{v}^ε from the EE and the corresponding equations for the corrector:

$$\left\{ \begin{array}{l} \frac{\partial v_1^\varepsilon}{\partial t} - \varepsilon \frac{\partial^2 v_1^\varepsilon}{\partial z^2} = \varepsilon \frac{\partial^2 u_1^0}{\partial z^2} + e.s.t., \quad \text{in } \Omega \times (0, T), \\ \frac{\partial v_2^\varepsilon}{\partial t} - \varepsilon \Delta v_2^\varepsilon + u_1^\varepsilon \frac{\partial v_2^\varepsilon}{\partial x} + v_1^\varepsilon \frac{\partial \Theta_2}{\partial x} + v_1^\varepsilon \frac{\partial u_2^0}{\partial x} = \varepsilon \Delta u_2^0 + e.s.t., \quad \text{in } \Omega \times (0, T), \\ \mathbf{v}^\varepsilon = 0, \quad \text{on } \Gamma, \\ \mathbf{v}^\varepsilon|_{t=0} = 0. \end{array} \right. \quad (4.27)$$

We multiply (4.27)₁ by v_1^ε and integrate over Ω to establish the following estimate:

$$\frac{1}{2} \frac{d}{dt} \|v_1^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \left\| \frac{\partial v_1^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \leq \kappa \varepsilon^2 + \kappa \|v_1^\varepsilon\|_{L^2(\Omega)}^2.$$

Thanks to Grönwall's inequality, we then obtain:

$$\|v_1^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}} \left\| \frac{\partial v_1^\varepsilon}{\partial z} \right\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon. \quad (4.28)$$

For the second component of \mathbf{v}^ε , v_2^ε , we multiply (4.27)₂ by v_2^ε and integrate over Ω . Then, using (4.24), the regularity of \mathbf{u}^0 , and (4.28) as well, we find that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v_2^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla v_2^\varepsilon\|_{L^2(\Omega)}^2 &\leq \kappa \varepsilon^2 + \kappa \|v_2^\varepsilon\|_{L^2(\Omega)}^2 + \left\| v_1^\varepsilon \left(\frac{\partial \Theta_2}{\partial x} + \frac{\partial u_2^0}{\partial x} \right) v_2^\varepsilon \right\|_{L^1(\Omega)} \\ &\leq \kappa \varepsilon^2 + \kappa \|v_2^\varepsilon\|_{L^2(\Omega)}^2 + \|v_1^\varepsilon\|_{L^2(\Omega)} \left\| \frac{\partial \Theta_2}{\partial x} + \frac{\partial u_2^0}{\partial x} \right\|_{L^\infty(\Omega)} \|v_2^\varepsilon\|_{L^2(\Omega)} \\ &\leq \kappa_T \varepsilon^2 + \kappa_T \|v_2^\varepsilon\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.29)$$

Above, we have used that, after integrating by parts, the third term on the left-hand side of (4.27)₂ integrates to zero, because u_1^ε does not depend on the variable x .

Using Grönwall's inequality again, we obtain from (4.29) that

$$\|v_2^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}} \|\nabla v_2^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon. \quad (4.30)$$

By combining the estimates above (4.19)₁ follows. Given the bounds on the corrector, (4.20) is a direct consequence of (4.19)₁.

To establish error bounds on the vorticity, (4.19)_{2,3,4}, we derived an initial-boundary value problem for ω^ε and use energy estimates. In doing so, we are confronted with the

problem of deriving a usable boundary condition for the vorticity. We follow here the approach of Lighthill. First, using the expression for $\boldsymbol{\omega}^\varepsilon$ given in (4.18), we derive the equations satisfied by $\boldsymbol{\omega}^\varepsilon$ from those for \mathbf{v}^ε , by differentiating (4.27)₂ in x and z , and (4.27)₁ in z :

$$\left\{ \begin{array}{l} \frac{\partial \omega_1^\varepsilon}{\partial t} - \varepsilon \Delta \omega_1^\varepsilon + u_1^\varepsilon \frac{\partial \omega_1^\varepsilon}{\partial x} \\ \quad = -\varepsilon \frac{\partial(\Delta u_2^0)}{\partial z} + \omega_2^\varepsilon \frac{\partial(u_2^0 + \Theta_2)}{\partial x} + v_1^\varepsilon \frac{\partial^2(u_2^0 + \Theta_2)}{\partial x \partial z} + \frac{\partial u_1^\varepsilon}{\partial z} \omega_3^\varepsilon + e.s.t. \quad \text{in } \Omega \times (0, T), \\ \frac{\partial \omega_2^\varepsilon}{\partial t} - \varepsilon \frac{\partial^2 \omega_2^\varepsilon}{\partial z^2} = \varepsilon \frac{\partial^3 u_1^0}{\partial z^3} + e.s.t. \quad \text{in } \Omega \times (0, T), \\ \frac{\partial \omega_3^\varepsilon}{\partial t} - \varepsilon \Delta \omega_3^\varepsilon + u_1^\varepsilon \frac{\partial \omega_3^\varepsilon}{\partial x} = \varepsilon \frac{\partial(\Delta u_2^0)}{\partial x} - v_1^\varepsilon \frac{\partial^2(u_2^0 + \Theta_2)}{\partial x^2} + e.s.t. \quad \text{in } \Omega \times (0, T). \end{array} \right. \quad (4.31)$$

Next, by restricting (4.27)_{1,2} on Γ , and using (4.27)_{3,4}, we find the boundary and initial conditions for $\boldsymbol{\omega}^\varepsilon$:

$$\left\{ \begin{array}{l} \frac{\partial \omega_1^\varepsilon}{\partial z} = \Delta u_2^0, \quad \text{on } \Gamma, \\ \frac{\partial \omega_2^\varepsilon}{\partial z} = -\frac{\partial^2 u_1^0}{\partial z^2}, \quad \text{on } \Gamma, \\ \omega_3^\varepsilon = 0, \quad \text{on } \Gamma, \\ \boldsymbol{\omega}^\varepsilon|_{t=0} = 0. \end{array} \right. \quad (4.32)$$

Above, we have used the fact that the terms of order *e.s.t.* in (4.27) vanishes on Γ (the explicit expression of these terms is given in (4.11) and (4.16)).

A standard energy estimate, using that u_1^ε is independent of x , the regularity and decay of the corrector Θ , the regularity of the EE solution \mathbf{u}^0 , and the bounds on \mathbf{v}^ε , gives that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega_3^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla \omega_3^\varepsilon\|_{L^2(\Omega)}^2 &\leq \kappa \varepsilon^2 + \kappa \|\omega_3^\varepsilon\|_{L^2(\Omega)}^2 + \left\| v_1^\varepsilon \left(\frac{\partial^2 \Theta_2}{\partial x^2} + \frac{\partial^2 u_2^0}{\partial x^2} \right) \omega_3^\varepsilon \right\|_{L^1(\Omega)} \\ &\leq \kappa \varepsilon^2 + \kappa \|\omega_3^\varepsilon\|_{L^2(\Omega)}^2 + \|v_1^\varepsilon\|_{L^2(\Omega)} \left\| \frac{\partial \Theta_2}{\partial x} + \frac{\partial u_2^0}{\partial x} \right\|_{L^\infty(\Omega)} \|\omega_3^\varepsilon\|_{L^2(\Omega)} \\ &\leq \kappa_T \varepsilon^2 + \kappa_T \|\omega_3^\varepsilon\|_{L^2(\Omega)}^2. \end{aligned}$$

Estimate (4.19)₄ now follows from (4.19)₁ by applying Grönwall's inequality again.

We proceed similarly for ω_2^ε and obtain:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega_2^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \left\| \frac{\partial \omega_2^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 &\leq \kappa \varepsilon^2 + \kappa \|\omega_2^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \left\| \frac{\partial^2 u_1^0}{\partial z^2} \omega_2^\varepsilon \right\|_{L^1(\Gamma)} \\ &\leq \kappa \varepsilon^2 + \kappa \|\omega_2^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \left\| \frac{\partial^2 u_1^0}{\partial z^2} \right\|_{L^2(\Gamma)} \|\omega_2^\varepsilon\|_{L^2(\Gamma)}. \end{aligned} \quad (4.33)$$

To apply Grönwall's inequality one more time, we need to estimate the third term on the right-hand side of (4.33). This estimate follows in turn from the regularity of \mathbf{u}^0 ,

and the trace theorem, noticing ω_2^ε depends on z alone:

$$\begin{aligned}
\varepsilon \left\| \frac{\partial^2 u_1^0}{\partial z^2} \right\|_{L^2(\Gamma)} \|\omega_2^\varepsilon\|_{L^2(\Gamma)} &\leq \kappa \varepsilon \|\omega_2^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\omega_2^\varepsilon\|_{H^1(\Omega)}^{\frac{1}{2}} \\
&\leq \kappa \varepsilon \|\omega_2^\varepsilon\|_{L^2(\Omega)} + \kappa \varepsilon \|\omega_2^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \left\| \frac{\partial \omega_2^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^{\frac{1}{2}} \\
&\leq \kappa \varepsilon^2 + \kappa \|\omega_2^\varepsilon\|_{L^2(\Omega)}^2 + \kappa \varepsilon^{\frac{3}{4}} \|\omega_2^\varepsilon\|_{L^2(\Omega)} + \kappa \varepsilon^{\frac{5}{4}} \left\| \frac{\partial \omega_2^\varepsilon}{\partial z} \right\|_{L^2(\Omega)} \\
&\leq \kappa \varepsilon^{\frac{3}{2}} + \kappa \|\omega_2^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{2} \varepsilon \left\| \frac{\partial \omega_2^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.34}$$

Combining (4.33) and (4.34), we obtain

$$\frac{d}{dt} \|\omega_2^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \left\| \frac{\partial \omega_2^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \leq \kappa \varepsilon^{\frac{3}{2}} + \kappa \|\omega_2^\varepsilon\|_{L^2(\Omega)}^2. \tag{4.35}$$

By Grönwall's inequality again, (4.19)₃ then follows from (4.35).

Again, similarly an energy estimate gives an *a priori* bound for ω_1^ε :

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\omega_1^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla \omega_1^\varepsilon\|_{L^2(\Omega)}^2 &\leq \kappa \varepsilon^2 + \kappa \|\omega_1^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|\Delta u_2^0\|_{L^2(\Gamma)} \|\omega_1^\varepsilon\|_{L^2(\Gamma)} \\
&\quad + \left\| \omega_2^\varepsilon \frac{\partial(u_2^0 + \Theta_2)}{\partial x} \omega_1^\varepsilon \right\|_{L^1(\Omega)} + \left| \int_{\Omega} v_1^\varepsilon \frac{\partial^2(u_2^0 + \Theta_2)}{\partial x \partial z} \omega_1^\varepsilon \right| \\
&\quad + \left\| \frac{\partial u_1^\varepsilon}{\partial z} \omega_3^\varepsilon \omega_1^\varepsilon \right\|_{L^1(\Omega)}.
\end{aligned} \tag{4.36}$$

We need to bound the last three terms on the right-hand side of the above expression.

We can estimate the third term on the right-hand side, by using again the regularity of \mathbf{u}^0 , and the trace theorem:

$$\begin{aligned}
\varepsilon \|\Delta u_2^0\|_{L^2(\Gamma)} \|\omega_1^\varepsilon\|_{L^2(\Gamma)} &\leq \kappa \varepsilon \|\omega_1^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\omega_1^\varepsilon\|_{H^1(\Omega)}^{\frac{1}{2}} \\
&\leq \kappa \varepsilon \|\omega_1^\varepsilon\|_{L^2(\Omega)} + \kappa \varepsilon \|\omega_1^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \omega_1^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \\
&\leq \kappa \varepsilon^2 + \kappa \|\omega_1^\varepsilon\|_{L^2(\Omega)}^2 + \kappa \varepsilon^{\frac{3}{4}} \|\omega_1^\varepsilon\|_{L^2(\Omega)} + \kappa \varepsilon^{\frac{5}{4}} \|\nabla \omega_1^\varepsilon\|_{L^2(\Omega)} \\
&\leq \kappa \varepsilon^{\frac{3}{2}} + \kappa \|\omega_1^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{4} \varepsilon \|\nabla \omega_1^\varepsilon\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.37}$$

Using the regularity of \mathbf{u}^0 , and the bounds for the corrector and ω_2^ε , we estimate the fourth term on the right-hand side of (4.36) as follows:

$$\begin{aligned}
\left\| \omega_2^\varepsilon \frac{\partial(u_2^0 + \Theta_2)}{\partial x} \omega_1^\varepsilon \right\|_{L^1(\Omega)} &\leq \|\omega_2^\varepsilon\|_{L^2(\Omega)} \left\| \frac{\partial(u_2^0 + \Theta_2)}{\partial x} \right\|_{L^\infty(\Omega)} \|\omega_1^\varepsilon\|_{L^2(\Omega)} \\
&\leq \kappa_T \varepsilon^{\frac{3}{4}} \|\omega_1^\varepsilon\|_{L^2(\Omega)} \\
&\leq \kappa_T \varepsilon^{\frac{3}{2}} + \kappa_T \|\omega_1^\varepsilon\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.38}$$

Next, we tackle the fifth term on the right-hand side of (4.36). Since $v_1^\varepsilon = 0$ on Γ , by integrating by parts we can write

$$\left| \int_{\Omega} v_1^\varepsilon \frac{\partial^2(u_2^0 + \Theta_2)}{\partial x \partial z} \omega_1^\varepsilon d\mathbf{x} \right| \leq \left\| \frac{\partial v_1^\varepsilon}{\partial z} \frac{\partial(u_2^0 + \Theta_2)}{\partial x} \omega_1^\varepsilon \right\|_{L^1(\Omega)} + \left\| v_1^\varepsilon \frac{\partial(u_2^0 + \Theta_2)}{\partial x} \frac{\partial \omega_1^\varepsilon}{\partial z} \right\|_{L^1(\Omega)}. \quad (4.39)$$

Then, we can proceed as for the fourth term, and obtain:

$$\begin{aligned} & \left| \int_{\Omega} v_1^\varepsilon \frac{\partial^2(u_2^0 + \Theta_2)}{\partial x \partial z} \omega_1^\varepsilon d\mathbf{x} \right| \\ & \leq \left\| \frac{\partial v_1^\varepsilon}{\partial z} \right\|_{L^2(\Omega)} \left\| \frac{\partial(u_2^0 + \Theta_2)}{\partial x} \right\|_{L^\infty(\Omega)} \|\omega_1^\varepsilon\|_{L^2(\Omega)} + \|v_1^\varepsilon\|_{L^2(\Omega)} \left\| \frac{\partial(u_2^0 + \Theta_2)}{\partial x} \right\|_{L^\infty(\Omega)} \left\| \frac{\partial \omega_1^\varepsilon}{\partial z} \right\|_{L^2(\Omega)} \\ & \leq \kappa \left\| \frac{\partial v_1^\varepsilon}{\partial z} \right\|_{L^2(\Omega)} \|\omega_1^\varepsilon\|_{L^2(\Omega)} + \kappa_T \varepsilon \left\| \frac{\partial \omega_1^\varepsilon}{\partial z} \right\|_{L^2(\Omega)} \\ & \leq \kappa \left\| \frac{\partial v_1^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 + \kappa \|\omega_1^\varepsilon\|_{L^2(\Omega)}^2 + \kappa_T \varepsilon + \frac{1}{4} \varepsilon \|\nabla \omega_1^\varepsilon\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.40)$$

Finally, we estimate the last term on the right-hand side of (4.36) as follows, once again utilizing the bounds for ω_3^ε and for the corrector Θ_1 :

$$\begin{aligned} \left\| \frac{\partial u_1^\varepsilon}{\partial z} \omega_3^\varepsilon \omega_1^\varepsilon \right\|_{L^1(\Omega)} & \leq \|\omega_2^\varepsilon \omega_3^\varepsilon \omega_1^\varepsilon\|_{L^1(\Omega)} + \left\| \frac{\partial u_1^0}{\partial z} \omega_3^\varepsilon \omega_1^\varepsilon \right\|_{L^1(\Omega)} + \left\| \frac{\partial \Theta_1}{\partial z} \omega_3^\varepsilon \omega_1^\varepsilon \right\|_{L^1(\Omega)} \\ & \leq \|\omega_2^\varepsilon\|_{L^\infty(\Omega)} \|\omega_3^\varepsilon\|_{L^2(\Omega)} \|\omega_1^\varepsilon\|_{L^2(\Omega)} + \left\| \frac{\partial u_1^0}{\partial z} \right\|_{L^\infty(\Omega)} \|\omega_3^\varepsilon\|_{L^2(\Omega)} \|\omega_1^\varepsilon\|_{L^2(\Omega)} \\ & \quad + \left\| \frac{\partial \Theta_1}{\partial z} \right\|_{L^\infty(\Omega)} \|\omega_3^\varepsilon\|_{L^2(\Omega)} \|\omega_1^\varepsilon\|_{L^2(\Omega)} \\ & \leq \kappa_T (\varepsilon \|\omega_2^\varepsilon\|_{L^\infty(\Omega)} + \varepsilon + (1 + t^{-\frac{1}{2}}) \varepsilon^{\frac{1}{2}}) \|\omega_1^\varepsilon\|_{L^2(\Omega)} \\ & \leq \kappa_T \varepsilon \|\omega_2^\varepsilon\|_{L^\infty(\Omega)}^2 + \kappa_T (1 + t^{-\frac{1}{2}}) \varepsilon + \kappa_T (1 + t^{-\frac{1}{2}}) \|\omega_1^\varepsilon\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.41)$$

Combining all the estimates above into (4.36), we deduce that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\omega_1^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla \omega_1^\varepsilon\|_{L^2(\Omega)}^2 \\ & \leq \kappa_T \varepsilon (1 + t^{-\frac{1}{2}} + \|\omega_2^\varepsilon\|_{L^\infty(\Omega)}^2) + \kappa \left\| \frac{\partial v_1^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 + \kappa_T (1 + t^{-\frac{1}{2}}) \|\omega_1^\varepsilon\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.42)$$

We need to estimate the L^∞ norm of ω_2^ε . This estimate, in turn, follows by applying the one-dimensional Agmon's inequality and the L^2 bounds on ω_2^ε , observing that ω_2^ε is a function of z only, which represents the distance to the boundary:

$$\int_0^T \|\omega_2^\varepsilon\|_{L^\infty(\Omega)}^2 dt \leq \kappa \int_0^T \|\omega_2^\varepsilon\|_{L^2(\Omega)} \left\| \frac{\partial \omega_2^\varepsilon}{\partial z} \right\|_{L^2(\Omega)} dt \leq \kappa_T \varepsilon. \quad (4.43)$$

Estimate (4.19)₂ now follows by an application of Grönwall's inequality one more time, with the integrating factor $\exp(-\kappa_T t - 2\kappa_T t^{1/2})$.

Since the approximate NS solution \mathbf{v}^ε vanishes on the boundary of Ω , Γ , we have that:

$$\|\mathbf{u}^\varepsilon\|_{H^k} \leq C_k \|\boldsymbol{\omega}^\varepsilon\|_{H^{k-1}}, \quad k \geq 0.$$

Therefore, estimate (4.19)₆ on the velocity follows from the corresponding bounds on the vorticity.

Finally, (4.19)₅ and (4.26) imply that $\|\operatorname{curl}(\mathbf{u}^\varepsilon - \mathbf{u}^0)\|_{L^\infty(0,T;L^1(\Omega))} \leq \kappa_T$. Weak convergence of the vorticity in (4.21) can then be established using Corollary C.2 in the Appendix.

□

5. PARALLEL PIPE FLOWS

In this section, we turn to the most complex class of symmetric flows, the so-called *parallel pipe flows* in an infinite, straight pipe.

Since our focus is on the behavior of the flow near walls, we consider the case of a pipe with annular cross section:

$$\Omega := \{(x, y, z) \in \mathbb{R}^3 \mid r_L < y^2 + z^2 < r_R\}, \quad 0 < r_L < r_R.$$

The case of a pipe with circular cross section is technically more difficult, because the estimates in cylindrical coordinates are weaker at the pipe axis. It can be treated, as in [45], by a two-step localization, near the boundary utilizing cylindrical coordinates, near the axis using Cartesian coordinates, or by applying a suitable form of Hardy's inequality, but we will not address this case here. We also periodize the channel in the direction of its axis, and denote the period by L . As in the case of the channel geometry, together with symmetry, periodicity ensures uniqueness for the solutions of the fluid equations, in particular by ensuring that the only steady pressure-driven flow is the trivial flow (see [71] for a more detailed discussion of this point).

We introduce cylindrical coordinates, $\boldsymbol{\xi} = (\phi, x, r)$, with associated orthonormal frame $\{\mathbf{e}_\phi, \mathbf{e}_x, \mathbf{e}_r\}$, so that the domain Ω corresponds in these coordinates to the parallelepiped

$$\tilde{\Omega} = \{\boldsymbol{\xi} = (\phi, x, r) \mid 0 \leq \phi < 2\pi, 0 < x < L, r_L \leq r < r_R\},$$

hence the notation of r_L and r_R for the inner and outer radii of the pipe. We employ this notation to emphasize comparison with the channel case.

We will call any three-dimensional flow a *parallel pipe flow* (PPF) if the velocity field is a vector field of the form:

$$\mathbf{F} = F_\phi(r)\mathbf{e}_\phi + F_x(\phi, r)\mathbf{e}_x. \quad (5.1)$$

In the special case of a planar PPF, i.e., when $F_x \equiv 0$, the flow reduces to a linear, 2D circularly symmetric flow in any cross-sections of the pipe.

Standard calculus equalities (see e.g [9]) imply that the gradient of each component of \mathbf{F} , F_ϕ or F_x , can be written as:

$$\nabla F_\phi = \frac{\partial F_\phi}{\partial r}\mathbf{e}_r, \quad \nabla F_x = \frac{1}{r}\frac{\partial F_x}{\partial \phi}\mathbf{e}_\phi + \frac{\partial F_x}{\partial r}\mathbf{e}_r. \quad (5.2)$$

The derivative of \mathbf{F} in the direction of the vector $\mathbf{n} = n_\phi\mathbf{e}_\phi + n_x\mathbf{e}_x + n_r\mathbf{e}_r$ is then given by:

$$\mathbf{n} \cdot \nabla \mathbf{F} = (\mathbf{n} \cdot \nabla F_\phi)\mathbf{e}_\phi + (\mathbf{n} \cdot \nabla F_x)\mathbf{e}_x + \left(-\frac{1}{r}n_\phi F_\phi\right)\mathbf{e}_r. \quad (5.3)$$

It is easy to verify that the divergence of \mathbf{F} is zero:

$$\operatorname{div} \mathbf{F} = 0, \quad (5.4)$$

and that the curl of \mathbf{F} takes the form:

$$\operatorname{curl} \mathbf{F} = \left(-\frac{\partial F_x}{\partial r} \right) \mathbf{e}_\phi + \left(\frac{1}{r} \frac{\partial(rF_\phi)}{\partial r} \right) \mathbf{e}_x + \left(\frac{1}{r} \frac{\partial F_x}{\partial \phi} \right) \mathbf{e}_r. \quad (5.5)$$

It is also straightforward, though tedious, to verify that the (vector) Laplacian of \mathbf{F} can be written as:

$$\Delta \mathbf{F} = \left(\Delta F_\phi - \frac{1}{r^2} F_\phi \right) \mathbf{e}_\phi + (\Delta F_x) \mathbf{e}_x, \quad (5.6)$$

where Laplacian of each component, F_ϕ or F_x , is given by:

$$\Delta F_\phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial F_\phi}{\partial r} \right), \quad \Delta F_x = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial F_x}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 F_x}{\partial \phi^2}. \quad (5.7)$$

These formulas will be used in deriving the equations satisfied by the NS and EE velocity fields and vorticities.

As in [45, 71], we will consider solutions of NSE and EE under this symmetry; that is, we assume the following form for the velocities, \mathbf{u}^ε and \mathbf{u}^0 , and for the associated pressures, p^ε and p^0 , respectively:

$$\mathbf{u}^\varepsilon = u_\phi^\varepsilon(r, t) \mathbf{e}_\phi + u_x^\varepsilon(\phi, r, t) \mathbf{e}_x, \quad p^\varepsilon = p^\varepsilon(r, t), \quad (5.8a)$$

$$\mathbf{u}^0 = u_\phi^0(r, t) \mathbf{e}_\phi + u_x^0(\phi, r, t) \mathbf{e}_x, \quad p^0 = p^0(r, t). \quad (5.8b)$$

We remark that any PPF is automatically incompressible, and that the components of its velocity field in the cross-section of the pipe have circular symmetry, and evolve independently from the third.

It can be shown (see [45, 71] for a proof), that the parallel pipe symmetry is preserved under both the NSE and EE evolution, provided that the data, the forcing \mathbf{f} and initial velocity \mathbf{u}^0 , are regular enough and have the same symmetry, i. e.,

$$\mathbf{f} = f_\phi(r, t) \mathbf{e}_\phi + f_x(\phi, r, t) \mathbf{e}_x, \quad \mathbf{u}_0 = u_{0,\phi}(r) \mathbf{e}_\phi + u_{0,x}(\phi, r) \mathbf{e}_x.$$

Then, the NSE (1.1) become the weakly coupled system:

$$\left\{ \begin{array}{l} \frac{\partial u_\phi^\varepsilon}{\partial t} - \varepsilon \Delta u_\phi^\varepsilon + \varepsilon \frac{1}{r^2} u_\phi^\varepsilon = f_\phi, \quad \text{in } \Omega \times (0, T), \\ \frac{\partial u_x^\varepsilon}{\partial t} - \varepsilon \Delta u_x^\varepsilon + \frac{1}{r} u_\phi^\varepsilon \frac{\partial u_x^\varepsilon}{\partial \phi} = f_x, \quad \text{in } \Omega \times (0, T), \\ -\frac{1}{r} (u_\phi^\varepsilon)^2 + \frac{\partial p^\varepsilon}{\partial r} = 0, \quad \text{in } \Omega \times (0, T), \\ u_x^\varepsilon \text{ is periodic in } \phi \text{ with period } 2\pi, \\ u_i^\varepsilon = 0, \quad i = \phi, x, \quad \text{on } \Gamma, \text{ i.e., at } r = r_L, r_R, \\ u_i^\varepsilon|_{t=0} = u_{0,i}, \quad i = \phi, x, \quad \text{in } \Omega, \end{array} \right. \quad (5.9)$$

and, similarly, the EE (1.2) become the weakly coupled system:

$$\left\{ \begin{array}{l} \frac{\partial u_\phi^0}{\partial t} = f_\phi, \quad \text{in } \Omega \times (0, T), \\ \frac{\partial u_x^0}{\partial t} + \frac{1}{r} u_\phi^0 \frac{\partial u_x^0}{\partial \phi} = f_x, \quad \text{in } \Omega \times (0, T), \\ -\frac{1}{r} (u_\phi^0)^2 + \frac{\partial p^0}{\partial r} = 0, \quad \text{in } \Omega \times (0, T), \\ u_x^0 \text{ is periodic in } \phi \text{ with period } 2\pi, \\ u_i^0|_{t=0} = u_{0,i}, \quad i = \phi, x, \quad \text{in } \Omega. \end{array} \right. \quad (5.10)$$

Differently than for parallel channel flows, the pressure cannot be taken constant here, but the pressure can be recovered up to a constant from knowledge of u^ϕ .

As in the case of a channel, we assume that the data, \mathbf{u}_0 and \mathbf{f} , are sufficiently regular, but ill-prepared in the sense that

$$\mathbf{u}_0 \in H \cap H^k(\Omega), \quad \mathbf{f} \in C(0, T; H \cap H^k(\Omega)), \quad \text{for a sufficiently large } k \geq 0, \quad (5.11)$$

but \mathbf{u}_0 is not assumed to vanish at the boundary nor \mathbf{f} is assumed necessarily compatible with \mathbf{u}_0 at $t = 0$. Again, this choice leads to the formation of an initial layer for NSE. Since the cross-sectional component of the inviscid solution is time independent, this initial layer affects the zero-viscosity limit, in particular vorticity production at the boundary in the limit.

Under the assumption (5.11) one can verify that there exists a unique global strong solution, which is also classical for positive time. Similarly, the limit problem (5.10) possesses a unique global, strong solution.

Since the tangential component of \mathbf{u}^0 need not vanish on Γ , boundary layers are expected to form on both components of the boundary, the inner and outer cylinders of the pipe.

In what follows, to highlight the effect of the curvature, we will leave the explicit dependence on r in many expressions, even though r will be a bounded function, bounded away from zero. Constants may depend on the geometry of the pipe through the pipe inner and outer radii, r_L, r_R .

5.1. Viscous approximation and convergence result. We will make the following ansatz for the approximate NS solution \mathbf{u}^ε :

$$\mathbf{u}^\varepsilon \cong \mathbf{u}^0 + \Theta, \quad (5.12)$$

where Θ is a corrector to the inviscid solution \mathbf{u}^0 . As for the channel case, this corrector depends on ε , but for ease of notation, we do not explicitly show it.

We assume that the corrector satisfies the same symmetry as the NSE and EE solutions, that is:

$$\Theta = \Theta_\phi(r, t)\mathbf{e}_\phi + \Theta_x(\phi, r, t)\mathbf{e}_x. \quad (5.13)$$

Effective equations for the corrector lead to the *weakly coupled parabolic system* below [45]:

$$\left\{ \begin{array}{l} \frac{\partial \Theta_\phi}{\partial t} - \varepsilon \frac{\partial^2 \Theta_\phi}{\partial r^2} = 0 \quad \text{in } \Omega \times (0, T), \\ \frac{\partial \Theta_x}{\partial t} - \varepsilon \frac{\partial^2 \Theta_x}{\partial r^2} + \Theta_\phi \frac{\partial \Theta_x}{\partial \phi} + u_\phi^0|_\Gamma \frac{\partial \Theta_x}{\partial \phi} + \frac{\partial u_x^0}{\partial \phi}|_\Gamma \Theta_\phi = 0 \quad \text{in } \Omega \times (0, T), \\ \Theta_i = -u_i^0 \quad \text{on } \Gamma, \quad i = \phi, x, \\ \Theta|_{t=0} = 0. \end{array} \right. \quad (5.14)$$

Again, for well-prepared data, these equations allow directly to establish good regularity and decay properties for the corrector, which in turn allow to study the zero-viscosity limit.

To handle the case of ill-prepared initial data, we follow the methodology introduced in Section 4. We first define Θ_ϕ as an explicit approximate solution of the 1D heat equation (5.14)₁ on a segment with the proposed boundary and initial conditions. Then, we define Θ_x as a solution of a drift-diffusion equation similar to (5.14)₂ with the proper boundary and initial conditions. The analysis proceeds parallel to that of channel flows. However, the curvature of the boundary has an effect here on the estimates for the approximate viscous solution.

As in the case of the channel, the corrector Θ_ϕ will be constructed from the exact solutions of the following one-dimensional heat equations on a half line, with boundary and initial data as in (5.14):

$$\left\{ \begin{array}{l} \Theta_{\phi,L}(r, t) = -2 u_\phi^0(r_L, 0) \operatorname{erfc} \left(\frac{r - r_L}{\sqrt{2\varepsilon t}} \right) - 2 \int_0^t \frac{\partial u_\phi^0}{\partial t}(r_L, s) \operatorname{erfc} \left(\frac{r - r_L}{\sqrt{2\varepsilon(t-s)}} \right) ds, \\ \Theta_{\phi,R}(r, t) = -2 u_\phi^0(r_R, 0) \operatorname{erfc} \left(\frac{r_R - r}{\sqrt{2\varepsilon t}} \right) - 2 \int_0^t \frac{\partial u_\phi^0}{\partial t}(r_R, s) \operatorname{erfc} \left(\frac{r_R - r}{\sqrt{2\varepsilon(t-s)}} \right) ds. \end{array} \right. \quad (5.15)$$

Again, these heat solutions satisfy the pointwise and L^p estimates for Φ in Lemmas A.1 and A.2 in the Appendix, with η replaced respectively by $r - r_L$ and $r_R - r$.

As before, we introduce a smooth cut-off function so as to localize the correctors near the inner and outer boundaries $r = r_L, r_R$, in order to enforce the boundary conditions exactly. Let

$$\sigma(r) = \begin{cases} 1, & r_L \leq r \leq r_L + a_0, \\ 0, & r \geq r_L + 2a_0, \end{cases} \quad (5.16)$$

for a fixed $0 < a_0 \ll r_R - r_L$.

Finally, we define Θ_ϕ as follows:

$$\Theta_\phi(r, t) = \sigma(r) \Theta_{\phi,L}(r, t) + \sigma(r_R - r) \Theta_{\phi,R}(r, t). \quad (5.17)$$

The equations for the corrector become then:

$$\left\{ \begin{array}{l} \frac{\partial \Theta_\phi}{\partial t} - \varepsilon \frac{\partial^2 \Theta_\phi}{\partial r^2} = -\varepsilon \left\{ 2\sigma'(r) \frac{\partial \Theta_{\phi,L}}{\partial r} + \sigma''(r) \Theta_{\phi,L} - 2\sigma'(r_R - r) \frac{\partial \Theta_{\phi,R}}{\partial r} + \sigma''(r_R - r) \Theta_{\phi,R} \right\} \quad \text{in } \Omega, \\ \Theta_\phi = -u_\phi^0 \quad \text{on } \Gamma, \\ \Theta_\phi|_{t=0} = 0. \end{array} \right. \quad (5.18)$$

The right-hand side of (5.18)₁ is an *e.s.t.* when considered as a term in (5.28) later in the section.

Having Θ_ϕ at hand, we can define the last component of the corrector Θ_x as follows. We first utilize the solutions, $\Theta_{x,L}$ and $\Theta_{x,R}$, to the following parabolic systems in periodic half spaces:

$$\left\{ \begin{array}{l} \frac{\partial \Theta_{x,L}}{\partial t} - \varepsilon \frac{\partial^2 \Theta_{x,L}}{\partial r^2} - \varepsilon \frac{\partial^2 \Theta_{x,L}}{\partial \phi^2} + \frac{1}{r} (\Theta_\phi + u_\phi^0) \frac{\partial \Theta_{x,L}}{\partial \phi} = -\frac{1}{r} \Theta_\phi \frac{\partial u_x^0}{\partial \phi}, \quad (0, 2\pi) \times (r_L, \infty), \\ \Theta_{x,L} \text{ is periodic in } \phi, \\ \Theta_{x,L} = -u_x^0, \quad r = r_L, \\ \Theta_{x,L} \rightarrow 0 \quad \text{as } r \rightarrow \infty, \\ \Theta_{x,L}|_{t=0} = 0, \end{array} \right. \quad (5.19)$$

and

$$\left\{ \begin{array}{l} \frac{\partial \Theta_{x,R}}{\partial t} - \varepsilon \frac{\partial^2 \Theta_{x,R}}{\partial r^2} - \varepsilon \frac{\partial^2 \Theta_{x,R}}{\partial \phi^2} + \frac{1}{r} (\Theta_\phi + u_\phi^0) \frac{\partial \Theta_{x,R}}{\partial \phi} = -\frac{1}{r} \Theta_\phi \frac{\partial u_x^0}{\partial \phi}, \quad (0, 2\pi) \times (-\infty, r_R), \\ \Theta_{x,R} \text{ is periodic in } \phi, \\ \Theta_{x,R} = -u_x^0, \quad r = r_R, \\ \Theta_{x,R} \rightarrow 0 \quad \text{as } r \rightarrow -\infty, \\ \Theta_{x,R}|_{t=0} = 0, \end{array} \right. \quad (5.20)$$

Then, using the cut-off function in (5.16), we define the second component of the corrector in the form,

$$\Theta_x(\phi, r, t) = \sigma(r) \Theta_{x,L}(\phi, r, t) + \sigma(r_R - r) \Theta_{x,R}(\phi, r, t), \quad (5.21)$$

which then satisfies:

$$\left\{ \begin{array}{l} \frac{\partial \Theta_x}{\partial t} - \varepsilon \frac{\partial^2 \Theta_x}{\partial r^2} - \varepsilon \frac{\partial^2 \Theta_x}{\partial \phi^2} + \frac{1}{r} (\Theta_\phi + u_\phi^0) \frac{\partial \Theta_x}{\partial \phi} \\ = -\frac{1}{r} \Theta_\phi \frac{\partial u_x^0}{\partial \phi} - \varepsilon \left\{ 2\sigma'(r) \frac{\partial \Theta_{x,L}}{\partial r} + \sigma''(r) \Theta_{x,L} - 2\sigma'(r_R - r) \frac{\partial \Theta_{x,R}}{\partial r} + \sigma''(r_R - r) \Theta_{x,R} \right\} \quad \text{in } \Omega, \\ \Theta_x = -u_x^0 \quad \text{on } \Gamma, \\ \Theta_x|_{t=0} = 0, \end{array} \right. \quad (5.22)$$

where the second term on the right hand side of (5.22)₁ is an *e.s.t.* as appearing in (5.31).

Some needed L^p -estimates on the corrector in terms of the viscosity ε are stated and proved separately in Section 5.2 below.

We next turn to estimating the error between the true NS solution and the approximate solution. We define again the error term as:

$$\mathbf{v}^\varepsilon = v_\phi^\varepsilon(r, t)\mathbf{e}_\phi + v_x^\varepsilon(\phi, r, t)\mathbf{e}_x := \mathbf{u}^\varepsilon - \mathbf{u}^0 - \Theta, \quad (5.23)$$

and, using (5.5), we compute its curl:

$$\begin{aligned} \boldsymbol{\omega}^\varepsilon &= \omega_\phi^\varepsilon(\phi, r, t)\mathbf{e}_\phi + \omega_x^\varepsilon(r, t)\mathbf{e}_x + \omega_r^\varepsilon(\phi, r, t)\mathbf{e}_r \\ &:= \operatorname{curl} \mathbf{v}^\varepsilon \\ &= -\frac{\partial v_x^\varepsilon}{\partial r}\mathbf{e}_\phi + \frac{1}{r}\frac{\partial(rv_\phi^\varepsilon)}{\partial r}\mathbf{e}_x + \frac{1}{r}\frac{\partial v_x^\varepsilon}{\partial \phi}\mathbf{e}_r. \end{aligned} \quad (5.24)$$

We give some convergence results for PPF below. These are proved in Section 5.3.

Theorem 5.1. *Under the regularity assumptions (5.11) on the data, for any fixed $0 < T < \infty$, the following estimates hold:*

$$\left\{ \begin{array}{l} \|\mathbf{v}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}}\|\nabla\mathbf{v}^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{3}{4}}, \\ \|\omega_\phi^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}}\|\nabla\omega_\phi^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{4}}, \\ \|\omega_x^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}}\|\nabla\omega_x^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{4}}, \\ \|\omega_r^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}}\|\nabla\omega_r^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{3}{4}}, \\ \|\boldsymbol{\omega}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}}\|\nabla\boldsymbol{\omega}^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{4}}, \\ \|\mathbf{v}^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} + \varepsilon^{\frac{1}{2}}\|\mathbf{v}^\varepsilon\|_{L^2(0,T;H^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{4}}. \end{array} \right. \quad (5.25)$$

where ε is the viscosity coefficient, \mathbf{v}^ε is the approximate NS solution, $\boldsymbol{\omega}^\varepsilon$ its vorticity, and κ_T is a constant independent of ε .

In particular, the parallel pipe viscous solution, \mathbf{u}^ε , converges to the corresponding inviscid solution, \mathbf{u}^0 , as viscosity vanishes, with rate:

$$\|\mathbf{u}^\varepsilon - \mathbf{u}^0\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{4}}. \quad (5.26)$$

Accumulation of vorticity at the boundary occurs in the limit as in (4.21).

As for the channel case, since the size of the corrector can be estimated, by the triangle inequality the rate of convergence to the inviscid solution is shown to be optimal.

Remark 5.2. We observe that the rates of convergence for pipe flows is slower than that for channel flows, which can be ascribed to the presence of a non-zero boundary curvature in the pipe geometry.

We also note that, since the gradient of the pressure is a function of the ϕ -component of the velocity (see (5.9)₃, (5.10)₃), from (5.26) we obtain convergence of the gradient of the pressure with rate:

$$\left\| r \frac{\partial(p^\varepsilon - p^0)}{\partial r} \right\|_{L^\infty(0,T;L^1(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{4}}.$$

Remark 5.3. By considering the evolution of the cross-sectional component of the velocity, $\mathbf{u}_\phi^\varepsilon$ and \mathbf{u}_ϕ^0 , and its vorticity, ω_x^ε , we recover the known bounds for CSF. Accumulation of vorticity at the boundary in the limit takes a particularly simple form in this case, i.e.,

$$\lim_{\varepsilon \rightarrow 0} (\operatorname{curl} \mathbf{u}^\varepsilon \cdot \mathbf{e}_x, \varphi)_{L^2(\Omega)} = (\operatorname{curl} \mathbf{u}^0 \cdot \mathbf{e}_x, \varphi)_{L^2(\Omega)} + (u_x^0, \varphi)_{L^2(\Gamma)}, \quad (5.27)$$

for all $\varphi \in C(\overline{\Omega})$, uniformly in $0 < t < T$ (cf. [64]). directions.

5.2. Estimates of the corrector Θ . In this section, we discuss needed estimates on the corrector $\Theta = \Theta_\phi(r, t)\mathbf{e}_\phi + \Theta_x(\phi, r, t)\mathbf{e}_x$, again using the results in Appendix A. Since the analysis is very similar to that for channel flows in Section 4.2, we omit the details of the proofs. We recall that each component of the corrector is a combination of correctors near each wall of the pipe using cut-off functions.

Bounds on the derivatives of Θ_ϕ follow by first observing that

$$(\text{the right-hand side of (5.18)}_1) = e.s.t., \quad (5.28)$$

since $\Theta_{\phi,L}$ and $\Theta_{\phi,R}$ satisfy the pointwise estimates in Lemma A.1 with η replaced by $r - r_L$ and $r_R - r$, respectively. Then, the L^p estimates in Lemma A.2 give that

$$\left\| \frac{\partial^m \Theta_\phi}{\partial r^m} \right\|_{L^p(\Omega)} \leq \kappa_T (1 + t^{\frac{1}{2p} - \frac{m}{2}}) \varepsilon^{\frac{1}{2p} - \frac{m}{2}}, \quad 1 \leq p \leq \infty, \quad 0 \leq m \leq 2, \quad 0 < t < T. \quad (5.29)$$

To derive estimates on the second component Θ_x of the corrector, we use the fact that $\Theta_{x,L}$ satisfies the parabolic system (A.6) in the domain $(0, L) \times (r_L, \infty)$ with $\tau = \phi$, $\eta = r - r_L$, and

$$\begin{cases} \mathcal{U} = \frac{1}{r} (\Theta_\phi + u_\phi^0), \\ G = -\frac{1}{r} \Theta_\phi \frac{\partial u_x^0}{\partial \phi}, \\ g = -u_x^0|_{r=r_L} \text{ (or } r_R). \end{cases}$$

(And similarly for $\Theta_{x,R}$ with $r - r_L$ replaced by $r_R - r$). Consequently, both $\Theta_{x,L}$ and $\Theta_{x,R}$ satisfies the estimates in Lemma A.4, which give:

$$\begin{cases} \left\| \frac{\partial^k \Theta_x}{\partial \phi^k} \right\|_{L^\infty(0,T;L^p(\Omega))} + \varepsilon^{\frac{1}{2p} + \frac{1}{4}} \left\| \nabla \frac{\partial^k \Theta_x}{\partial \phi^k} \right\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{2p}}, \quad 1 \leq p \leq \infty, \\ \left\| \frac{\partial^{k+1} \Theta_x}{\partial \phi^k \partial r} \right\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}} \left\| \nabla \frac{\partial^{k+1} \Theta_x}{\partial \phi^k \partial r} \right\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{-\frac{1}{4}}, \\ \left\| \frac{\partial^{k+1} \Theta_x}{\partial \phi^k \partial r} \right\|_{L^\infty(0,T;L^1(\Omega))} \leq \kappa_T, \end{cases} \quad (5.30)$$

with $k \geq 0$, for a constant, κ_T , depending on T and the other data, but independent of ε . The L^1 bound of $\partial^k \Theta_x / \partial \phi^k$ is obtained by the estimates in Lemma A.4 and the continuity of L^p norm in a bounded interval. Moreover, performing an energy estimate on Θ_x using, as test function, any derivative of the cut-off function σ , yields

$$(\text{the second term on the right hand side of (5.22)}_1) = e.s.t.. \quad (5.31)$$

Finally, the bounds on the derivatives of the corrector plus the explicit form of $\text{curl } \Theta$ (see (5.24)) imply that

$$\|\text{curl } \Theta\|_{L^\infty(0,T;L^1(\Omega))} \leq \kappa_T. \quad (5.32)$$

5.3. Proof of Theorem 5.1. We recall the notation from the Introduction: κ, κ_T will denote generic constants that may depend on the data or time, but not on ε , and may change from line to line.

To prove (5.25)₁, using (5.6), (5.9), (5.10), (5.18), (5.22), and (5.23), we write the equation for \mathbf{v}^ε :

We begin by deriving the equations satisfied by the components of $\mathbf{v}^\varepsilon, v_\phi^\varepsilon$ and v_x^ε . We utilize the the form (5.6) for the Laplacean of a parallel pipe velocity field, and isolate terms that are negligible in ε , exploiting the estimates for the corrector obtained above. We therefore have:

$$\left\{ \begin{array}{l} \frac{\partial v_\phi^\varepsilon}{\partial t} - \varepsilon \Delta v_\phi^\varepsilon + \varepsilon \frac{1}{r^2} v_\phi^\varepsilon = \varepsilon \Delta u_\phi^0 - \varepsilon \frac{1}{r^2} u_\phi^0 + R^\varepsilon(\Theta_\phi) + e.s.t., \quad \text{in } \Omega, \\ \frac{\partial v_x^\varepsilon}{\partial t} - \varepsilon \Delta v_x^\varepsilon + \frac{1}{r} u_\phi^\varepsilon \frac{\partial v_x^\varepsilon}{\partial \phi} + v_\phi^\varepsilon \left(\frac{\partial \Theta_x}{\partial \phi} + \frac{\partial u_x^0}{\partial \phi} \right) = \varepsilon \Delta u_x^0 + R^\varepsilon(\Theta_x) + e.s.t., \quad \text{in } \Omega, \\ \mathbf{v}^\varepsilon = 0, \quad \text{on } \Gamma, \\ \mathbf{v}^\varepsilon|_{t=0} = 0, \end{array} \right. \quad (5.33)$$

where

$$\left\{ \begin{array}{l} R^\varepsilon(\Theta_\phi) = \frac{\partial \Theta_\phi}{\partial t} + \varepsilon \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Theta_\phi}{\partial r} \right) - \frac{1}{r^2} \Theta_\phi \right\} \\ R^\varepsilon(\Theta_x) = -\varepsilon \frac{1}{r} \frac{\partial \Theta_x}{\partial r} - \varepsilon \frac{1}{r^2} \frac{\partial \Theta_x}{\partial \phi^2}. \end{array} \right. \quad (5.34)$$

The bound for v_ϕ can be easily obtained via standard energy estimates. In fact, by multiplying the first equation in (5.33) by v_ϕ^ε , and integrating over Ω by parts, we deduce that

$$\frac{d}{dt} \|v_\phi^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla v_\phi^\varepsilon\|_{L^2(\Omega)}^2 \leq \kappa \varepsilon \left| (\Delta \mathbf{u}_0 \cdot \mathbf{e}_\phi, v_\phi^\varepsilon)_{L^2(\Omega)} \right| + \kappa \left| \int_{r_L}^{r_R} R^\varepsilon(\Theta_\phi) v_\phi^\varepsilon r \, dr \right|, \quad (5.35)$$

for some positive constant κ , where we used that $\mathbf{u}_0 \cdot \mathbf{e}_\phi$ is time independent, and that both $R(\theta_\phi)$ and v_ϕ^ε are radial. The estimates on the corrector Θ_ϕ of the previous section give:

$$\left| \int_{r_L}^{r_R} R^\varepsilon(\Theta_\phi) v_\phi^\varepsilon r \, dr \right| \leq \kappa_T t^{-\frac{1}{4}} \varepsilon^{\frac{3}{4}} \|\sqrt{r} v_\phi^\varepsilon\|_{L^2(r_L, r_R)} \leq \kappa_T t^{-\frac{1}{2}} \varepsilon^{\frac{3}{2}} + \|\mathbf{v}^\varepsilon\|_{L^2(\Omega)}^2, \quad (5.36)$$

from which it follows, by applying Cauchy's inequality on $(\Delta \mathbf{u}_0 \cdot \mathbf{e}_\phi, v_\phi^\varepsilon)_{L^2(\Omega)}$, that

$$\frac{d}{dt} \|\mathbf{v}^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla \mathbf{v}^\varepsilon\|_{L^2(\Omega)}^2 \leq \kappa_T (1 + t^{-\frac{1}{2}}) \varepsilon^{\frac{3}{2}} + 2 \|\mathbf{v}^\varepsilon\|_{L^2(\Omega)}^2, \quad (5.37)$$

where κ_T depends on the data, but not on ε . Grönwall's inequality then gives:

$$\|v_\phi^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}} \|\nabla v_\phi^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{3}{4}}. \quad (5.38)$$

We proceed similarly for the component of the velocity along the axis. We multiply (5.33)₂ by v_x^ε and integrate it over Ω . Using the fact that v_ϕ^ε is independent of the variable ϕ , we obtain:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|v_x^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla v_x^\varepsilon\|_{L^2(\Omega)}^2 \\ & \leq \|v_\phi^\varepsilon\|_{L^2(\Omega)} \left\| \frac{\partial \Theta_x}{\partial \phi} + \frac{\partial u_x^0}{\partial \phi} \right\|_{L^\infty(\Omega)} \|v_x^\varepsilon\|_{L^2(\Omega)} + (\varepsilon \|\Delta u_x^0\|_{L^2(\Omega)} + \|R^\varepsilon(\Theta_x)\|_{L^2(\Omega)}) \|v_x^\varepsilon\|_{L^2(\Omega)}. \end{aligned}$$

Again, Cauchy's inequality, the estimates on the corrector, and the bound on v_ϕ^ε in (5.38) yield:

$$\frac{1}{2} \frac{d}{dt} \|v_x^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla v_x^\varepsilon\|_{L^2(\Omega)}^2 \leq \kappa \varepsilon^{\frac{3}{4}} \|v_x^\varepsilon\|_{L^2(\Omega)} \leq \kappa \varepsilon^{\frac{3}{2}} + \|v_x^\varepsilon\|_{L^2(\Omega)}^2.$$

and by Grönwall's:

$$\|v_x^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}} \|\nabla v_x^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{3}{4}}, \quad (5.39)$$

which together with (5.38) gives (5.25)₁. Estimate (5.26) follows from (5.25)₁ by the triangle inequality, given the estimates on the corrector Θ .

To verify (5.25)_{2,3,4}, we once gain follow the idea of Lighthill. We differentiate (5.33)_i, $i = 1, 2$, in r , and divide (5.33)₂ by r after differentiating it in ϕ . We then find the equations for ω^ε , utilizing the explicit form for the pipe vorticity given in (5.24):

$$\left\{ \begin{array}{l} \frac{\partial \omega_\phi^\varepsilon}{\partial t} - \varepsilon \Delta \omega_\phi^\varepsilon + \frac{1}{r} u_\phi^\varepsilon \frac{\partial \omega_\phi^\varepsilon}{\partial \phi} \\ \quad = -\varepsilon \frac{\partial(\Delta u_x^0)}{\partial r} - \frac{\partial(R^\varepsilon(\Theta_x))}{\partial r} + \left(\omega_x^\varepsilon - \frac{1}{r} v_\phi^\varepsilon\right) \frac{\partial(u_x^0 + \Theta_x)}{\partial \phi} + v_\phi^\varepsilon \frac{\partial^2(u_x^0 + \Theta_x)}{\partial \phi \partial r} \\ \quad - \frac{1}{r} u_\phi^\varepsilon \omega_r^\varepsilon + \frac{\partial u_\phi^\varepsilon}{\partial r} \omega_r^\varepsilon + e.s.t. \quad \text{in } \Omega, \\ \frac{\partial \omega_x^\varepsilon}{\partial t} - \varepsilon \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \omega_x^\varepsilon}{\partial r} \right) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \text{ (right-hand side of (5.33))}_1 \right) \quad \text{in } \Omega, \\ \frac{\partial \omega_r^\varepsilon}{\partial t} - \varepsilon \Delta \omega_r^\varepsilon - \varepsilon \frac{1}{r^2} \omega_r^\varepsilon - 2\varepsilon \frac{1}{r} \frac{\partial \omega_r^\varepsilon}{\partial r} + \frac{1}{r} u_\phi^\varepsilon \frac{\partial \omega_r^\varepsilon}{\partial \phi} \\ \quad = \varepsilon \frac{1}{r} \frac{\partial(\Delta u_x^0)}{\partial \phi} + \frac{1}{r} \frac{\partial(R^\varepsilon(\Theta_x))}{\partial \phi} - \frac{1}{r} v_\phi^\varepsilon \frac{\partial^2(u_x^0 + \Theta_x)}{\partial \phi^2} \quad \text{in } \Omega. \end{array} \right. \quad (5.40)$$

The boundary and initial conditions for ω^ε can be obtained by restricting (5.33)_{1,2} on Γ and using (5.33)_{3,4}:

$$\left\{ \begin{array}{l} \frac{\partial \omega_\phi^\varepsilon}{\partial r} + \omega_\phi^\varepsilon = \Delta u_x^0 \quad \text{on } \Gamma, \\ \frac{\partial \omega_x^\varepsilon}{\partial r} = -\varepsilon^{-1} \text{ (right-hand side of (5.33))}_1 \text{ up to } e.s.t. \text{ terms) } \quad \text{on } \Gamma, \\ \omega_r^\varepsilon = 0 \quad \text{on } \Gamma, \\ \omega^\varepsilon|_{t=0} = 0. \end{array} \right. \quad (5.41)$$

Above, we have used the fact that the *e.s.t.* terms in (5.33) vanish on Γ (see the right-hand side of (5.18) and (5.22)).

We first estimate ω_x^ε , which is precisely the scalar vorticity of the cross-sectional velocity in the pipe. Taking ω_x^ε as test function and integrating by parts gives:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega_x^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla \omega_x^\varepsilon\|_{L^2(\Omega)}^2 &= \varepsilon \left(\omega_x^\varepsilon \frac{\partial \omega_x^\varepsilon}{\partial r} \right) \Big|_{\Gamma} \\ &+ \int_{r_L}^{r_R} \left\{ \operatorname{curl} (\varepsilon (\Delta \mathbf{u}_0 \cdot \mathbf{e}_\phi + R^\varepsilon(\Theta_\phi)) \mathbf{e}_\phi) \right\} r \omega_x^\varepsilon dr + e.s.t, \end{aligned} \quad (5.42)$$

identifying the two-dimensional curl with a scalar function, and exploiting again that the cross-sectional inviscid velocity is stationary, and that $u_{0,\phi}$, ω_x^ε , $R^\varepsilon(\Theta_\phi)$ are radial. We can bound the first term on the right-hand side above as follows:

$$\varepsilon \left| \left(\omega_x^\varepsilon(r_*, t) \frac{\partial \omega_x^\varepsilon(r_*, t)}{\partial r} \right) \right| \leq \varepsilon \left| \frac{\partial \omega_x^\varepsilon}{\partial r}(r_*, t) \right| |\omega_x^\varepsilon(r_*, t)| \leq \kappa(1+t^{-\frac{1}{2}}) \varepsilon^{\frac{1}{2}} |\omega_x^\varepsilon(r_*, t)|, \quad * = L, R,$$

where we have used the Neumann boundary condition (5.41)₂ for ω_x^ε and the estimates for the corrector to bound the radial derivative of ω_x^ε on Γ . But $|\omega_x^\varepsilon(r_*, t)| = (2\pi)^{-1/2} \|\omega_x^\varepsilon(t)\|_{L^2(\Gamma)}$, $* = L, R$, which is estimated using the bounds on the trace in Corollary B.2 with $p = 2$. We then apply a modified version of Young's inequality with three factors (more precisely we apply Lemma B.3 with $p = 4/3$, $q = 4$, $p_1 = 3/2$, $p_2 = 3$) and obtain:

$$\begin{aligned} \varepsilon \left| \left(\frac{\partial \omega_x^\varepsilon \omega_x^\varepsilon}{\partial r} \Big|_{\Gamma} \right) \right| &\leq \kappa(1+t^{-\frac{1}{2}}) \varepsilon^{\frac{1}{2}} \|\omega_x^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \omega_x^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \\ &= \kappa \left[(1+t^{-\frac{1}{2}})^{\frac{2}{3}} \varepsilon^{\frac{1}{4}} \right] \left[(1+t^{-\frac{1}{2}})^{\frac{1}{3}} \|\omega_x^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \right] \left[2^{-\frac{3}{4}} \varepsilon^{\frac{1}{4}} \|\nabla \omega_x^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \right] \\ &\leq \kappa(1+t^{-\frac{2}{3}}) \varepsilon^{\frac{1}{2}} + \kappa(1+t^{-\frac{2}{3}}) \|\omega_x^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2} \|\nabla \omega_x^\varepsilon\|_{L^2(\Omega)}^2. \end{aligned} \quad (5.43)$$

Similarly, the regularity of \mathbf{u}_0 , the estimates for the corrector, and standard inequalities give for the second term on the right-hand side of (5.42):

$$\begin{aligned} &\left| \int_{r_L}^{r_R} \operatorname{curl} (\varepsilon (\Delta \mathbf{u}_0 \cdot \mathbf{e}_\phi + R^\varepsilon(\Theta_\phi)) \mathbf{e}_\phi) r \omega_x^\varepsilon dr \right| \\ &\leq \varepsilon \left\{ \kappa \|\operatorname{curl} \Delta \mathbf{u}_0\|_{L^2(\Omega)} + \sum_{m=0}^2 \left\| \sqrt{r} \frac{\partial^m \Theta}{\partial r^m} \right\|_{L^2(0,1)} \right\} \kappa \|\omega_x^\varepsilon\|_{L^2(\Omega)} \\ &\leq \kappa(1+t^{-\frac{3}{4}}) \varepsilon^{\frac{1}{4}} \|\omega_x^\varepsilon\|_{L^2(\Omega)} \\ &\leq \kappa(1+t^{-\frac{3}{4}}) \varepsilon^{\frac{1}{2}} + \kappa(1+t^{-\frac{3}{4}}) \|\omega_x^\varepsilon\|_{L^2(\Omega)}^2. \end{aligned} \quad (5.44)$$

Combining the estimates above into (5.42), we have for ε small enough that

$$\frac{d}{dt} \|\omega_x^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla \omega_x^\varepsilon\|_{L^2(\Omega)}^2 \leq \kappa(1+t^{-\frac{3}{4}}) \varepsilon^{\frac{1}{2}} + \kappa(1+t^{-\frac{3}{4}}) \|\omega_x^\varepsilon\|_{L^2(\Omega)}^2, \quad (5.45)$$

from which the bound on ω_x^ε follows from Grönwall's lemma with the integrating factor $\exp(-\kappa t - 4\kappa t^{1/4})$:

$$\|\omega_x^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon^{\frac{1}{2}} \|\nabla \omega_x^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{4}}. \quad (5.46)$$

We proceed in an entirely analogous fashion for the radial component of the vorticity, multiplying (5.40)₃ by ω_r^ε and integrating over Ω by parts using the homogeneous

Dirichlet condition on ω_r^ε :

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\omega_r^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla \omega_r^\varepsilon\|_{L^2(\Omega)}^2 \\
& \leq \varepsilon \left\| \frac{1}{r} \omega_r^\varepsilon \right\|_{L^2(\Omega)}^2 + \varepsilon \left\| \frac{1}{r} \frac{\partial(\Delta u_x^0)}{\partial \phi} \right\|_{L^2(\Omega)} \|\omega_r^\varepsilon\|_{L^2(\Omega)} + \left\| \frac{\partial(R^\varepsilon(\Theta_x))}{\partial \phi} \right\|_{L^2(\Omega)} \|\omega_r^\varepsilon\|_{L^2(\Omega)} \\
& \quad + \|v_\phi^\varepsilon\|_{L^2(\Omega)} \left\| \frac{1}{r} \frac{\partial^2(u_x^0 + \Theta_x)}{\partial \phi^2} \right\|_{L^\infty(\Omega)} \|\omega_r^\varepsilon\|_{L^2(\Omega)} \\
& \leq \kappa_T \varepsilon^{\frac{3}{2}} + \kappa_T \|\omega_r^\varepsilon\|_{L^2(\Omega)}^2.
\end{aligned} \tag{5.47}$$

As before, (5.25)₄ follows by applying Grönwall's inequality.

A similar integration by parts leads to the following energy estimate for ω_ϕ^ε , using again that u_ϕ^ε is independent of ϕ :

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla \omega_\phi^\varepsilon\|_{L^2(\Omega)}^2 \\
& \leq \varepsilon \left\| \frac{\partial \omega_\phi^\varepsilon}{\partial r} \omega_\phi^\varepsilon \right\|_{L^1(\Gamma)} + \varepsilon \left\| \frac{\partial(\Delta u_x^0)}{\partial r} \right\|_{L^2(\Omega)} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} + \left\| \frac{\partial(R^\varepsilon(\Theta_x))}{\partial r} \right\|_{L^2(\Omega)} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} \\
& \quad + \left\{ \|\omega_x^\varepsilon\|_{L^2(\Omega)} \left\| \frac{\partial(u_x^0 + \Theta_x)}{\partial \phi} \right\|_{L^\infty(\Omega)} + \|v_\phi^\varepsilon\|_{L^2(\Omega)} \left\| \frac{1}{r} \frac{\partial(u_x^0 + \Theta_x)}{\partial \phi} \right\|_{L^\infty(\Omega)} \right\} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} \\
& \quad + \left| \int_0^{2\pi} \int_0^L \int_{r_L}^{r_R} v_\phi^\varepsilon \frac{\partial^2(u_x^0 + \Theta_x)}{\partial \phi \partial r} \omega_\phi^\varepsilon r dr dx d\phi \right| \\
& \quad + \left\| -\frac{1}{r} u_\phi^\varepsilon + \frac{\partial u_\phi^\varepsilon}{\partial r} \right\|_{L^\infty(\Omega)} \|\omega_r^\varepsilon\|_{L^2(\Omega)} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}.
\end{aligned} \tag{5.48}$$

Above, we have also used that the integrals containing the fourth and fifth term on the left-hand side of (5.40)₃ vanish.

We next estimate each term on the right-hand side of (5.48) in order to apply Grönwall's Lemma. The second term is already in the appropriate form. The first term can be bounded as follows, using the boundary condition for ω_ϕ^ε (5.41)₁:

$$\varepsilon \left\| \frac{\partial \omega_\phi^\varepsilon}{\partial r} \omega_\phi^\varepsilon \right\|_{L^1(\Gamma)} \leq \varepsilon \left\| \frac{\partial \omega_\phi^\varepsilon}{\partial r} \right\|_{L^2(\Gamma)} \|\omega_\phi^\varepsilon\|_{L^2(\Gamma)} \leq \varepsilon \|\omega_\phi^\varepsilon\|_{L^2(\Gamma)}^2 + \kappa \varepsilon \|\omega_\phi^\varepsilon\|_{L^2(\Gamma)}. \tag{5.49}$$

Standard estimates then give:

$$\begin{aligned}
\varepsilon \|\omega_\phi^\varepsilon\|_{L^2(\Gamma)}^2 & \leq \kappa \varepsilon \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} \|\omega_\phi^\varepsilon\|_{H^1(\Omega)} \\
& \leq \kappa \varepsilon \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}^2 + \kappa \varepsilon \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} \|\nabla \omega_\phi^\varepsilon\|_{L^2(\Omega)} \\
& \leq \kappa \varepsilon \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{8} \varepsilon \|\nabla \omega_\phi^\varepsilon\|_{L^2(\Omega)}^2,
\end{aligned} \tag{5.50}$$

$$\begin{aligned}
\varepsilon \|\omega_\phi^\varepsilon\|_{L^2(\Gamma)} &\leq \kappa \varepsilon \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\omega_\phi^\varepsilon\|_{H^1(\Omega)}^{\frac{1}{2}} \\
&\leq \kappa \varepsilon \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} + \kappa \varepsilon \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \omega_\phi^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \\
&\leq \kappa \varepsilon \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} + \kappa \varepsilon \|\nabla \omega_\phi^\varepsilon\|_{L^2(\Omega)} \\
&\leq \kappa \varepsilon + \kappa \varepsilon \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{8} \varepsilon \|\nabla \omega_\phi^\varepsilon\|_{L^2(\Omega)}^2,
\end{aligned} \tag{5.51}$$

from which it follows that

$$\varepsilon \left\| \frac{\partial \omega_\phi^\varepsilon}{\partial r} \omega_\phi^\varepsilon \right\|_{L^1(\Gamma)} \leq \kappa \varepsilon + \kappa \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{4} \varepsilon \|\nabla \omega_\phi^\varepsilon\|_{L^2(\Omega)}^2. \tag{5.52}$$

We bound the third term on the right-hand side of (5.48), employing the explicit form of $R^\varepsilon(\Theta_x)$ and the estimates on the corrector:

$$\begin{aligned}
\left\| \frac{\partial(R^\varepsilon(\Theta_x))}{\partial r} \right\|_{L^2(\Omega)} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} &\leq \left\{ \kappa_T \varepsilon^{\frac{3}{4}} + \varepsilon \left\| \frac{\partial^2 \Theta_x}{\partial r^2} \right\|_{L^2(\Omega)} \right\} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} \\
&\leq \kappa_T \varepsilon^{\frac{3}{2}} + \kappa \varepsilon \left\| \frac{\partial^2 \Theta_x}{\partial r^2} \right\|_{L^2(\Omega)}^2 + \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}^2.
\end{aligned} \tag{5.53}$$

The fourth and fifth terms on the right-hand side of (5.48) are readily estimated, thanks to the bounds on \mathbf{v}^ε and ω_x^ε already established, (5.25)₁ and (5.25)₃:

$$\begin{aligned}
\left\{ \|\omega_x^\varepsilon\|_{L^2(\Omega)} \left\| \frac{\partial(u_x^0 + \Theta_x)}{\partial \phi} \right\|_{L^\infty(\Omega)} + \|v_\phi^\varepsilon\|_{L^2(\Omega)} \left\| \frac{1}{r} \frac{\partial(u_x^0 + \Theta_x)}{\partial \phi} \right\|_{L^\infty(\Omega)} \right\} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} \\
\leq \kappa \varepsilon^{\frac{1}{4}} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} \leq \kappa \varepsilon^{\frac{1}{2}} + \kappa \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}^2.
\end{aligned} \tag{5.54}$$

After integrating by parts in the r direction, we write the sixth term on the right-hand side of (5.48) in the form, using that v_ϕ^ε vanishes on the boundary:

$$\begin{aligned}
&\left| \int_0^{2\pi} \int_0^L \int_{r_L}^{r_R} v_\phi^\varepsilon \frac{\partial^2(u_x^0 + \Theta_x)}{\partial \phi \partial r} \omega_\phi^\varepsilon r \, dr dx d\phi \right| \\
&= \left| \int_0^{2\pi} \int_0^L \int_{r_L}^{r_R} \frac{\partial(u_x^0 + \Theta_x)}{\partial \phi} \frac{\partial(v_\phi^\varepsilon \omega_\phi^\varepsilon r)}{\partial r} \, dr dx d\phi \right| \\
&\leq \left\| \frac{\partial(u_x^0 + \Theta_x)}{\partial \phi} \frac{\partial v_\phi^\varepsilon}{\partial r} \omega_\phi^\varepsilon \right\|_{L^1(\Omega)} + \left\| \frac{\partial(u_x^0 + \Theta_x)}{\partial \phi} v_\phi^\varepsilon \frac{\partial \omega_\phi^\varepsilon}{\partial r} \right\|_{L^1(\Omega)} + \left\| \frac{\partial(u_x^0 + \Theta_x)}{\partial \phi} v_\phi^\varepsilon \omega_\phi^\varepsilon \frac{1}{r} \right\|_{L^1(\Omega)} \\
&\leq \left\| \frac{\partial(u_x^0 + \Theta_x)}{\partial \phi} \right\|_{L^\infty(\Omega)} \left\{ \left\| \frac{\partial v_\phi^\varepsilon}{\partial r} \right\|_{L^2(\Omega)} + \left\| \frac{v_\phi^\varepsilon}{r} \right\|_{L^2(\Omega)} \right\} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} \\
&\quad + \left\| \frac{\partial(u_x^0 + \Theta_x)}{\partial \phi} \right\|_{L^\infty(\Omega)} \|v_\phi^\varepsilon\|_{L^2(\Omega)} \left\| \frac{\partial \omega_\phi^\varepsilon}{\partial r} \right\|_{L^2(\Omega)} \\
&\leq \kappa \left\| \frac{\partial v_\phi^\varepsilon}{\partial r} \right\|_{L^2(\Omega)} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} + \kappa \varepsilon^{\frac{1}{2}} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} + \kappa \varepsilon^{\frac{3}{4}} \left\| \frac{\partial \omega_\phi^\varepsilon}{\partial r} \right\|_{L^2(\Omega)} \\
&\leq \kappa \left\| \frac{\partial v_\phi^\varepsilon}{\partial r} \right\|_{L^2(\Omega)}^2 + \kappa \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}^2 + \kappa \varepsilon^{\frac{1}{2}} + \frac{1}{4} \varepsilon \|\nabla \omega_\phi^\varepsilon\|_{L^2(\Omega)}^2,
\end{aligned} \tag{5.55}$$

where we employed estimate (5.38) and the fact that $0 < r_L < r < r_R < \infty$ to bound $\|v_\phi^\varepsilon/r\|_{L^2(\Omega)}$.

We bound the last term on the right-hand side of (5.48), by first writing it in terms of \mathbf{v}^ε and $\boldsymbol{\omega}^\varepsilon$, exploiting the explicit form of the curl for a parallel pipe flow, and then using the estimates on the corrector:

$$\begin{aligned} & \left\| -\frac{1}{r} u_\phi^\varepsilon + \frac{\partial u_\phi^\varepsilon}{\partial r} \right\|_{L^\infty(\Omega)} \\ & \leq 2 \left\| \frac{1}{r} v_\phi^\varepsilon \right\|_{L^\infty(\Omega)} + \left\| \frac{1}{r} (u_\phi^0 + \Theta_\phi) \right\|_{L^\infty(\Omega)} + \|\omega_x^\varepsilon\|_{L^\infty(\Omega)} + \left\| \frac{\partial(u_\phi^0 + \Theta_\phi)}{\partial r} \right\|_{L^\infty(\Omega)} \\ & \leq \kappa \|v_\phi^\varepsilon\|_{L^\infty(\Omega)} + \|\omega_x^\varepsilon\|_{L^\infty(\Omega)} + \kappa_T (1 + t^{-\frac{1}{2}}) \varepsilon^{-\frac{1}{2}}. \end{aligned} \tag{5.56}$$

Then, Poincaré's, the one-dimensional Agmon's inequalities, and the bounds on v_ϕ^ε and ω_x^ε , (5.25)₁ and (5.25)₃, yield:

$$\|v_\phi^\varepsilon\|_{L^\infty(\Omega)} \leq \kappa \|v_\phi^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|v_\phi^\varepsilon\|_{H^1(\Omega)}^{\frac{1}{2}} \leq \kappa \varepsilon^{\frac{3}{8}} \|\nabla v_\phi^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}}, \tag{5.57}$$

$$\begin{aligned} \|\omega_x^\varepsilon\|_{L^\infty(\Omega)} & \leq \kappa \|\omega_x^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\omega_x^\varepsilon\|_{H^1(\Omega)}^{\frac{1}{2}} \leq \kappa \|\omega_x^\varepsilon\|_{L^2(\Omega)} + \kappa \|\omega_x^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \omega_x^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \\ & \leq \kappa \varepsilon^{\frac{1}{4}} + \kappa \varepsilon^{\frac{1}{8}} \|\nabla \omega_x^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}}. \end{aligned} \tag{5.58}$$

Putting together these estimates finally gives the following bound for the last term on the right-hand side of (5.48):

$$\begin{aligned} & \left\| -\frac{1}{r} u_\phi^\varepsilon + \frac{\partial u_\phi^\varepsilon}{\partial r} \right\|_{L^\infty(\Omega)} \|\omega_r^\varepsilon\|_{L^2(\Omega)} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} \\ & \leq \kappa_T \left\{ 1 + \varepsilon^{\frac{3}{8}} \|\nabla v_\phi^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} + (1 + t^{-\frac{1}{2}}) \varepsilon^{-\frac{1}{2}} + \varepsilon^{\frac{1}{4}} + \varepsilon^{\frac{1}{8}} \|\nabla \omega_x^\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \right\} \varepsilon^{\frac{3}{4}} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)} \\ & \leq \kappa_T (1 + t^{-\frac{1}{2}}) \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}^2 + \kappa_T \left(t^{-\frac{1}{2}} + \varepsilon^{\frac{3}{2}} + \varepsilon^{\frac{7}{4}} \|\nabla v_\phi^\varepsilon\|_{L^2(\Omega)} + \varepsilon^{\frac{5}{4}} \|\nabla \omega_x^\varepsilon\|_{L^2(\Omega)} \right) \varepsilon^{\frac{1}{2}} \\ & \leq \kappa_T (1 + t^{-\frac{1}{2}}) \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}^2 + \kappa_T (1 + t^{-\frac{1}{2}}) \varepsilon^{\frac{1}{2}} + \kappa_T \varepsilon^{\frac{7}{2}} \|\nabla v_\phi^\varepsilon\|_{L^2(\Omega)}^2 + \kappa_T \varepsilon^{\frac{5}{2}} \|\nabla \omega_x^\varepsilon\|_{L^2(\Omega)}^2. \end{aligned} \tag{5.59}$$

Combining all previous bounds, we deduce from (5.48) that

$$\begin{aligned} \frac{d}{dt} \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla \omega_\phi^\varepsilon\|_{L^2(\Omega)}^2 & \leq \kappa_T (1 + t^{-\frac{1}{2}}) \|\omega_\phi^\varepsilon\|_{L^2(\Omega)}^2 + \kappa_T (1 + t^{-\frac{1}{2}}) \varepsilon^{\frac{1}{2}} \\ & \quad + \kappa \varepsilon \left\| \frac{\partial^2 \Theta_x}{\partial r^2} \right\|_{L^2(\Omega)}^2 + \kappa_T \varepsilon^{\frac{7}{2}} \|\nabla v_\phi^\varepsilon\|_{L^2(\Omega)}^2 + \kappa_T \varepsilon^{\frac{5}{2}} \|\nabla \omega_x^\varepsilon\|_{L^2(\Omega)}^2, \end{aligned} \tag{5.60}$$

from which (5.25)₂ follows by applying Grönwall's Lemma once again, since

$$\kappa_T \int_0^T \left(\varepsilon \left\| \frac{\partial^2 \Theta_x}{\partial r^2} \right\|_{L^2(\Omega)}^2 + \varepsilon^{\frac{7}{2}} \|\nabla v_\phi^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon^{\frac{5}{2}} \|\nabla \omega_x^\varepsilon\|_{L^2(\Omega)}^2 \right) dt \leq \kappa_T \varepsilon^{\frac{1}{2}}.$$

This last inequality in turns follows from the estimates on the correctors and the bounds already established on \mathbf{v}^ε and $\boldsymbol{\omega}^\varepsilon$.

As for the case of channel flows, the bounds on the vorticity in (5.25) imply that:

$$\|\operatorname{curl}(\mathbf{u}^\varepsilon - \mathbf{u}^0)\|_{L^\infty(0,T;L^1(\Omega))} \leq \kappa_T.$$

Then, weak convergence of the vorticity with accumulation at the boundary as a vortex sheet as in (4.21) follows again Corollary C.2 in the Appendix.

The proof of Theorem 5.1 is complete.

APPENDIX A. ONE-DIMENSIONAL HEAT EQUATIONS

In this Appendix we discuss mostly known results on 1D and 2D heat equations with possible drift. These results, in turn, are used throughout the paper to derive decay and regularity estimates for the boundary layer correctors. In fact, due to the weakly non-linear nature of the flows considered here, the corrector can be taken to be linear (cf. the approach using layer potentials for a heat equation with drift in [73]).

A.1. On the one-dimensional heat equation with small diffusivity. We consider the following boundary-value problem for the heat equation on a half line:

$$\left\{ \begin{array}{l} \frac{\partial \Phi}{\partial t} - \varepsilon \frac{\partial^2 \Phi}{\partial \eta^2} = 0, \quad \eta, t > 0, \\ \Phi = g(t), \quad \eta = 0, \\ \Phi \rightarrow 0, \quad \text{as } \eta \rightarrow \infty, \\ \Phi = 0, \quad t = 0. \end{array} \right. \quad (\text{A.1})$$

Above, the diffusivity ε is a fixed, strictly positive parameter, and $g(t)$ is the boundary data, assumed sufficiently smooth. The incompatibility between the boundary data, which need not vanish at $t = 0$, and the initial data leads to the formation of an initial layer.

The solution of (A.1) is explicitly given by the following formula (see e.g. the classical reference [22]):

$$\Phi(\eta, t) = 2g(0) \operatorname{erfc}\left(\frac{\eta}{\sqrt{2\varepsilon t}}\right) + 2 \int_0^t \frac{\partial g}{\partial t}(s) \operatorname{erfc}\left(\frac{\eta}{\sqrt{2\varepsilon(t-s)}}\right) ds, \quad (\text{A.2})$$

where erfc is the complimentary error function on \mathbb{R}_+ ,

$$\operatorname{erfc}(z) := \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-y^2/2} dy, \quad (\text{A.3})$$

which satisfies

$$\operatorname{erfc}(0) = \frac{1}{2}, \quad \operatorname{erfc}(\infty) = 0.$$

We recall known estimates on Φ (for a proof in the context of boundary layer analysis, we refer to [35, 34]).

Lemma A.1. *Let $g \in W^{1,\infty}(0, T)$, $0 < T < \infty$. Then, the following pointwise estimates hold for $\eta > 0$ and $0 < t < T$,*

$$\left\{ \begin{array}{l} |\Phi(t, \eta)| \leq \kappa_T e^{-\eta^2/(4\varepsilon t)}, \\ \left| \frac{\partial \Phi}{\partial \eta}(t, \eta) \right| \leq \kappa_T \varepsilon^{-\frac{1}{2}} (1 + t^{-\frac{1}{2}}) e^{-\eta^2/(4\varepsilon t)}, \\ \left| \frac{\partial^2 \Phi}{\partial \eta^2}(t, \eta) \right| \leq \kappa_T (\varepsilon t)^{-1} \frac{\eta}{\sqrt{\varepsilon t}} e^{-\eta^2/(4\varepsilon t)} + \kappa_T \varepsilon^{-1} \int_0^t s^{-1} \frac{\eta}{\sqrt{\varepsilon s}} e^{-\eta^2/(4\varepsilon s)} ds, \end{array} \right.$$

where κ_T depends on T and the data g , but is independent of ε .

Lemma A.2. *Assume again that $g \in W^{1,\infty}(0, T)$, $0 < T < \infty$. Then, for $1 \leq p \leq \infty$ and $0 \leq m \leq 2$,*

$$\left\| \frac{\partial^m \Phi}{\partial \eta^m}(t) \right\|_{L^p_\eta(0, \infty)} \leq \kappa_T (1 + t^{\frac{1}{2p} - \frac{m}{2}}) \varepsilon^{\frac{1}{2p} - \frac{m}{2}},$$

for $0 < t < T$, with a constant κ_T depending on T and the data g , but independent of ε .

The next result is utilized in particular in establishing concentration of vorticity at the boundary in the vanishing viscosity limit. We include a proof for the reader's convenience.

Lemma A.3. *Under the hypotheses of Lemma A.1 and A.2,*

$$\frac{\partial \Phi(\cdot, t)}{\partial \eta} \xrightarrow{\varepsilon \rightarrow 0} 2g(t)\delta_0,$$

weakly-* in the space of Radon measures on \mathbb{R} , pointwise in $0 < t < T$. That is,

$$\lim_{\varepsilon \rightarrow 0} \left(\frac{\partial \Phi(\cdot, t)}{\partial \eta}, \varphi \right)_{L^2_\eta(\mathbb{R})} = 2g(t)\varphi(0),$$

for all $\varphi \in C_c(\mathbb{R})$, the space of continuous, compactly supported functions on \mathbb{R} .

Proof. We observe that we can write $\frac{\partial \Phi}{\partial \eta}$ as

$$\frac{\partial \Phi(\eta, t)}{\partial \eta} = 2g(0)K_\varepsilon(\eta, t) + 2 \int_0^t \frac{\partial g(s)}{\partial s} K_\varepsilon(\eta, t-s) ds, \quad (\text{A.4})$$

where

$$K_\varepsilon(\eta, t) := \frac{\partial \operatorname{erfc}(\eta/\sqrt{2\varepsilon t})}{\partial \eta} = -\frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{\varepsilon t}} e^{-\eta^2/(4\varepsilon t)}. \quad (\text{A.5})$$

Since the family $\{K_\varepsilon(\cdot, t)\}_\varepsilon$ is a classical approximation of the identity in the variable η , the first term on the right-hand side of (A.5) converges in the sense of distributions, and also weakly-* in the space of Radon measures, to $2g(0)\varphi(0)$. It is therefore enough to identify the weak-* limit of the second term on the right-hand side of (A.5) as $2(g(t) - g(0))\varphi(0)$, as $\varepsilon \rightarrow 0$.

To this end, we fix $\varphi \in C_c(\mathbb{R})$, and observe that the regularity of g implies

$$\begin{aligned} \left(2 \int_0^t \frac{\partial g(s)}{\partial s} K_\varepsilon(\eta, t-s) ds, \varphi(\eta) \right)_{L^2_\eta(\Omega)} &= 2 \int_{\mathbb{R}} \int_0^t \frac{\partial g(s)}{\partial s} K_\varepsilon(\eta, t-s) ds \varphi(\eta) d\eta \\ &= 2 \int_0^t \frac{\partial g(s)}{\partial s} \int_{\mathbb{R}} K_\varepsilon(\eta, t-s) \varphi(\eta) d\eta ds. \end{aligned}$$

We can conclude the proof if we can bring the limit $\varepsilon \rightarrow 0$ inside the integrals. To justify this step, we first assume that $\varphi \in C^1_c(\mathbb{R})$ and note that, for each $\varepsilon > 0$,

$$\begin{aligned} \left| \int_{\mathbb{R}} K_\varepsilon(\eta, t-s) \varphi(\eta) d\eta \right| &= \int_{\mathbb{R}} \frac{\partial \operatorname{erfc}(\eta/\sqrt{2\varepsilon(t-s)})}{\partial \eta} \varphi(\eta) d\eta \\ &= \int_{\mathbb{R}} \operatorname{erfc} \left(\frac{\eta}{\sqrt{2\varepsilon(t-s)}} \right) \varphi'(\eta) d\eta \leq \int_{\mathbb{R}} |\varphi'(\eta)| d\eta \leq \kappa_\varphi, \end{aligned}$$

since $|\operatorname{erfc}| \leq 1$. hence, we are justified in writing:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left(2 \int_0^t \frac{\partial g(s)}{\partial s} K_\varepsilon(\eta, t-s) ds, \varphi(\eta) \right)_{L^2_\eta(\Omega)} &= 2 \int_0^t \frac{\partial g(s)}{\partial s} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} K_\varepsilon(\eta, t-s) \varphi(\eta) d\eta ds \\ &= 2(g(t) - g(0)) \varphi(0), \end{aligned}$$

which gives the desired result for $\varphi \in C^1_C(\mathbb{R})$. Using the density of $C^1_C(\mathbb{R})$ in $C_C(\mathbb{R})$ then completes the proof. \square

A.2. On a drift-diffusion equation with small diffusivity. In this subsection, we derive various estimates for a drift-diffusion equation in a periodic channel, uniformly in the diffusivity ε .

We consider the following initial-boundary value problem:

$$\left\{ \begin{array}{l} \frac{\partial \Psi}{\partial t} - \varepsilon \frac{\partial^2 \Psi}{\partial \tau^2} - \varepsilon \frac{\partial^2 \Psi}{\partial \eta^2} + \mathcal{U}(\eta, t) \frac{\partial \Psi}{\partial \tau} = G(\tau, \eta, t), \quad 0 < \tau < L_\tau, \eta, t > 0, \\ \Psi \text{ is periodic in } \tau \text{ with period } L_\tau, \\ \Psi = g(\tau, t), \quad \eta = 0, \\ \Psi \rightarrow 0, \quad \text{as } \eta \rightarrow \infty, \\ \Psi = 0, \quad t = 0. \end{array} \right. \quad (\text{A.6})$$

Above, $\varepsilon > 0$ represents again the diffusivity, and $\mathcal{U}(\eta)$, $G(\tau, \eta, t)$, and $g(\tau, t)$ are sufficiently smooth data in the indicated variables such that, for a given $0 < T < \infty$ and for all $0 < t < T$,

$$\left\{ \begin{array}{l} \mathcal{U}|_{\eta=0} = 0, \quad \|G|_{\eta=0}\|_{L^\infty((0, L_\tau) \times (0, T))} \leq \kappa_T, \\ \left\| \frac{\partial^m \mathcal{U}}{\partial \eta^m}(\cdot, t) \right\|_{L^p(0, \infty)} \leq \kappa_T (1 + t^{\frac{1}{2p} - \frac{m}{2}}) (1 + \varepsilon^{\frac{1}{2p} - \frac{m}{2}}), \\ \left\| \frac{\partial^{k+m} G}{\partial \tau^k \partial \eta^m}(\cdot, \cdot, t) \right\|_{L^p((0, L_\tau) \times (0, \infty))} \leq \kappa_T (1 + t^{\frac{1}{2p} - \frac{m}{2}}) \varepsilon^{\frac{1}{2p} - \frac{m}{2}}, \end{array} \right. \quad (\text{A.7})$$

for $1 \leq p \leq \infty$, $k \geq 0$, and $0 \leq m \leq 2$. As before, the boundary data is assumed incompatible with the initial condition in the sense that $g(\tau, 0)$ may not necessarily vanish.

To estimate Ψ solution of (A.6), we will utilize the solution of (A.1) with $g(t)$ replaced by $g(\tau, t)$, denoted by $\Phi(t, \tau, \eta)$. Thanks to Lemma A.2, we have that

$$\left\| \frac{\partial^{k+m} \Phi}{\partial \tau^k \partial \eta^m}(t, \cdot, \cdot) \right\|_{L^p((0, L_\tau) \times (0, \infty))} \leq \kappa_T (1 + t^{\frac{1}{2p} - \frac{m}{2}}) \varepsilon^{\frac{1}{2p} - \frac{m}{2}}, \quad 1 \leq p \leq \infty, \quad k \geq 0, \quad 0 \leq m \leq 2, \quad (\text{A.8})$$

which, in turn, yields the following estimates for Ψ .

Lemma A.4. *Assuming that the data \mathcal{U} , G , and g satisfy (A.7) and are sufficiently regular, we have for all $k \in \mathbb{Z}_+$, $1 < p \leq \infty$,*

$$\left\{ \begin{array}{l} \left\| \frac{\partial^k \Psi}{\partial \tau^k} \right\|_{L^\infty(0,T;L^p((0,L_\tau) \times (0,\infty)))} + \varepsilon^{\frac{1}{2p} + \frac{1}{4}} \left\| \nabla \frac{\partial^k \Psi}{\partial \tau^k} \right\|_{L^2(0,T;L^2((0,L_\tau) \times (0,\infty)))} \leq \kappa_T \varepsilon^{\frac{1}{2p}}, \\ \left\| \frac{\partial^{k+1} \Psi}{\partial \tau^k \partial \eta} \right\|_{L^\infty(0,T;L^2((0,L_\tau) \times (0,\infty)))} + \varepsilon^{\frac{1}{2}} \left\| \nabla \frac{\partial^{k+1} \Psi}{\partial \tau^k \partial \eta} \right\|_{L^2(0,T;L^2((0,L_\tau) \times (0,\infty)))} \leq \kappa_T \varepsilon^{-\frac{1}{4}}, \\ \left\| \frac{\partial^{k+1} \Psi}{\partial \tau^k \partial \eta} \right\|_{L^\infty(0,T;L^1((0,L_\tau) \times (0,\infty)))} \leq \kappa_T, \end{array} \right. \quad (\text{A.9})$$

with $k \geq 0$, for a constant κ_T depending on T and the data, but independent of ε .

Proof. We denote:

$$\tilde{\Psi} := \Psi - \Phi, \quad (\text{A.10})$$

and observe that $\tilde{\Psi}$ satisfies the following initial-boundary value problem:

$$\left\{ \begin{array}{l} \frac{\partial \tilde{\Psi}}{\partial t} - \varepsilon \frac{\partial^2 \tilde{\Psi}}{\partial \tau^2} - \varepsilon \frac{\partial^2 \tilde{\Psi}}{\partial \eta^2} + \mathcal{U}(\eta) \frac{\partial \tilde{\Psi}}{\partial \tau} = G + \varepsilon \frac{\partial^2 \Phi}{\partial \tau^2} - \mathcal{U}(\eta) \frac{\partial \Phi}{\partial \tau}, \quad 0 < \tau < L_\tau, \quad \eta, t > 0, \\ \tilde{\Psi} \text{ is periodic in } \tau \text{ with period } L_\tau, \\ \tilde{\Psi} = 0, \quad \eta = 0, \\ \tilde{\Psi} \rightarrow 0, \quad \text{as } \eta \rightarrow \infty, \\ \tilde{\Psi} = 0, \quad t = 0. \end{array} \right. \quad (\text{A.11})$$

To prove (A.9)₁, we multiply (A.11)₁ by $\tilde{\Psi}^{p-1}$ where $p > 1$ is a simple fraction q/r with an even integer q . Then, by integrating over $(0, L_\tau) \times (0, \infty)$, we find that

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \|\tilde{\Psi}\|_{L^p((0,L_\tau) \times (0,\infty))}^p + \varepsilon(p-1) \int_0^\infty \int_0^{L_\tau} |\nabla \tilde{\Psi}|^2 \tilde{\Psi}^{p-2} d\tau d\eta \\ &= \int_0^\infty \int_0^{L_\tau} \left(G + \varepsilon \frac{\partial^2 \Phi}{\partial \tau^2} - \mathcal{U}(\eta) \frac{\partial \Phi}{\partial \tau} \right) \tilde{\Psi}^{p-1} d\tau d\eta \\ &\leq \left\{ \int_0^\infty \int_0^{L_\tau} \left(G + \varepsilon \frac{\partial^2 \Phi}{\partial \tau^2} - \mathcal{U}(\eta) \frac{\partial \Phi}{\partial \tau} \right)^p d\tau d\eta \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \int_0^{L_\tau} \tilde{\Psi}^p d\tau d\eta \right\}^{\frac{p-1}{p}} \\ &\leq \kappa \left\| G + \varepsilon \frac{\partial^2 \Phi}{\partial \tau^2} - \mathcal{U}(\eta) \frac{\partial \Phi}{\partial \tau} \right\|_{L^p((0,L_\tau) \times (0,\infty))}^p + \kappa \|\tilde{\Psi}\|_{L^p((0,L_\tau) \times (0,\infty))}^p. \end{aligned} \quad (\text{A.12})$$

The hypotheses on the data (A.7) and the bounds (A.8) on $\tilde{\Psi}$ then give:

$$\frac{1}{p} \frac{d}{dt} \|\tilde{\Psi}\|_{L^p((0,L_\tau) \times (0,\infty))}^p + \varepsilon(p-1) \|\tilde{\Psi}^{\frac{p-2}{2}} \nabla \tilde{\Psi}\|_{L^2((0,L_\tau) \times (0,\infty))}^2 \leq \kappa \varepsilon^{\frac{1}{2}} + \kappa \|\tilde{\Psi}\|_{L^p((0,L_\tau) \times (0,\infty))}^p, \quad (\text{A.13})$$

which implies that

$$\left\{ \begin{array}{l} \|\Psi\|_{L^\infty(0,T;L^p((0,L_\tau) \times (0,\infty)))} \leq \kappa_T \varepsilon^{\frac{1}{2p}}, \\ \|\nabla \Psi\|_{L^2(0,T;L^2((0,L_\tau) \times (0,\infty)))} \leq \kappa_T \varepsilon^{-\frac{1}{4}}. \end{array} \right. \quad (\text{A.14})$$

given the continuity of the L^p norm in p , the estimate (A.14)₁ is valid for any $1 < p \leq \infty$.

Next, since any tangential derivative of satisfies an equation similar to (A.11), one can verify (A.9)₁ for $k > 0$ in an analogous manner.

To show (A.9)₂, using the regularity on the data and Φ , we restrict the equation for $\tilde{\Psi}$ to $\eta = 0$ and find that:

$$-\varepsilon \frac{\partial^2 \tilde{\Psi}}{\partial \eta^2} = G - \varepsilon \frac{\partial^2 g}{\partial \tau^2}, \quad \text{at } \eta = 0, \quad (\text{A.15})$$

using also that that $\Phi|_{\eta=0} = g$.

Hence, after differentiating (A.11) in η , we obtain an equation for $\partial \tilde{\Psi} / \partial \eta$ supplemented by a Neumann boundary condition:

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) - \varepsilon \frac{\partial^2}{\partial \tau^2} \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) - \varepsilon \frac{\partial^2}{\partial \eta^2} \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) + \mathcal{U} \frac{\partial}{\partial \tau} \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \\ \quad = -\frac{\partial \mathcal{U}}{\partial \eta} \tilde{\Psi} + \frac{\partial}{\partial \eta} \left(G + \varepsilon \frac{\partial^2 \Phi}{\partial \tau^2} - \mathcal{U} \frac{\partial \Phi}{\partial \tau} \right), \quad 0 < \tau < L_\tau, \eta, t > 0, \\ \frac{\partial \tilde{\Psi}}{\partial \eta} \text{ is periodic in } \tau \text{ with period } L_\tau, \\ \frac{\partial^2 \tilde{\Psi}}{\partial \eta^2} = -\varepsilon^{-1} G + \frac{\partial^2 g}{\partial \tau^2}, \quad \eta = 0, \\ \frac{\partial \tilde{\Psi}}{\partial \eta} \rightarrow 0, \quad \text{as } \eta \rightarrow \infty, \\ \frac{\partial \tilde{\Psi}}{\partial \eta} = 0, \quad t = 0. \end{array} \right. \quad (\text{A.16})$$

We next multiply (A.16)₁ by $\partial \tilde{\Psi} / \partial \eta$ and integrate over $(0, L_\tau) \times (0, \infty)$:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^p((0, L_\tau) \times (0, \infty))}^2 + \varepsilon \left\| \nabla \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^p((0, L_\tau) \times (0, \infty))}^2 &= \varepsilon \int_0^{L_\tau} \left[\frac{\partial^2 \tilde{\Psi}}{\partial \eta^2} \frac{\partial \tilde{\Psi}}{\partial \eta} \right]_{\eta=0} d\tau \\ &+ \int_0^\infty \int_0^{L_\tau} \left(-\frac{\partial \mathcal{U}}{\partial \eta} \tilde{\Psi} + \frac{\partial}{\partial \eta} \left(G + \varepsilon \frac{\partial^2 \Phi}{\partial \tau^2} - \mathcal{U} \frac{\partial \Phi}{\partial \tau} \right) \right) \frac{\partial \tilde{\Psi}}{\partial \eta} d\tau d\eta. \end{aligned} \quad (\text{A.17})$$

Using Lemma B.1, we estimate the first term on the right-hand side of (A.17) as follows:

$$\begin{aligned}
\varepsilon \left| \int_0^{L_\tau} \left[\frac{\partial^2 \tilde{\Psi}}{\partial \eta^2} \frac{\partial \tilde{\Psi}}{\partial \eta} \right]_{\eta=0} d\tau \right| &\leq \left| \int_0^{L_\tau} \left[\left(-G + \varepsilon \frac{\partial^2 g}{\partial \tau^2} \right) \frac{\partial \tilde{\Psi}}{\partial \eta} \right]_{\eta=0} d\tau \right| \\
&\leq \kappa \left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \Big|_{\eta=0} \right\|_{L^2(0, L_\tau)} \leq \kappa \left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^2((0, L_\tau) \times (0, \infty))}^{\frac{1}{2}} \left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{H^1((0, L_\tau) \times (0, \infty))}^{\frac{1}{2}} \\
&\leq \kappa \left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^2((0, L_\tau) \times (0, \infty))} + \kappa \left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^2((0, L_\tau) \times (0, \infty))}^{\frac{1}{2}} \left\| \nabla \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^2((0, L_\tau) \times (0, \infty))}^{\frac{1}{2}} \\
&\leq \kappa + \left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^2((0, L_\tau) \times (0, \infty))}^2 + \kappa \varepsilon^{-\frac{1}{2}} + \kappa \varepsilon^{\frac{1}{2}} \left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^2((0, L_\tau) \times (0, \infty))} \left\| \nabla \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^2((0, L_\tau) \times (0, \infty))} \\
&\leq \kappa \varepsilon^{-\frac{1}{2}} + \left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^2((0, L_\tau) \times (0, \infty))}^2 + \frac{1}{2} \varepsilon \left\| \nabla \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^2((0, L_\tau) \times (0, \infty))}^2.
\end{aligned} \tag{A.18}$$

We can then estimate the second term on the right-hand side of (A.17) as follows:

$$\begin{aligned}
&\left| \int_0^\infty \int_0^{L_\tau} \left(-\frac{\partial \mathcal{U}}{\partial \eta} \tilde{\Psi} + \frac{\partial}{\partial \eta} \left(G + \varepsilon \frac{\partial^2 \Phi}{\partial \tau^2} - \mathcal{U} \frac{\partial \Phi}{\partial \tau} \right) \right) \frac{\partial \tilde{\Psi}}{\partial \eta} d\tau d\eta \right| \\
&\leq \left\{ \begin{array}{l} \left\| \frac{\partial \mathcal{U}}{\partial \eta} \right\|_{L^\infty((0, L_\tau) \times (0, \infty))} \|\tilde{\Psi}\|_{L^2((0, L_\tau) \times (0, \infty))} \\ + \left\| \frac{\partial G}{\partial \eta} \right\|_{L^2((0, L_\tau) \times (0, \infty))} + \varepsilon \left\| \frac{\partial^3 \Phi}{\partial \eta \partial \tau^2} \right\|_{L^2((0, L_\tau) \times (0, \infty))} \\ + \left\| \frac{\partial \mathcal{U}}{\partial \eta} \right\|_{L^\infty((0, L_\tau) \times (0, \infty))} \left\| \frac{\partial \Phi}{\partial \tau} \right\|_{L^2((0, L_\tau) \times (0, \infty))} \\ + \|\mathcal{U}\|_{L^\infty((0, L_\tau) \times (0, \infty))} \left\| \frac{\partial^2 \Phi}{\partial \eta \partial \tau} \right\|_{L^2((0, L_\tau) \times (0, \infty))} \end{array} \right\} \left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^2((0, L_\tau) \times (0, \infty))} \\
&\leq \kappa_T (1 + t^{-\frac{1}{2}}) \varepsilon^{-\frac{1}{4}} \left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^2((0, L_\tau) \times (0, \infty))} \\
&\leq \kappa_T (1 + t^{-\frac{1}{2}}) \varepsilon^{-\frac{1}{2}} + \kappa_T (1 + t^{-\frac{1}{2}}) \left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^2((0, L_\tau) \times (0, \infty))}^2.
\end{aligned} \tag{A.19}$$

Combining (A.17)-(A.19), we find that

$$\begin{aligned}
&\frac{d}{dt} \left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^p((0, L_\tau) \times (0, \infty))}^2 + \varepsilon \left\| \nabla \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^p((0, L_\tau) \times (0, \infty))}^2 \\
&\leq \kappa_T (1 + t^{-\frac{1}{2}}) \varepsilon^{-\frac{1}{2}} + \kappa_T (1 + t^{-\frac{1}{2}}) \left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^2((0, L_\tau) \times (0, \infty))}^2.
\end{aligned} \tag{A.20}$$

Using the Gronwall's Lemma with the integrating factor $\exp(-\kappa_T t - 2\kappa_T t^{1/2})$, we deduce that

$$\left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^\infty(0, T; L^2((0, L_\tau) \times (0, \infty)))} + \varepsilon^{\frac{1}{2}} \left\| \nabla \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^2(0, T; L^2((0, L_\tau) \times (0, \infty)))} \leq \kappa_T \varepsilon^{-\frac{1}{4}},$$

and (A.9)₂ with $k = 0$ follows from the uniform bounds (A.8) on Φ . (A.9)₂ for $k > 0$ can be verified in an analogous manner.

To verify (A.9)₃, we introduce a standard convex regularization of the absolute value, \mathcal{F}_λ , $\lambda > 0$, defined as

$$\mathcal{F}_\lambda(x) = \sqrt{\lambda^2 + x^2}. \quad (\text{A.21})$$

and obtain:

$$\begin{aligned} & \frac{d}{dt} \int_0^{L_\tau} \int_0^\infty \mathcal{F}_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) d\eta d\tau \\ &= \int_0^{L_\tau} \int_0^\infty \mathcal{F}'_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \frac{\partial}{\partial t} \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) d\eta d\tau \\ &= \varepsilon \int_0^{L_\tau} \int_0^\infty \mathcal{F}'_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \Delta \frac{\partial \tilde{\Psi}}{\partial \eta} d\eta d\tau - \int_0^{L_\tau} \int_0^\infty \mathcal{F}'_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \mathcal{U} \frac{\partial}{\partial \tau} \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) d\eta d\tau \\ & \quad + \int_0^{L_\tau} \int_0^\infty \mathcal{F}'_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) (\text{right-hand side of (A.16)})_1 d\eta d\tau. \end{aligned} \quad (\text{A.22})$$

Now convexity of \mathcal{F}_λ implies that

$$\begin{aligned} \Delta \left(\mathcal{F}_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \right) &= \mathcal{F}'_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \Delta \frac{\partial \tilde{\Psi}}{\partial \eta} + \mathcal{F}''_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \left\{ \left(\frac{\partial^2 \tilde{\Psi}}{\partial \tau \partial \eta} \right)^2 + \left(\frac{\partial^2 \tilde{\Psi}}{\partial \eta^2} \right)^2 \right\} \\ &\geq \mathcal{F}'_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \Delta \frac{\partial \tilde{\Psi}}{\partial \eta}. \end{aligned}$$

Then, integrating by parts, the first term on the right-hand side of (A.22) can be estimated as

$$\begin{aligned} & \varepsilon \int_0^{L_\tau} \int_0^\infty \mathcal{F}'_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \Delta \frac{\partial \tilde{\Psi}}{\partial \eta} d\eta d\tau \\ & \leq \varepsilon \int_0^{L_\tau} \int_0^\infty \Delta \left(\mathcal{F}_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \right) d\eta d\tau = \varepsilon \int_0^{L_\tau} \left[\mathcal{F}'_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \frac{\partial^2 \tilde{\Psi}}{\partial \eta^2} \right]_{\eta=0} d\tau. \end{aligned} \quad (\text{A.23})$$

Using periodicity in the τ -direction and the fact that \mathcal{U} is a function in η only, we observe that the second term on the right-hand side of (A.22) is identically zero:

$$\begin{aligned} & \int_0^{L_\tau} \int_0^\infty \mathcal{F}'_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \mathcal{U} \frac{\partial}{\partial \tau} \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) d\eta d\tau \\ &= \int_0^\infty \int_0^{L_\tau} \frac{\partial}{\partial \tau} \left(\mathcal{F}_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \right) \mathcal{U} d\tau d\eta = \int_0^\infty \left[\mathcal{F}_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \mathcal{U} \right]_{\tau=0}^{\tau=L_\tau} d\eta = 0. \end{aligned} \quad (\text{A.24})$$

We can finally conclude that:

$$\begin{aligned} & \frac{d}{dt} \int_0^{L_\tau} \int_0^\infty \mathcal{F}_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) d\eta d\tau \\ & \leq \varepsilon \int_0^{L_\tau} \left[\mathcal{F}'_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \frac{\partial^2 \tilde{\Psi}}{\partial \eta^2} \right]_{\eta=0} d\tau + \int_0^{L_\tau} \int_0^\infty \mathcal{F}'_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) (\text{right-hand side of (A.16)})_1 d\eta d\tau. \end{aligned} \quad (\text{A.25})$$

The equation for $\tilde{\Psi}$ at the boundary $\eta = 0$ yields a uniform bound in ε on $\varepsilon \partial^2 \tilde{\Psi}_2 / \partial \eta^2|_{\eta=0}$ in $L^\infty(0, T; L^1(0, L_\tau))$. It follows that the right-hand side of (A.16) is in $L^\infty(0, T; L^1((0, L_\tau) \times (0, \infty)))$ uniformly in ε . Then, since $|\mathcal{F}'_\lambda(\cdot)| \leq 1$ in \mathbb{R} as well, we conclude that

$$\frac{d}{dt} \int_0^{L_\tau} \int_0^\infty \mathcal{F}_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) d\eta d\tau \leq \kappa_T, \quad \text{independent of } \lambda.$$

Since the integral of \mathcal{F}_λ over $(0, L_\tau) \times (0, \infty)$ is positive at each time and $\partial \tilde{\Psi} / \partial \eta = 0$ at $t = 0$, we see that

$$\lim_{\lambda \rightarrow 0} \int_0^{L_\tau} \int_0^\infty \mathcal{F}_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) d\eta d\tau \leq \kappa_T, \quad \text{uniformly in } 0 < t < T. \quad (\text{A.26})$$

Using (A.8) and (A.9)₂, thanks to the Lebesgue dominated convergence theorem, we deduce from (A.26) that

$$\left\| \frac{\partial \tilde{\Psi}}{\partial \eta} \right\|_{L^1((0, L_\tau) \times (0, \infty))} = \int_0^{L_\tau} \int_0^\infty \lim_{\lambda \rightarrow 0} \mathcal{F}_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) = \lim_{\lambda \rightarrow 0} \int_0^{L_\tau} \int_0^\infty \mathcal{F}_\lambda \left(\frac{\partial \tilde{\Psi}}{\partial \eta} \right) \leq \kappa_T,$$

independent of ε , t , and λ , uniformly in time $0 < t < T$. Hence (A.9)₃ with $k = 0$ follows from (A.8) and the inequality above. Equation (A.9)₃ for $k > 0$ can be proved similarly as well. \square

APPENDIX B. A FEW AUXILIARY RESULTS

In this Appendix, we collect a few auxiliary results, which are needed for the analysis of previous sections.

Lemma B.1 below contains a well-known trace inequality, mostly used in the special case where $p = q = q' = 2$. A complete proof of this fact and Corollary B.2 can be found in [56].

Lemma B.1 (Trace lemma). *Let $p \in (1, \infty)$, $q \in [1, \infty]$, and let q' be Hölder conjugate to q . Then, there exists a constant $C = C(\Omega)$ such that for all $f \in W^{1,p}(\Omega)$,*

$$\|f\|_{L^p(\Gamma)} \leq C \|f\|_{L^{(p-1)q}(\Omega)}^{1-\frac{1}{p}} \|f\|_{W^{1,q'}(\Omega)}^{\frac{1}{p}}.$$

If, in addition, $f \in W^{1,p}(\Omega)$ has mean zero or $f \in W_0^{1,p}(\Omega)$,

$$\|f\|_{L^p(\Gamma)} \leq C \|f\|_{L^{(p-1)q}(\Omega)}^{1-\frac{1}{p}} \|\nabla f\|_{L^{q'}(\Omega)}^{\frac{1}{p}}.$$

Corollary B.2. *For any $\mathbf{v} \in H$,*

$$\|\mathbf{v}\|_{L^2(\Gamma)} \leq C \|\mathbf{v}\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \mathbf{v}\|_{L^2(\Omega)}^{\frac{1}{2}}$$

and for any $\mathbf{v} \in V \cap H^2(\Omega)$,

$$\|\text{curl } \mathbf{v}\|_{L^2(\Gamma)} \leq C \|\text{curl } \mathbf{v}\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \text{curl } \mathbf{v}\|_{L^2(\Omega)}^{\frac{1}{2}}.$$

We recall that $H = \{\mathbf{v} \in L^2(\Omega) \mid \text{div } \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}$, and $V = \{\mathbf{v} \in H_0^1(\Omega) \mid \text{div } \mathbf{v} = 0\}$.

Lastly, we state for the reader's convenience Young's inequality for multiple factors:

Lemma B.3. *Let $a, b, c \geq 0$, $p, q, p_1, q_1 \geq 1$ with $p^{-1} + q^{-1} = p_1^{-1} + p_2^{-1} = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

$$abc \leq \frac{a^{pp_1}}{pp_1} + \frac{b^{pp_2}}{pp_2} + \frac{c^q}{q}.$$

APPENDIX C. VORTICITY ACCUMULATION ON THE BOUNDARY

We close the Appendices with a short discussion of a result that is used to derive a quantitative estimate for the vorticity production at the boundary that persists in the vanishing viscosity limit.

It is shown in [54] that the classical vanishing viscosity limit, that is, convergence of the NSE solution to an EE solution in the energy norm, holds if and only if vorticity accumulates on the boundary in the manner described in Theorem C.1. (The specific 3D form of this condition is derived in [56].) In Theorem C.1, μ is the measure supported on Γ for which $\mu|_\Gamma$ corresponds to the normalized Lebesgue measure on Γ (arc length in 2D, surface area in 3D). Then μ is also a member of $H^1(\Omega)^*$, the dual space to $H^1(\Omega)$.

Theorem C.1. *Let Ω be a bounded domain in \mathbb{R}^2 or \mathbb{R}^3 of class C^2 . Then,*

$$\begin{aligned} 2D : \quad & \mathbf{u}^\varepsilon \rightarrow \mathbf{u}^0 \text{ in } L^\infty(0, T; H) \\ & \iff \operatorname{curl} \mathbf{u}^\varepsilon \rightarrow \operatorname{curl} \mathbf{u}^0 - (\mathbf{u}^0 \cdot \boldsymbol{\tau})\mu \text{ in } L^\infty(0, T; (H^1(\Omega)^2)^*), \\ 3D : \quad & \mathbf{u}^\varepsilon \rightarrow \mathbf{u}^0 \text{ in } L^\infty(0, T; H) \\ & \iff \operatorname{curl} \mathbf{u}^\varepsilon \rightarrow \operatorname{curl} \mathbf{u}^0 + (\mathbf{u}^0 \times \mathbf{n})\mu \text{ in } L^\infty(0, T; (H^1(\Omega)^3)^*). \end{aligned}$$

The following simple corollary of Theorem C.1 is derived in [56].

Corollary C.2. *Under the hypotheses of Theorem C.1, if $\mathbf{u}^\varepsilon \rightarrow \mathbf{u}^0$ in $L^\infty(0, T; H)$ and $\|\operatorname{curl} \mathbf{u}^\varepsilon - \operatorname{curl} \mathbf{u}^0\|_{L^\infty(0, T; L^1(\Omega))} \leq \kappa_T$ then*

$$\begin{aligned} 2D : \quad & \operatorname{curl} \mathbf{u}^\varepsilon \rightarrow \operatorname{curl} \mathbf{u}^0 - (\mathbf{u}^0 \cdot \boldsymbol{\tau})\mu \text{ in } L^\infty(0, T; \mathcal{M}(\overline{\Omega})), \\ 3D : \quad & \operatorname{curl} \mathbf{u}^\varepsilon \rightarrow \operatorname{curl} \mathbf{u}^0 + (\mathbf{u}^0 \times \mathbf{n})\mu \text{ in } L^\infty(0, T; \mathcal{M}(\overline{\Omega})), \end{aligned}$$

where $\mathcal{M}(\overline{\Omega})$ is the space of Radon measures on $\overline{\Omega}$,

Above, $\boldsymbol{\tau}$ is the unit tangent vector to the boundary, defined as $J\mathbf{n}$, where \mathbf{n} is the unit outer normal, and J is rotation counterclockwise by $\pi/2$.

The regularity of the boundary $\partial\Omega$ in the results above is sufficient for the applications in this manuscript, but it is not expected to be optimal. In fact, results of De Giorgi on weak convergence of gradients suggest that similar statements hold for much rougher domains, namely sets of finite perimeter. We do not investigate this point further in this work.

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¹ DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISVILLE, 328 NATURAL SCIENCES BUILDING, LOUISVILLE, KY 40292, U.S.A.

² DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, 900 UNIVERSITY AVE., RIVERSIDE, CA 92521, U.S.A.

³ INSTITUTO DE MATEMATICA, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, CAIXA POSTAL 68530, 21941-909, RIO DE JANEIRO, RJ, BRAZIL

⁴ DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNIVERSITY PARK, PA 16802, U.S.A.

E-mail address: gungmin.gie@louisville.edu

E-mail address: kelliher@math.ucr.edu

E-mail address: mlopes@im.ufrj.br

E-mail address: alm24@psu.edu

E-mail address: hlopes@im.ufrj.br