

PROPAGATION OF REGULARITY OF LEVEL SETS FOR A CLASS OF ACTIVE TRANSPORT EQUATIONS

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ABSTRACT. We prove that for any $\alpha \in (0, 1)$, the $C^{1,\alpha}$ regularity of level sets for solutions to a class of active transport equations is propagated over the existence time of the solution. This extends a recent result of Bertozzi, Garnett, Laurent, and Verdera for patch boundary regularity for the aggregation equation.

CONTENTS

1. Introduction	1
2. Main results	4
3. Notation, conventions, and definitions	7
4. Sufficient families	9
5. Serfati's linear algebra lemma	10
6. Lemmas involving the velocity gradient	11
7. Preparing the initial data	13
8. Pushforwards and transport	15
9. Propagation of striated regularity: proof of Theorem 2.3	21
10. Removability of singularities: proof of Theorem 2.5	30
11. Patch boundary regularity: comparison to [2]	31
Acknowledgements	32
References	32

1. INTRODUCTION

1.1. **Active transport equations.** We consider active transport equations of the form,

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = F(\rho), \\ v = K * \rho, \\ \rho(0) = \rho_0 \end{cases} \quad (1.1)$$

in \mathbb{R}^d , $d \geq 2$. We assume that the function $F: \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable with F' locally Lipschitz and $F(0) = 0$. To fix terminology, we refer to ρ as the density of an unspecified substance (though we impose no requirement on the sign of ρ) and v as the velocity field.

Throughout this paper we fix $\alpha \in (0, 1)$.

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Our purpose is to show that if (subsets of) the level sets of the initial density (that is, sets on which ρ_0 is constant) have $C^{1,\alpha}$ regularity as a curve in 2D, surface in 3D, or hypersurface for $d \geq 4$, then they retain that regularity for the existence time of the weak solution. Hence, we employ a kernel that neither smooths the density nor introduces additional singularities, as smoothing would trivialize the problem and adding singularities would prevent the propagation of level set regularity. In short, this means that the kernel must impart to v one more derivative of regularity than that possessed by ρ . A not quite completely general form of such a kernel is

$$K = R\nabla\Phi,$$

where Φ is the fundamental solution of the Laplacian, or Newtonian potential (so $\Delta\Phi = \delta$), and R is a rotation matrix.

In the 2D special case in which $F \equiv 0$ and R gives rotation counterclockwise by 90° (so $K = \nabla^\perp\Phi$), we obtain the 2D Euler equations without forcing:

$$\begin{cases} \partial_t\omega + v \cdot \nabla\omega = 0, \\ v = \nabla^\perp\Phi * \omega, \\ \omega(0) = \omega_0, \end{cases} \quad (1.2)$$

where we write ω in place of ρ , and where $\nabla^\perp := (-\partial_2, \partial_1)$. Here, $\omega = \text{curl } v := \partial_1 v^2 - \partial_2 v^1$ is the vorticity and $\text{div } v = 0$. This was the first instance of an active scalar equation for which the propagation of *striated regularity*, a generalization of the level set regularity in this paper, was demonstrated—by Chemin [7, 8]. This gave as a special case that the regularity of the boundary of a vortex patch (initial vorticity the characteristic function of a bounded domain) is propagated over time (also shown in [3]).

1.2. Aggregation equation. References [7, 8, 20] and all subsequent studies of the vortex patch problem ([11, 14, 12, 1], for instance) very much rely upon v being divergence-free, both to derive identities related to transport equations and to obtain estimates in negative Hölder spaces of the regularity of solutions to the resulting transport equations, estimates that do not hold when $\text{div } v \neq 0$. In this paper, we treat the complementary case corresponding to $R = -I$, in which $\text{curl } v = 0$ but $\text{div } v \neq 0$, choosing

$$K = -\nabla\Phi. \quad (1.3)$$

Hence, we have a type of gradient flow, in which $\text{div } v = -\rho$.

This is the kernel appearing in the aggregation equation with Newtonian potential (the inviscid form of a limiting case of the Keller-Segel equation),

$$\begin{cases} \partial_t\rho + v \cdot \nabla\rho = \rho^2, \\ v = -\nabla\Phi * \rho, \\ \rho(0) = \rho_0. \end{cases} \quad (1.4)$$

Then $\text{div}(\rho v) = v \cdot \nabla\rho + \rho \text{div } v = v \cdot \nabla\rho - \rho^2$, so $(1.4)_1$ can be written in the usual divergence form, $\partial_t\rho + \text{div}(\rho v) = 0$. We see that $F(\rho) = \rho^2$ in (1.4) , but we work with a general F , as it creates no significant additional difficulties and, in fact, the structure of the associated transport equations are a little clearer in the more general form. It also makes more transparent the comparison to the approach in [2], in which (1.4) is transformed into $(1.1, 1.3)$ with $F = \rho(\rho - 1) \equiv 0$ for patch data (see Section 11).

To get a feel for the problem, let us consider the situation somewhat formally at first. Let us assume that v possesses a unique flow map, η ; that is, $\eta: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies

$$\partial_t \eta(t, x) = v(t, \eta(t, x)), \quad \eta(0, x) = x. \quad (1.5)$$

Then (1.1)_{1,3} can be written as

$$\frac{d}{dt} \rho(t, \eta(t, x)) = F(\rho(t, \eta(t, x))), \quad \rho(0, x) = \rho_0(x) \quad (1.6)$$

or in integral form as

$$\rho(t, \eta(t, x)) = \rho_0(x) + \int_0^t F(\rho(s, \eta(s, x))) ds. \quad (1.7)$$

Fixing $x \in \mathbb{R}^d$ and defining $\bar{\rho}(t) = \rho(t, \eta(t, x))$, we have $\bar{\rho}'(t) = F(\bar{\rho}(t))$, $\bar{\rho}(0) = \rho_0(x)$. Since F is locally Lipschitz (we will also need F' locally Lipschitz later in the proof of Proposition 8.2), there exists a unique solution at least up to the possibly infinite time,

$$T^* = T^*(\|\rho_0\|_{L^\infty}, F). \quad (1.8)$$

If $F \geq 0$ then it is easy to see that T^* is a decreasing function of its first argument.

For example, when $F(\rho) = \rho^2$ as in (1.4), we find (see, for instance, the discussion in [4]),

$$\bar{\rho}(t) = \rho(t, \eta(t, x)) = \frac{\rho_0(x)}{1 - \rho_0(x)t},$$

and $T^* = \|\rho_0^+\|_{L^\infty}^{-1}$, where ρ_0^+ is the positive part of ρ_0 . Solutions are global-in-time if $\rho_0 \leq 0$.

These formal considerations can be made rigorous and lead to Theorem 1.2, below.

Definition 1.1. Fix $T > 0$ and let $v: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a velocity field that is log-Lipschitz in space. Let $\eta: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be its unique flow map (as in (1.5)). We say that ρ is a Lagrangian solution to (1.1, 1.3) with initial density ρ_0 if (1.7) holds and $\rho(t, x) = -\operatorname{div} v(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$.

Existence of weak solutions for the particular kind of striated regularity we will impose on the initial data will appear as part of the proof of Theorem 2.3. In its proof, however, we use the following well-posedness result, applied to smooth initial data.

Theorem 1.2. Let $\rho_0 \in L^\infty(\mathbb{R}^d)$ have compact support, $K_0 \subseteq \mathbb{R}^d$. Fix $T > 0$ with $T < T^*$. There exists a unique Lagrangian solution to (1.1, 1.3), and $\rho(t)$ remains compactly supported in some $K(T) \subseteq \mathbb{R}^d$, with $K(T)$ depending only upon T , K_0 , and $\|\rho_0\|_{L^\infty}$. Moreover,

$$\|\rho(t)\|_{L^p} \leq C \|\rho_0\|_{L^p} \quad (1.9)$$

for all $p \in [1, \infty]$ and $t \in (0, T)$, where $C = C(T, p, K_0, \|\rho_0\|_{L^\infty})$.

If ρ_0 also lies in $C^{k, \alpha}(\mathbb{R}^d)$ for some $k \geq 0$ then $\rho \in L^\infty(0, T; C^{k, \alpha}(\mathbb{R}^d))$. When $k \geq 1$, $\rho \in C^k(0, T; C^{k, \alpha}(\mathbb{R}^d))$ and is also the unique, classical Eulerian solution.

Proof. The proof in the case of the aggregation equation ($F(\rho) = \rho^2$) is given in [4]; for a slightly more general case, see [10]. Both proofs are easily adapted to allow the general form in (1.1)₁. We note here only that we imposed the restriction that $F(0) = 0$ so that the compact support of $\rho(t)$ will be maintained for the full time of existence. \square

We also have the following formulation of a weak Eulerian solution to (1.1, 1.3):

Definition 1.3. We say that $\rho \in C([0, T]; L^2(\mathbb{R}^d))$ is a weak Eulerian solution to (1.1, 1.3) with initial density $\rho_0 \in L^2(\mathbb{R}^d)$ if $\rho(0) = \rho_0$ and

$$\int_{(0, T) \times \mathbb{R}^d} (\rho \partial_t \varphi + \rho v \cdot \nabla \varphi + (F(\rho) - \rho^2) \varphi) = 0$$

for all $\varphi \in C_c^\infty((0, T) \times \mathbb{R}^d)$. That is, ρ satisfies (1.1, 1.3) as a distribution on $(0, T) \times \mathbb{R}^d$.

2. MAIN RESULTS

By *patch data* in the context of active scalar equations we mean that the active scalar, ρ in the case of (1.1), is the characteristic function of a domain. The interest in “patch problems,” by which we mean the question of whether the regularity of the patch boundary is maintained over time, seems to have originated in [22], where Yudovich first formulated it for the 2D Euler equations. Interest was renewed by the survey paper [16], and the question was answered affirmatively for the 2D Euler equations by Chemin in [7, 8].

Actually, Chemin did much more in [7, 8] than address the patch problem. He showed, roughly speaking, that initial vorticity having C^α regularity in a set of directions measured by a family of C^α vector fields that together foliate the plane maintain C^α regularity in the directions of the pushed forward vector fields, which themselves maintain their C^α regularity. This kind of regularity he called *striated*. In the case of patch data for a $C^{1, \alpha}$ boundary, the key C^α vector field would be a tangent vector field on the boundary, extended somewhat arbitrarily to be C^α throughout the plane. The pushforward of this vector field would remain tangent to the boundary of the patch, and hence imply the $C^{1, \alpha}$ regularity of the patch boundary for all time.

Ideally, we would prove a result for the aggregation equation as general as that proved by Chemin for the 2D Euler equations. This is not, however, possible for reasons that we will explain subsequently. What we can show, however, is the following more limited result (stated more precisely and more fully in Theorem 2.3):

Main result roughly stated: Assume that the level sets of ρ_0 are composed of level sets of $C^{1, \alpha}$ functions, ψ_1, \dots, ψ_N (in the language of Definition 2.1, each ψ_j is level-set compatible with ρ_0). Then the level sets of $\rho(t)$ are composed of the level sets of the transported functions, $\psi_1(t), \dots, \psi_N(t)$, whose level sets remain $C^{1, \alpha}$ for all time. Very roughly speaking, this says that if the level sets of ρ_0 are $C^{1, \alpha}$ then the level sets of $\rho(t)$ remain $C^{1, \alpha}$ for all time.

For patch data, if $\rho_0 = \mathbb{1}_\Omega$, where Ω is a bounded domain with a $C^{1, \alpha}$ boundary, we can use one $C^{1, \alpha}$ function, ψ_1 , that defines $\partial\Omega$ in that $\partial\Omega = \psi_1^{-1}(0)$. If there are multiple patches, possibly nested, then we simply use one such function for each patch.

So in Theorem 2.3, we assume there are N non-overlapping domains of irregularities in ρ_0 characterized in this manner by ψ_1, \dots, ψ_N , but on the remainder of the plane, ρ_0 is $C^{1, \alpha}$. A difficulty that arises, however, is that the transported functions, $\psi_1(t), \dots, \psi_N(t)$, are only Lipschitz continuous, even though their level sets turn out to be $C^{1, \alpha}$. To obtain this regularity, we need to introduce another set of vector fields, Z_1, \dots, Z_N , which are parallel to $\nabla \psi_j$ for all time and which remain C^α . As applied to the patch problem, then, we use the C^α regularity of each Z_j , which will be normal to the patch boundary, to give the $C^{1, \alpha}$ regularity of the boundary.

A precise statement of all this is in Definition 2.1 and Theorem 2.3.

Definition 2.1. Let $f, g \in L^\infty(\mathbb{R}^d)$ and U be an open subset of \mathbb{R}^d . Recalling that measurable functions are really equivalence classes of functions defined only up to sets of measure 0, we

say that f is level-set-compatible with g on U if for some representatives of their respective equivalence classes, which we will continue to call f and g , each level set of f is contained in a level set of g , up to sets of measure zero. (So, for instance, any f is compatible with a constant function.)

Definition 2.2. We say that an initial compactly supported density $\rho_0 \in L^\infty(\mathbb{R}^d)$ is suitable if there exists a collection $(\psi_j)_{j=1}^N$ of $C^{1,\alpha}(\mathbb{R}^d)$ -functions with the following properties:

- Each ψ_j is level-set-compatible with ρ_0 on U_j , the interior of $\text{supp } \psi_j$.
- For each $j \leq N$, U_j is connected and intersects no other U_k .
- There exists an open $V \subseteq \mathbb{R}^d$, which, together with the U_j 's, cover \mathbb{R}^d .
- On V , ρ_0 is $C^{1,\alpha}$.
- There exists $r_0 > 0$ such that for all j and all $x \in \partial U_j$, the ball $B_{r_0}(x) \subseteq V$.
- $|\nabla \psi_j| \geq C_0$ on $U_j \setminus V$ for some fixed $C_0 > 0$.

Theorem 2.3. Let ρ_0 be a suitable initial density, as in Definition 2.2, and let ρ be the unique Lagrangian solution to (1.1, 1.3) given by Theorem 1.2 on $[0, T]$ for some $T < T^*$. Then ρ is also a weak Eulerian solution to (1.1, 1.3) as in Definition 1.3 and the following hold:

- (a) $\nabla v \in L^\infty((0, T) \times \mathbb{R}^d)$ with $\|\nabla v(t)\|_{L^\infty} \leq Ce^{Ct}$, where C depends only upon ρ_0 .
(b) Let $\psi_j(t)$ be ψ_j transported by the flow map for v . Then for all $t \in [0, T]$,

$$\|\nabla \psi_j(t)\|_{L^\infty} \leq \|\nabla \psi_j(0)\|_{L^\infty} e^{e^{C(T)}t}, \quad (2.1)$$

$\psi_j(t)$ is level-set-compatible with $\rho(t)$ on the interior of $\text{supp } \psi_j(t)$ and the level sets of $\psi_j(t)$ are $C^{1,\alpha}$. Here, $C(T)$ depends continuously on F , ρ_0 , and T .

- (c) For each j there exists a solution $Z_j \in L^\infty(0, T; C^\alpha(\mathbb{R}^d))$ to

$$\partial_t Z_j + v \cdot \nabla Z_j = -Z_j \cdot \nabla v - \rho Z_j, \quad Z_j(0) = Y_{0,d}^j, \quad (2.2)$$

and $Z_j(t)$ is non-vanishing and parallel to $\nabla \psi_j(t)$ for all $t \in [0, T]$. Here, $Y_{0,d}^{(j)} = \wedge_{i < d} Y_{0,i}^{(j)}$ is defined in Section 3 using a sufficient family of vector field in Definition 4.2.

If $\rho_0 = c\mathbb{1}_\Omega$, then we see that $\rho(t) = c(t)\mathbb{1}_{\Omega_t}$, where $\Omega_t = \eta(t, \Omega)$ and the value of $c(t)$ is derived from (1.6); that is, a patch remains a patch over time. Corollary 2.4 shows that the boundary of a patch maintains $C^{1,\alpha}$ regularity, extending the result in [2] to a general F .

Corollary 2.4. Assume that $\rho_0 = c\mathbb{1}_\Omega$ for some constant c , where Ω is a bounded domain in \mathbb{R}^d with $C^{1,\alpha}$ boundary and having a finite number of connected components. Let $\rho = c(t)\mathbb{1}_{\Omega_t}$ be the unique Lagrangian solution to (1.1, 1.3) given by Theorem 1.2. Then $\partial\Omega_t$ is $C^{1,\alpha}$ for all $t \in (0, T^*)$.

Were ρ_0 in $C^{1,\alpha}$ on all of \mathbb{R}^d , Theorem 1.2 would give $\nabla v(t)$ in $C^\alpha(\mathbb{R}^d)$ for all time, whereas for the solutions given by Theorem 2.3, $\nabla v(t)$ is only in $L^\infty(\mathbb{R}^d)$. Theorem 2.5 shows that $\nabla v(t)$ is $C^{1,\alpha}$ in the same directions in which $\rho(t)$ is $C^{1,\alpha}$. For patch data, this is no surprise, since ∇v is C^∞ inside and outside the patch, but for more general initial data it is perhaps somewhat surprising, since v is recovered from ρ globally by (1.1)₂, and physically one views the active scalar ρ as carrying the irregularities in the solution.

A version of Theorem 2.5 for 2D Euler is due to Serfati [20].

Theorem 2.5. Let ρ, v, η be as in Theorem 2.3. There exists a matrix field $A \in L^\infty(0, T; C^\alpha(\mathbb{R}^d))$ such that for all $t \in [0, T]$,

$$\nabla v(t) - \rho(t)A(t) \in C^\alpha(\mathbb{R}^d). \quad (2.3)$$

The matrix A of Theorem 2.5 has a very simple form: see Remark 10.1.

From Theorems 2.3 and 2.5 we will establish Theorem 2.6.

Theorem 2.6. *Let ρ be the solution to (1.1, 1.3) given in Theorem 2.3. If $\rho_0 \in C^{k,\alpha}(W)$ for some open $W \in \mathbb{R}^d$, where $k = 0$ or 1 , then $\rho(t) \in C^{k,\alpha}(\eta(t, W))$ for all $t \in [0, T]$ with a norm bounded over $[0, T]$.*

2.1. Overview. A logical starting point for any analysis of the patch problem or its generalizations is to describe the level sets using a function that has greater regularity than ρ_0 itself (which may have no regularity in directions perpendicular to its level sets), transport this function by the flow map, and use its regularity to indicate the regularity of the level sets at time t . In Theorem 2.3, this could be ψ_j , which has $C^{1,\alpha}$ regularity, except that $\nabla\psi_j(t)$ fails to have $C^{1,\alpha}$ regularity for $t > 0$.

For an aggregation patch, this difficulty is overcome in [2] by adding a carefully chosen forcing term to the transport equation for ψ_j , which leads to an evolution equation for $\nabla\psi_j$. The special forcing term cancels a singularity that would otherwise appear in this evolution equation and allows the authors of [2] to show that the C^α regularity of the boundary persists over time. (In Section 11 we give a more in-depth explanation.)

As we will see, the evolution equation in [2] is actually that of the wedge product of $d - 1$ linearly independent vector fields initially tangent to the patch boundary that are each separately pushed forward by the flow map. This is in accordance with the approach taken by Chemin for the 2D Euler equations in [8] and in higher dimension by Danchin in [11], used to eliminate the apparent singularity that occurs in the evolution equation for $\nabla\psi_j$.

What allows the approach in [8, 11] to work is that the pushforward of any vector field that is tangent to the level sets initially remains tangent to the transported level sets for all time. Hence, if we construct a complete set of vector fields, (Y_1, \dots, Y_{d-1}) , tangent to the level sets, we can push these forward by the flow to serve as a framework against which to assess the regularity of the level sets. Topological considerations make it clear that one set of vector fields will usually not suffice, but rather an entire family will be needed. The idea of using such a family goes back to Bony [5] (see the remarks at the end of Chapter 5 of [9]).

In treating the initial data, however, we will encounter a substantial complication that does not appear for the Euler equations, and it is this complication that makes it impossible to treat irregularities in the initial data as general as those treated for the Euler equations.

To explain this complication, we must first define what we mean by $Y \cdot \nabla\rho$ when ρ lies only in L^∞ with $Y \in C^\alpha(\mathbb{R}^d)$ and $\operatorname{div} Y \in L^\infty(\mathbb{R}^d)$, so $Y \cdot \nabla\rho$ does not make sense as a pointwise product. We interpret this product in the sense of distributions as

$$Y \cdot \nabla\rho := \operatorname{div}(\rho Y) - \rho \operatorname{div} Y.$$

On the right-hand side, $\operatorname{div}(\rho Y)$ is a distributional derivative (and so a distribution) and $\rho \operatorname{div} Y$ is a regular distribution. Equivalently, we can define $Y \cdot \nabla\rho$ by its action on test functions:

$$(Y \cdot \nabla\rho, \varphi) = -(\rho, \operatorname{div}(\varphi Y)) \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^d). \quad (2.4)$$

Now, suppose that $Y \cdot \nabla\rho = 0$. We would like to interpret this as meaning that ρ is constant a.e. in directions given by Y , as it would if Y and ρ were sufficiently smooth: the unique flow map for Y would determine its directions and $Y \cdot \nabla\rho = 0$ would mean that ρ is constant along the flow lines. (Or, the flow lines form level sets of ρ .) But Y is only C^α , so flow maps exist for it but need not be unique. And even were Y smooth enough to give

unique flow lines, $Y \cdot \nabla \rho = 0$ would only mean that $(\rho, \operatorname{div}(\varphi Y)) = 0$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, and it is not easy to show that this means that ρ is constant a.e. along the flow lines.

The true difficulty with $Y \cdot \nabla \rho$, though, lies not in its interpretation, but in preparing the initial data in such a way that $Y_0 \cdot \nabla \rho_{0,n}$ —where $(\rho_{0,n})$ is a sequence of approximating, more regular initial densities—is controlled by $Y_0 \cdot \nabla \rho_0$ in the appropriate norm. In treating the 2D Euler equations (1.2) in [7, 8], the appropriate norm is the negative Hölder space, $C^{\alpha-1}$, which allows $\omega_{0,n}$ to be prepared in the standard way by mollification, leading easily to the bound, $\|\operatorname{div}(\omega_{0,n} Y_0)\|_{C^{\alpha-1}} \leq C \|\operatorname{div}(\omega_0 Y_0)\|_{C^{\alpha-1}}$. This norm is appropriate, because $\operatorname{div}(\omega_0 Y_0)$ is purely transported by the flow (so it is used instead of $Y_0 \cdot \nabla \omega_0$), and the velocity field is divergence-free, which allows the propagation of regularity in the $C^{\alpha-1}$ norm.

As we will see in Section 8, however, neither $Y_0 \cdot \nabla \rho_0$ nor $\operatorname{div}(\rho_0 Y_0)$ are purely transported by the flow for (1.1, 1.3). We can easily propagate the L^∞ norm of $Y_0 \cdot \nabla \rho_0$ and $\operatorname{div} Y_0$, but simple mollification of ρ_0 to obtain $\rho_{n,0}$ will not lead to any control on $Y_0 \cdot \nabla \rho_{n,0}$ in L^∞ , so this is of no use. Conceptually, we avoid these difficulties by using a family of $C^{1,\alpha}$ hypersurfaces, both to regularize the initial data so that $\rho_{0,n} \in C^{1,\alpha}$ and $Y_0 \cdot \nabla \rho_{0,n}$ is bounded in L^∞ , and to construct an appropriate family of vector fields against which to measure the striated regularity as they are pushed forward by the flow map. In actuality, we never define these hypersurfaces, using only their normal vector fields given by the family of functions $(\nabla \psi_j)$.

Beyond preparing the initial data, the core difficulty in proving the propagation of striated regularity, even for patch data, is showing that the velocity gradient remains Lipschitz over time. Although our manner of proof is adapted from [7, 8], this core difficulty is handled, as it is in [1], by adapting Serfati's [20]. This appears in our use of Lemma 5.1 to establish (9.5), the key bound that closes the bounds that yield $\nabla v \in L^\infty((0, T) \times \mathbb{R}^d)$.

Finally, we mention that in 2D, if we assume that $\operatorname{div} Y_0 = 0$, a unique *classical* flow map for Y_0 does exist and the corresponding transport equation has a unique solution, as shown in [6]. (Such uniqueness does not extend to higher dimension or to non-autonomous systems.) This would allow us to prepare the initial data in 2D under the assumption that $\mathcal{Y}_0 \in C^\alpha(\mathbb{R}^2)$, $\operatorname{div} \mathcal{Y}_0 = 0$, $\rho_0 \in L^\infty(\mathbb{R}^2)$, and $\mathcal{Y}_0 \cdot \nabla \rho_0 = 0$. Here, \mathcal{Y} is a collection of vector fields, a *sufficient family*, as defined as in [8, 11] (or see Definition 4.1, below). We do not pursue this in detail, here, however, as we are interested in a dimension-independent result.

2.2. Organization of this paper. In Section 3, we fix some notation and make a few definitions, leaving the definition of a sufficient family to Section 4. In Section 5, we give a linear algebra lemma due to Serfati that will be used in our proof of Theorem 2.3 to obtain a refined estimate on ∇v in L^∞ . Section 6 includes a number of lemmas centered around ∇v . In Section 7, we discuss the preparation of the initial data. In Section 8, we derive the transport equations for the pushforward of a vector field by a gradient vector field. In Section 9, we prove Theorem 2.3 and Corollary 2.4; in Section 10, we prove Theorems 2.5 and 2.6. In Section 11, we compare our proof to the proof in [2] of the propagation of regularity of the boundary of an aggregation patch.

3. NOTATION, CONVENTIONS, AND DEFINITIONS

We define

$M_{m \times n}(\mathbb{R})$ = the space of all $m \times n$ real matrices,

M_j^i = the element at the i -th row, j -th column of $M \in M_{m \times n}(\mathbb{R})$,

M_j = the j -th column of $M \in M_{m \times n}(\mathbb{R})$,

$$M \cdot N = \sum_{i,j} M_j^i N_j^i = \sum_j M_j \cdot N_j \text{ for all } M, N \in M_{m \times n}(\mathbb{R}),$$

where repeated indices appearing in upper/lower index pairs are summed over, but no summation occurs if the indices are both upper or both lower.

We write $|v|$ for the Euclidean norm of $v = (v^1, v^2, \dots, v^d)$, $|v|^2 = (v^1)^2 + (v^2)^2 + \dots + (v^d)^2$. For $M \in M_{m \times n}(\mathbb{R})$, we use the operator norm,

$$|M| := \max_{|v|=1} |Mv|. \quad (3.1)$$

For M in $M_{d \times d}(\mathbb{R})$, let $\text{cofac } M$ be its cofactor matrix; thus, $(\text{cofac } M)_j^i = (-1)^{i+j}$ times the (i, j) -minor of M . Form M in $M_{d \times d}(\mathbb{R})$ by letting its first $d-1$ columns be Y_1, \dots, Y_{d-1} and choosing its last column arbitrarily. We can define $\wedge_{i < d} Y_i$ to be the last column of $(\text{cofac } M)$, which we note is independent of the last column of M ; that is,

$$\wedge_{i < d} Y_i := \text{cofac} \begin{pmatrix} Y_1 & Y_2 & \dots & Y_{d-1} & \cdot \end{pmatrix}_d.$$

(We are really making a natural identification between the $(d-1)$ st exterior power, $\Lambda^{d-1} \mathbb{R}^d$, which is one-dimensional, and \mathbb{R}^d itself.) In 2D and 3D,

$$\begin{aligned} \wedge_{i < 2} Y_i &= Y_1^\perp := (-Y_1^2, Y_1^1), & d=2, \\ \wedge_{i < 3} Y_i &= Y_1 \times Y_2, & d=3. \end{aligned}$$

Definition 3.1 (Hölder spaces). *Let $U \subseteq \mathbb{R}^d$ be open. Then $C^\alpha(U)$ is the space of all measurable functions for which*

$$\|f\|_{C^\alpha(U)} := \|f\|_{L^\infty(U)} + \|f\|_{\dot{C}^\alpha(U)} < \infty, \quad \|f\|_{\dot{C}^\alpha(U)} := \sup_{\substack{x, y \in U \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Note that

$$\begin{aligned} \|f \circ g\|_{\dot{C}^\alpha} &\leq \|f\|_{\dot{C}^\alpha} \|\nabla g\|_{L^\infty}^\alpha, & \|f \circ g\|_{C^\alpha} &\leq \|f\|_{L^\infty} + \|f\|_{\dot{C}^\alpha} \|\nabla g\|_{L^\infty}^\alpha, \\ \|fg\|_{C^\alpha} &\leq \|f\|_{C^\alpha} \|g\|_{C^\alpha}, & \|1/f\|_{\dot{C}^\alpha} &\leq \frac{\|f\|_{\dot{C}^\alpha}}{(\inf|f|)^\alpha}, \\ \|f \circ g\|_{\dot{C}^\alpha} &\leq \|f\|_{lip} \|g\|_{C^\alpha}. \end{aligned} \quad (3.2)$$

The last inequality uses the homogeneous Lipschitz norm; if f is in C^1 then $\|f\|_{lip} = \|f'\|_{L^\infty}$.

Definition 3.2 (Radial cutoff functions). *We make an arbitrary, but fixed, choice of a radially symmetric function $a \in C_c^\infty(\mathbb{R}^d)$ taking values in $[0, 1]$ with $a = 1$ on $B_1(0)$ and $a = 0$ on $B_2(0)^C$. For $r > 0$, we define the rescaled cutoff function, $a_r(x) = a(x/r)$, and for $r, h > 0$ we define*

$$\mu_{rh} = a_r(1 - a_h).$$

When using the cutoff function μ_{rh} , we will be fixing r while taking $h \rightarrow 0$, in which case we can safely assume that h is sufficiently smaller than r so that μ_{rh} vanishes outside of $(h, 2r)$ and equals 1 identically on $(2h, r)$. It will then follow that

$$\begin{cases} |\nabla \mu_{rh}(x)| \leq Ch^{-1} \leq C|x|^{-1} & \text{for } |x| \in (h, 2h), \\ |\nabla \mu_{rh}(x)| \leq Cr^{-1} \leq C|x|^{-1} & \text{for } |x| \in (r, 2r), \\ \nabla \mu_{rh} \equiv 0 & \text{elsewhere.} \end{cases} \quad (3.3)$$

Hence, we also have $|\nabla \mu_{rh}(x)| \leq C|x|^{-1}$ everywhere.

Lemma 3.3. *Let $r \in (0, 1]$. For all $f \in \dot{C}^\alpha(\mathbb{R}^d)$, $g \in L^\infty(\mathbb{R}^d)$, we have*

$$\left| \int \nabla[\mu_{rh} \nabla \Phi](x-y)(f(x) - f(y))g(y) dy \right| \leq C\alpha^{-1} \|f\|_{\dot{C}^\alpha} \|g\|_{L^\infty} r^\alpha. \quad (3.4)$$

For all $f \in L^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned} \left| \int (\mu_{rh} \nabla \mathcal{F}_d)(x-y)f(y) dy \right| &= \left| \int (\mu_{rh} \nabla \mathcal{F}_d)(x-y)(f(y) - f(x)) dy \right| \\ &\leq C\alpha^{-1} \|f\|_{L^\infty} r^\alpha. \end{aligned} \quad (3.5)$$

Proof. See Lemma 3.4 of [1]. (In [1], a negative Hölder space norm on f was used in place of L^∞ in the bound in (3.5), so the bound here, sufficient for our purposes, is not optimal.) \square

Definition 3.4. $LL = LL(\mathbb{R}^d)$ is the space of bounded log-Lipschitz vector fields with

$$\|g\|_{LL} := \|g\|_{L^\infty} + \sup \left\{ \frac{|g(x) - g(y)|}{-|x - y| \log |x - y|} : x, y \in \mathbb{R}^d, 0 < |x - y| \leq e^{-1} \right\}.$$

4. SUFFICIENT FAMILIES

We adapt the concept of a sufficient family as defined in [8, 11] to better suit the manner in which we prepare the initial data, by organizing the vector fields into *frames*.

Definition 4.1. Let $\mathbf{Y} = (Y_1, \dots, Y_d)$ be an ordered set of C^α vector fields on \mathbb{R}^d with $Y_d = \wedge_{j < d} Y_j$ (see Section 3) and where $\operatorname{div} Y_1, \dots, \operatorname{div} Y_{d-1} \in C^\alpha(\mathbb{R}^d)$. We call \mathbf{Y} a frame.

Let

$$\mathcal{Y} = (\mathbf{Y}^{(j)})_{j \in \Lambda} = ((Y_1^{(j)}, \dots, Y_d^{(j)})_{j \in \Lambda}, \quad (4.1)$$

be a family of frames indexed by Λ , which in our applications will be finite.

For any function f on vector fields (such as div or the identity map), define

$$f(\mathcal{Y}) := ((f(Y_1^{(j)}), \dots, f(Y_{d-1}^{(j)}))_{j \in \Lambda}$$

and define, for any Banach space X ,

$$\|f(\mathcal{Y})\|_X := \sup_{j \in \Lambda, k < d} \|f(Y_k^{(j)})\|_X.$$

Note that we exclude the final vector in each frame. If $\|f(\mathcal{Y})\|_X < \infty$ we say that $f(\mathcal{Y}) \in X$.

Define

$$I(\mathcal{Y}) := \inf_{x \in \mathbb{R}^d} \sup_{j \in \Lambda} |Y_d^{(j)}(x)|. \quad (4.2)$$

Definition 4.2. With \mathcal{Y} as above, we say that \mathcal{Y} is a sufficient family if

$$\mathcal{Y} \in C^\alpha(\mathbb{R}^d), \quad \operatorname{div} \mathcal{Y} \in L^\infty(\mathbb{R}^d), \quad \text{and } I(\mathcal{Y}) > 0.$$

Let

$$\mathcal{Y}_0 = (\mathbf{Y}_0^{(j)})_{j \in \Lambda} = ((Y_{0,1}^{(j)}, \dots, Y_{0,d}^{(j)})_{j \in \Lambda}$$

be a sufficient family. For any $Y_{0,k}^{(j)}$, $k < d$, define its pushforward $Y_k^{(j)}(t)$ (see Section 8 for more on pushforwards) by

$$Y_k^{(j)}(t, \eta(t, x)) := (Y_{0,k}^{(j)}(x) \cdot \nabla) \eta(t, x). \quad (4.3)$$

(We assume here that $\nabla\eta \in L^\infty((0, T) \times \mathbb{R}^d)$.) We then let $Y_d^{(j)}(t) = \wedge_{k < d} Y_k^{(j)}(t)$ and define the pushforward $\mathcal{Y}(t)$ of \mathcal{Y}_0 as

$$\mathcal{Y}(t) = (\mathbf{Y}^{(j)}(t))_{j \in \Lambda} = ((Y_1^{(j)}(t), \dots, Y_d^{(j)}(t)))_{j \in \Lambda}.$$

5. SERFATI'S LINEAR ALGEBRA LEMMA

Lemma 5.1 is a version of a linear algebra lemma due to Serfati, appearing in various forms in [18, 19, 20]. The version here is a minor refinement of the version appearing in [1].

Lemma 5.1. *For any symmetric $B \in M_{d \times d}(\mathbb{R})$, $d \geq 1$, we have*

$$|B| \leq \frac{P(Y_1, \dots, Y_{d-1})}{|\wedge_{i < d} Y_i|^4} \sum_{i=1}^{d-1} |BY_i| + |\operatorname{tr} B|, \quad (5.1)$$

where Y_1, \dots, Y_{d-1} are any linearly independent vectors in \mathbb{R}^d and P is a homogeneous polynomial in Y_1, \dots, Y_{d-1} of degree $n_d := 4d - 5$.

Proof. Setting $Y_d = \wedge_{k < d} Y_k$ as in our definition of a frame, form the matrix M by column as

$$M := [Y_1 \quad \cdots \quad Y_{d-1} \quad Y_d].$$

Note that Y_d is the last column of \underline{M} , the cofactor matrix of M (as in our definition of the wedge product itself in Section 1). Expanding about the last column of M , we see that $|Y_d|^2 = \det M \neq 0$, because we assumed that Y_1, \dots, Y_{d-1} are linearly independent.

Now,

$$M \underline{M}^T = \underline{M} M^T = \det M I,$$

from which it follows that

$$B = \frac{\underline{M}}{(\det M)^2} A \underline{M}^T, \quad A := M^T B M. \quad (5.2)$$

Then, noting that $A_j^i = Y_i \cdot BY_j$, we can write,

$$A = \begin{pmatrix} Y_1 \cdot BY_1 & \cdots & Y_1 \cdot BY_d \\ & \ddots & \\ & & Y_{d-1} \cdot BY_1 & \cdots & Y_{d-1} \cdot BY_d \\ Y_d \cdot BY_1 & \cdots & Y_d \cdot BY_d \end{pmatrix}.$$

Because B is symmetric, so is A . Hence, we can replace $Y_j \cdot BY_d$ in the final column with $Y_d \cdot BY_j$, eliminating BY_d from all but $A_d^d = Y_d \cdot BY_d$. For A_d^d , we calculate,

$$A_d^d = Y_d \cdot BY_d = \underline{M}_d \cdot BY_d = \sum_{i=1}^d \underline{M}_i \cdot BY_i - \sum_{i=1}^{d-1} \underline{M}_i \cdot BY_i,$$

where we used that $\underline{M}_d = Y_d$. But,

$$\sum_{i=1}^d \underline{M}_i \cdot BY_i = \sum_{i=1}^d (\underline{M}^T B M)_i^i = \operatorname{tr}(\underline{M}^T B M) = \operatorname{tr}(M \underline{M}^T B) = \det M \operatorname{tr} B,$$

so

$$Y_d \cdot BY_d = \det M \operatorname{tr} B - \sum_{i=1}^{d-1} \underline{M}_i \cdot BY_i.$$

We conclude that

$$A = E + \det M (\operatorname{tr} B) D,$$

where

$$E := \begin{pmatrix} Y_1 \cdot BY_1 & \cdots & Y_1 \cdot (BY)_{d-1} & Y_d \cdot BY_1 \\ & & \vdots & \\ & & \vdots & \\ & & \vdots & \\ Y_{d-1} \cdot BY_1 & \cdots & Y_{d-1} \cdot (BY)_{d-1} & Y_d \cdot BY_{d-1} \\ Y_d \cdot BY_1 & \cdots & Y_d \cdot (BY)_{d-1} & -\sum_{i=1}^{d-1} \underline{M}_i \cdot BY_i \end{pmatrix}, \quad D := \begin{pmatrix} 0 & \cdots & 0 \\ & & \vdots \\ & & \vdots \\ & & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{pmatrix}.$$

Now,

$$(\underline{M} \underline{D} \underline{M}^T)^i_j = \underline{M}_k^i \underline{D}_\ell^k \underline{M}_\ell^j = \underline{M}_d^i \underline{M}_d^j = Y_d^i Y_d^j,$$

or,

$$\underline{M} \underline{D} \underline{M}^T = |Y_d|^2 (\operatorname{tr} B) Y_d \otimes Y_d.$$

Hence,

$$\left| \det M (\operatorname{tr} B) \frac{\underline{M}}{|Y_d|^4} \underline{D} \underline{M}^T \right| \leq (\operatorname{tr} B) \frac{|Y_d \otimes Y_d|}{|Y_d|^2} \leq \operatorname{tr} B.$$

As can be easily seen, the square of the operator norm of a square matrix is bounded above by the sum of the squares of the norms of the columns or of the rows. From this, (5.2), the bound above, and the form in which we have expressed E , (5.1) follows. The degree and the homogeneity of P follow from scaling, or from summing the degrees of the factors making it up. \square

Lemma 5.2. *Let U be an open subset of \mathbb{R}^d , $d \geq 2$. Assume that (Y_1, \dots, Y_d) is a frame on U (so $Y_d = \wedge_{i < d} Y_i$) with $\|Y_d\|_{C^\alpha} \geq C_0 > 0$. For any symmetric $B \in M_{d \times d}(\mathbb{R})$, we have*

$$\|B\|_{C^\alpha} \leq C_0^{-8} Q(\|Y_1\|_{C^\alpha}, \dots, \|Y_{d-1}\|_{C^\alpha}, \|BY_1\|_{C^\alpha}, \dots, \|BY_{d-1}\|_{C^\alpha}) + \|\operatorname{tr} B\|_{C^\alpha},$$

for some polynomial, Q , the C^α norms being over U .

Proof. The result follows from applying (3.2) to the decomposition of B given in the proof of Lemma 5.1. \square

6. LEMMAS INVOLVING THE VELOCITY GRADIENT

Throughout this section, we assume that

$$\rho \in L^1 \cap L^\infty(\mathbb{R}^d), \quad v = -\nabla \Phi * \rho.$$

Proposition 6.1 is the extension of (7.12) of [2] to a general ρ .

Proposition 6.1. *We have,*

$$\nabla v(x) = -\frac{\rho(x)}{d} I - \text{p. v.} \int \nabla \nabla \Phi(x-y) \rho(y) dy.$$

Proof. Let $H\Phi$ be the distributional Hessian matrix of Φ . Then,

$$H\Phi = \text{p. v. } \nabla \nabla \Phi + \frac{1}{d} \delta_0 I,$$

where δ_0 is the Dirac delta function at the origin. Using that $v = -\nabla \Phi * \rho$ completes the proof. \square

The next proposition is the analog of Proposition 4.2 of [1].

Proposition 6.2. *Let Y be a vector field in $C^\alpha(\mathbb{R}^d)$. Then*

$$\text{p. v. } \int \nabla \nabla \Phi(x-y) Y(y) \rho(y) dy = -\nabla \Phi * \text{div}(\rho Y)(x) - \frac{\rho(x) Y(x)}{d}.$$

Proof. The proof is fairly standard, so we give only an outline. We assume first that $\rho \in C^\infty(\mathbb{R}^d)$. Noting that

$$[\nabla \nabla \Phi(x-y) Y(y)]^j = (\nabla \nabla \Phi)^{jk}(x-y) Y^k(y) = \partial_k \partial_j \Phi(x-y) Y^k(y) = \nabla \partial_j \Phi(x-y) \cdot Y(y),$$

integration by parts gives

$$\begin{aligned} \left[\text{p. v. } \int \nabla \nabla \Phi(x-y) Y(y) \rho(y) dy \right]^j &= \text{p. v. } \int \nabla \partial_j \Phi(x-y) \cdot Y(y) \rho(y) dy \\ &= -\lim_{\delta \rightarrow 0} \int_{B_\delta^c(x)} \partial_j \Phi(x-y) \text{div}(\rho Y)(y) dy + \lim_{\delta \rightarrow 0} \int_{\partial B_\delta(x)} \partial_j \Phi(x-y) \rho(y) (Y(y) \cdot \mathbf{n}) dy, \end{aligned}$$

where \mathbf{n} is the unit outer vector on $\partial B_\delta(x)$. The area integral converges in the limit to $\partial_j \Phi * \text{div}(\rho Y)$ since $\nabla \Phi$ is locally integrable. For the boundary integral, one replaces $Y(y)$ by $Y(x)$, in which case the boundary integral converges to $-\rho(x) Y^j(x)/d$. Using that $|Y(x) - Y(y)| \leq \|Y\|_{\dot{C}^\alpha} |x-y|^\alpha$, the error in replacing $Y(y)$ by $Y(x)$ is easily seen to vanish as $\delta \rightarrow 0$. The result then follows by a density argument. \square

Corollary 6.3. *Let $Y \in C^\alpha(\mathbb{R}^d)$. Then*

$$Y(x) \cdot \nabla v(x) = -\text{p. v. } \int \nabla \nabla \Phi(x-y) [Y(x) - Y(y)] \rho(y) dy + \nabla \Phi * \text{div}(\rho Y)(x).$$

Moreover,

$$\begin{aligned} \left\| \text{p. v. } \int \nabla \nabla \Phi(x-y) [Y(x) - Y(y)] \rho(y) dy \right\|_{C^\alpha} &\leq CV(\rho) \|Y\|_{\dot{C}^\alpha}, \\ \|Y \cdot \nabla v(t)\|_{C^\alpha} &\leq CV(\rho) \|Y\|_{\dot{C}^\alpha} + \|\text{div}(\rho Y)\|_{L^\infty}, \end{aligned}$$

where

$$V(\rho) := \|\rho\|_{L^\infty} + \left\| \text{p. v. } \int \nabla \nabla \Phi(\cdot - y) \rho(y) dy \right\|_{L^\infty} \quad (6.1)$$

Proof. The expression for $Y(x) \cdot \nabla v(x)$ follows from comparing the expressions in Proposition 6.1 and Proposition 6.2. The C^α -bound on the integral follows from fairly standard singular integral estimates (for instance, by applying Lemma 3.3 of [1] with the kernel $\rho(y) \nabla \nabla \Phi(x-y) [Y(x) - Y(y)]$.) For the bound on the convolution $\nabla \Phi * \text{div}(\rho Y)$, we first note

$$\|\nabla \Phi * \text{div}(\rho Y)\|_{\dot{C}^\alpha} \leq C \|\nabla \Phi * \text{div}(\rho Y)\|_{LL}$$

because

$$\frac{-|x-y| \log |x-y|}{|x-y|^\alpha} \leq C$$

when $|x - y| \leq e^{-1}$. We then apply the classical estimate $\|\nabla\Phi * f\|_{LL} \leq C \|f\|_{L^\infty}$, which holds for all $d \geq 2$ (see [17, Lemma 8.1] for a 2D proof), to obtain

$$\|\nabla\Phi * \operatorname{div}(\rho Y)\|_{LL} \leq C \|\operatorname{div}(\rho Y)\|_{L^\infty}.$$

□

The estimate $\|\nabla\Phi * f\|_{LL} \leq C \|f\|_{L^\infty}$ will also be used later in the proof of Theorem 2.3 in Section 9.2, (iii).

Corollary 6.3 is the extension to a general ρ of Lemma 7.3 of [2]. We note that the term $\nabla\Phi * \operatorname{div}(\rho Y)(x)$ appearing here is cancelled in the setting of [2] by the additional term $-\rho Y$ appearing in (7.6) of [2].

Remark 6.4. For any i, j ,

$$(\nabla v)_j^i = \partial_j v^i = -\partial_j(\partial_i \Phi * \rho) = -(\partial_j \partial_i \Phi) * \rho = -(\partial_i \partial_j \Phi) * \rho = (\nabla v)_i^j,$$

so ∇v is symmetric, a fact will use in the following sections.

7. PREPARING THE INITIAL DATA

In Proposition 7.1 we prepare the initial data and construct a sufficient family in such a way as to allow us to adapt the machinery created by Chemin in [7, 8] to obtain the propagation of striated regularity.

Proposition 7.1. *Let ρ_0 be a suitable initial density, as in Definition 2.2, and let ψ_j and U_j be as in that definition. There exists a sufficient family \mathcal{Y}_0 and a sequence $(\rho_{0,n})_{n \in \mathbb{N}}$ of functions in $C^{1,\alpha}(\mathbb{R}^d)$, such that:*

- (i) Each $\rho_{0,n}$ is supported in a fixed compact $K \subseteq \mathbb{R}^d$;
- (ii) $\rho_{0,n} \rightarrow \rho_0$ in $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty)$;
- (iii) $\|\rho_{0,n}\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}$ for all $n \in \mathbb{N}$;
- (iv) $\operatorname{div} \mathcal{Y}_0 = 0$;
- (v) $\|\mathcal{Y}_0 \cdot \nabla \rho_{0,n}\|_{L^\infty} \leq C$ for all $n \in \mathbb{N}$.

The constant in (v) depends upon $\|\rho_0\|_{L^\infty}$ and (ψ_j) .

Proof. Because we deal only with initial data in this proof, we drop the time zero subscripts, writing ρ and \mathcal{Y} , for instance, for ρ_0 and \mathcal{Y}_0 . Recall from Definition 2.2 that U_j is the interior of $\operatorname{supp} \psi_j$.

Since U_j is connected, $\psi_j(U_j) = (a_j, b_j)$ for some $-\infty < a_j < b_j < \infty$. Because ψ_j is level-set-compatible with ρ on U_j as in Definition 2.1, $\rho \circ \psi_j^{-1}: (a_j, b_j) \rightarrow \mathbb{R}$ is well-defined: for any $r \in (a_j, b_j)$, $\rho \circ \psi_j^{-1}(r) = \rho(x)$, where x is any point in U_j for which $\psi_j(x) = r$. For any $n \in \mathbb{N}$, we can then define

$$\rho_{j,n}: U_j \rightarrow \mathbb{R}, \quad \rho_{j,n} = (\mu_{1/n} * \mathcal{E}_0(\rho \circ \psi_j^{-1})) \circ \psi_j,$$

where \mathcal{E}_0 is extension by zero from the domain (a_j, b_j) to all of \mathbb{R} , and $(\mu_\varepsilon)_{\varepsilon \geq 1}$ is a family of 1D mollifiers with $\mu_\varepsilon(\cdot) = \varepsilon^{-d} \mu(\cdot/\varepsilon)$ and μ_1 supported is on $(-1, 1)$. Because $\mu_{1/n} * \mathcal{E}_0(\rho \circ \psi_j^{-1}) \in C^\infty(\mathbb{R})$ and $\psi_j \in C^{1,\alpha}(\mathbb{R}^d)$, we see that $\rho_{j,n} \in C^{1,\alpha}(\mathbb{R}^d)$.

Any level set of ψ_j maps to a level set of $\rho_{j,n}$, so ψ_j is level-set compatible with $\rho_{j,n}$ on U_j .

Let r_0 be as in Definition 2.2 and define

$$U_j' = \{x \in U_j: \operatorname{dist}(x, \partial U_j) > r_0/2\}.$$

(Note that U_j' could be empty, though only if $U_j \subseteq V$, in which case we disregard U_j entirely.)

Construct a partition of unity $(\varphi_j)_{j=1}^{N+1}$ with the property that for all $j \leq N$, $\varphi_j \equiv 1$ on U'_j and $\varphi_j \equiv 0$ on $\mathbb{R}^d \setminus U_j$. Then let $\varphi_{N+1} = 1 - \sum_{j \leq N} \varphi_j$. Note that at each point in its support, $\varphi_{N+1} = 1 - \varphi_j$ for one j .

For any $n \in \mathbb{N}$, define ρ_n by

$$\rho_n := \sum_{j=1}^N \varphi_j \rho_{j,n} + \varphi_{N+1} \rho.$$

Noting that $\rho \in C^{1,\alpha}(V)$ and $\text{supp } \varphi_{N+1} \subseteq V$, we see that $\rho_n \in C^{1,\alpha}(\mathbb{R}^d)$, and the ρ_n are supported in a common compact $K \subset \mathbb{R}^d$ since ρ is compactly supported, giving (i). Basic properties of (1D) mollification give (ii) and (iii).

It remains to construct a sufficient family \mathcal{Y} satisfying (iv) and (v). In 2D, we need only $N + 1$ frames. We define the j th frame, $Y^{(j)} = (Y_1^{(j)}, Y_2^{(j)})$, by choosing

$$Y_1^{(j)} = \nabla^\perp \psi_j, \quad Y_2^{(j)} = \nabla \psi_j,$$

noting that $\text{div } Y_1^{(j)} = 0$, giving (iv). For (v), ψ_j is level-set compatible with $\rho_{j,n}$ on U_j , so $Y_1^{(j)} \cdot \nabla \rho_{j,n} = 0$. Also, $Y_1^{(j)}$ vanishes on U_k for $k \neq j$. Hence,

$$\begin{aligned} Y_1^{(j)} \cdot \nabla \rho_n &= \sum_{k=1}^N \left(\varphi_k Y_1^{(j)} \cdot \nabla \rho_{k,n} + \rho_{k,n} Y_1^{(j)} \cdot \nabla \varphi_k \right) + \varphi_{N+1} Y_1^{(j)} \cdot \nabla \rho + \rho Y_1^{(j)} \cdot \nabla \varphi_{N+1} \\ &= \rho_{j,n} Y_1^{(j)} \cdot \nabla \varphi_j + \varphi_{N+1} Y_1^{(j)} \cdot \nabla \rho + \rho Y_1^{(j)} \cdot \nabla \varphi_{N+1} \\ &= -\rho_{j,n} Y_1^{(j)} \cdot \nabla \varphi_{N+1} + \varphi_{N+1} Y_1^{(j)} \cdot \nabla \rho + \rho Y_1^{(j)} \cdot \nabla \varphi_{N+1}. \end{aligned}$$

Since also ρ is in $C^{1,\alpha}(\text{supp } \varphi_{N+1})$, we see that each of the terms above is bounded in L^∞ , from which (v) follows.

Finally, since ρ is $C^{1,\alpha}$ on U_{N+1} , we can set $Y_1^{(N+1)} = \nabla^\perp(\varphi_{N+1} \mathbf{e}_1)$, $Y_2^{(N+1)} = (Y_1^{(N+1)})^\perp$, and we can see that the family of frames, $\mathcal{Y} = (Y^{(1)}, \dots, Y^{(N+1)})$, is sufficient.

The argument for $d \geq 3$ is similar to that for 2D, though now we need multiple frames to “cover” each U_j —these frames are given by applying Lemma 7.3, below, with $W = U_j$, $W' = U'_j$, and $\psi = \psi_j$ to provide a family of frames \mathcal{Z}_j . The full family of frames is obtained as the union, $\mathcal{Y} = \mathcal{Z}_1 \cup \dots \cup \mathcal{Z}_N$ with one final frame added to cover U_{N+1} by applying Lemma 7.3, again with $W = V$, $W' = V \setminus \cup_j U'_j$, and $\psi = \varphi_{N+1} x_1$. \square

It is shown by Foote in [13] that for a bounded domain $\Omega \subseteq \mathbb{R}^d$, if $\partial\Omega$ is C^k for $k \geq 2$ then there is a tubular neighborhood U of the boundary in the sense that every point in U has a unique closest point on $\partial\Omega$. Moreover, although the projection map $P: U \rightarrow \partial\Omega$ to this closest point is only C^{k-1} , the signed distance function, $\delta(x) = \pm \text{dist}(x, \partial\Omega)$, chosen positive (negative) if x is outside (inside) is C^k on U . The very simple and clean proofs in [13] extend, with almost no change, to a $C^{1,\alpha}$ boundary, from which we can conclude that δ is $C^{1,\alpha}$ and P is C^α on U . Using this δ , it is not hard to apply Proposition 7.1 to obtain the following:

Corollary 7.2. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with $C^{1,\alpha}$ boundary and a finite number of connected components. Then $\rho_0 := c \mathbf{1}_\Omega$, where c is a constant, is a suitable initial density as in Definition 2.2.*

We used the following lemma in the proof of Proposition 7.1.

Lemma 7.3. *Let W and $W' \subseteq W$ be nonempty open subsets of \mathbb{R}^d . Let $\psi \in C^{1,\alpha}(\mathbb{R}^d)$ with $|\nabla\psi| \geq C_0 > 0$ on W' . Then there exists a family of frames,*

$$\mathcal{Z} = ((Z_1^{(k)}, \dots, Z_d^{(k)})_{k=1}^d,$$

with each $Z_j^{(k)}$ in $C^\alpha(W)$ and for which $\operatorname{div} Z_j^{(k)} = Z_j^{(k)} \cdot \nabla\psi = 0$ for all $j < d$. Moreover, $\mathcal{I}|_{W'}(\mathcal{Z}) > 0$, where $\mathcal{I}|_{W'}$ is defined as in (4.2), but restricted to W' .

Proof. To illustrate the approach, we first show how to choose \mathcal{Z} when $d = 3$. We set

$$\begin{aligned} Z_1^{(1)} &= (-\partial_2\psi, \partial_1\psi, 0), & Z_2^{(1)} &= (-\partial_3\psi, 0, \partial_1\psi), \\ Z_1^{(2)} &= (\partial_2\psi, -\partial_1\psi, 0), & Z_2^{(2)} &= (0, -\partial_3\psi, \partial_2\psi), \\ Z_1^{(3)} &= (\partial_3\psi, 0, -\partial_1\psi), & Z_2^{(3)} &= (0, \partial_3\psi, -\partial_2\psi). \end{aligned}$$

We also set $Z_3^{(k)} = Z_1^{(k)} \wedge Z_2^{(k)}$ for each k . Then, $\operatorname{div} Z_j^{(k)} = 0$ and $Z_j^{(k)} \cdot \nabla\psi = 0$ for all $j < 3$, $k \leq 3$. Moreover, for any $x_0 \in W'$, at least one component of $\nabla\psi(x_0)$, $\partial_k\psi(x_0)$, must have $|\partial_k\psi|(x_0) \geq C_0/\sqrt{3}$. Since $\nabla\psi$ is continuous, $\partial_k\psi \geq C_0/2\sqrt{3}$ on some open set containing x_0 . Moreover, it is clear that $Z_1^{(k)}$ and $Z_2^{(k)}$ are linearly independent, and that locally to x_0 , $|Z_3^{(k)}| = |\nabla\psi|^2 \geq C_0^2$ is bounded away from zero. This implies that $\mathcal{I}|_{W'}(\mathcal{Z}) > 0$.

We next take $d = 4$. We set

$$\begin{aligned} Z_1^{(1)} &= (-\partial_2\psi, \partial_1\psi, 0, 0), & Z_2^{(1)} &= (-\partial_3\psi, 0, \partial_1\psi, 0), & Z_3^{(1)} &= (-\partial_4\psi, 0, 0, \partial_1\psi), \\ Z_1^{(2)} &= (\partial_2\psi, -\partial_1\psi, 0, 0), & Z_2^{(2)} &= (0, -\partial_3\psi, \partial_2\psi, 0), & Z_3^{(2)} &= (0, -\partial_4\psi, 0, \partial_2\psi), \\ Z_1^{(3)} &= (\partial_3\psi, 0, -\partial_1\psi, 0), & Z_2^{(3)} &= (0, \partial_3\psi, -\partial_2\psi, 0), & Z_3^{(3)} &= (0, 0, -\partial_4\psi, \partial_3\psi), \\ Z_1^{(4)} &= (\partial_4\psi, 0, 0, -\partial_1\psi), & Z_2^{(4)} &= (0, \partial_4\psi, 0, -\partial_2\psi), & Z_3^{(4)} &= (0, 0, \partial_4\psi, -\partial_3\psi). \end{aligned}$$

We also set $Z_4^{(k)} = Z_1^{(k)} \wedge Z_2^{(k)} \wedge Z_3^{(k)}$ for each k . We see that on all of W , $\operatorname{div} Z_j^{(k)} = 0$ and $Z_j^{(k)} \cdot \nabla\psi = 0$ for all $j < 4$, $k \leq 4$. Moreover, for any $x_0 \in W'$, at least one component of $\nabla\psi(x_0)$, $\partial_k\psi(x_0)$, must have $|\partial_k\psi|(x_0) \geq C_0/2$. Since $\nabla\psi$ is continuous, $\partial_k\psi \geq C_0/4$ on some open set containing x_0 . Moreover, it is clear that $Z_1^{(k)}$, $Z_2^{(k)}$, $Z_3^{(k)}$ are linearly independent, and that locally to x_0 , $|Z_4^{(k)}| = |\nabla\psi|^2 \geq C_0^2$ is bounded away from zero. This implies that $\mathcal{I}|_{W'}(\mathcal{Z}) > 0$.

The extension to any $d \geq 2$ is then clear. \square

8. PUSHFORWARDS AND TRANSPORT

We establish the basic transport equations for Y , $\operatorname{div} Y$, and $Y \cdot \nabla\rho$ in Proposition 8.1, and use them (in part) to establish the bounds in Proposition 8.2.

Proposition 8.1. *Assume that ρ is a solution to (1.1, 1.3) for which the velocity, v , is at least Lipschitz continuous in space uniformly in time and that $\nabla\rho(t)$ is at least continuous in time and space. Let η be the corresponding unique flow map for v as in (1.5). Define the pushforward of the vector field $Y_0 \in C^\alpha(\mathbb{R}^d)$ by*

$$Y(t, \eta(t, x)) = Y_0(x) \cdot \nabla\eta(t, x). \quad (8.1)$$

Then,

$$\partial_t Y + v \cdot \nabla Y = Y \cdot \nabla v, \quad (8.2)$$

$$\partial_t \operatorname{div} Y + v \cdot \nabla \operatorname{div} Y = -Y \cdot \nabla \rho, \quad (8.3)$$

$$\partial_t (Y \cdot \nabla \rho) + v \cdot \nabla (Y \cdot \nabla \rho) = F'(\rho) Y \cdot \nabla \rho. \quad (8.4)$$

Proof. By our assumptions, $\operatorname{div}(\rho Y)$ and $Y \cdot \nabla \rho$ are continuous in time and space. The identity in (8.2) is classical. Taking the divergence of each side of (8.2), we obtain

$$\partial_t \operatorname{div} Y + v \cdot \nabla \operatorname{div} Y = Y \cdot \nabla \operatorname{div} v = -Y \cdot \nabla \rho,$$

giving (8.3).

Using (1.6) and (8.2), we have

$$\begin{aligned} \frac{d}{dt}(\rho Y)(t, \eta(t, x)) &= \rho(t, \eta(t, x)) \frac{d}{dt} Y(t, \eta(t, x)) + Y(t, \eta(t, x)) \frac{d}{dt} \rho(t, \eta(t, x)) \\ &= \rho(t, \eta(t, x))(Y \cdot \nabla v)(t, \eta(t, x)) + F(\rho(t, \eta(t, x))) Y(t, \eta(t, x)), \end{aligned} \quad (8.5)$$

which we write more compactly as

$$\partial_t(\rho Y) + v \cdot \nabla(\rho Y) = \rho Y \cdot \nabla v + F(\rho) Y.$$

Noting that the derivation of (8.3) from (8.2) did not use the specific definition of Y in (8.1), the same argument applies with ρY in place of Y . We have, however, the additional term on the right-hand side,

$$\operatorname{div}(F(\rho) Y) = Y \cdot \nabla(F(\rho)) + F(\rho) \operatorname{div} Y = F'(\rho) Y \cdot \nabla \rho + F(\rho) \operatorname{div} Y,$$

so we conclude that

$$\partial_t \operatorname{div}(\rho Y) + v \cdot \nabla \operatorname{div}(\rho Y) = -\rho Y \cdot \nabla \rho + F'(\rho) Y \cdot \nabla \rho + F(\rho) \operatorname{div} Y. \quad (8.6)$$

Then, since $\operatorname{div}(\rho Y) = Y \cdot \nabla \rho + \rho \operatorname{div} Y$, we have

$$\begin{aligned} \partial_t \operatorname{div}(\rho Y) + v \cdot \nabla \operatorname{div}(\rho Y) &= \partial_t(Y \cdot \nabla \rho) + \rho(\partial_t \operatorname{div} Y + v \cdot \nabla \operatorname{div} Y) + (\partial_t \rho + v \cdot \nabla \rho) \operatorname{div} Y + v \cdot \nabla(Y \cdot \nabla \rho) \\ &= \partial_t(Y \cdot \nabla \rho) + v \cdot \nabla(Y \cdot \nabla \rho) - \rho Y \cdot \nabla \rho + F(\rho) \operatorname{div} Y, \end{aligned}$$

where we used (1.1)₁ and (8.3). By (8.6) it follows that

$$\partial_t(Y \cdot \nabla \rho) + v \cdot \nabla(Y \cdot \nabla \rho) = F'(\rho) Y \cdot \nabla \rho,$$

which is (8.4). \square

Proposition 8.2. *Assume that ρ_0 is a suitable initial density as in Definition 2.2, let $(\rho_{0,n})$ and the sufficient family \mathcal{Y}_0 be given by Proposition 7.1, let (ρ_n) be the corresponding $C^1(0, T; C^{1,\alpha}(\mathbb{R}^d))$ -solutions to (1.1, 1.3), and let \mathcal{Y}_n be the pushforward of \mathcal{Y} by the flow map, η_n . If $\operatorname{div} \mathcal{Y}_0 = \mathcal{Y}_0 \cdot \nabla \rho_{0,n} = 0$ on the open set W then $\operatorname{div} \mathcal{Y}_n(t) = \mathcal{Y}_n \cdot \nabla \rho_n = 0$ on $\eta_n(t, W)$ for all $t \in [0, T]$. Moreover,*

$$\begin{aligned} \|\mathcal{Y}_n \cdot \nabla \rho_n(t)\|_{L^\infty} &\leq \|\mathcal{Y}_0 \cdot \nabla \rho_0\|_{L^\infty} e^{C(T)t}, \\ \|\operatorname{div} \mathcal{Y}_n\|_{L^\infty} &\leq \|\mathcal{Y}_0 \cdot \nabla \rho_0\|_{L^\infty} e^{C(T)t}. \end{aligned} \quad (8.7)$$

Proof. Fix $x \in W$ and let Y_0 be one of the first $d-1$ vector fields in any frame in \mathcal{Y}_0 . Define

$$g(t) := Y_n \cdot \nabla \rho_n(t, \eta_n(t, x)), \quad h(t) := \operatorname{div} Y_n(t, \eta_n(t, x)), \quad \bar{\rho}(t) := \rho_n(t, \eta_n(t, x)),$$

where $Y_n = Y_n(t)$ is the pushforward of Y_0 by the flow map, η_n . Then from (8.3) and (8.4) we see that $g(t)$, $h(t)$ solve the ODEs,

$$\begin{aligned} \dot{g}(t) &= F'(\bar{\rho}(t))g(t), & g(0) &= Y_0 \cdot \nabla \rho_{0,n}(x), \\ \dot{h}(t) &= -g(t), & h(0) &= \operatorname{div} Y_0(x). \end{aligned}$$

Along flow lines, ρ is Lipschitz continuous (as we can see from (1.7)), so $\bar{\rho}$ is locally Lipschitz. Also, since $h(0) = 0$ for all $x \in \mathbb{R}^d$ (by Proposition 7.1) and $g(0) = 0$ for $x \in W$, we see that $Y \cdot \nabla \rho(t, x) = \operatorname{div} Y(t, x) = 0$ for $x \in \eta(t, W)$.

Since we assumed that F' is locally Lipschitz, we can integrate the equations for g then h to obtain (after calculating that $(\log g(t))' = F'(\bar{\rho}(t))$ and using that $h(0) \equiv 0$ on \mathbb{R}^d),

$$g(t) = g(0) \exp \int_0^t F'(\bar{\rho}(s)) ds, \quad h(t) = - \int_0^t g(s) ds.$$

Since $\rho \in L^\infty([0, T] \times \mathbb{R}^2)$, $|F'(\bar{\rho}(s))| \leq C(T)$. This gives the simple bounds,

$$|g(t)| \leq e^{C(T)t} |g(0)|, \quad |h(t)| \leq te^{C(T)t} |g(0)|,$$

from which, along with (v) of Proposition 7.1, (8.7) follows. \square

Proposition 8.3, which we will apply in the proof of Theorem 2.6, gives estimates on the propagation of local regularity of the density. Whereas in Proposition 8.2 we needed to work with regular solutions to make sense of the pushforward of \mathcal{Y}_0 and to make sense of the term $Y \cdot \nabla \rho$ appearing in the transport equations, in Proposition 8.3, we need only the regularity of a weak solution. Without, however, higher regularity of the flow map, such as that obtained in Theorem 2.3, the bounds will all be infinite.

Proposition 8.3. *Let ρ be a weak Lagrangian solution to (1.1, 1.3) with initial density ρ_0 , as in Definition 1.1. Then for any open set $W \subseteq \mathbb{R}^d$,*

$$\begin{aligned} \|\rho(t)\|_{C^\alpha(\eta(t, W))} &\leq \|\rho_0\|_{C^\alpha(W)} e^{C(T)t + \alpha J(t)}, \\ \|\nabla \rho(t)\|_{L^\infty(\eta(t, W))} &\leq \|\nabla \rho_0\|_{L^\infty(W)} e^{C(T)t + J(t)}, \\ \|\nabla \rho(t)\|_{C^\alpha(\eta(t, W))} &\leq \|\nabla \rho_0\|_{C^\alpha(W)} e^{C(T)t + J(t)} \|\nabla \eta^{-1}(t)\|_{C^\alpha(\eta(t, W))}, \end{aligned} \quad (8.8)$$

where $J(t) := \int_0^t \|\nabla v(s)\|_{L^\infty} ds$ and $C(T)$ depends continuously on F , $\|\rho_0\|_{L^\infty}$, and T .

Proof. Before proceeding further, observe that

$$\|\nabla \eta(t)\|_{L^\infty}, \|\nabla \eta^{-1}(t)\|_{L^\infty} \leq e^{J(t)}. \quad (8.9)$$

The bound on the forward flow map follows directly from the integral form of (1.5),

$$\eta(t, x) = x + \int_0^t v(s, \eta(s, x)) ds, \quad (8.10)$$

while the bound on the inverse flow map is derived later in Lemma 8.4.

Let

$$L(s, x) := \rho(s, \eta(s, x)).$$

Then (1.7) can be expressed in the form,

$$L(t, x) = \rho_0(x) + \int_0^t F(L(s, x)) ds. \quad (8.11)$$

Applying Osgood's lemma (see ([9, Lemma 5.2.1]), we see that

$$\int_{|\rho_0(x)|}^{|L(t, x)|} \frac{dr}{|F(r)|} \leq t.$$

This gives the finite time, T^* , up to which $L(t, x)$ remains bounded by some constant $C(T)$: this is the origin of (1.9). That is, $\|L(t)\|_{L^\infty} = \|\rho(t)\|_{L^\infty} \leq C(T)$, and there exists a bounded $E \subseteq (0, T) \times \mathbb{R}^d$, such that

$$\text{image}(L) \subseteq E.$$

That E is bounded will be important when applying (3.2), which we do multiple times below.

Returning to (8.11), we can put the bound on $\|L(t)\|_{L^\infty}$ in the form

$$\|L(t)\|_{L^\infty} = \|\rho(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{C(T)t}, \quad (8.12)$$

though we have no effective method for calculating $C(T)$, which also depends upon ρ_0 , without an explicit choice of F .

We now bound $L(t)$ in $C^\alpha(W)$ and $C^{1+\alpha}(W)$. So assume $x \in W$. Then from (8.11),

$$\|L(t)\|_{\dot{C}^\alpha(W)} \leq \|\rho_0\|_{\dot{C}^\alpha(W)} + \int_0^t \|F'\|_{L^\infty(E)} \|L(s)\|_{\dot{C}^\alpha(W)} ds,$$

where we used (3.2). Thus, using also (8.12),

$$\|L(t)\|_{C^\alpha(W)} \leq \|\rho_0\|_{C^\alpha(W)} e^{C(T)t}$$

follows from Grönwall's lemma.

Similarly, applying ∇ to (8.11) gives

$$\nabla L(t, x) = \nabla \rho_0(x) + \int_0^t F'(L(s, x)) \nabla L(s, x) ds. \quad (8.13)$$

Hence,

$$\|\nabla L(t)\|_{L^\infty(W)} \leq \|\nabla \rho_0\|_{L^\infty(W)} + \int_0^t \|F'\|_{L^\infty(E)} \|\nabla L(s)\|_{L^\infty(W)} ds$$

and Grönwall's lemma then gives

$$\|\nabla L(t)\|_{L^\infty(W)} \leq \|\nabla \rho_0\|_{L^\infty(W)} e^{Ct}. \quad (8.14)$$

We can also use (8.13) to bound the C^α norm. First, observe that by (3.2),

$$\begin{aligned} \|F'(L(s, x))\|_{C^\alpha(W)} &= \|F'(L(s, x))\|_{L^\infty} + \|F'(L(s, x))\|_{\dot{C}^\alpha} \\ &\leq C(T) + \|F'\|_{lip(E)} \|L(s)\|_{\dot{C}^\alpha(W)} \leq C(T), \end{aligned}$$

by our earlier bound on $\|L(t)\|_{\dot{C}^\alpha(W)}$. Hence,

$$\begin{aligned} \|\nabla L(t)\|_{C^\alpha(W)} &\leq \|\nabla \rho_0\|_{C^\alpha(W)} + \int_0^t \|F'(L(s, x))\|_{C^\alpha(W)} \|\nabla L(s)\|_{C^\alpha(W)} ds \\ &\leq \|\nabla \rho_0\|_{C^\alpha(W)} + C(T) \int_0^t \|\nabla L(s)\|_{C^\alpha(W)} ds. \end{aligned}$$

Then Grönwall's lemma combined with (8.14) give

$$\|\nabla L(t)\|_{C^\alpha(W)} \leq \|\nabla \rho_0\|_{C^\alpha(W)} e^{C(T)t}.$$

Then (3.2) also gives

$$\begin{aligned} \|\rho(t)\|_{\dot{C}^\alpha(\eta(t, W))} &= \|L(t, \eta^{-1}(t, \cdot))\|_{\dot{C}^\alpha(\eta(t, W))} \leq \|L(t)\|_{\dot{C}^\alpha(\eta(t, W))} \|\nabla \eta^{-1}(t)\|_{L^\infty}^\alpha \\ &\leq \|\rho_0\|_{\dot{C}^\alpha(W)} e^{C(T)t + \alpha J(t)}. \end{aligned}$$

And $\nabla \rho(t) = \nabla(L(t, \eta^{-1}(t, x))) = (\nabla L)(t, \eta^{-1}(t, x)) \cdot \nabla \eta^{-1}(t, x)$. Thus,

$$\|\nabla \rho(t)\|_{L^\infty(\eta(t, W))} \leq \|\nabla L(t)\|_{L^\infty(W)} \|\nabla \eta^{-1}(t)\|_{L^\infty(\eta(t, W))} \leq \|\nabla \rho_0\|_{L^\infty(W)} e^{C(T)t + J(t)}.$$

Because

$$\begin{aligned} \|\nabla(L(t, \eta^{-1}(t, \cdot)))\|_{C^\alpha(\eta(t, W))} &\leq \|\nabla(L(t, \eta^{-1}(t, \cdot)))\|_{L^\infty(\eta(t, W))} + \|\nabla(L(t, \eta^{-1}(t, \cdot)))\|_{\dot{C}^\alpha(\eta(t, W))} \\ &\leq \|\nabla\rho(t)\|_{L^\infty(\eta(t, W))} + \|\nabla L(t)\|_{C^\alpha(W)} \|\nabla\eta^{-1}(t)\|_{L^\infty}^\alpha, \end{aligned}$$

we also have (with the bounds above),

$$\begin{aligned} \|\nabla\rho(t)\|_{C^\alpha(\eta(t, W))} &\leq \|\nabla(L(t, \eta^{-1}(t, \cdot)))\|_{C^\alpha(W)} \|\nabla\eta^{-1}(t)\|_{C^\alpha(\eta(t, W))} \\ &\leq \|\nabla\rho_0\|_{C^\alpha(W)} e^{C(T)t+J(t)} \|\nabla\eta^{-1}(t)\|_{C^\alpha(\eta(t, W))}, \end{aligned}$$

from which (8.8) follows. \square

The following lemma gives estimates for the inverse flow map. As a general rule, the bounds on spatial (though not temporal) regularity of an inverse flow map are the same as those for the forward flow map, just more involved to derive for non-autonomous flows.

Lemma 8.4. *Let η be the flow map on $[0, T] \times \mathbb{R}^d$ for a vector v with $v, \nabla v \in L^\infty((0, T) \times \mathbb{R}^d)$. Then*

$$\|\nabla\eta^{-1}(t)\|_{L^\infty} \leq \exp \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau. \quad (8.15)$$

If, further, $v(t) \in C^{1,\alpha}(\eta(t, W))$ for some open $W \subseteq \mathbb{R}^d$ with a uniform bound over $t \in [0, T]$ then

$$\|\nabla\eta^{-1}(t)\|_{C^\alpha(\eta(t, W))} \leq e^{e^{C(T)t}}, \quad (8.16)$$

where we have not been explicit about the constant $C(T)$.

Proof. Suppose that a particle moving under the flow map is at position x at time t . Let $\mu(\tau; t, x)$ be the position of that same particle at time $t - \tau$, where $0 \leq \tau \leq t$. Then

$$\eta^{-1}(t, x) = \mu(t; t, x), \quad x = \mu(0; t, x)$$

and

$$\frac{d}{d\tau} \mu(\tau; t, x) = -v(t - \tau, \mu(\tau; t, x)).$$

By the fundamental theorem of calculus,

$$\mu(s; t, x) - x = \int_0^s \frac{d}{d\tau} \mu(\tau; t, x) d\tau,$$

or,

$$\mu(s; t, x) = x - \int_0^s v(t - \tau, \mu(\tau; t, x)) d\tau.$$

Taking the spatial gradient, we have

$$\nabla\mu(s; t, x) = I - \int_0^s (\nabla v)(t - \tau, \mu(\tau; t, x)) \nabla\mu(\tau; t, x) d\tau. \quad (8.17)$$

Hence,

$$\|\nabla\mu(s; t, \cdot)\|_{L^\infty} \leq 1 + \int_0^s \|\nabla v(t - \tau)\|_{L^\infty} \|\nabla\mu(\tau; t, \cdot)\|_{L^\infty} d\tau.$$

It follows from Grönwall's lemma that

$$\|\nabla\mu(s; t, \cdot)\|_{L^\infty} \leq \exp \int_0^s \|\nabla v(t - \tau)\|_{L^\infty} d\tau \quad (8.18)$$

for all $s \in [0, t]$. Setting $s = t$ gives (8.15).

Now we apply (3.2) to (8.17), with x restricted to lie in $\eta(t, W)$, noting that $\mu(\tau; t, x) \in \eta(t - \tau, W)$. This gives

$$\begin{aligned} \|\nabla\mu(s; t, x)\|_{C^\alpha(\eta(t-s, W))} &\leq 1 + \int_0^s (\|\nabla v(t - \tau)\|_{L^\infty} + \|\nabla v(t - \tau)\|_{\dot{C}^\alpha} \|\nabla\mu(\tau; t, \cdot)\|_{L^\infty}^\alpha) \\ &\quad \|\nabla\mu(\tau; t, \cdot)\|_{C^\alpha(\eta(t-\tau, W))} d\tau \\ &\leq 1 + \int_0^s (\|\nabla v(t - \tau)\|_{C^\alpha} \|\nabla\mu(\tau; t, \cdot)\|_{L^\infty}^\alpha) \|\nabla\mu(\tau; t, \cdot)\|_{C^\alpha(\eta(t-\tau, W))} d\tau \\ &\leq 1 + C(T)e^{Ct} \int_0^s \|\nabla\mu(\tau; t, \cdot)\|_{C^\alpha(\eta(t-\tau, W))} d\tau, \end{aligned}$$

where we applied (8.18). Grönwall's lemma gives

$$\|\nabla\mu(s; t, x)\|_{C^\alpha(\eta(t-s, W))} \leq \exp(C(T)e^{Cs}),$$

and setting $s = t$ gives (8.16). \square

The following proposition will be used in the proof of Theorem 2.3 in Section 9.

Proposition 8.5. *Let Y_0 be a frame pushed forward to Y by the flow map for the solution to (1.1, 1.3). Then*

$$\partial_t Y_d + v \cdot \nabla Y_d = -Y_d \cdot \nabla v - \rho Y_d.$$

Proof. Recall that by the definition of a frame, $Y_d(t) = \wedge_{k < d} Y_k(t)$. A direct computation from our definition of the wedge product in Section 3 (∇v need not be symmetric) gives

$$\partial_t Y_d + v \cdot \nabla Y_d = -Y_d \cdot \nabla v + (\operatorname{div} v) Y_d.$$

The results follows since $\operatorname{div} v = -\rho$. (The special case when $\operatorname{div} v = 0$ appears in the proof of Proposition 4.1 of [11].) \square

Proposition 8.6. *Let ϕ be ϕ_0 (purely) transported by the flow, $\partial_t \phi + v \cdot \nabla \phi = 0$. Let Y be Y_0 pushed forward according to (4.3) and let $W_0 = \nabla \phi_0$ and $W = \nabla \phi$. Assume that $W_0 \cdot Y_0 = 0$. Then, $W \cdot Y = 0$ for all time.*

Proof. We have

$$\partial_t \nabla \phi + \nabla(v \cdot \nabla \phi) = 0.$$

But

$$\begin{aligned} (\nabla(v \cdot \nabla \phi))^i &= \partial_i(v^j \partial_j \phi) = v^j \cdot \partial_j \partial_i \phi + \partial_i v^j \partial_j \phi = v \cdot \nabla(\nabla \phi)^i + ((\nabla v)^T \nabla \phi)^i \\ &= (v \cdot \nabla \nabla \phi)^i + (\nabla v \nabla \phi)^i, \end{aligned}$$

since ∇v is symmetric. Hence,

$$\partial_t \nabla \phi + v \cdot \nabla \nabla \phi = -\nabla v \nabla \phi,$$

or

$$\frac{d}{dt} (W(t, \eta(t, x))) = (\partial_t W + v \cdot \nabla W)(t, \eta(t, x)) = -(W \cdot \nabla v)(t, \eta(t, x)).$$

Assume that Y is obtained from (4.3). Then, since,

$$\frac{d}{dt} (Y(t, \eta(t, x))) = (\partial_t Y + v \cdot \nabla Y)(t, \eta(t, x)) = (Y \cdot \nabla v)(t, \eta(t, x)),$$

we have

$$\begin{aligned} \frac{d}{dt} ((W \cdot Y)(t, \eta(t, x))) &= W(t, \eta(t, x)) \cdot \frac{d}{dt} (Y(t, \eta(t, x))) + Y(t, \eta(t, x)) \cdot \frac{d}{dt} (W(t, \eta(t, x))) \\ &= (W \cdot (Y \cdot \nabla v) - Y \cdot (W \cdot \nabla v))(t, \eta(t, x)). \end{aligned}$$

But, for two vector fields, Y and Z , we have

$$Y \cdot (Z \cdot \nabla v) - Z \cdot (Y \cdot \nabla v) = Y^i (Z^j \partial_j v^i) - Z^j (Y^i \partial_i v^j) = 0,$$

again using that ∇v is symmetric. Therefore,

$$\frac{d}{dt} (W \cdot Y)(t, \eta(t, x)) = 0.$$

We conclude that if $W_0 \cdot Y_0 = 0$ then $W \cdot Y = 0$ for all time. \square

We will find Proposition 8.7 useful when changing variables using the flow map:

Proposition 8.7. *Let ρ, η be as in Proposition 8.1. Then $\det \nabla \eta(t, x)$, the Jacobian determinant of the map $x \rightarrow \eta(t, x)$ (which is positive), is bounded by*

$$\exp \left(- \int_0^t |\rho(s, \eta(s, x))| ds \right) \leq \det \nabla \eta(t, x) \leq \exp \left(\int_0^t |\rho(s, \eta(s, x))| ds \right).$$

The same bound applies to $\det \nabla \eta^{-1}(t, x)$.

Proof. Fix $x \in \mathbb{R}^d$ and let $f(t) = \det \nabla \eta(t, x)$, $\bar{\rho}(t) = \rho(t, \eta(t, x))$. Then (see, for instance, page 3 of [9])

$$f'(t) = \operatorname{div} v(t, \eta(t, x)) f(t) = -\bar{\rho}(t) f(t).$$

It follows that $(\log f(t))' = -\bar{\rho}(t)$, so since also $f(0) = 1$,

$$\log f(t) = - \int_0^t \bar{\rho}(s) ds \implies f(t) = \exp \left(- \int_0^t \bar{\rho}(s) ds \right).$$

This gives the bounds on $\det \nabla \eta(t, x)$. Since $\eta(\cdot, x)$ is a diffeomorphism, the same bounds apply on its inverse. \square

9. PROPAGATION OF STRIATED REGULARITY: PROOF OF THEOREM 2.3

It would be quite convenient if we could implement the proof of Theorem 2.3 using the weak solution provided by Theorem 1.2 directly. The problem is that to define the pushforward of Y_0 , we need v to at least be Lipschitz so that $\nabla \eta$ is defined (and bounded), whereas we only know a priori that the weak solution has a log-Lipschitz v . Beyond this, we need higher regularity than v Lipschitz to justify the transport equations in (8.3) and (8.4). Hence, we will need to use a sequence of approximating regular solutions, (ρ_n) , and, as part of the proof, show that there is a uniform bound on (∇v_n) in $L^\infty((0, T) \times \mathbb{R}^d)$ so that for the limiting weak solution, ∇v will lie in $L^\infty((0, T) \times \mathbb{R}^d)$.

This is much as in the proof of the propagation of striated regularity of the vorticity for the 2D Euler equations following any of the existing approaches in [7, 8, 20, 1]. A key difference arises, however, in the transport equations in (8.3) and (8.4): for the 2D Euler equations these are pure transport, whereas we must deal with a troublesome right-hand side. Also, proof that the sequence of approximating solutions converges to a weak solution to (1.1, 1.3) will require more work than for the Euler equations because the velocity fields are not divergence-free.

We will first establish in Proposition 9.1 a series of estimates for solutions to (1.1, 1.3) with the initial data regularized as in Proposition 7.1. These estimates will control \mathcal{Y} and $\mathcal{Y} \cdot \nabla v$ in terms of the initial data and the quantity,

$$V(t) := \|\rho_0\|_{L^\infty} + \left\| \text{p. v.} \int_{\mathbb{R}^d} \nabla \nabla \Phi(x-y) \rho(y) dy \right\|_{L_x^\infty(\mathbb{R}^d)}.$$

The key part of the proof of Proposition 9.1 lies in establishing a useful bound on $V(t)$ itself so that we can ultimately close the estimates.

We give the proof of Proposition 9.1 in Section 9.1, using it to prove Theorem 2.3 in Section 9.2. In Section 9.3 we give the proof of Corollary 2.4.

9.1. Uniform estimates on approximating solutions. In this subsection we establish Proposition 9.1, a series of uniform estimates for the approximating regular solutions to (1.1, 1.3). The estimates in (9.1) through (9.4) are based primarily upon the gradient flow structure of the transport equations in Section 8. Obtaining the estimates in (9.4) and (9.5) will consume most of our effort.

Proposition 9.1. *Let (ρ_n) be the sequence of approximating $C^{1,\alpha}$ -solutions to (1.1, 1.3) for the suitable initial density ρ_0 , as in Proposition 8.2. Let v_n , η_n , and \mathcal{Y}_n the corresponding velocity field, flow map, and pushforward of \mathcal{Y}_0 . Define,*

$$V_n(t) := \|\rho_n\|_{L^\infty} + \left\| \text{p. v.} \int_{\mathbb{R}^d} \nabla \nabla \Phi(x-y) \rho_n(y) dy \right\|_{L_x^\infty(\mathbb{R}^d)}.$$

For all sufficiently large n , the following estimates hold for all $t \in [0, T]$:

$$\|\nabla v_n(t)\|_{L^\infty} \leq V_n(t), \tag{9.1}$$

$$\|\mathcal{Y}_n(t)\|_{L^\infty} \leq \|\mathcal{Y}_0\|_{L^\infty} e^{\int_0^t V_n(s) ds}, \tag{9.2}$$

$$\|\mathcal{Y}_n \cdot \nabla v_n(t)\|_{C^\alpha} \leq C(T) (V_n(t) \|\mathcal{Y}_n(t)\|_{C^\alpha} + 1), \tag{9.3}$$

$$\|\mathcal{Y}_n(t)\|_{C^\alpha} \leq C(T) \|\mathcal{Y}_0\|_{C^\alpha} e^{\int_0^t C(T) V_n(s) ds}, \tag{9.4}$$

where $C(T)$ depends continuously upon F , $\|\rho_0\|_{L^\infty}$ and T . Moreover,

$$V_n(t) \leq C(T)(1 - \log r) + C(T)e^{C(T) \int_0^t V_n(s) ds} r^\alpha \text{ for any } r \in (0, 1]. \tag{9.5}$$

Proof. Because $\|\rho_{0,n}\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}$, we see that all solutions will exist on the time interval $[0, T]$. It also follows that (v_n) is uniformly bounded in the log-Lipschitz norm.

To streamline notation, we **drop the subscript n** throughout the rest of this subsection.

In what follows, Y will always be a vector field among the first $d-1$ vector fields of a frame in \mathcal{Y} . Except in the proof of (9.5), this frame will be arbitrary. Taking the supremum of the bounds over all the frames gives the final estimates expressed in terms of \mathcal{Y} .

Equation (9.1) follows immediately from Proposition 6.1. Equation (9.2) follows from (8.2) and Grönwall's Lemma. By Corollary 6.3,

$$\begin{aligned} \|Y \cdot \nabla v(t)\|_{C^\alpha} &\leq CV(t) \|Y\|_{C^\alpha} + \|\text{div}(\rho Y)\|_{L^\infty} \\ &\leq CV(t) \|Y\|_{C^\alpha} + \|Y \cdot \nabla \rho\|_{L^\infty} + \|\rho \text{div} Y\|_{L^\infty} \leq CV(t) \|Y\|_{C^\alpha} + C(T), \end{aligned}$$

where we used (8.7). This gives (9.3).

The derivations of (9.4) and (9.5) are more involved. We argue much as in Sections 10.2 and 10.3 of [1], which follows an argument in [20], but the differences with [1] are too subtle to merely outline, so we include their complete derivations.

(9.4): Bound on Y in C^α . We write (8.2)₁ as

$$\frac{d}{dt}Y(t, \eta(t, x)) = (Y \cdot \nabla v)(t, \eta(t, x)),$$

and integrate in time, to obtain

$$Y(t, \eta(t, x)) = Y_0(x) + \int_0^t (Y \cdot \nabla v)(s, \eta(s, x)) ds.$$

This leads to the simple bound,

$$\|Y(t)\|_{L^\infty} \leq \|Y_0\|_{L^\infty} + \int_0^t \|(Y \cdot \nabla v)(s)\|_{L^\infty} ds. \quad (9.6)$$

Using the inverse flow map, we also have

$$Y(t, x) = Y_0(\eta^{-1}(t, x)) + \int_0^t (Y \cdot \nabla v)(s, \eta(s, \eta^{-1}(t, x))) ds. \quad (9.7)$$

First, let us estimate $\|\nabla(\eta(s, \eta^{-1}(t, x)))\|_{L^\infty}$. We start with

$$\partial_s \eta(s, \eta^{-1}(t, x)) = u(s, \eta(s, \eta^{-1}(t, x))).$$

Then

$$\partial_s \nabla(\eta(s, \eta^{-1}(t, x))) = \nabla u(s, \eta(s, \eta^{-1}(t, x))) \nabla(\eta(s, \eta^{-1}(t, x))).$$

Integrating in time and using $\nabla(\eta(s, \eta^{-1}(t, x)))|_{s=t} = I$, the identity matrix, we have

$$\nabla(\eta(s, \eta^{-1}(t, x))) = I - \int_s^t \nabla u(\tau, \eta(\tau, \eta^{-1}(t, x))) \nabla(\eta(\tau, \eta^{-1}(t, x))) d\tau.$$

By Grönwall's lemma, then,

$$\|\nabla(\eta(s, \eta^{-1}(t, x)))\|_{L^\infty} \leq \exp \int_s^t \|\nabla u(\tau)\|_{L^\infty} d\tau \leq \exp \int_s^t V(\tau) d\tau.$$

Thus, applying (3.2) and (9.3) to (9.7), and using (3.2) and (8.9), and adding the bound in (9.6), we see that

$$\begin{aligned} \|Y(t)\|_{C^\alpha} &\leq \|Y_0\|_{C^\alpha} \|\nabla \eta^{-1}(t)\|_{L^\infty}^\alpha + \int_0^t \|(Y \cdot \nabla v)(s)\|_{C^\alpha} \|\nabla \eta(s, \eta^{-1}(t, x))\|_{L_x^\infty}^\alpha ds \\ &\leq \|Y_0\|_{C^\alpha} \exp\left(\alpha \int_0^t V(\tau) d\tau\right) \\ &\quad + C(T) \int_0^t (V(s) \|Y(s)\|_{C^\alpha} + 1) \exp\left(\alpha \int_s^t V(\tau) d\tau\right) ds. \end{aligned}$$

Letting

$$y(t) = \|Y(t)\|_{C^\alpha} \exp\left(-\int_0^t V(\tau) d\tau\right),$$

we see that

$$\begin{aligned}
y(t) &\leq \|Y_0\|_{C^\alpha} + C(T) \exp\left(-\int_0^t V(\tau) d\tau\right) \int_0^t \left(V(s) y(s) \exp\int_0^s V(\tau) d\tau + 1\right) \\
&\quad \exp\left(\alpha \int_s^t V(\tau) d\tau\right) ds \\
&\leq \|Y_0\|_{C^\alpha} + C(T) \exp\left(-\int_0^t V(\tau) d\tau\right) \int_0^t V(s) y(s) \exp\int_0^s V(\tau) d\tau ds \\
&\quad + C(T) \exp\left(-\int_0^t V(\tau) d\tau\right) \int_0^t \exp\left(\alpha \int_s^t V(\tau) d\tau\right) ds \\
&\leq \|Y_0\|_{C^\alpha} + C(T) \int_0^t V(s) y(s) ds + C(T) \leq C(T) + C(T) \int_0^t V(s) y(s) ds.
\end{aligned}$$

It follows from Grönwall's lemma that $y(t) \leq \|Y_0\|_{C^\alpha} \exp\left(C(T) \int_0^t V(s) ds\right)$ and hence that

$$\|Y(t)\|_{C^\alpha} \leq \|Y_0\|_{C^\alpha} \exp\left(C(T) \int_0^t V(s) ds\right),$$

which is (9.4).

(9.5): Bound on V . Fix $x \in \mathbb{R}^d$. We start by splitting the principal value integral part of ∇v (as in Proposition 6.1) into two parts as

$$\begin{aligned}
& - \text{p. v.} \int \nabla \nabla \Phi(x-y) \rho(y) dy \\
& = - \text{p. v.} \int \nabla(a_r \nabla \Phi)(x-y) \rho(y) dy - \text{p. v.} \int \nabla((1-a_r) \nabla \Phi)(x-y) \rho(y) dy,
\end{aligned} \tag{9.8}$$

where $r \in (0, 1]$ is arbitrary.

On the support of $\nabla(1-a_r) = -\nabla a_r$, $|x-y| \leq 2r$, so

$$|\nabla((1-a_r) \nabla \Phi)| \leq |(1-a_r) \nabla \nabla \Phi| + |(-\nabla a_r) \otimes \nabla \Phi| \leq C|x-y|^{-d}. \tag{9.9}$$

Hence, one term in (9.8) is easily bounded, using only that $\rho(t) \in L^1 \cap L^\infty$, by

$$\begin{aligned}
& \left| \text{p. v.} \int \nabla((1-a_r) \nabla \Phi)(x-y) \rho(t, y) dy \right| \leq C \int_{B_r^C(x)} |x-y|^{-d} |\rho(t, y)| dy \\
& \leq C \int_r^1 \frac{\|\rho(t)\|_{L^\infty}}{\lambda^d} \lambda^{d-1} d\lambda + C \| |x-\cdot|^{-d} \|_{L^\infty(B_1^C(x))} \|\rho_0\|_{L^1} \\
& \leq -C(T) \log r \|\rho_0\|_{L^\infty} + C(T) \|\rho_0\|_{L^1} \leq C(T)(-\log r + 1).
\end{aligned} \tag{9.10}$$

We used (1.9) to obtain the constant, $C(T)$.

For the other term in (9.8), letting μ_{rh} be as in Definition 3.2, we can write

$$\left| \text{p. v.} \int \nabla(a_r \nabla \Phi)(x-y) \rho(t, y) dy \right| = \left| \lim_{h \rightarrow 0} \nabla(\mu_{rh} \nabla \Phi) * \rho(t, x) \right| = \lim_{h \rightarrow 0} |B|,$$

where

$$B(t, x) := \nabla [(\mu_{rh} \nabla \Phi) * \rho](x).$$

Because \mathcal{Y}_0 is a sufficient family, there exists a frame \mathbf{Y}_0 in \mathcal{Y}_0 so that $|Y_{0,d}(x)| \geq I(\mathcal{Y}_0)/2$. Applying Lemma 5.1 to this frame gives, at (t, x) ,

$$|B| \leq \frac{P(Y_1, \dots, Y_{d-1})}{|Y_d|^4} \sum_{i=j}^{d-1} |BY_j| + \text{tr } B,$$

where P has degree n_d . Moreover, from Proposition 8.5, it follows, using (reverse) Grönwall's Lemma, that

$$|Y_d(t, \eta(t, x))| \geq |Y_{0,d}(x)| e^{-\int_0^t (V(s) + \|\rho(s)\|_{L^\infty}) ds} \geq I(\mathcal{Y}_0) e^{-t\|\rho_0\|_{L^\infty}} e^{-\int_0^t V(s) ds}. \quad (9.11)$$

In light of the bound in (9.2), we have (recall from Lemma 5.1 that $n_d = 4d - 5$)

$$\begin{aligned} |B| &\leq C \frac{\|\mathcal{Y}(t)\|_{L^\infty}^{n_d}}{I(\mathcal{Y}_0)^4} e^{2\int_0^t (V(s) + \|\rho(s)\|_{L^\infty}) ds} \sum_{i=j}^{d-1} |BY_j| + |\text{tr } B| \\ &\leq C(T) e^{(n_d+4)\int_0^t V(s) ds} \sum_{i=j}^{d-1} |BY_j| + |\text{tr } B|. \end{aligned} \quad (9.12)$$

We first compute $\text{tr } B$. We have,

$$\text{tr } B = \sum_{j=1}^d [\partial_j \mu_{rh} \partial_j \Phi] * \rho + [\mu_{rh} \Delta \Phi] * \rho = \sum_{j=1}^d [\partial_j \mu_{rh} \partial_j \Phi] * \rho,$$

using $\Delta \Phi = \delta_0$ and $\mu_{rh}(0) = 0$ to remove the one term. By (3.3), we have for $j = 1, 2, \dots, d$,

$$\begin{aligned} |[\partial_j \mu_{rh} \partial_j \Phi] * \rho| &\leq \frac{C}{r} \int_{r < |x-y| < 2r} \frac{|\rho(t, y)|}{|x-y|^{d-1}} dy + \frac{C}{h} \int_{h < |x-y| < 2h} \frac{|\rho(t, y)|}{|x-y|^{d-1}} dy \\ &\leq \frac{C}{r} \int_r^{2r} \frac{\|\rho(t)\|_{L^\infty}}{\lambda^{d-1}} \lambda^{d-1} d\lambda + \frac{C}{h} \int_h^{2h} \frac{\|\rho(t)\|_{L^\infty}}{\lambda^{d-1}} \lambda^{d-1} d\lambda \\ &= C \|\rho(t)\|_{L^\infty} \leq C(T), \end{aligned}$$

where we used (1.9). Thus,

$$\lim_{h \rightarrow 0} |\text{tr } B| \leq C(T).$$

We next estimate $|BY_j|$. Because

$$B_l^k = \partial_l [\mu_{rh} \partial_k \Phi] * \rho,$$

we have, for $j < d$,

$$\begin{aligned} (BY_j)^k &= B_l^k Y_j^l = (\partial_l [\mu_{rh} \partial_k \Phi] * \rho) Y_j^l \\ &= (\partial_l [\mu_{rh} \partial_k \Phi] * \rho) Y_j^l - \partial_l [\mu_{rh} \partial_k \Phi] * (\rho Y_j^l) + (\mu_{rh} \partial_k \Phi) * \text{div}(\rho Y_j) \\ &=: \sum_{l=1}^d d_l^k + (\mu_{rh} \partial_k \Phi) * \text{div}(\rho Y_j). \end{aligned}$$

By Lemma 3.3,

$$|d_l^k| = \left| \left[\int \nabla [\mu_{rh} \nabla \Phi](x-y) (Y_j(x) - Y_j(y)) \rho(y) dy \right]_l^k \right| \leq C \|Y_j(t)\|_{\dot{C}^\alpha} \|\rho(t)\|_{L^\infty} r^\alpha,$$

and using (1.9) and (9.4), we see that

$$\sum_{l=1}^d \lim_{h \rightarrow 0} |d_l^k| \leq C \|Y_j(t)\|_{C^\alpha} \|\rho_0\|_{L^\infty} r^\alpha \leq C e^{C(T) \int_0^t V(s) ds} r^\alpha.$$

Also by Lemma 3.3 along with (8.7),

$$\begin{aligned} \lim_{h \rightarrow 0} |(\mu_{rh} \partial_k \Phi) * \operatorname{div}(\rho Y_j)| &= \lim_{h \rightarrow 0} \left| \int_{\mathbb{R}^d} (\mu_{rh} \nabla \Phi)(x-y) \operatorname{div}(\rho Y_j)(y) dy \right| \\ &\leq C \|\operatorname{div}(\rho Y_j)\|_{L^\infty} r^\alpha \leq C \|\mathcal{Y}_0 \cdot \nabla \rho_0\|_{L^\infty} e^{C(T)t} r^\alpha \leq C(T) r^\alpha. \end{aligned}$$

Thus, returning to (9.12), and noting that our bounds are independent of h , we have (now including the explicit dependence of B on t and x),

$$\lim_{h \rightarrow 0} |B(t, x)| \leq C(T) r^\alpha e^{C(T) \int_0^t V(s) ds} + C(T).$$

Combined with (9.10), which applies for all $x \in \mathbb{R}^d$, we obtain,

$$\begin{aligned} V(t) &\leq C(1 - \log r) \|\rho_0\|_{L^1 \cap L^\infty} + \sup_{Y_0 \in \mathcal{Y}_0} \sup_{x \in \mathbb{R}^d} \lim_{h \rightarrow 0} |B(t, x)| \\ &\leq C(T)(1 - \log r) + C(T) e^{C(T) \int_0^t V(s) ds} r^\alpha, \end{aligned}$$

which is (9.5). □

9.2. Proof of Theorem 2.3. To prove Theorem 2.3, we will first close the bound on $V_n(t)$ as in Section 10.4 of [1] to obtain a uniform bound on it, and so on $\|\nabla v_n\|_{L^\infty}$. We will then show that our sequence of approximate solutions converges to a solution to the aggregation equation having the same uniform bounds.

Closing the bounds. Setting

$$r = r_n = \exp\left(-\frac{C(T)}{\alpha} \int_0^t V_n(s) ds\right),$$

we have

$$1 - \log r = 1 + \frac{C(T)}{\alpha} \int_0^t V_n(s) ds, \quad r^\alpha = \exp\left(-C(T) \int_0^t V_n(s) ds\right).$$

The bound in (9.5) thus yields the estimate,

$$V_n(t) \leq C(T) + \frac{C(T)}{\alpha} \int_0^t V_n(s) ds. \tag{9.13}$$

By Grönwall's lemma, we conclude that

$$\|\nabla v_n(t)\|_{L^\infty} \leq V_n(t) \leq C(T) e^{C(T)\alpha^{-1}t}$$

so also, by virtue of (9.4),

$$\|\mathcal{Y}_n(t)\|_{C^\alpha} \leq C(T) \exp\left(C(T) e^{C(T)\alpha^{-1}t}\right). \tag{9.14}$$

Convergence to a solution. What is left to show is that the sequence of regular solutions (ρ_n) converges to a weak solution ρ and that key bounds in Proposition 9.1 continue to hold for the limiting solution. In proving this, we adapt the corresponding arguments for the 2D Euler equations, combining aspects of the arguments in Section 8.2 of [17], pages 105-106 of [9], and Section 10.5 of [1]. This consists of the following steps:

- (i) Prove that the sequence of forward flow maps (η_n) converges to a function η and that the sequence of inverse flow maps (η_n^{-1}) converges to η^{-1} in $L^\infty([0, T] \times \mathbb{R}^d)$.
- (ii) Define $\rho(t, x)$ to satisfy (1.7) and show that $\rho_n \rightarrow \rho$ in $L^\infty([0, T]; L^p(\mathbb{R}^d))$ for any $p \in [1, \infty)$.
- (iii) Define $v = -\nabla\Phi * \rho$ and show that η is, in fact, the flow map for v . Hence, ρ is a Lagrangian solution to (1.1, 1.3).
- (iv) Show that $\nabla v \in L^\infty([0, T] \times \mathbb{R}^d)$, giving (a) in Theorem 2.3.
- (v) Show that $\mathcal{Y} \in L^\infty(0, T; C^\alpha(\mathbb{R}^d))$.
- (vi) Show that ρ is also a weak Eulerian solution.
- (vii) Show (b) and (c) in Theorem 2.3.

(i). By (8.9),

$$\|\nabla\eta_n(t)\|_{L^\infty}, \|\nabla\eta_n^{-1}(t)\|_{L^\infty} \leq J(t) \leq e^{V_n(t)} \leq C(T).$$

Moreover, for all $t_1, t_2 \in [0, T]$,

$$|\eta_n(t_1, x) - \eta_n(t_2, x)| \leq \|v_n\|_{L^\infty([0, T] \times \mathbb{R}^d)} |t_1 - t_2| \leq C(T) |t_1 - t_2|.$$

That is, the sequence (η_n) is Lipschitz-continuous in time and space uniformly in n , so the Arzela-Ascoli theorem shows that a subsequence converges uniformly on compact subsets to a Lipschitz-continuous in time and space function, η . The analogous argument gives the same type of convergence for the inverse flow maps η_n^{-1} to η^{-1} (though with lower time regularity, an issue that will not affect us). With the compact support of the initial data it is easy to see that a subsequence converges in $L^\infty([0, T] \times \mathbb{R}^d)$. Once we prove that in the limit we obtain an Eulerian solution to (1.1, 1.3), for which we have uniqueness, it will follow that these convergences are for the full sequence; for now, we will simply reindex as needed to avoid additional notation.

(ii). For any fixed $x \in \mathbb{R}^d$, let $\bar{\rho}(t, x)$ be the solution to the single-dimensional ODE, $\partial_t \bar{\rho}(t, x) = F(\bar{\rho}(t, x))$, $\bar{\rho}(0, x) = \rho_0(x)$ on $[0, T]$, and define ρ on $[0, T] \times \mathbb{R}^d$ by $\rho(t, x) = \bar{\rho}(t, \eta^{-1}(t, x))$. It follows that (1.7) holds. Therefore,

$$\begin{aligned} \|\rho(t, \eta(t, \cdot)) - \rho_n(t, \eta_n(t, \cdot))\| &\leq \|\rho_0 - \rho_{0,n}\| + \int_0^t \|F(\rho(s, \eta(s, \cdot))) - F(\rho_n(s, \eta_n(s, \cdot)))\| ds \\ &\leq \|\rho_0 - \rho_{0,n}\| + C(T) \int_0^t \|\rho(s, \eta(s, \cdot)) - \rho_n(s, \eta_n(s, \cdot))\| ds, \end{aligned}$$

where we note that $\|\cdot\|$ is the L^2 norm and we used that $\rho(\eta(s, \cdot))$ and $\rho_n(\eta_n(s, \cdot))$ are supported in some fixed compact set for all $s \in [0, T]$ because of the compact support of the initial data and the uniform bound on (ρ_n) in $L^\infty([0, T] \times \mathbb{R}^d)$. But $\|\rho_0 - \rho_{0,n}\| \rightarrow 0$ as $n \rightarrow \infty$ by (ii) of Proposition 7.1 with $p = 2$, so Grönwall's lemma gives that

$$\|\rho(t, \eta(t, \cdot)) - \rho_n(t, \eta_n(t, \cdot))\| \leq \|\rho_0 - \rho_{0,n}\| e^{C(T)t} \rightarrow 0. \quad (9.15)$$

A similar argument applied to the inverse flow map gives

$$\begin{aligned} \|\rho(t) - \rho_n(t)\| &\leq \|\rho_0(\eta^{-1}(t, \cdot) - \rho_{0,n}(\eta_n^{-1}(t, \cdot)))\| \\ &\quad + \int_0^t \|F(\rho(s, \eta(s, \eta^{-1}(s, \cdot)))) - F(\rho_n(s, \eta_n(s, \eta_n^{-1}(s, \cdot))))\| ds. \end{aligned} \quad (9.16)$$

Now,

$$\begin{aligned} &\|\rho_0(\eta^{-1}(t, \cdot) - \rho_{0,n}(\eta_n^{-1}(t, \cdot)))\| \\ &\leq \|\rho_0(\eta^{-1}(t, \cdot) - \rho_0(\eta_n^{-1}(t, \cdot)))\| + \|\rho_0(\eta_n^{-1}(t, \cdot) - \rho_{0,n}(\eta_n^{-1}(t, \cdot)))\|. \end{aligned}$$

The first term vanishes in the limit by Lemma 9.3, below, the second vanishes because $\rho_{0,n} \rightarrow \rho_0$ in L^2 and we have a uniform bound on the Jacobian, $\det \nabla \eta_n^{-1}$, by Proposition 8.7. For the time integral in (9.16), we have

$$\begin{aligned} &\|F(\rho(s, \eta(s, \eta^{-1}(s, \cdot)))) - F(\rho_n(s, \eta_n(s, \eta_n^{-1}(s, \cdot))))\| \\ &\leq C(T) \|\rho(s, \eta(s, \eta^{-1}(s, \cdot))) - \rho_n(s, \eta_n(s, \eta_n^{-1}(s, \cdot)))\| \\ &\leq C(T) \|\rho(s, \eta(s, \eta_n^{-1}(s, \cdot))) - \rho_n(s, \eta_n(s, \eta_n^{-1}(s, \cdot)))\| \\ &\quad + C(T) \|\rho(s, \eta(s, \eta^{-1}(s, \cdot))) - \rho(s, \eta(s, \eta_n^{-1}(s, \cdot)))\|. \end{aligned}$$

Again using that the Jacobian for the inverse map η_n^{-1} is bounded, we have

$$\|\rho(s, \eta(s, \eta_n^{-1}(s, \cdot))) - \rho_n(s, \eta_n(s, \eta_n^{-1}(s, \cdot)))\| \leq C(T) \|\rho(s, \eta(s, \cdot)) - \rho_n(s, \eta_n(s, \cdot))\|,$$

which vanishes in the limit by (9.15). Letting $f(s, x) = \rho(s, \eta(s, x))$, we can write,

$$\|\rho(s, \eta(s, \eta^{-1}(s, \cdot))) - \rho(s, \eta(s, \eta_n^{-1}(s, \cdot)))\| = \|f(\eta^{-1}(s, \cdot)) - f(\eta_n^{-1}(s, \cdot))\|,$$

which vanishes in the limit by Lemma 9.3. We conclude from the dominated convergence theorem that the time integral in (9.16) vanishes in the limit. Noting that the arguments above using the L^2 norm apply for any $p \in [1, \infty)$, we see that

$$\rho_n(t) \rightarrow \rho(t) \text{ in } L^\infty(0, T; L^p(\mathbb{R}^d)) \text{ for any } p \in [1, \infty). \quad (9.17)$$

(iii). Define $v = -\nabla \Phi * \rho$. Then since $\rho_n(t)$ and $\rho(t)$ are compactly supported in some common $K(T) \subseteq \mathbb{R}^d$ for all $t \in [0, T]$, we have, for any $p \in (2, \infty)$ and $p' = p/(p-1)$,

$$\|v_n - v\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^d))} \leq \|\nabla \Phi\|_{L^{p'}(2K(T))} \|\rho_n - \rho\|_{L^\infty(0, T; L^p(\mathbb{R}^d))} \rightarrow 0. \quad (9.18)$$

We see in the same way that (v_n) is uniformly bounded in $L^\infty([0, T] \times \mathbb{R}^d)$; hence, we can apply the dominated convergence theorem to give, for any $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$\eta(t, x) - x = \lim_{n \rightarrow \infty} (\eta_n(t, x) - x) = \lim_{n \rightarrow \infty} \int_0^t v_n(s, \eta_n(s, x)) ds = \int_0^t \lim_{n \rightarrow \infty} v_n(s, \eta_n(s, x)) ds.$$

But,

$$\begin{aligned} &|v_n(s, \eta_n(s, x)) - v(s, \eta(s, x))| \\ &\leq |v_n(s, \eta_n(s, x)) - v(s, \eta_n(s, x))| + |v(s, \eta_n(s, x)) - v(s, \eta(s, x))| \\ &\leq \|v_n(s) - v(s)\|_{L^\infty} + |v(s, \eta_n(s, x)) - v(s, \eta(s, x))|. \end{aligned}$$

The first term on the right vanishes by (9.18); the second vanishes because v has a log-Lipschitz modulus of continuity and $\eta_n \rightarrow \eta$ in $L^\infty([0, T] \times \mathbb{R}^d)$. In the limit, then, we see

that

$$\eta(t, x) - x = \int_0^t v(s, \eta(s, x)) ds.$$

We conclude that η is the flow map for v and hence ρ is a Lagrangian solution to (1.1, 1.3).

(iv). Using the Arzela-Ascoli theorem as in (i), we know that for all $t \in [0, T]$ the sequence (v_n) converges uniformly on compact subsets to a Lipschitz-continuous vector field. It follows from (iii) and the uniqueness of limits that that limit is $v(t)$; hence, $\nabla v \in L^\infty([0, T] \times \mathbb{R}^d)$.

(v). Let $Y_0 = Y_{0,k}^{(j)}$, $k < d$, be any vector field from among the first $d - 1$ vector fields in a frame of \mathcal{Y}_0 . We can write (8.1) in the form, $Y_0 \cdot \nabla \eta_n = Y_n \circ \eta_n$. Then by (8.9) and (9.4) with (3.2),

$$\|Y_0 \cdot \nabla \eta_n\|_{C^\alpha(\mathbb{R}^d)} \leq \|Y_n\|_{C^\alpha(\mathbb{R}^d)} \|\nabla \eta_n\|_{L^\infty(\mathbb{R}^d)}^\alpha \leq C(T) \|\mathcal{Y}_0\|_{C^\alpha} \exp\left(C(T) \int_0^t V_n(s) ds\right),$$

so $Y_0 \cdot \nabla \eta_n$ is uniformly bounded in $C^\alpha(\mathbb{R}^d)$. But C^α is compactly embedded in C^β for all $\beta < \alpha$ so a subsequence of $(Y_0 \cdot \nabla \eta_n(t))$ converges in $C^\beta(\mathbb{R}^d)$ to some $f(t)$ for all $\beta < \alpha$, and it is easy to see that $f(t) \in C^\alpha(\mathbb{R}^d)$.

To show that $f(t) = Y_0 \cdot \nabla \eta(t)$, we need only show convergence of $Y_0 \cdot \nabla \eta_n(t) \rightarrow Y_0 \cdot \nabla \eta(t)$ in some weaker sense. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$. As in (2.4) (noting that $Y(t), \operatorname{div} Y(t) \in C^\alpha$),

$$(Y_0 \cdot \nabla \eta_n(t), \varphi) = -(\eta_n(t), \operatorname{div}(\varphi Y_0)) \rightarrow -(\eta(t), \operatorname{div}(\varphi Y_0)) = (Y_0 \cdot \nabla \eta(t), \varphi), \quad (9.19)$$

by the strong convergence of η_n to η we established earlier. Hence, $Y_0 \cdot \nabla \eta_n(t) \rightarrow Y_0 \cdot \nabla \eta(t)$ as distributions, so we can conclude that $f(t) = Y_0 \cdot \nabla \eta(t)$. Since $f(t) \in C^\alpha(\mathbb{R}^d)$, we see that

$$Y(t) = (Y_0 \cdot \nabla \eta) \circ \eta^{-1} = f(t) \circ \eta^{-1}$$

lies in $C^\alpha(\mathbb{R}^d)$ by (3.2).

(vi). That ρ is a weak Eulerian solution to (1.1, 1.3) as in Definition 1.3 follows easily from the convergences we proved of the sequences (ρ_n) and (v_n) .

(vii). Now let $\psi_j(t)$ be ψ_j transported by the flow map. Because $\psi_j(t, x) = \psi_j(0)(\eta^{-1}(t, x))$, we have $\|\psi_j(t)\|_{L^\infty} = \|\psi_j(0)\|_{L^\infty}$. Taking the gradient of the transport equation, $\partial_t \psi_j + v \cdot \nabla \psi_j = 0$, we have

$$\partial_t \nabla \psi_j + \nabla(v \cdot \nabla \psi_j) = 0.$$

(Note that we have assumed insufficient regularity on ψ_j for the second term here. We could proceed either by using the sequence of approximating flow maps as we did above, or by making a separate approximation argument. For simplicity, however, we present this somewhat formal approach.) But

$$\begin{aligned} (\nabla(v \cdot \nabla \psi_j))^i &= \partial_i(v^j \partial_j \psi_j) = v^j \cdot \partial_j \partial_i \psi_j + \partial_i v^j \partial_j \psi_j = v \cdot \nabla(\nabla \psi_j)^i + ((\nabla v)^T \nabla \psi_j)^i \\ &= (v \cdot \nabla \nabla \psi_j)^i + (\nabla v \nabla \psi_j)^i, \end{aligned}$$

since ∇v is symmetric. Hence,

$$\partial_t \nabla \psi_j + v \cdot \nabla \nabla \psi_j = -\nabla v \nabla \psi_j, \quad (9.20)$$

which we can rewrite as

$$\nabla \psi_j(t, x) = \nabla \psi_j(0, \eta^{-1}(t, x)) - \int_0^t (\nabla v \nabla \psi_j)(s, \eta(s, \eta^{-1}(t, x))) ds.$$

Therefore,

$$\|\nabla\psi_j(t)\|_{L^\infty} \leq \|\nabla\psi_j(0)\|_{L^\infty} + \int_0^t \|\nabla v(s)\|_{L^\infty} \|\nabla\psi_j(s)\|_{L^\infty} ds.$$

Applying Grönwall's lemma gives (2.1).

Noting that the level sets of both ρ_0 and ψ_j are transported by the flow map, we see that $\psi_j(t)$ remains level-set-compatible to $\rho(t)$ on the interior of $\text{supp } \psi_j(t)$. Finally, note that $Z_j \in L^\infty(0, T; C^\alpha(\mathbb{R}^d))$ and is non-vanishing, which follow from Proposition 8.5 since $Z_j = Y_{d,j}$ and each frame lies in $C^\alpha(\mathbb{R}^d)$. Moreover, by Proposition 8.6, Z_j is parallel to $\nabla\psi_j(t)$ for all $t \in [0, T]$. Therefore, we see that the level sets of $\psi_j(t)$ retain their $C^{1,\alpha}$ regularity \square

Remark 9.2. *It follows, much as in the proof of (v) above, that $\mathcal{Y} \cdot \nabla v \in C^\alpha(\mathbb{R}^d)$.*

In the proof above of Theorem 2.3, we used Lemma 9.3, a version of continuity of the L^p norm with respect to translation.

Lemma 9.3. *Let $f \in L^p(\mathbb{R}^d)$, $p \in [1, \infty)$, be compactly supported, let (g_n) be a sequence of bounded homeomorphisms on \mathbb{R}^d , and let g be a bounded homeomorphism on \mathbb{R}^d . Assume that $g_n \rightarrow g$ uniformly on compact subsets of \mathbb{R}^d . Then*

$$f \circ g_n \rightarrow f \circ g \text{ in } L^p(\mathbb{R}^d) \text{ as } n \rightarrow \infty.$$

Proof. This is a simple adaptation of the classical proof in which g_n is translation and g is the identity. See, for instance, Theorem 8.19 p. 134-135 of [21]. Or see Lemma 8.2 of [15] for an explicit proof. \square

9.3. Proof of Corollary 2.4. Apply Corollary 7.2 to Theorem 2.3. Since each $Z_j(t)$ is parallel to $\nabla\psi_j(t)$, each is normal to the boundary; because $Z_j(t) \in C^\alpha$ it follows that $\partial\Omega(t)$ is $C^{1,\alpha}$. \square

10. REMOVABILITY OF SINGULARITIES: PROOF OF THEOREM 2.5

We establish in this section the higher regularity of the corrected velocity gradient, constructing the matrix A of Theorem 2.5.

Proof of Theorem 2.5. Let \mathcal{Y}_0 be the sufficient family given by Proposition 7.1 with \mathcal{Y} its pushforward constructed in the proof of Theorem 2.3. First, we argue locally. Let $\mathbf{Y} = (Y_1, \dots, Y_d)$ be any frame in \mathcal{Y} for which $|Y_d| > I(\mathcal{Y})/2$ on some nonempty open $\Omega \subseteq \mathbb{R}^d$. Define

$$A := -\frac{1}{|Y_d|^2} Y_d \otimes Y_d = -\widehat{Y}_d \otimes \widehat{Y}_d \text{ on } \Omega, \quad (10.1)$$

where $\widehat{Y}_d = Y_d/|Y_d|$, recalling that $Y_d = \wedge_{k < d} Y_k$.

For all i, k ,

$$[(Y_d \otimes Y_d)Y_k]^i = Y_d^i Y_d^j Y_k^j = Y_d^i (Y_d \cdot Y_k) = \delta_{dk} Y_d^i |Y_d|^2,$$

so

$$AY_k = 0 \text{ for } k < d, \quad AY_d = -Y_d.$$

For $k < d$, then,

$$(\nabla v - \rho A)Y_k = \nabla v Y_k - \rho AY_k = \nabla v Y_k \in C^\alpha,$$

since $\nabla v Y_k \in C^\alpha$ as noted in Remark 9.2. Because $\text{tr } A = -1$,

$$\text{tr}(\nabla v - \rho A) = \text{div } v - \rho \text{tr } A = -\rho(1 + \text{tr } A) = 0.$$

Since $\nabla v - \rho A$ is symmetric and traceless, it follows from Lemma 5.2 that $\nabla v - \rho A \in C^\alpha(\Omega)$.

The result then follows by using a partition of unity and the compact support of ρ_0 to combine each local version of A . \square

Remark 10.1. *Because $Y_d^{(j)}$ is the wedge product of $d - 1$ vector fields lying in C^α , it also lies in C^α . Nonetheless, if u is any nonvanishing vector field parallel to Y_d on U_j , even a highly irregular one, then we see from (10.1) that $A_j = -\widehat{u} \otimes \widehat{u}$, where $\widehat{u} = u/|u|$. In particular, transporting ψ_j to give $\psi_j(t)$ as in Theorem 2.3, we can use $u = \nabla \psi_j$ locally, so $A = -\widehat{\nabla \psi_j} \otimes \widehat{\nabla \psi_j}$, even though $\nabla \psi_j(t)$ is only in L^∞ .*

Proof of Theorem 2.6. The result follows from (8.8)₁ for $k = 0$ and the bound on $J(t)$ that follows from the proof of Theorem 2.3 (or follows easily from (a) of its statement).

Now assume $k = 1$. By Theorem 2.5, $\nabla v - \rho A \in C^\alpha(\eta(t, W))$. But $A \in C^\alpha(\mathbb{R}^d)$ and $\rho \in C^\alpha(\eta(t, W))$ by the $k = 0$ result. We conclude that $\nabla v \in C^\alpha(\eta(t, W))$ for all $t \in [0, T]$. The result then follows from (8.8)₃ and Lemma 8.4. \square

11. PATCH BOUNDARY REGULARITY: COMPARISON TO [2]

We now give a brief explanation of how Corollary 2.4 is established in [2] then compare it to our approach.

The authors of [2] first make the transformation,

$$s := -\frac{1}{T} \log \left(1 - \frac{t}{T} \right), \quad \bar{\rho}(s, x) := T(T - t) \rho(t, x), \quad \bar{v}(s, x) := T(T - t) v(t, x).$$

Relabeling to return to t, ρ, v , we obtain (for $F(\rho) = \rho^2$), in place of (1.1, 1.3),

$$\partial_t \rho + v \cdot \nabla \rho = \rho(\rho - T), \quad v = -\nabla \Phi * \rho, \quad \rho(0) = \rho_0.$$

Hence, $F(\rho) = \rho^2$ is transformed to $F(\rho) = \rho(\rho - T)$, the negative of the logistic map, without changing anything else. It is easy to see that if $0 \leq \rho_0 \leq T$ then $0 \leq \rho(t) \leq T$ for all $t \in [0, T]$. Moreover, when $\rho_0 = \mathbb{1}_\Omega$, since $T^* = 1$, we can set $T = 1$, and it is easy to see that the forcing vanishes for all $t \in [0, T]$. Hence, for the special case of patch data, the density is purely transported in the transformed variables. All further arguments in [2] apply then to $\partial_t \rho + v \cdot \nabla \rho = 0$. Or, in our notation, the authors treat (1.1, 1.3) in the special case, $F \equiv 0$.

Now, even though $\psi_j(0) \in C^{1,\alpha}(\mathbb{R}^d)$, it is clear from (9.20) that we cannot propagate more than the Lipschitz regularity of ψ_j . The authors of [2] get around this difficulty by evolving each $\psi_j(0)$ to obtain the function $\phi(t)$ with the nonsingular forcing term $-\phi \mathbb{1}_{\Omega_t} = -\phi \rho$:

$$\partial_t \phi + v \cdot \nabla \phi = -\phi \rho, \quad \phi(0) = \phi_0 := \psi_j(0).$$

Here, ϕ_0 defines one component of $\partial\Omega$, where it vanishes and has non-vanishing gradient.

Setting $Z = \nabla \phi$, and using that $\phi \nabla \rho \equiv 0$, a calculation like that in (9.20) gives

$$\partial_t Z + v \cdot \nabla Z = -\nabla v Z - \rho Z. \tag{11.1}$$

A priori, we only know that $Z(t) \in L^\infty$, just as with $\nabla \psi_j$ of Theorem 2.3. But the term $-\rho Z$, in effect, cancels the term $\nabla \Phi * \text{div}(\rho Z)$ appearing in Corollary 6.3. Hence,

$$\partial_t Z(t, x) + (v \cdot \nabla Z)(t, x) = \int_{\Omega_t} \nabla^2 \phi(x - y) (Z(t, x) - Z(y, t)) dy, \quad x \in \mathbb{R}^d,$$

which implies that

$$\|Z(t)\|_{C^\alpha} \leq \|\nabla\phi_0\|_{C^\alpha} \exp\left(C \int_0^t (1 + \|\nabla v(s)\|_{L^\infty}) ds\right).$$

This gives the $C^{1,\alpha}$ regularity of the patch boundary.

In light of (2.2) and Proposition 8.5, we see that the authors of [2] are using the last vector field in a frame without ever employing the first $d - 1$ vector fields. They are able to do this because they can use the transport properties of ϕ to explicitly solve for what we call Z (in terms of the flow map) inside and outside Ω_t and ultimately show that it has higher regularity in the normal direction than at first appears. It is critical in their argument, however, that ϕ vanish on $\partial\Omega$, thereby eliminating the singularity that would otherwise be in $-\nabla(\phi\rho)$. Their approach could be extended, for example, to nested sums of vortex patches without great difficulty, simply by choosing ϕ_0 to have a successive series of zeroes at the boundary of each vortex patch. (F would no longer vanish entirely, but it would take on only discrete values, a minor complication.) But this would not work for initial densities whose level sets are $C^{1+\alpha}$ but which have no regularity at all in directions perpendicular to the level sets, which we allow in Theorem 2.3. Any choice of ϕ_0 that vanishes on these level sets would necessarily vanish on open sets, and hence the level sets would not be regular points of ϕ_0 .

Hence, it appears unavoidable to bring in the idea of pushing forward a frame, then using the C^α regularity of the vector fields in the tangential direction to obtain the C^α regularity of Z . This is the idea that Chemin [8] brought to bear on the vortex patch problem for the 2D Euler equations.

Another key difference between [2] and this paper is that Picard iteration on parametrizations of the boundary of the patch is used in [2], which avoids the need to regularize the initial data—an issue that was perhaps the most troublesome in our approach. But applying Picard iteration with the initial data in Theorem 2.3 has its own, seemingly insurmountable difficulties, especially in face of the complicated structure of the transport equations in Section 8.

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