

THE VORTEX PATCHES OF SERFATI

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ABSTRACT. In 1993, two proofs of the persistence of regularity of the boundary of a classical vortex patch for the 2D Euler equations were published, one by Chemin in [5] (announced in 1991 in [4]) the other by Bertozzi and Constantin in [2]. Chemin, in fact, proved a more general result, extending it further in his 1995 text [7] showing, roughly, that vorticity initially having discontinuities only in directions normal to a family of vector fields that together foliate the plane continue to be so characterized by the time-evolved vector fields. A different, four-page “elementary” proof of Chemin’s 1993 result was published in 1994 by Ph. Serfati [22], who also gave a fuller characterization of the velocity gradient’s regularity. We give a detailed version of Serfati’s proof along with an extension of it to a family of vector fields that reproduces the 1995 result of Chemin.

In the late 1980s into the early 1990s there was a great deal of interest in determining whether a vortex patch having a smooth boundary at time zero continues to have a smooth boundary for all time as it evolves under the 2D Euler equations. Majda had suggested in [20] the possibility of singularities forming in finite time. Existing numerical evidence showed that the boundary typically deforms dramatically over time, and hinted at the development of such singularities. The announcement in 1991 [4] and the two 1993 papers [5, 2] came, then, as a surprise to many, showing as they did that the boundary remains regular for all time.

In 1994, another proof of the persistence of regularity of a vortex patch was published by Ph. Serfati in the four-page paper, [22]. Like Chemin’s [4], it was published in a journal devoted primarily to short announcements, but unlike [4], it was never followed by a full-length publication. In this highly condensed form much is omitted that would aid the reader in understanding, and much is left to the reader to decipher. It’s opaqueness has kept [22] from having an influence on subsequent developments in two-dimensional fluid mechanics. One of our purposes here is to present our interpretation of Serfati’s argument in a detailed enough form to make it accessible, for it is not only an elegant approach to the vortex patch problem, but some of its ideas, buried for two decades, have potential applications to problems of current interest.

Chemin proved a more general result in [7] of which the persistence of regularity of the boundary of a vortex patch was a special case. He employed a family of vector fields and showed, speaking roughly, that if the initial vorticity is C^α in the direction of this family for some $\alpha > 0$ then this property holds true for all time. A second purpose of this work is to show that if one extends Serfati’s hypotheses on the initial data by employing a family of vector fields then one obtains the same result as [7]. We also reinterpret this result as showing that if the initial velocity is $C^{1+\alpha}$ in the direction of the family of vector fields then this property holds true for all time.

Finally, Serfati also showed that the gradient of the velocity is C^α after being corrected by a C^α multiple of the vorticity. We give a different proof of this result (which was one sentence in [22]) and show that it yields an improved estimate on the local propagation of Hölder regularity of the velocity.

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1. INTRODUCTION AND STATEMENTS OF RESULTS

The Euler equations (without forcing) in velocity form can be written,

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = 0, \\ \operatorname{div} u = 0, \end{cases} \quad (1.1)$$

where u is the velocity field and p is the pressure. The operator $u \cdot \nabla = u^i \partial_i$, where we follow the usual convention that repeated indices are summed over. These equations model the flow of an incompressible inviscid fluid.

By introducing the 2D vorticity,

$$\omega = \partial_1 u^2 - \partial_2 u^1,$$

we obtain the vorticity formulation,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u = K * \omega. \end{cases} \quad (1.2)$$

Here,

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \quad x^\perp := (-x_2, x_1), \quad (1.3)$$

is the Biot-Savart kernel, which can also be written

$$K = \nabla^\perp \mathcal{F}, \quad \mathcal{F}(x) = \frac{1}{2\pi} \log |x|, \quad \nabla^\perp := (-\partial_2, \partial_1),$$

\mathcal{F} being the fundamental solution to the Laplacian.

Let $\eta(t, x)$ be the flow map associated to the velocity field u , so that

$$\partial_t \eta(t, x) = u(t, \eta(t, x)), \quad \eta(0, x) = x. \quad (1.4)$$

Then (1.2) tells us that the vorticity is transported by the flow map, so that

$$\omega(t, x) = \omega_0(\eta^{-1}(t, x)) \quad (1.5)$$

is the vorticity of the solution to the Euler equations at time t , where ω_0 is the initial vorticity.

All this presupposes that sufficiently regular solutions exist and are unique. In fact, it all can be made sense of for initial vorticity in $L^1 \cap L^\infty$, in which case the vorticity remains in $L^1 \cap L^\infty$, as first shown by Yudovich in [28]. One must, however, use a weak formulation of (1.1) or (1.2), though (1.2)₂ and (1.5) continue to hold.

If the vorticity is initially the characteristic function of a bounded domain, it will remain so for all time as the Euler solution evolves, since $\eta(t, \cdot)$ is a diffeomorphism. A (classical) vortex patch is such a bounded domain. So if

$$\omega_0 = \mathbb{1}_\Omega, \quad (1.6)$$

where Ω is a bounded domain, then by (1.5),

$$\omega(t) = \mathbb{1}_{\Omega_t}, \quad \Omega_t := \eta(t, \Omega).$$

The bounded domain, Ω_t , is the vortex patch at time t .

The regularity of the boundary of Ω will be specified using a parameter, α .

Throughout this paper we fix $\alpha \in (0, 1)$.

We can now state the result of [5, 2] more precisely.

Theorem 1.1. [4, 5, 2] *Let Ω be a bounded domain whose boundary is the image of a simple closed curve $\gamma_0 \in C^{1+\alpha}(\mathbb{S}^1)$ and let ω_0 be as in (1.6). There exists a unique solution u to the 2D Euler equations, with*

$$\nabla u(t) \in L^\infty(\mathbb{R}^2), \quad \gamma(t, \cdot) := \eta(t, \gamma_0(\cdot)) \in C^{1+\alpha}(\mathbb{S}^1) \text{ for all } t \in \mathbb{R}.$$

In [5], Chemin proves a more general result of which Theorem 1.1 is a corollary. We show in Section 9 that Serfati's result in [23] is equivalent to that in [5]. To describe Serfati's result, we must first make some definitions. Let Σ be a closed subset of \mathbb{R}^2 and Y_0 be a C^α -vector field in \mathbb{R}^2 . Let $\mathcal{N}_\delta(\Sigma) := \{x \in \mathbb{R}^2 : d(x, \Sigma) < \delta\}$. Serfati assumes the following initial conditions:

$$\begin{cases} \omega_0 = \omega_0^1 + \omega_0^2 \in (L^1 \cap L^\infty)(\mathbb{R}^2), & \omega_0^1 = 0 \text{ on } \Sigma^C, \omega_0^2 \in C^\alpha(\mathbb{R}^2), \\ Y_0 \in C^\alpha(\mathbb{R}^2), & |Y_0| \geq c > 0 \text{ on } \mathcal{N}_{\delta_0}(\Sigma), \delta_0 > 0, \\ K * \operatorname{div}(\omega_0 Y_0) \in C^\alpha(\mathbb{R}^2). \end{cases} \quad (1.7)$$

We streamline these conditions to

$$\begin{cases} \omega_0 \in C^\alpha(\mathbb{R}^2 \setminus \Sigma) \cap (L^1 \cap L^\infty)(\mathbb{R}^2), \\ Y_0 \in C^\alpha(\mathbb{R}^2), & |Y_0| \geq c > 0 \text{ on } \mathcal{N}_{\delta_0}(\Sigma), \delta_0 > 0, \\ \operatorname{div}(\omega_0 Y_0) \in C^{\alpha-1}(\mathbb{R}^2), \\ \operatorname{div} Y_0 \in C^\alpha(\mathbb{R}^2). \end{cases} \quad (1.8)$$

(For the negative index Hölder space, $C^{\alpha-1}(\mathbb{R}^2)$, see Definition 2.1.)

We show in Appendix A that (1.7)₃ is equivalent to (1.8)₃ (since $\omega_0 Y_0 \in L^\infty$). Also, we added the condition in (1.8)₄, which is missing in [22], though present in [5, 7], as it is necessary in the proof of the convergence of the approximate solutions (see, however, Remark 5.1). Such convergence is not addressed by Serfati in [22]. This is the only place this condition is required. (See also Remark 1.3.)

We define the pushforward of Y_0 by

$$Y(t, \eta(t, x)) := (Y_0(x) \cdot \nabla) \eta(t, x). \quad (1.9)$$

This is just the Jacobian of the diffeomorphism, $\eta(t, \cdot)$, multiplied by Y_0 . Equivalently,

$$Y(t, x) = \eta(t)_* Y_0(t, x) := (Y_0(\eta^{-1}(t, x)) \cdot \nabla) \eta(t, \eta^{-1}(t, x)).$$

We will make frequent use of constants of the form,

$$c_\alpha = C(\omega_0, Y_0) \alpha^{-1}, \quad C_\alpha = C(\omega_0, Y_0) \alpha^{-1} (1 - \alpha)^{-1}, \quad (1.10)$$

where $C(\omega_0, Y_0)$ is a constant that depends upon only ω_0 and Y_0 . The values of these constants can vary from expression to expression and even between two occurrences within the same expression.

Theorem 1.2 (Serfati [22]). *Assume that ω_0 is an initial vorticity for which there exists some vector field Y_0 so that (1.8) is satisfied and let ω in $L^\infty(\mathbb{R}; (L^1 \cap L^\infty)(\mathbb{R}^2))$ be the unique solution to the Euler equations. We have,*

$$\|\nabla u(t)\|_{L^\infty} \leq c_\alpha e^{c_\alpha t}, \quad \|Y(t)\|_{C^\alpha} \leq C_\alpha e^{e^{c_\alpha t}}. \quad (1.11)$$

Moreover, there exists a matrix $A(t) \in C^\alpha(\mathbb{R}^2)$ such that

$$\nabla u(t) - \omega(t)A(t) \in C^\alpha(\mathbb{R}^2) \text{ for all time.} \quad (1.12)$$

(An explicit form for the matrix A is given in (6.3).)

Remark 1.3. *The first part of Theorem 1.2 giving (1.11) is the same result as that of Chemin in [5]. In [5], though, Chemin assumes $\operatorname{div} Y_0 = 0$ (dropping this restriction in [7]). Serfati does not state this restriction on $\operatorname{div} Y_0$; the present authors, however, were unable to determine from Serfati's proof whether or not he meant to do so. Given that he did not add the required condition $\operatorname{div} Y_0 \in C^\alpha$, it seems likely that he did intend to do so. We will show in our version of the proof that, in any case, $\operatorname{div} Y_0 = 0$ is not required to complete the proof.*

A classical vortex patch satisfies (1.8), as we show in Section 7. We describe other examples satisfying (1.8) in Section 10.

A number of additional useful facts follow from the proof of Theorem 1.2 or are simple consequences of it. We summarize these facts in Theorems 1.4 and 1.5.

Theorem 1.4. *Let ω_0, Y_0 be as in Theorem 1.2. Then*

$$\|\operatorname{div} Y(t)\|_{C^\alpha} \leq \|\operatorname{div} Y_0\|_{C^\alpha} e^{e^{c\alpha t}}, \quad (1.13)$$

$$\|\operatorname{div}(\omega Y)(t)\|_{C^{\alpha-1}} \leq C_\alpha e^{e^{c\alpha t}}, \quad (1.14)$$

$$\|(Y \cdot \nabla)u(t)\|_{C^\alpha} \leq C_\alpha e^{e^{c\alpha t}}, \quad (1.15)$$

$$\|\nabla \eta(t)\|_{L^\infty}, \|\nabla \eta^{-1}(t)\|_{L^\infty} \leq e^{e^{c\alpha t}}, \quad (1.16)$$

$$\|A(t)\|_{C^\alpha}, \|\nabla u(t) - \omega(t)A(t)\|_{C^\alpha} \leq C_\alpha e^{e^{c\alpha t}}. \quad (1.17)$$

Suppose that Y_0 is divergence-free, and let ϕ_0 be a stream function¹ for Y_0 ; that is, $Y_0 = \nabla^\perp \phi_0$. Let ϕ be ϕ_0 transported by the flow, so that $\phi(t, x) := \phi_0(\eta^{-1}(t, x))$. Further, suppose that γ_0 is a $C^{1+\alpha}$ level curve of ϕ_0 and let $\gamma(t, \cdot) = \eta(t, \gamma_0(\cdot))$. Then $\gamma(t)$ is a $C^{1+\alpha}$ level curve of $\phi(t)$ with

$$\|\gamma(t)\|_{C^{1+\alpha}} \leq C_\alpha e^{e^{c\alpha t}}. \quad (1.18)$$

The bounds in (1.13) through (1.17) are proven as part of the proof of Theorem 1.2. We prove (1.18) in Section 7. The bound in (1.15) means that u remains $C^{1+\alpha}$ -smooth in the direction of Y . (For a classical vortex patch, it means that ∇u is discontinuous only across the boundary.)

A simple consequence of Theorem 1.2 is the local propagation of Hölder regularity stated in Theorem 1.5. Critical to its proof is Serfati's construction of the matrix A ; (1.20) of Theorem 1.5 does not follow from [5] or [7], which has no analog of A .

Theorem 1.5. *Let ω_0, Y_0 be as in Theorem 1.2. If $\omega_0 \in C^\beta(U)$ for some open subset U of \mathbb{R}^2 and $\beta \in [0, 1)$ then $\omega(t) \in C^\beta(U)$ for all t , with*

$$\|\omega(t)\|_{C^\beta(U_t)} \leq \|\omega_0\|_{C^\beta(U)} e^{e^{c\alpha t}}, \quad (1.19)$$

where $U_t = \eta(t, U)$. Further,

$$\|\nabla u(t)\|_{C^\alpha(U_t)} \leq C_\alpha e^{e^{c\alpha t}}. \quad (1.20)$$

Proof. For any $x, y \in U_t$,

$$\frac{|\omega(t, x) - \omega(t, y)|}{|x - y|^\beta} = \frac{|\omega_0(\eta^{-1}(t, x)) - \omega_0(\eta^{-1}(t, y))|}{|\eta^{-1}(t, x) - \eta^{-1}(t, y)|^\beta} \left(\frac{|\eta^{-1}(t, x) - \eta^{-1}(t, y)|}{|x - y|} \right)^\beta.$$

¹For any continuous divergence-free vector field a stream function always exists and is unique up to an additive constant.

Together with (1.16) this gives (1.19) (a bound that holds for any Lipschitz velocity field). The bound in (1.20) then follows from (1.17). \square

Theorem 1.5 improves, for initial data satisfying (1.8), existing estimates of local propagation of Hölder regularity for bounded initial vorticity. For instance, Proposition 8.3 of [21] would only give $\nabla u(t) \in C_{loc}^\alpha(U_t)$.

In [7], Chemin extends the result he established in [5] by employing a family of vector fields in whose the direction the initial vorticity has regularity. We do the same for Serfati's initial conditions in Section 8, yielding the same result as Chemin. Moreover, we show in Section 9 that (1.8)₃ is equivalent to

$$Y_0 \cdot \nabla u_0 \in C^\alpha(\mathbb{R}^2),$$

meaning the initial velocity field has $C^{1+\alpha}$ regularity in the direction of Y_0 . By (1.15), this regularity persists for all time.

In outline, Serfati's proof involves showing that the vorticity is transported over time in such a manner that its discontinuities are characterized by $Y(t)$. The regularity of $Y(t)$ is shown to be retained over all time, its estimate being inextricably entwined with an estimate on $\nabla u(t)$ in L^∞ . At this high level, Serfati's approach is the same as that of Chemin in [5] and Constantin and Bertozzi in [2].

Thus, though Serfati's approach is novel in many ways, it clearly owes much to both [5] and [2]. Like Chemin, Serfati proves a more general result involving the persistence of tangential regularity along a vector field, for which a vortex patch is a special case. Like Bertozzi and Constantin, Serfati uses estimates on singular integrals, some of them in much the same manner (Corollary B.3, for instance, a special case of which is used in [2]). Unlike [2], however, Serfati uses no "geometric lemma" and there is also no clear analog in [5, 2] of Serfati's linear algebra lemma, Lemma 3.1.

More concisely, one could say that the setup of the problem and the use of transport estimates is much like that of [5] while the estimates involving the gradient of the velocity are more like that of [2], but the most difficult to estimate term is bounded in an entirely novel way. Plus, Serfati characterizes the gradient of the velocity more fully. (Also see the comments at the end of Section 9.)

There has been a number of papers since [22] related to the regularity of the boundary of vortex patches. Among those treating 2D vortex patches are [9, 6, 10, 13, 24], which study the inviscid limit (see [24] for historical comments as well); [3, 12, 11, 14], which study vortex patch boundaries having singularities; [15, 16] for vortex patches in a bounded domain. None of these, however, have used techniques from [22]: they are all intellectual descendants of either [5, 7] or [2] (or both).

This paper is organized as follows. In Section 2, we fix some notation and make a few definitions. In Section 3, we state six key lemmas we will need. In Section 4, we study the transport equations of Y and a related vector field R , as well as the propagation of regularity of $\operatorname{div}(\omega Y)$. In Section 5, we prove the first part of Theorem 1.2, the bounds in (1.11), and also prove (1.13) through (1.16). In Section 6, we prove the second part of Theorem 1.2, the existence of the matrix A satisfying (1.12), along with the proof of (1.17). In Section 7, we consider the case of a classical vortex patch, showing how Theorem 1.1 follows from Theorem 1.2, along the way proving (1.18). In Section 8, we describe an extension of Serfati's result to a family of vector fields like those of Chemin in [7], in Section 9 showing that the resulting hypotheses on the initial data are equivalent to those of [7]. In Section 10, we discuss several examples of initial data that satisfy the hypotheses of Theorem 1.2. In

Appendix A, we prove the lemmas stated in Section 3. In Appendix B, we detail some calculations involving ∇u . Finally, in Appendix C, we discuss our use of weak transport equations.

2. NOTATION, CONVENTIONS, AND DEFINITIONS

We define

$$\nabla u := \begin{pmatrix} \partial_1 u^1 & \partial_2 u^1 \\ \partial_1 u^2 & \partial_2 u^2 \end{pmatrix} = Du,$$

the Jacobian matrix of u , and define the gradient of other vector fields in the same manner.

We follow the common convention that the gradient and divergence operators apply only to the spatial variables.

Another common convention we follow is that constants denoted by C depend only on the quantities specified in the context or stated explicitly, such as in $C(\omega_0, Y_0)$, but do not depend on other parameters, such as α or ε . In a series of inequalities, the value of C can vary with each inequality.

All vectors are to be treated as column vectors for linear algebra operations, even when written in the form (v^1, v^2) .

We write $|v|$ for the Euclidean norm of $v = (v^1, v^2)$, $|v|^2 = (v^1)^2 + (v^2)^2$. For a 2×2 matrix, $M = (M_{ij})$, we use the operator norm,

$$|M| := \max_{|v|=1} |Mv|. \quad (2.1)$$

Of course, all norms on finite-dimensional spaces are equivalent, so the choice of matrix norm just affects the values of constants. Our choice has the convenient properties, however, that it is sub-multiplicative, gives the identity matrix norm 1, and

$$|M| = \sqrt{\max \text{ eigenvalue of } MM^*} \leq \left(\sum_{i,j} M_{ij}^2 \right)^{\frac{1}{2}} \leq \sqrt{2} |M|, \quad (2.2)$$

the first inequality being strict when M is nonsingular.

If X is a function space, we define

$$\|v\|_X := \| |v| \|_X, \quad \|M\|_X := \| |M| \|_X.$$

Definition 2.1 (Hölder and Lipschitz spaces). *Let $\alpha \in (0, 1)$ and $U \subseteq \mathbb{R}^d$, $d \geq 1$, be open. Then $C^\alpha(U)$ is the space of all measurable functions for which*

$$\|f\|_{C^\alpha(U)} := \|f\|_{L^\infty(U)} + \|f\|_{\dot{C}^\alpha(U)} < \infty, \quad \|f\|_{\dot{C}^\alpha(U)} := \sup_{\substack{x, y \in U \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

For $\alpha = 1$, we obtain the Lipschitz space, which is not called C^1 but rather $Lip(U)$. We also define $lip(U)$ for the homogeneous space. Explicitly, then,

$$\|f\|_{Lip(U)} := \|f\|_{L^\infty(U)} + \|f\|_{lip(U)}, \quad \|f\|_{lip(U)} := \sup_{\substack{x, y \in U \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|}.$$

For any positive integer k , $C^{k+\alpha}(U)$ is the space of k -times continuously differentiable functions on U for which

$$\|f\|_{C^{k+\alpha}(U)} := \sum_{|\beta| \leq k} \|D^\beta f\|_{L^\infty(U)} + \sum_{|\beta|=k} \|D^\beta f\|_{C^\alpha(U)} < \infty.$$

We define the negative Hölder space, $C^{\alpha-1}(U)$, by

$$\begin{aligned} C^{\alpha-1}(U) &= \{f + \operatorname{div} v : f, v \in C^\alpha(U)\}, \\ \|h\|_{C^{\alpha-1}(U)} &= \inf\{\|f\|_{C^\alpha(U)} + \|v\|_{C^\alpha(U)} : h = f + \operatorname{div} v; f, v \in C^\alpha(U)\}. \end{aligned}$$

It follows immediately from the definition of $C^{\alpha-1}$ that

$$\|\operatorname{div} v\|_{C^{\alpha-1}} \leq \|v\|_{C^\alpha}. \quad (2.3)$$

We also have the elementary inequalities,

$$\begin{aligned} \|f \circ g\|_{\dot{C}^\alpha} &\leq \|f\|_{\dot{C}^\alpha} \|\nabla g\|_{L^\infty}^\alpha, \\ \|fg\|_{C^\alpha} &\leq \|f\|_{C^\alpha} \|g\|_{C^\alpha}. \end{aligned} \quad (2.4)$$

Definition 2.2 (The inf “norm”). For any measurable subset $\Lambda \subseteq \mathbb{R}^2$ and measurable function f on Λ , we define

$$\|f\|_{\inf(\Lambda)} = \inf_{x \in \Lambda} |f(x)|.$$

Definition 2.3. For any $\Lambda \subseteq \mathbb{R}^2$, we define $\mathcal{N}_\delta(\Lambda) := \{x \in \mathbb{R}^2 : d(x, \Lambda) < \delta\}$, where d is the Euclidean distance in \mathbb{R}^2 .

Definition 2.4 (Radial cutoff functions). We make an arbitrary, but fixed, choice of a radially symmetric function $a \in C_c^\infty(\mathbb{R}^2)$ taking values in $[0, 1]$ with $a = 1$ on $B_1(0)$ and $a = 0$ on $B_2(0)^C$. For $r > 0$, we define the rescaled cutoff function, $a_r(x) = a(x/r)$, and for $r, h > 0$ we define

$$\mu_{rh} = a_r(1 - a_h).$$

Remark 2.5. When using the cutoff function μ_{rh} we will be fixing r while taking $h \rightarrow 0$, in which case we can safely assume that h is sufficiently smaller than r so that μ_{rh} vanishes outside of $(h, 2r)$ and equals 1 identically on $(2h, r)$. It will then follow that

$$\begin{cases} |\nabla \mu_{rh}(x)| \leq Ch^{-1} \leq C|x|^{-1} & \text{for } |x| \in (h, 2h), \\ |\nabla \mu_{rh}(x)| \leq Cr^{-1} \leq C|x|^{-1} & \text{for } |x| \in (r, 2r), \\ \nabla \mu_{rh} \equiv 0 & \text{elsewhere.} \end{cases}$$

Hence, also, $|\nabla \mu_{rh}(x)| \leq C|x|^{-1}$ everywhere.

Definition 2.6 (Mollifier). Let $\rho \in C_c^\infty(\mathbb{R}^2)$ with $\rho \geq 0$ have $\|\rho\|_{L^1} = 1$. For $\varepsilon > 0$, define $\rho_\varepsilon(\cdot) = (\varepsilon^{-2})\rho(\cdot/\varepsilon)$.

Definition 2.7 (Principal value integral). For any measurable integral kernel, $L: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, and any measurable function, $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, define the integral transform $L[f]$ by

$$L[f](x) := \text{p. v.} \int_{\mathbb{R}^2} L(x, y) f(y) dy := \lim_{h \rightarrow 0^+} \int_{|x-y| > h} L(x, y) f(y) dy,$$

whenever the limit exists.

When dealing with algebraic manipulations of principal value integrals, we will find it convenient to introduce the notation in Definition 2.8.

Definition 2.8.

$$f \tilde{*} g(x) := \text{p. v.} \int f(x-y)g(y) dy.$$

3. KEY LEMMAS

Lemmas 3.1 to 3.6 are six key lemmas we will need in the proof of Theorem 1.2. Lemmas 3.1 and 3.2 are the two lemmas of [22]: a simple, if seemingly unmotivated, linear algebra lemma and an estimate on an integral transform that includes singular integrals. Lemma 3.3 is a variant on Lemma 3.2, while Lemma 3.4 gives explicit estimates on the four kernels to which we will apply Lemmas 3.2 and 3.3. Lemma 3.5 is used to establish the equivalence of (1.7)₃ and (1.8)₃, and Lemma 3.6 is the form of Gronwall's lemma that we will need. We give the proof of Lemma 3.4 in this section and defer the proof of the other lemmas (except for Gronwall's lemma, which is classical) to Appendix A.

Lemma 3.1. *Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible matrix. For any 2×2 symmetric matrix B ,*

$$|B| \leq 2 \frac{|M|}{|\det M|} |BM_1| + |\text{tr } B|,$$

where $M_1 = (a, c)^T$ is the first column of M .

Lemma 3.2. *Let $L: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be an integral kernel for which*

$$\|L\|_* := \sup_{x, y \in \mathbb{R}^2} \left\{ |x-y|^2 |L(x, y)| + |x-y|^3 |\nabla_x L(x, y)| \right\} < \infty$$

and for which

$$\left| \text{p. v.} \int_{\mathbb{R}^2} L(x, y) dy \right| < \infty \text{ for all } x \in \mathbb{R}^2. \quad (3.1)$$

Let $L[f]$ be as in Definition 2.7. Then

$$\begin{aligned} \|L[f - f(x)](x)\|_{\dot{C}^\alpha} &= \left\| \text{p. v.} \int_{\mathbb{R}^2} L(x, y) [f(y) - f(x)] dy \right\|_{\dot{C}^\alpha} \\ &\leq C\alpha^{-1}(1-\alpha)^{-1} \|L\|_* \|f\|_{\dot{C}^\alpha}. \end{aligned} \quad (3.2)$$

If

$$\text{p. v.} \int_{\mathbb{R}^2} L(\cdot, y) dy \equiv 0 \quad (3.3)$$

then

$$\|L[f]\|_{\dot{C}^\alpha} \leq C\alpha^{-1}(1-\alpha)^{-1} \|L\|_* \|f\|_{\dot{C}^\alpha}. \quad (3.4)$$

The inequality in (3.4) is a classical result relating a Dini modulus of continuity of f to a singular integral operator applied to f in the special case where the modulus of continuity is $r \mapsto Cr^\alpha$. (See, for instance, the lemma in [18], and note that applying that lemma to a C^α function gives the same factor of $\alpha^{-1}(1-\alpha)^{-1}$ that appears in Lemma 3.2. This reflects the fact that the integral transform in (3.2) applied to a C^1 -function gives only a log-Lipschitz function, and applied to a C^0 -function yields no modulus of continuity.)

Lemma 3.3 allows us to bound the full C^α norm.

Lemma 3.3. *Let L be as in Lemma 3.2 and suppose further that*

$$\|L\|_{**} := \|L\|_* + \sup_{x \in \mathbb{R}^2} \|L(x, \cdot)\|_{L^1(B_1(x)^c)} < \infty.$$

Then the conclusions of Lemma 3.2 hold with each \dot{C}^α replaced by C^α and $\|L\|_$ replaced by $\|L\|_{**}$.*

We shall apply Lemma 3.2 to the kernel L_2 of Lemma 3.4 and apply Lemma 3.3 to the kernels L_1 , L_3 , and L_4 . Note that for L_2 , L_3 , and L_4 , we are actually applying Lemma 3.2 to each of their components.

Lemma 3.4. *Consider the four kernels,*

- (1) $L_1(x, y) = \rho_\varepsilon(x - y)\omega_0(y)$;
- (2) $L_2(x, y) = \nabla(a_r K)(x - y)$ for some fixed $r > 0$;
- (3) $L_3(x, y) = \nabla K(x - y)\omega(y)$, where $\omega \in C_C^\infty(\mathbb{R}^2)$;
- (4) $L_4(x, y) = \rho_\varepsilon(x - y)\nabla u_0(y)$.

*Here, K is the Biot-Savart kernel of (1.3). Then $\|L_1\|_{**} \leq C\|\omega_0\|_{L^\infty}$ for C independent of ε ; L_2 satisfies (3.3) with $\|L_2\|_* \leq C$ independently of r ; $\|L_3\|_{**} \leq CV(\omega)$ with*

$$V(\omega) := \|\omega\|_{L^\infty} + \left\| \text{p. v.} \int \nabla K(x - y)\omega(y) dy \right\|_{L^\infty}; \quad (3.5)$$

*and $\|L_4\|_{**} \leq C\|\nabla u_0\|_{L^\infty}$ for C independent of ε .*

Proof. The bounds on the $*$ -norms of L_1 , L_2 , and L_4 are easily verified, the key points being their L^1 -bound uniform in x , the decay of $K(x - y)$ and $\nabla_x K(x - y)$, and the scaling of $\rho_\varepsilon(x - y)$ and $\nabla_x \rho_\varepsilon(x - y)$ in terms of ε . For L_3 , $\|L_3\|_*$ is bounded as for L_2 , with the p. v. integral in (3.5) coming from the final term in $\|L\|_{**}$. \square

Lemma 3.5. *Let $\alpha \in (0, 1)$ and $Z \in L^\infty(\mathbb{R}^2)$. Then, $\text{div } Z \in C^{\alpha-1}(\mathbb{R}^2)$ if and only if $\nabla \mathcal{F} * \text{div } Z \in C^\alpha(\mathbb{R}^2)$ (equivalently, $K * \text{div } Z \in C^\alpha(\mathbb{R}^2)$). Moreover,*

$$\|\text{div } Z\|_{C^{\alpha-1}} \leq \|\nabla \mathcal{F} * \text{div } Z\|_{C^\alpha} \leq C(\|Z\|_{L^\infty} + \|\text{div } Z\|_{C^{\alpha-1}}). \quad (3.6)$$

Lemma 3.6 (Gronwall's lemma and reverse Gronwall's lemma). *Suppose $h \geq 0$ is a continuous nondecreasing or nonincreasing function on $[0, T]$, $g \geq 0$ is an integrable function on $[0, T]$, and*

$$f(t) \leq h(t) + \int_0^t g(s)f(s) ds \quad \text{or} \quad f(t) \geq h(t) - \int_0^t g(s)f(s) ds$$

for all $t \in [0, T]$. Then

$$f(t) \leq h(t) \exp \int_0^t g(s) ds \quad \text{or} \quad f(t) \geq h(t) \exp \left(- \int_0^t g(s) ds \right),$$

respectively, for all $t \in [0, T]$.

4. APPROXIMATE SOLUTIONS AND TRANSPORT EQUATIONS

We first regularize the initial data by setting $u_{0,\varepsilon} = \rho_\varepsilon * u_0$, where ρ_ε is the standard mollifier of Definition 2.6, letting ε range over values in $(0, 1]$. It follows that $\omega_{0,\varepsilon} = \rho_\varepsilon * \omega_0$. Then there exists a solution, $\omega_\varepsilon(t) \in C^\infty(\mathbb{R}^2)$, to the Euler equations (1.2) for all time with C^∞ velocity field, u_ε ([19, 27] or see Theorem 4.2.4 of [7]). These solutions converge to a solution $\omega(t)$ of (1.2). (We say more about convergence in Section 5.6.)

The flow map, η_ε , is given in (1.4) with u_ε in place of u . Moreover, all the L^p -norms of ω_ε are conserved over time with

$$\|\omega_\varepsilon(t)\|_{L^p} = \|\omega_{\varepsilon,0}\|_{L^p} \leq \|\omega_0\|_{L^p} \leq \|\omega_0\|_{L^1 \cap L^\infty} =: \|\omega_0\|_{L^1} + \|\omega_0\|_{L^\infty} \quad (4.1)$$

for all $p \in [1, \infty]$. Also,

$$\|u_\varepsilon(t)\|_{L^\infty} \leq C \|\omega_0\|_{L^1 \cap L^\infty} \quad (4.2)$$

(see Proposition 8.2 of [21]) so $\|u_\varepsilon\|_{L^\infty(\mathbb{R} \times \mathbb{R}^2)}$ is uniformly bounded in ε .

For the most of the proof we will use these smooth solutions, passing to the limit as $\varepsilon \rightarrow 0$ in the final steps in Section 5.6.

We let

$$Y_\varepsilon(t, \eta_\varepsilon(t, x)) = Y_0(x) \cdot \nabla \eta_\varepsilon(t, x) \quad (4.3)$$

be the pushforward of Y_0 under the flow map η_ε (as in (1.9)). (Note the slight notational collision between Y_ε and Y_0 and ω_ε and ω_0 ; this should not, however, cause any confusion.)

Standard calculations show that

$$\partial_t Y_\varepsilon + u_\varepsilon \cdot \nabla Y_\varepsilon = Y_\varepsilon \cdot \nabla u_\varepsilon \quad (4.4)$$

and that

$$\begin{aligned} \partial_t \operatorname{div} Y_\varepsilon + u_\varepsilon \cdot \nabla \operatorname{div} Y_\varepsilon &= 0, \\ \partial_t \operatorname{div}(\omega_\varepsilon Y_\varepsilon) + u_\varepsilon \cdot \nabla \operatorname{div}(\omega_\varepsilon Y_\varepsilon) &= 0, \end{aligned} \quad (4.5)$$

the latter equality using that the vorticity is transported by the flow map. Hence,

$$\begin{aligned} \operatorname{div} Y_\varepsilon(t, x) &= \operatorname{div} Y_0(\eta_\varepsilon^{-1}(t, x)), \\ \operatorname{div}(\omega_\varepsilon Y_\varepsilon)(t, x) &= \operatorname{div}(\omega_{0,\varepsilon} Y_0)(\eta_\varepsilon^{-1}(t, x)). \end{aligned} \quad (4.6)$$

Remark 4.1. *Actually, the transport equations in (4.4) and (4.5), and others we will state later, are satisfied in a weak sense, since Y_0 and $\operatorname{div}(\omega_{0,\varepsilon} Y_0)$ only lie in C^α . We refer to Definition 3.13 of [1] for the notion of weak transport. With the exception of the use of Theorem 3.19 of [1] in the proof of Lemma 4.3, we will treat all transport equations as though they are satisfied in a strong sense, however, justifying such use in Appendix C. (See also Remark 4.4.)*

We can also write (4.5) as

$$\begin{aligned} \frac{d}{dt} Y_\varepsilon(t, \eta_\varepsilon(t, x)) &= (Y_\varepsilon \cdot \nabla u_\varepsilon)(t, \eta_\varepsilon(t, x)), \\ \frac{d}{dt} \operatorname{div}(\omega_\varepsilon Y_\varepsilon)(t, \eta_\varepsilon(t, x)) &= 0. \end{aligned} \quad (4.7)$$

Define the initial vector field

$$R_{0,\varepsilon} = \omega_{0,\varepsilon} Y_0 + \rho_\varepsilon * \nabla \mathcal{F} * \operatorname{div}(\omega_0 Y_0) - \rho_\varepsilon * (\omega_0 Y_0) \quad (4.8)$$

and observe that

$$\operatorname{div} R_{0,\varepsilon} = \operatorname{div}(\omega_{0,\varepsilon} Y_0) + \operatorname{div} \left(\rho_\varepsilon * (\nabla \mathcal{F} * \operatorname{div}(\omega_0 Y_0) - \omega_0 Y_0) \right) = \operatorname{div}(\omega_{0,\varepsilon} Y_0),$$

where we used that $\Delta \mathcal{F}$ is the Dirac delta function. Hence, pushing forward $R_{0,\varepsilon}$ will have the same effect on $\operatorname{div}(\omega_{0,\varepsilon} Y_0)$ as does pushing forward $\omega_{0,\varepsilon} Y_0$ itself; that is, letting

$$R_\varepsilon(t, \eta_\varepsilon(t, x)) = R_{0,\varepsilon}(x) \cdot \nabla \eta_\varepsilon(t, x),$$

we have

$$\partial_t R_\varepsilon + u_\varepsilon \cdot \nabla R_\varepsilon = R_\varepsilon \cdot \nabla u_\varepsilon$$

and

$$\operatorname{div}(\omega_\varepsilon Y_\varepsilon)(t, x) = \operatorname{div} R_\varepsilon(t, x) = \operatorname{div} R_{0,\varepsilon}(\eta_\varepsilon^{-1}(t, x)) = \operatorname{div}(\omega_{0,\varepsilon} Y_0)(\eta_\varepsilon^{-1}(t, x)). \quad (4.9)$$

Although R_ε and $\omega_\varepsilon Y_\varepsilon$ have the same divergence, we will see in the proof of Theorem 1.2 that R_ε is bounded in C^α uniformly in ε in (5.11), which is not true of $\omega_\varepsilon Y_\varepsilon$. We will take advantage of this fact in the estimate in (5.19).

Lemma 4.2. *The vector field $R_{0,\varepsilon}$, defined in (4.8), is in $C^\alpha(\mathbb{R}^2)$, with*

$$\|R_{0,\varepsilon}\|_{C^\alpha} \leq C_\alpha,$$

uniformly over ε in $(0, 1]$, where C_α is as in (1.10).

Proof. We rewrite $R_{0,\varepsilon}$ in the form,

$$R_{0,\varepsilon} = \rho_\varepsilon * \nabla \mathcal{F} * \operatorname{div}(\omega_0 Y_0) + [(\rho_\varepsilon * \omega_0) Y_0 - \rho_\varepsilon * (\omega_0 Y_0)].$$

Since $\nabla \mathcal{F} * \operatorname{div}(\omega_0 Y_0) \in C^\alpha(\mathbb{R}^2)$ by Lemma 3.5 (noting that $\omega_0 Y_0 \in L^\infty$), we have

$$\|\rho_\varepsilon * \nabla \mathcal{F} * \operatorname{div}(\omega_0 Y_0)\|_{C^\alpha} \leq C \|\nabla \mathcal{F} * \operatorname{div}(\omega_0 Y_0)\|_{C^\alpha} \leq C(\omega_0, Y_0). \quad (4.10)$$

Since $Y_0 \in C^\alpha(\mathbb{R}^2)$, applying Lemma 3.3 with the kernel L_1 of Lemma 3.4, we have

$$\begin{aligned} \|(\rho_\varepsilon * \omega_0) Y_0 - \rho_\varepsilon * (\omega_0 Y_0)\|_{C^\alpha} &= \left\| \int_{\mathbb{R}^2} \rho_\varepsilon(x-y) \omega_0(y) [Y_0(x) - Y_0(y)] dy \right\|_{C^\alpha} \\ &\leq C(\omega_0, Y_0) (\alpha^{-1}(1-\alpha)^{-1}) = C_\alpha. \end{aligned} \quad (4.11)$$

This completes the proof. \square

Finally, we prove the propagation of regularity of $\operatorname{div}(\omega_\varepsilon Y_\varepsilon)$.

Lemma 4.3. *We have $\operatorname{div}(\omega_\varepsilon Y_\varepsilon)(t) \in C^{\alpha-1}(\mathbb{R}^2)$ with*

$$\|\operatorname{div}(\omega_\varepsilon Y_\varepsilon)(t)\|_{C^{\alpha-1}} \leq C_\alpha \exp \int_0^t \|\nabla u(s)\|_{L^\infty} ds.$$

Proof. Noting that $C^{\alpha-1}(\mathbb{R}^2)$ is equivalent to the Besov space $B_{\infty,\infty}^{\alpha-1}(\mathbb{R}^2)$, Theorem 3.14 of [1] applied to the weak transport equation in (4.5)₂ (see Remark 4.1) gives

$$\|\operatorname{div}(\omega_\varepsilon Y_\varepsilon)(t)\|_{C^{\alpha-1}} \leq C \|\operatorname{div}(\omega_{0,\varepsilon} Y_0)\|_{C^{\alpha-1}} \exp \int_0^t \|\nabla u(s)\|_{L^\infty} ds.$$

We must still, however, bound $\|\operatorname{div}(\omega_{0,\varepsilon} Y_0)\|_{C^{\alpha-1}}$ uniformly in ε .

From the triangle inequality,

$$\|\operatorname{div}(\omega_{0,\varepsilon} Y_0)\|_{C^{\alpha-1}} \leq \|\operatorname{div}(\omega_{0,\varepsilon} Y_0) - \rho_\varepsilon * \operatorname{div}(\omega_0 Y_0)\|_{C^{\alpha-1}} + \|\rho_\varepsilon * \operatorname{div}(\omega_0 Y_0)\|_{C^{\alpha-1}}.$$

Now,

$$\|\operatorname{div}(\omega_{0,\varepsilon} Y_0) - \rho_\varepsilon * \operatorname{div}(\omega_0 Y_0)\|_{C^{\alpha-1}} \leq \|\omega_{0,\varepsilon} Y_0 - \rho_\varepsilon * (\omega_0 Y_0)\|_{C^\alpha} \leq C_\alpha,$$

the first inequality following from (2.3), the second from (4.11). Also,

$$\begin{aligned} \|\rho_\varepsilon * \operatorname{div}(\omega_0 Y_0)\|_{C^{\alpha-1}} &\leq C \|\nabla \mathcal{F} * (\rho_\varepsilon * \operatorname{div}(\omega_0 Y_0))\|_{C^\alpha} = C \|\rho_\varepsilon * (\nabla \mathcal{F} * \operatorname{div}(\omega_0 Y_0))\|_{C^\alpha} \\ &\leq C \|\nabla \mathcal{F} * \operatorname{div}(\omega_0 Y_0)\|_{C^\alpha} \leq C (\|\omega_0 Y_0\|_{L^\infty} + \|\operatorname{div}(\omega_0 Y_0)\|_{C^{\alpha-1}}). \end{aligned}$$

For the first inequality we applied Lemma 3.5, for the second inequality we used $\|\rho_\varepsilon * f\|_{C^\alpha} \leq \|f\|_{C^\alpha}$, and for the third we applied Lemma 3.5 once more. Hence,

$$\|\operatorname{div}(\omega_{0,\varepsilon}Y_0)\|_{C^{\alpha-1}} \leq C_\alpha + (\|\omega_0Y_0\|_{L^\infty} + \|\operatorname{div}(\omega_0Y_0)\|_{C^{\alpha-1}}) \leq C_\alpha.$$

□

Remark 4.4. *It would be natural to let $Y_{0,\varepsilon} = \rho_\varepsilon * Y_0$ and pushforward $Y_{0,\varepsilon}$ rather than Y_0 in the definition of Y_ε , and also use $Y_{0,\varepsilon}$ rather than Y_0 in the definition of $R_{0,\varepsilon}$. This would allow us to use transport equations purely in strong form. It is the bound in (4.10), however, that prevents us from doing this, as the equivalent bound with $Y_{0,\varepsilon}$ in place of Y_0 may not hold true. Instead, we take the approach described in Appendix C.*

5. PROOF OF SERFATI'S THEOREM PART I

In this section we prove the first part of Theorem 1.2; namely, (1.11). Before proceeding to the fairly long and technical proof, let us first sketch the overall strategy.

The proof hinges on the transport equation for Y_ε in (4.4) and the identity in Corollary B.3. Together, they allow us to relate the four quantities,

$$\begin{aligned} Q_1 &:= \|Y_\varepsilon \cdot \nabla u_\varepsilon\|_{C^\alpha}, & Q_2 &:= \|\nabla u\|_{L^\infty}, \\ Q_3 &:= \|Y_\varepsilon\|_{C^\alpha}, & Q_4 &:= \|K * \operatorname{div}(\omega_\varepsilon Y_\varepsilon)\|_{C^\alpha}. \end{aligned}$$

A bound on Q_4 comes essentially for free via Lemma 4.3, so we may as well take it as given. From (4.4) we obtain a bound on Q_3 in terms of Q_1 , and from Corollary B.3 we obtain a bound on Q_1 in terms of Q_2 and Q_3 . Obtaining these estimates will occupy Sections 5.1 to 5.3, and will also involve estimates on the flow map η_ε .

At this point, we could close the estimates if we could obtain a bound on Q_2 in terms of Q_3 . This is the subject of Section 5.4, which will lead in Section 5.5 to a bound on Q_2 in terms of itself. Once we have a bound on Q_2 , we easily obtain a bound on Q_1 and Q_3 . In Section 5.6, we show that in the limit we obtain (1.11).

Note that a coarse bound on Q_2 in terms of Q_1 and Q_3 is easily derived (along the lines of (9.1)), but is inadequate: It will take a great deal more work to obtain in Section 5.4 a tight enough bound that Gronwall's lemma can be successfully applied in Section 5.5.

5.1. Preliminary estimate of $\|\nabla u_\varepsilon(t)\|_{L^\infty}$. By the expression for ∇u_ε in Lemma B.1,

$$\|\nabla u_\varepsilon(t)\|_{L^\infty} \leq V_\varepsilon(t),$$

where

$$V_\varepsilon(t) := \|\omega_0\|_{L^\infty} + \left\| \text{p. v.} \int \nabla K(\cdot - y) \omega_\varepsilon(t, y) dy \right\|_{L^\infty}. \quad (5.1)$$

Here, we used (4.1) to replace $\|\omega_\varepsilon(t)\|_{L^\infty}$ by $\|\omega_0\|_{L^\infty}$ in the first term.

5.2. Estimate of $\|\nabla \eta_\varepsilon(t)\|_{L^\infty}$ and $\|\nabla \eta_\varepsilon^{-1}(t)\|_{L^\infty}$. As in (1.4), the defining equation for η_ε is

$$\partial_t \eta_\varepsilon(t, x) = u_\varepsilon(t, \eta_\varepsilon(t, x)), \quad \eta_\varepsilon(0, x) = x, \quad (5.2)$$

or, in integral form,

$$\eta_\varepsilon(t, x) = x + \int_0^t u_\varepsilon(s, \eta_\varepsilon(s, x)) ds. \quad (5.3)$$

This immediately implies that

$$\|\nabla\eta_\varepsilon(t)\|_{L^\infty} \leq \exp \int_0^t V_\varepsilon(s) ds. \quad (5.4)$$

Similarly,

$$\|\nabla\eta_\varepsilon^{-1}(t)\|_{L^\infty} \leq \exp \int_0^t V_\varepsilon(s) ds. \quad (5.5)$$

The bound in (5.5) does not follow as immediately as that in (5.4) because the flow is not autonomous. For the details, see, for instance, the proof of Lemma 8.2 p. 318-319 of [21] (applying the argument there to $\nabla\eta_\varepsilon^{-1}$ rather than to η_ε^{-1}).

Applying Lemma 3.6 to (5.3) yields

$$|x - y| \exp \left(- \int_0^t V_\varepsilon(s) ds \right) \leq |\eta_\varepsilon(t, x) - \eta_\varepsilon(t, y)| \leq |x - y| \exp \int_0^t V_\varepsilon(s) ds.$$

Hence, for any $\delta_0 > 0$,

$$\mathcal{N}_{\delta_t^-}(\eta_\varepsilon(t, \Sigma)) \subseteq \eta_\varepsilon(t, \mathcal{N}_{\delta_0}(\Sigma)) \subseteq \mathcal{N}_{\delta_t^+}(\eta_\varepsilon(t, \Sigma)), \quad \delta_t^\pm := \delta_0 \exp \left(\pm \int_0^t V_\varepsilon(s) ds \right). \quad (5.6)$$

5.3. Estimate of Y_ε and R_ε . Taking the inner product of (4.7)₁ with $Y_\varepsilon(t, \eta_\varepsilon(t, x))$ gives

$$\frac{d}{dt} Y_\varepsilon(t, \eta_\varepsilon(t, x)) \cdot Y_\varepsilon(t, \eta_\varepsilon(t, x)) = (Y_\varepsilon \cdot \nabla u_\varepsilon)(t, \eta_\varepsilon(t, x)) \cdot Y_\varepsilon(t, \eta_\varepsilon(t, x)).$$

The left-hand side equals

$$\frac{1}{2} \frac{d}{dt} |Y_\varepsilon(t, \eta_\varepsilon(t, x))|^2$$

so

$$\begin{aligned} \left| \frac{d}{dt} |Y_\varepsilon(t, \eta_\varepsilon(t, x))|^2 \right| &\leq 2 \|\nabla u_\varepsilon(t, \eta_\varepsilon(t, \cdot))\|_{L^\infty} |Y_\varepsilon(t, \eta_\varepsilon(t, x))|^2 \\ &= 2 \|\nabla u_\varepsilon(t)\|_{L^\infty} |Y_\varepsilon(t, \eta_\varepsilon(t, x))|^2 \leq 2V_\varepsilon(t) |Y_\varepsilon(t, \eta_\varepsilon(t, x))|^2. \end{aligned}$$

It follows that

$$\frac{d}{dt} |Y_\varepsilon(t, \eta_\varepsilon(t, x))|^2 \leq 2V_\varepsilon(t) |Y_\varepsilon(t, \eta_\varepsilon(t, x))|^2.$$

Similarly,

$$\frac{d}{dt} |Y_\varepsilon(t, \eta_\varepsilon(t, x))|^2 \geq -2V_\varepsilon(t) |Y_\varepsilon(t, \eta_\varepsilon(t, x))|^2.$$

Integrating in time and applying Lemma 3.6 gives

$$|Y_0(x)| e^{-\int_0^t \|\nabla u_\varepsilon(s)\|_{L^\infty} ds} \leq |Y_\varepsilon(t, \eta_\varepsilon(t, x))| \leq |Y_0(x)| e^{\int_0^t \|\nabla u_\varepsilon(s)\|_{L^\infty} ds}.$$

Taking the L^∞ norm in x , we conclude that

$$\|Y_\varepsilon(t)\|_{L^\infty} \leq \|Y_0\|_{L^\infty} e^{\int_0^t V_\varepsilon(s) ds}. \quad (5.7)$$

Also, for any measurable set $\Lambda \subseteq \mathbb{R}^2$,

$$\|Y_\varepsilon(t)\|_{\inf(\eta_\varepsilon(t, \Lambda))} \geq \|Y_0\|_{\inf(\Lambda)} e^{-\int_0^t V_\varepsilon(s) ds}. \quad (5.8)$$

The estimate for R_ε corresponding to (5.7) is, using Lemma 4.2,

$$\|R_\varepsilon(t)\|_{L^\infty} \leq C(\omega_0, Y_0) e^{\int_0^t V_\varepsilon(s) ds}. \quad (5.9)$$

Integrating (4.7)₁ in time and substituting $\eta_\varepsilon^{-1}(t, x)$ for x yields

$$Y_\varepsilon(t, x) = Y_0(\eta_\varepsilon^{-1}(t, x)) + \int_0^t (Y_\varepsilon \cdot \nabla u_\varepsilon)(s, \eta_\varepsilon(s, \eta_\varepsilon^{-1}(t, x))) ds.$$

Taking the \dot{C}^α norm and applying (2.4)₁, we have

$$\|Y_\varepsilon(t)\|_{\dot{C}^\alpha} \leq \|Y_0\|_{\dot{C}^\alpha} \|\nabla \eta_\varepsilon^{-1}(t)\|_{L^\infty}^\alpha + \int_0^t \|(Y_\varepsilon \cdot \nabla u_\varepsilon)(s)\|_{\dot{C}^\alpha} \|\nabla(\eta_\varepsilon(s, \eta_\varepsilon^{-1}(t, x)))\|_{L^\infty}^\alpha ds.$$

Now, by Corollary B.3, we have

$$\begin{aligned} Y_\varepsilon \cdot \nabla u_\varepsilon(s, x) &= \text{p. v.} \int \nabla K(x - y) \omega_\varepsilon(s, y) [Y_\varepsilon(s, x) - Y_\varepsilon(s, y)] dy \\ &\quad + K * \text{div}(\omega_\varepsilon Y_\varepsilon)(s, x) =: \text{I} + \text{II} \end{aligned}$$

with

$$\|\text{I}\|_{C^\alpha} \leq C \|Y_\varepsilon(s)\|_{C^\alpha} V_\varepsilon(s).$$

By Lemmas 3.5 and 4.3, we have

$$\|\text{II}\|_{C^\alpha} \leq C_\alpha \exp \int_0^s V_\varepsilon(\tau) d\tau.$$

It follows that

$$\|Y_\varepsilon \cdot \nabla u_\varepsilon(t)\|_{C^\alpha} \leq \|Y_\varepsilon(t)\|_{C^\alpha} V_\varepsilon(t) + C_\alpha \exp \int_0^t V_\varepsilon(\tau) d\tau. \quad (5.10)$$

To estimate $\|\nabla(\eta_\varepsilon(s, \eta_\varepsilon^{-1}(t, x)))\|_{L^\infty}$, we start with

$$\partial_\tau \eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t, x)) = u_\varepsilon(\tau, \eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t, x))),$$

which follows from (5.2). Applying the spatial gradient and the chain rule gives

$$\partial_\tau \nabla(\eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t, x))) = \nabla u_\varepsilon(\tau, \eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t, x))) \nabla(\eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t, x))).$$

Integrating in time and using $\nabla(\eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t, x)))|_{\tau=t} = I^{2 \times 2}$, the identity matrix, we have

$$\nabla(\eta_\varepsilon(s, \eta_\varepsilon^{-1}(t, x))) = I^{2 \times 2} - \int_s^t \nabla u_\varepsilon(\tau, \eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t, x))) \nabla(\eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t, x))) d\tau.$$

By Lemma 3.6, then,

$$\|\nabla(\eta_\varepsilon(s, \eta_\varepsilon^{-1}(t, x)))\|_{L^\infty} \leq \exp \int_s^t \|\nabla u_\varepsilon(\tau)\|_{L^\infty} d\tau \leq \exp \int_s^t V_\varepsilon(\tau) d\tau.$$

These bounds with (5.5), and accounting for (5.7), give

$$\begin{aligned} \|Y_\varepsilon(t)\|_{C^\alpha} &\leq \|Y_0\|_{C^\alpha} \exp\left(\alpha \int_0^t V_\varepsilon(s) ds\right) \\ &\quad + \int_0^t \left[\|Y_\varepsilon(s)\|_{C^\alpha} V_\varepsilon(s) + C_\alpha \exp \int_0^s V_\varepsilon(\tau) d\tau \right] \exp\left(\alpha \int_s^t V_\varepsilon(\tau) d\tau\right) ds \\ &\leq (\|Y_0\|_{C^\alpha} + C_\alpha t) \exp \int_0^t V_\varepsilon(s) ds + \int_0^t \|Y_\varepsilon(s)\|_{C^\alpha} V_\varepsilon(s) \left[\exp \int_s^t V_\varepsilon(\tau) d\tau \right] ds. \end{aligned}$$

Letting

$$y_\varepsilon(t) = \|Y_\varepsilon(t)\|_{C^\alpha} \exp\left[-\int_0^t V_\varepsilon(s) ds\right]$$

it follows that y_ε satisfies the inequality,

$$y_\varepsilon(t) \leq \|Y_0\|_{C^\alpha} + C_\alpha t + \int_0^t V_\varepsilon(s) y_\varepsilon(s) ds.$$

Therefore, by Lemma 3.6, we obtain

$$y_\varepsilon(t) \leq (\|Y_0\|_{C^\alpha} + C_\alpha t) \exp\left(\int_0^t V_\varepsilon(s) ds\right) \leq C_\alpha(1+t) \exp\left(\int_0^t V_\varepsilon(s) ds\right)$$

and thus, with the similar bound for R_ε ,

$$\|Y_\varepsilon(t)\|_{C^\alpha}, \|R_\varepsilon(t)\|_{C^\alpha} \leq C_\alpha(1+t) \exp\left(2 \int_0^t V_\varepsilon(s) ds\right). \quad (5.11)$$

5.4. Refined estimate of ∇u_ε . We split the second term in V_ε in (5.1) into two parts, as (see Definition 2.8 for the meaning of $\tilde{*}$)

$$\begin{aligned} \text{p. v. } \int \nabla K(x-y) \omega_\varepsilon(t, y) dy &= \nabla K \tilde{*} \omega_\varepsilon(t, x) \\ &= \nabla(a_r K) \tilde{*} \omega_\varepsilon(t, x) + \nabla((1-a_r)K) \tilde{*} \omega_\varepsilon(t, x), \end{aligned} \quad (5.12)$$

where $r \in (0, 1]$ will be chosen later (in (5.17)).

On the support of $\nabla(1-a_r) = -\nabla a_r$, $|x-y| \leq 2r$, so

$$|\nabla((1-a_r)K)| \leq |(1-a_r)\nabla K| + |\nabla a_r \otimes K| \leq C|x-y|^{-2}. \quad (5.13)$$

Hence, one term in (5.12) is easily bounded by

$$\begin{aligned} |\nabla((1-a_r)K) \tilde{*} \omega_\varepsilon(t, x)| &\leq C \int_{B_r^C(x)} |x-y|^{-2} |\omega_\varepsilon(t, y)| dy \\ &\leq C \int_r^1 \frac{\|\omega_\varepsilon\|_{L^\infty}}{\rho^2} \rho d\rho + C \| |x-\cdot|^{-2} \|_{L^\infty(B_1^C(x))} \|\omega_{\varepsilon,0}\|_{L^1} \\ &\leq -C \log r \|\omega_0\|_{L^\infty} + C \|\omega_0\|_{L^1} \leq C(-\log r + 1) \|\omega_0\|_{L^1 \cap L^\infty}. \end{aligned}$$

For the other term in (5.12), we decompose the vorticity as follows. Let δ_0 be as in (1.8)₂ and take a smooth function χ such that $\chi = 1$ on $\mathcal{N}_{\delta_0/4}(\Sigma)$ and $\chi = 0$ on $\mathcal{N}_{\delta_0/2}(\Sigma)^C$ (see Definition 2.3). Let b be a smooth function such that $b = 1$ on $\mathcal{N}_{3\delta_0/4}(\Sigma)$ and $b = 0$ on $\mathcal{N}_{\delta_0}(\Sigma)^C$. Then $|Y_0| \geq c$ on $\text{supp } b$ by (1.8)₂, so

$$|Y_0(\eta_\varepsilon^{-1})| \geq c \text{ on } \text{supp } b(\eta_\varepsilon^{-1}) \supseteq \text{supp } \chi(\eta_\varepsilon^{-1}). \quad (5.14)$$

We now let

$$\omega_{0,\varepsilon}^1 = \chi \omega_{0,\varepsilon}, \quad \omega_{0,\varepsilon}^2 = (1-\chi) \omega_{0,\varepsilon},$$

and then let

$$\omega_\varepsilon^1(t, x) = \omega_{0,\varepsilon}^1(\eta_\varepsilon^{-1}(t, x)), \quad \omega_\varepsilon^2(t, x) = \omega_{0,\varepsilon}^2(\eta_\varepsilon^{-1}(t, x)). \quad (5.15)$$

The vorticity ω_ε^1 is the ‘‘bad vorticity,’’ in that it is transported from a neighborhood of the set Σ on which the initial vorticity fails to be C^α . By contrast, ω_ε^2 is the ‘‘good vorticity’’ since for all sufficiently small $\varepsilon > 0$, we have

$$\|\omega_{0,\varepsilon}^2\|_{C^\alpha} \leq C \|\omega_0\|_{C^\alpha(\mathbb{R}^2 \setminus \Sigma)} = C(\omega_0). \quad (5.16)$$

By (5.6),

$$\text{supp } \omega_\varepsilon^1 \subseteq \mathcal{N}_{\delta/2}(\eta_\varepsilon(t, \Sigma)), \quad \delta = \delta_t^+ := \delta_0 \exp \int_0^t V_\varepsilon(s) ds.$$

Now we split the other term in (5.12) into three parts, as

$$\begin{aligned} \nabla(a_r K) \tilde{*} \omega_\varepsilon &= (1-b)(\eta_\varepsilon^{-1}) \nabla(a_r K) \tilde{*} \omega_\varepsilon^1 + (1-b)(\eta_\varepsilon^{-1}) \nabla(a_r K) \tilde{*} \omega_\varepsilon^2 \\ &\quad + b(\eta_\varepsilon^{-1}) \nabla(a_r K) \tilde{*} \omega_\varepsilon \\ &=: \text{III}_1 + \text{III}_2 + \text{III}_3. \end{aligned}$$

Now choose

$$r = \min \left\{ 1, \frac{\delta_0}{8} \exp \left(-C' \int_0^t V_\varepsilon(s) ds \right) \right\}, \quad (5.17)$$

leaving the choice of the constant $C' > 1$ until later (see (5.22)). We have $r < \delta_t^-/8$, where δ_t^- is defined in (5.6), so $\nabla(a_r K) \tilde{*} \omega_\varepsilon^1$ is supported on $\mathcal{N}_\delta(\eta_\varepsilon(t, \Sigma))$ for all $\varepsilon < \delta/8$. Hence, $\text{III}_1 = 0$.

Noting that the bound in (5.13) applies also to $|\nabla(a_r K)|$, we have

$$\begin{aligned} \left| \lim_{h \rightarrow 0} \text{III}_2 \right| &= \left| (1-b)(\eta_\varepsilon^{-1}) \lim_{h \rightarrow 0} \int_{\mathbb{R}^2} \nabla(a_r K)(x-y) [\omega_\varepsilon^2(y) - \omega_\varepsilon^2(x)] dy \right| \\ &\leq C \|\omega_\varepsilon^2(t)\|_{\dot{C}^\alpha} \int_{\mathbb{R}^2} |x-y|^\alpha |\nabla(a_r K)(x-y)| dy \\ &\leq C \|\omega_{0,\varepsilon}^2\|_{\dot{C}^\alpha} \|\nabla(\eta_\varepsilon^{-1})\|_{L^\infty}^\alpha \int_{|x-y| \leq 2r} |x-y|^{\alpha-2} dy \\ &\leq C(\omega_0) \alpha^{-1} \|\nabla \eta_\varepsilon^{-1}\|_{L^\infty}^\alpha r^\alpha. \end{aligned}$$

In the second inequality we used (2.4)₁ and in the third we used (5.16).

To estimate III_3 , we will find it slightly more convenient to use $\nabla \mathcal{F}$ in place of $K = \nabla^\perp \mathcal{F}$, the norms that result being identical. Letting μ_{rh} be as in Definition 2.4, by virtue of Lemma B.4, we can write

$$|\text{III}_3| = \left| b(\eta_\varepsilon^{-1}) \lim_{h \rightarrow 0} \nabla(\mu_{rh} K) \tilde{*} \omega_\varepsilon \right| = \lim_{h \rightarrow 0} |b(\eta_\varepsilon^{-1}) B|,$$

where

$$B = \nabla [\mu_{rh} \nabla \mathcal{F}] * \omega_\varepsilon.$$

Because $\nabla [\mu_{rh} \nabla \mathcal{F}]$ is not in L^1 uniformly in $h > 0$, we cannot estimate B in L^∞ directly. Instead, we will apply Lemma 3.1 with

$$M = (Y_\varepsilon \quad ((\nabla \eta_\varepsilon)^T Y_0^\perp) \circ \eta_\varepsilon^{-1}),$$

so that $M_1 = Y_\varepsilon$.

From

$$\begin{aligned} Y_\varepsilon \circ \eta^\varepsilon &= \nabla \eta_\varepsilon \begin{pmatrix} Y_0^1 \\ Y_0^2 \end{pmatrix} = \begin{pmatrix} \partial_1 \eta_\varepsilon^1 & \partial_2 \eta_\varepsilon^1 \\ \partial_1 \eta_\varepsilon^2 & \partial_2 \eta_\varepsilon^2 \end{pmatrix} \begin{pmatrix} Y_0^1 \\ Y_0^2 \end{pmatrix} = \begin{pmatrix} Y_0^1 \partial_1 \eta_\varepsilon^1 + Y_0^2 \partial_2 \eta_\varepsilon^1 \\ Y_0^1 \partial_1 \eta_\varepsilon^2 + Y_0^2 \partial_2 \eta_\varepsilon^2 \end{pmatrix}, \\ (\nabla \eta_\varepsilon)^T Y_0^\perp &= \begin{pmatrix} \partial_1 \eta_\varepsilon^1 & \partial_1 \eta_\varepsilon^2 \\ \partial_2 \eta_\varepsilon^1 & \partial_2 \eta_\varepsilon^2 \end{pmatrix} \begin{pmatrix} -Y_0^2 \\ Y_0^1 \end{pmatrix} = \begin{pmatrix} -Y_0^2 \partial_1 \eta_\varepsilon^1 + Y_0^1 \partial_1 \eta_\varepsilon^2 \\ -Y_0^2 \partial_2 \eta_\varepsilon^1 + Y_0^1 \partial_2 \eta_\varepsilon^2 \end{pmatrix}, \end{aligned}$$

we have

$$M = \begin{pmatrix} Y_0^1 \partial_1 \eta_\varepsilon^1 + Y_0^2 \partial_2 \eta_\varepsilon^1 & -Y_0^2 \partial_1 \eta_\varepsilon^1 + Y_0^1 \partial_1 \eta_\varepsilon^2 \\ Y_0^1 \partial_1 \eta_\varepsilon^2 + Y_0^2 \partial_2 \eta_\varepsilon^2 & -Y_0^2 \partial_2 \eta_\varepsilon^1 + Y_0^1 \partial_2 \eta_\varepsilon^2 \end{pmatrix} \circ \eta_\varepsilon^{-1}.$$

A direct computation yields

$$\det M(t, x) = |Y_0|^2 (\eta_\varepsilon^{-1}(t, x)) \det \nabla \eta_\varepsilon (\eta_\varepsilon^{-1}(t, x)) = |Y_0|^2 (\eta_\varepsilon^{-1}(t, x)),$$

since $\det \nabla \eta_\varepsilon (\eta_\varepsilon^{-1}(t, x)) = 1$.

For the rest of the analysis of the matrix B , we restrict our analysis to $\text{supp } b(\eta_\varepsilon^{-1})$, on which III_3 is supported. By (5.14), then, we have

$$\det M = |Y_0|^2 (\eta_\varepsilon^{-1}(t, x)) \geq c^2 > 0. \quad (5.18)$$

Hence, M is invertible, and applying Lemma 3.1, we have

$$|B| \leq C \frac{\|Y_\varepsilon\|_{L^\infty} + \|Y_0\|_{L^\infty} \|\nabla \eta_\varepsilon\|_{L^\infty}}{c^2} |BM_1| + |\text{tr } B|.$$

We now compute $\text{tr } B$. We have,

$$\begin{aligned} \text{tr } B &= [\partial_1 \mu_{rh} \partial_1 \mathcal{F}] * \omega_\varepsilon + [\partial_2 \mu_{rh} \partial_2 \mathcal{F}] * \omega_\varepsilon + [\mu_{rh} \Delta \mathcal{F}] * \omega_\varepsilon \\ &= [\partial_1 \mu_{rh} \partial_1 \mathcal{F}] * \omega_\varepsilon + [\partial_2 \mu_{rh} \partial_2 \mathcal{F}] * \omega_\varepsilon, \end{aligned}$$

using $\Delta \mathcal{F} = \delta_0$ and $\mu_{rh}(0) = 0$ to remove the last term.

But, referring to Remark 2.5, for $j = 1, 2$, we have

$$\begin{aligned} |[\partial_j \mu_{rh} \partial_j \mathcal{F}] * \omega_\varepsilon| &\leq \frac{C}{r} \int_{r < |x-y| < 2r} \frac{|\omega_\varepsilon(t, y)|}{|x-y|} dy + \frac{C}{h} \int_{h < |x-y| < 2h} \frac{|\omega_\varepsilon(t, y)|}{|x-y|} dy \\ &\leq \frac{C}{r} \int_r^{2r} \frac{\|\omega_\varepsilon(t)\|_{L^\infty}}{\rho} \rho d\rho + \frac{C}{h} \int_h^{2h} \frac{\|\omega_\varepsilon(t)\|_{L^\infty}}{\rho} \rho d\rho \\ &= C \|\omega_\varepsilon(t)\|_{L^\infty} \end{aligned}$$

so that

$$\lim_{h \rightarrow 0} |\text{tr } B| \leq C \|\omega_0\|_{L^\infty}.$$

We next estimate $|BM_1|$. Because

$$B = \begin{pmatrix} \partial_1 [\mu_{rh} \partial_1 \mathcal{F}] * \omega_\varepsilon & \partial_2 [\mu_{rh} \partial_1 \mathcal{F}] * \omega_\varepsilon \\ \partial_1 [\mu_{rh} \partial_2 \mathcal{F}] * \omega_\varepsilon & \partial_2 [\mu_{rh} \partial_2 \mathcal{F}] * \omega_\varepsilon \end{pmatrix}$$

we have

$$BM_1 = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} := \begin{pmatrix} (\partial_1 [\mu_{rh} \partial_1 \mathcal{F}] * \omega_\varepsilon) Y_\varepsilon^1 + (\partial_2 [\mu_{rh} \partial_1 \mathcal{F}] * \omega_\varepsilon) Y_\varepsilon^2 \\ (\partial_1 [\mu_{rh} \partial_2 \mathcal{F}] * \omega_\varepsilon) Y_\varepsilon^1 + (\partial_2 [\mu_{rh} \partial_2 \mathcal{F}] * \omega_\varepsilon) Y_\varepsilon^2 \end{pmatrix}.$$

We now decompose F_1 and F_2 into two parts as $F_k = d_k + e_k$, where

$$\begin{aligned} d_k &= \sum_{j=1}^2 (\partial_j [\mu_{rh} \partial_k \mathcal{F}] * \omega_\varepsilon) Y_\varepsilon^j - \partial_j [\mu_{rh} \partial_k \mathcal{F}] * (\omega_\varepsilon Y_\varepsilon^j), \\ e_k &= \partial_1 [\mu_{rh} \partial_k \mathcal{F}] * (\omega_\varepsilon Y_\varepsilon^1) + \partial_2 [\mu_{rh} \partial_k \mathcal{F}] * (\omega_\varepsilon Y_\varepsilon^2) = \text{div} \left(\mu_{rh} \partial_k \mathcal{F} * (\omega_\varepsilon Y_\varepsilon) \right). \end{aligned}$$

As we can see from Remark 2.5, $|\mu_{rh} \nabla \mathcal{F}(x-y)| \leq C|x-y|^{-2}$, so

$$\begin{aligned} \sum_{k=1,2} \left| \lim_{h \rightarrow 0} d_k \right| &\leq 2 \left| \lim_{h \rightarrow 0} \int_{\mathbb{R}^2} \nabla [\mu_{rh} \nabla \mathcal{F}](x-y)(Y_\varepsilon(x) - Y_\varepsilon(y))\omega_\varepsilon(y) dy \right| \\ &\leq C \|Y_\varepsilon(t)\|_{\dot{C}^\alpha} \|\omega_\varepsilon(t)\|_{L^\infty} \int_{|x-y| \leq 2r} |x-y|^{\alpha-2} dy \\ &\leq C\alpha^{-1} \|Y_\varepsilon(t)\|_{C^\alpha} \|\omega_0\|_{L^\infty} r^\alpha \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1,2} \left| \lim_{h \rightarrow 0} e_k \right| &\leq 2 \left| \lim_{h \rightarrow 0} \int_{\mathbb{R}^2} [\mu_{rh} \nabla \mathcal{F}](x-y) \operatorname{div}(\omega_\varepsilon Y_\varepsilon)(y) dy \right| \\ &= 2 \left| \lim_{h \rightarrow 0} \int_{\mathbb{R}^2} [\mu_{rh} \nabla \mathcal{F}] [\operatorname{div}(\omega_\varepsilon Y_\varepsilon)(y) - \operatorname{div}(\omega_\varepsilon Y_\varepsilon)(x)] dy \right| \\ &= 2 \left| \lim_{h \rightarrow 0} \int_{\mathbb{R}^2} [\mu_{rh} \nabla \mathcal{F}] [\operatorname{div} R_\varepsilon(y) - \operatorname{div} R_\varepsilon(x)] dy \right| \tag{5.19} \\ &= 2 \left| \lim_{h \rightarrow 0} \int_{\mathbb{R}^2} \nabla [\mu_{rh} \nabla \mathcal{F}] [R_\varepsilon(y) - R_\varepsilon(x)] dy \right| \\ &\leq C \|R_\varepsilon(t)\|_{\dot{C}^\alpha} \int_{|x-y| \leq 2r} |x-y|^{\alpha-2} dy \\ &\leq C\alpha^{-1} \|R_\varepsilon(t)\|_{C^\alpha} r^\alpha, \end{aligned}$$

where we used (4.9) and the regularity of R_ε in (5.11). Thus,

$$\begin{aligned} \lim_{h \rightarrow 0} |B| &\leq C\alpha^{-1} \frac{\|Y_\varepsilon\|_{L^\infty} + \|Y_0\|_{L^\infty} \|\nabla \eta_\varepsilon\|_{L^\infty}}{c^2} (\|Y_\varepsilon\|_{C^\alpha} \|\omega_0\|_{L^\infty} + \|R_\varepsilon\|_{C^\alpha}) r^\alpha \\ &\quad + C \|\omega_0\|_{L^\infty}. \end{aligned} \tag{5.20}$$

Collecting all the bounds we have obtained so far, we conclude that

$$\begin{aligned} V_\varepsilon(t) &\leq C(1 - \log r) \|\omega_0\|_{L^1 \cap L^\infty} + C(\omega_0)\alpha^{-1} \|\nabla \eta_\varepsilon^{-1}(t)\|_{L^\infty}^\alpha r^\alpha \\ &\quad + \frac{C}{c^2 \alpha} \frac{\|Y_\varepsilon(t)\|_{L^\infty} + \|Y_0\|_{L^\infty} \|\nabla \eta_\varepsilon(t)\|_{L^\infty}}{\|Y_\varepsilon(t)\|_{\inf(\operatorname{supp} b(\eta_\varepsilon^{-1}))}} (\|\omega_0\|_{L^\infty} \|Y_\varepsilon(t)\|_{C^\alpha} + \|R_\varepsilon(t)\|_{C^\alpha}) r^\alpha. \end{aligned} \tag{5.21}$$

5.5. Closing the estimates using Gronwall's lemma. We have, from (5.17), that

$$\begin{aligned} 1 - \log r &\leq \left| 1 - \log \frac{\delta_0}{8} \right| + C' \int_0^t V_\varepsilon(s) ds \leq C(Y_0) + C' \int_0^t V_\varepsilon(s) ds, \\ r^\alpha &\leq \frac{\delta_0^\alpha}{8^\alpha} \exp\left(-C'\alpha \int_0^t V_\varepsilon(s) ds\right) \leq C(Y_0) \exp\left(-C'\alpha \int_0^t V_\varepsilon(s) ds\right). \end{aligned}$$

Returning to (5.21), then, these bounds on $1 - \log r$ and r^α , along with the bounds in (5.4), (5.5), and (5.7) to (5.9), and (5.11), yield the estimate,

$$\begin{aligned} V_\varepsilon(t) &\leq C(\omega_0, Y_0) + C' C(\omega_0, Y_0) \int_0^t V_\varepsilon(s) ds + \frac{C(\omega_0, Y_0)}{\alpha} \exp\left(\alpha(1 - C') \int_0^t V_\varepsilon(s) ds\right) \\ &\quad + C_\alpha(1 + t) \exp\left(\alpha(4 - C') \int_0^t V_\varepsilon(s) ds\right) \\ &\leq C_\alpha(1 + t) + \frac{C(\omega_0, Y_0)}{\alpha} \int_0^t V_\varepsilon(s) ds \end{aligned}$$

as long as we choose

$$C' = 4\alpha^{-1}. \quad (5.22)$$

We note that, as required, $C' > 1$.

By Lemma 3.6, we conclude that

$$\|\nabla u_\varepsilon(t)\|_{L^\infty} \leq V_\varepsilon(t) \leq C_\alpha(1 + t)e^{C(\omega_0, Y_0)\alpha^{-1}t} \leq C_\alpha e^{c_\alpha t}.$$

If $\alpha > 1/2$, we can apply the above bound with $1/2$ in place of α , eliminating the factor of $(1 - \alpha)^{-1}$ that appear in C_α . This gives

$$\|\nabla u_\varepsilon(t)\|_{L^\infty} \leq V_\varepsilon(t) = C(\omega_0, Y_0)\alpha^{-1}e^{c_\alpha t} \leq c_\alpha e^{c_\alpha t}. \quad (5.23)$$

Then

$$\int_0^t V_\varepsilon(s) ds < \frac{c_\alpha}{c_\alpha} e^{c_\alpha t} = e^{c_\alpha t}$$

so by virtue of (5.11),

$$\|Y_\varepsilon(t)\|_{C^\alpha}, \|R_\varepsilon(t)\|_{C^\alpha} \leq C_\alpha e^{c_\alpha t}. \quad (5.24)$$

It follows from (5.10) that $\|Y_\varepsilon \cdot \nabla u_\varepsilon(t)\|_{C^\alpha}$ has this same bound.

5.6. Convergence of approximate solutions. In this section we show that in the limit as $\varepsilon \rightarrow 0$, the estimates in (1.11) hold for the solution ω to (1.2) with velocity u . For the delicate parts of the proof we follow the argument on pages 105-106 of [7], but beginning in a slightly different manner.

That the approximate solutions (u_ε) converge to the solution u for bounded initial vorticity is by now classical (see Section 8.2 of [21], for instance). Because (∇u_ε) is uniformly bounded in L^∞ , however, we can obtain stronger convergence, as follows.

Fix $T > 0$. It follows from [23], under the assumption only that the initial vorticity and velocity are both in L^∞ , that ∇p_ε is bounded in $L^\infty([0, T] \times \mathbb{R}^2)$ (see [17] for details). Then since $\partial_t u_\varepsilon = -u_\varepsilon \cdot \nabla u_\varepsilon - \nabla p_\varepsilon$ it follows that $(\partial_t u_\varepsilon)$ is bounded uniformly in ε . So, in fact, (u_ε) is a bounded equicontinuous family on $[0, T] \times L$ for any compact subset L of \mathbb{R}^2 with a Lipschitz modulus of continuity in time and space. By the Arzela-Ascoli theorem, a subsequence converges uniformly on $[0, T] \times L$ to some \bar{u} , with $\bar{u} = u$ by the uniqueness of limits. This also shows that $\nabla u \in L^\infty([0, T] \times \mathbb{R}^2)$, with the bound in (1.11). (Here and in what follows, we take subsequences as necessary without relabeling the indices.) It then follows that $\nabla \eta$, and for that matter $\nabla \eta_\varepsilon$, are in $L^\infty([0, T] \times \mathbb{R}^2)$.

It follows from Lemma 5.2, which we prove below, that u_ε decays in space uniformly in time and in ε . This same uniform decay applies to u by the convergence we showed above. Thus, in fact, $u_\varepsilon \rightarrow u$ in $L^\infty([0, T] \times \mathbb{R}^2)$ since we can control the size of $|u_\varepsilon - u|$ outside of a

sufficiently large compact subset. By interpolation it follows that $u_\varepsilon \rightarrow u$ in $L^\infty(0, T; C^\beta(\mathbb{R}^2))$ for all $\beta < 1$.

From (5.3), then, we can estimate,

$$\begin{aligned} & |\eta_\varepsilon(t, x) - \eta(t, x)| \\ & \leq \int_0^t |u_\varepsilon(s, \eta_\varepsilon(s, x)) - u(s, \eta_\varepsilon(s, x))| ds + \int_0^t |u(s, \eta_\varepsilon(s, x)) - u(s, \eta(s, x))| ds \\ & \leq \int_0^t \|u_\varepsilon(s) - u(s)\|_{L^\infty} + \int_0^t \|\nabla u\|_{L^\infty([0, T] \times \mathbb{R}^2)} |\eta_\varepsilon(s, x) - \eta(s, x)| ds. \end{aligned}$$

It follows from Lemma 3.6 that $\eta_\varepsilon - \eta \rightarrow 0$ in $L^\infty([0, T] \times \mathbb{R}^2)$ and, similarly, that $\eta_\varepsilon^{-1} - \eta^{-1} \rightarrow 0$ in $L^\infty([0, T] \times \mathbb{R}^2)$. By interpolation it follows that $\eta_\varepsilon - \eta \rightarrow 0$ in $L^\infty(0, T; C^\beta(\mathbb{R}^2))$ for all $\beta < 1$.

We now argue along the lines of pages 105-106 of [7].

We can write (4.3) as

$$Y_0 \cdot \nabla \eta_\varepsilon = Y_\varepsilon \circ \eta_\varepsilon.$$

By (2.4)₂ and (5.24), then, $Y_0 \cdot \nabla \eta_\varepsilon$ is uniformly bounded in $L^\infty(0, T; C^\alpha(\mathbb{R}^2))$. But $C^\alpha(\mathbb{R}^2)$ is compactly embedded in $C^\beta(\mathbb{R}^2)$ for all $\beta < \alpha$ so a subsequence of $(Y_0 \cdot \nabla \eta_\varepsilon)$ converges in $L^\infty(0, T; C^\beta(\mathbb{R}^2))$ to some f for all $\beta < \alpha$, and it is easy to see that $f \in L^\infty(0, T; C^\alpha(\mathbb{R}^2))$.

To show that $f = Y_0 \cdot \nabla \eta$, we need only show convergence of $Y_0 \cdot \nabla \eta_\varepsilon \rightarrow Y_0 \cdot \nabla \eta$ in some weaker sense. To do this, observe that

$$(Y_0 \cdot \nabla \eta_\varepsilon)^j = Y_0 \cdot \nabla \eta_\varepsilon^j = \operatorname{div}(\eta_\varepsilon^j Y_0) - \eta_\varepsilon^j \operatorname{div} Y_0.$$

But $\eta_\varepsilon - \eta \rightarrow 0$ in $L^\infty(0, T; C^\beta(\mathbb{R}^2))$ for all $\beta < 1$ as we showed above so $\eta_\varepsilon^j Y_0 - \eta^j Y_0 \rightarrow 0$ in $L^\infty(0, T; C^\alpha(\mathbb{R}^2))$. And, by assumption (1.8)₄, $\eta_\varepsilon^j \operatorname{div} Y_0 - \eta^j \operatorname{div} Y_0 \rightarrow 0$ in $L^\infty(0, T; C^\alpha(\mathbb{R}^2))$. By the definition of negative Hölder spaces in Definition 2.1 it follows that $Y_0 \cdot \nabla \eta_\varepsilon \rightarrow Y_0 \cdot \nabla \eta$ in $L^\infty(0, T; C^{\alpha-1}(\mathbb{R}^2))$. Hence, $f = Y_0 \cdot \nabla \eta$, so we can conclude that $Y_0 \cdot \nabla \eta \in L^\infty(0, T; C^\alpha(\mathbb{R}^2))$ and $Y_0 \cdot \nabla \eta_\varepsilon \rightarrow Y_0 \cdot \nabla \eta$ in $L^\infty(0, T; C^\beta(\mathbb{R}^2))$ for all $\beta < \alpha$.

Then, since $Y_\varepsilon = (Y_0 \cdot \nabla \eta_\varepsilon) \circ \eta_\varepsilon^{-1}$ and $Y = (Y_0 \cdot \nabla \eta) \circ \eta^{-1}$ (see (1.9) and (4.3)), we have,

$$\begin{aligned} \|Y_\varepsilon - Y\|_{L^\infty} & \leq \|(Y_0 \cdot \nabla \eta_\varepsilon) \circ \eta_\varepsilon^{-1} - (Y_0 \cdot \nabla \eta_\varepsilon) \circ \eta^{-1}\|_{L^\infty} \\ & \quad + \|(Y_0 \cdot \nabla \eta_\varepsilon) \circ \eta^{-1} - (Y_0 \cdot \nabla \eta) \circ \eta^{-1}\|_{L^\infty} \\ & \leq \|Y_0 \cdot \nabla \eta_\varepsilon\|_{C^\alpha} \|\eta_\varepsilon^{-1} - \eta^{-1}\|_{L^\infty}^\alpha + \|Y_0 \cdot \nabla \eta_\varepsilon - Y_0 \cdot \nabla \eta\|_{L^\infty} \\ & \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \end{aligned}$$

where we used (2.4)₁. Here the L^∞ norms are over $[0, T] \times \mathbb{R}^2$ for any fixed $T > 0$. Arguing as for $Y_0 \cdot \nabla \eta$, it also follows that $Y \in L^\infty(0, T; C^\alpha(\mathbb{R}^2))$ and that the bound on $Y(t)$ in (1.11) holds. Then (1.16) follows from (1.11) as in (5.4) and (5.5).

The proofs of (1.13) and (1.14), which we suppress, follow much the same course as the bounds above. Finally,

$$(Y_\varepsilon \cdot \nabla u_\varepsilon)^j = \operatorname{div}(u_\varepsilon^j Y_\varepsilon) - u_\varepsilon^j \operatorname{div} Y_\varepsilon,$$

and given that we now know that $Y_\varepsilon \rightarrow Y$ in $C^\beta(\mathbb{R}^2)$ for all $\beta < \alpha$ with $Y \in C^\alpha(\mathbb{R}^2)$, (1.15) can be proved much the way we proved the convergence of $Y_0 \cdot \nabla \eta_\varepsilon \rightarrow Y_0 \cdot \nabla \eta$, above (taking advantage of (1.13)).

This completes the proof of the first part of Theorem 1.2.

Remark 5.1. *Had we only assumed that $\operatorname{div} Y_0 \in C^{\alpha'}(\mathbb{R}^2)$ for some $\alpha' \in (0, \alpha]$ then the argument above that showed $Y_0 \cdot \nabla \eta_\varepsilon \rightarrow Y_0 \cdot \nabla \eta$ in $L^\infty(0, T; C^{\alpha-1}(\mathbb{R}^2))$ would yield $Y_0 \cdot \nabla \eta_\varepsilon \rightarrow Y_0 \cdot \nabla \eta$ in $L^\infty(0, T; C^{\alpha'-1}(\mathbb{R}^2))$. This would be sufficient to conclude that $f = Y_0 \cdot \nabla \eta$, and the proof would proceed unchanged.*

Lemma 5.2. *The approximating solutions, u_ε , decay in space uniformly in time and in ε .*

Proof. Let $|x| > 2$ and $R = |x|/2$. From (1.2)₂, we can write

$$u_\varepsilon = \left(\int_{B_1(x)} + \int_{B_R(x) \setminus B_1(x)} + \int_{B_R(x)^c} \right) K(x-y) \omega_\varepsilon(y) dy.$$

Fixing p, q Hölder conjugate with $p \in [1, 2)$ and applying Hölder's inequality to each of the three terms above, we have

$$\begin{aligned} |u_\varepsilon(t, x)| &\leq \left(\|K(x-\cdot)\|_{L^p(B_1(x))} \|\omega_\varepsilon(t)\|_{L^q(B_1(x))} \right. \\ &\quad + \|K(x-\cdot)\|_{L^\infty(B_R(x) \setminus B_1(x))} \|\omega_\varepsilon(t)\|_{L^1(B_R(x) \setminus B_1(x))} \\ &\quad \left. + \|K(x-\cdot)\|_{L^\infty(B_R(x)^c)} \|\omega_\varepsilon(t)\|_{L^1(B_R(x)^c)} \right) \\ &\leq C \|\omega_\varepsilon(t)\|_{(L^1 \cap L^q)(B_R(x))} + CR^{-1} \|\omega_\varepsilon(t)\|_{L^1(\mathbb{R}^2)} \\ &\leq C \|\omega_\varepsilon(t)\|_{(L^1 \cap L^q)(B_{|x|/2}(0)^c)} + C \|\omega_0\|_{L^1(\mathbb{R}^2)} |x|^{-1}. \end{aligned} \tag{5.25}$$

We claim that for constants $C_1, C_2 > 0$,

$$\begin{aligned} \|\omega_\varepsilon(t)\|_{(L^1 \cap L^q)(B_{|x|/2}(0)^c)} &\leq \|\omega_{0,\varepsilon}\|_{(L^1 \cap L^q)(B_{|x|/2-C_1T}(0)^c)} \\ &\leq \|\omega_0\|_{(L^1 \cap L^q)(B_{|x|/2-C_1T-C_2}(0)^c)} \end{aligned}$$

for all $|x|/2 \geq C_1T + C_2$. The first inequality holds because $\|u_\varepsilon\|_{L^\infty([0,T] \times \Omega)} \leq C_1 = \|\omega_0\|_{L^1 \cap L^\infty}$ by (4.2) and the vorticity is transported by the flow map. Then since $\omega_{0,\varepsilon} = \rho_\varepsilon * \omega_0$ and ρ_ε is supported within a ball of radius $C_2\varepsilon$, and we have assumed that $\varepsilon \leq 1$, the second inequality holds as well. Thus, ω_ε decays in space uniformly in time and in ε . It then follows from (5.25) that u_ε decays in space uniformly in time and in ε . \square

6. PROOF OF SERFATI'S THEOREM PART II

In this section we prove the second part of Theorem 1.2, finding a matrix $A \in C^\alpha(\mathbb{R}^2)$ such that $\nabla u - \omega A \in C^\alpha(\mathbb{R}^2)$.

We start with the expression

$$\nabla u_\varepsilon(x) = \frac{\omega_\varepsilon(x)}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \text{p.v.} \int \nabla K(x-y) \omega_\varepsilon(y) dy \tag{6.1}$$

for ∇u_ε given by Lemma B.1. This expression may not hold in the limit as $\varepsilon \rightarrow 0$, but to remove from ∇u the discontinuities inherent in ω we will clearly need, before taking ε to zero, to subtract from ∇u_ε its antisymmetric part, $\omega_\varepsilon A^{(1)}$, where

$$A^{(1)} := \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

What remains, then, is the principal value integral in (6.1): the symmetric part of ∇u_ε . We first show that away from $\eta_\varepsilon(t, \Sigma)$, the symmetric part of ∇u_ε has C^α regularity.

Let δ_0 be as in (1.8)₂ and $r > 0$ and the cutoff function χ be as in Section 5.4. Then

$$\begin{aligned}
& (1 - \chi(\eta_\varepsilon^{-1}(x))) \text{p. v.} \int \nabla K(x - y) \omega_\varepsilon(y) dy \\
&= (1 - \chi(\eta_\varepsilon^{-1}(x))) \text{p. v.} \int_{\mathbb{R}^2} \nabla(a_r K)(x - y) \omega_\varepsilon(y) dy \\
&\quad + (1 - \chi(\eta_\varepsilon^{-1}(x))) \int_{\mathbb{R}^2} \nabla((1 - a_r)K)(x - y) \omega_\varepsilon(y) dy \\
&=: I_1 + I_2.
\end{aligned}$$

By (3.3) in Lemma 3.2 applied with the kernel L_2 of Lemma 3.4, we have $I_1 \in C^\alpha(\mathbb{R}^2)$ with a C^α bound that is uniform in ε . (Note that in the integrals above, y remains bounded away from $\eta_\varepsilon(\Sigma, t)$ on the support of $(1 - \chi(\eta_\varepsilon^{-1}(x)))$.) By (1.19) we have the ε -independent bound,

$$\|I_1\|_{C^\alpha} \leq \|\omega_0\|_{C^\alpha(\Sigma^C)} e^{e^{c\alpha t}}.$$

Since the kernel $\nabla((1 - a_r)K)$ is smooth and bounded, I_2 is smooth, lying in $C^k(\mathbb{R}^2)$ for all k with a norm that depends only upon $\|\omega_\varepsilon\|_{L^1} = \|\omega_0\|_{L^1}$. In particular, $\|I_2\|_{C^\alpha} \leq C(\omega_0)$ with a norm independent of ε . (Cutting off with χ was not needed to conclude this.)

We will complete the proof by establishing the following lemma:

Lemma 6.1. *We have,*

$$\chi(\eta_\varepsilon^{-1}) \text{p. v.} \int_{\mathbb{R}^2} \nabla K(x - y) \omega_\varepsilon(y) dy = \omega_\varepsilon B_\varepsilon + D_\varepsilon,$$

where

$$B_\varepsilon = \frac{\chi(\eta_\varepsilon^{-1})}{2|Y_\varepsilon|^2} \begin{pmatrix} 2Y_\varepsilon^1 Y_\varepsilon^2 & (Y_\varepsilon^2)^2 - (Y_\varepsilon^1)^2 \\ (Y_\varepsilon^2)^2 - (Y_\varepsilon^1)^2 & -2Y_\varepsilon^1 Y_\varepsilon^2 \end{pmatrix}$$

and D_ε is presented in the proof. B_ε and D_ε lie in $C^\alpha(\mathbb{R}^2)$ with

$$\|B_\varepsilon(t)\|_{C^\alpha}, \|D_\varepsilon(t)\|_{C^\alpha} \leq C_\alpha e^{e^{c\alpha t}}. \quad (6.2)$$

Before proving Lemma 6.1, we show how it completes the proof of the second part of Theorem 1.2. Let

$$\bar{A}_\varepsilon = A^{(1)} + B_\varepsilon,$$

which lies in C^α with a C^α norm that is uniform in ε . Then

$$\begin{aligned}
\nabla u_\varepsilon - \omega_\varepsilon \bar{A}_\varepsilon &= \text{p. v.} \int \nabla K(x - y) \omega_\varepsilon(y) dy - \omega_\varepsilon B_\varepsilon \\
&= I_1 + I_2 + \chi(\eta_\varepsilon^{-1}) \text{p. v.} \int_{\mathbb{R}^2} \nabla K(x - y) \omega_\varepsilon(y) dy - \omega_\varepsilon B_\varepsilon \\
&= I_1 + I_2 + D_\varepsilon
\end{aligned}$$

lies in C^α with a C^α norm that is uniform in ε . The final equality is where we used Lemma 6.1.

This shows that \bar{A}_ε is a candidate for our matrix A (in the limit), but for aesthetic reasons we prefer to apply the cutoff function χ to $A^{(1)}$ as well, using

$$A_\varepsilon = \chi(\eta_\varepsilon^{-1}) A^{(1)} + B_\varepsilon.$$

This is valid, since the bound in (1.19) was established without using the matrix A , so we know at this point that $(1 - \chi(\eta_\varepsilon^{-1}))\omega_\varepsilon$ lies in C^α with a C^α norm that is uniform in ε . A simple calculation shows that

$$A_\varepsilon = \frac{\chi(\eta_\varepsilon^{-1})}{|Y_\varepsilon|^2} \begin{pmatrix} Y_\varepsilon^1 Y_\varepsilon^2 & -(Y_\varepsilon^1)^2 \\ (Y_\varepsilon^2)^2 & -Y_\varepsilon^1 Y_\varepsilon^2 \end{pmatrix}.$$

The bounds in (1.17) then follow from (6.2), the bounds on I_1 and I_2 , above, and Lemma 6.2.

An examination of each of the components of D_ε shows, arguing as in Section 5.6, that for some subsequence, $(\varepsilon_k)_{k=1}^\infty$, $D_{\varepsilon_k} \rightarrow D$ in $L^\infty(0, T; C^\beta)$ for all $\beta < \alpha$ and that D lies in $L^\infty(0, T; C^\alpha)$. Similarly, A_{ε_k} converges in C^β for all $\beta < \alpha$ to

$$A = \frac{\chi(\eta^{-1})}{|Y|^2} \begin{pmatrix} Y^1 Y^2 & -(Y^1)^2 \\ (Y^2)^2 & -Y^1 Y^2 \end{pmatrix}, \quad (6.3)$$

which lies in C^α , and also in the limit (1.17) holds.

Proof of Lemma 6.1. The proof comes down to understanding the regularity of the principal value integral in (6.1) on the support of $\chi(\eta_\varepsilon^{-1})$. To do this, we forcefully inject into this integral the vector field Y_ε , which characterizes the discontinuities in the vorticity field. This approach has already been used in our application of Corollary B.3 in Section 5.3. Now, however, the injection of Y_ε will be deeper. Loosely speaking, Y_ε was injected linearly in Corollary B.3; now, we will inject Y_ε quadratically.

Using the notation in Definition 2.8, we can write

$$\text{p. v.} \int \nabla K(x - y)\omega_\varepsilon(y) dy = \begin{pmatrix} \partial_1 K^1 \tilde{*} \omega_\varepsilon & \partial_1 K^2 \tilde{*} \omega_\varepsilon \\ \partial_2 K^1 \tilde{*} \omega_\varepsilon & \partial_2 K^2 \tilde{*} \omega_\varepsilon \end{pmatrix}. \quad (6.4)$$

Now, $\text{div } K = 0$ so

$$\partial_1 K^1 \tilde{*} \omega_\varepsilon = -\partial_2 K^2 \tilde{*} \omega_\varepsilon.$$

Also,

$$\text{curl } K = -\text{div } K^\perp = -\text{div}(\nabla^\perp \mathcal{F})^\perp = \Delta \mathcal{F} = \delta.$$

But since the $\tilde{*}$ operator avoids the origin in the integrand, we have

$$\partial_1 K^2 \tilde{*} \omega_\varepsilon = \partial_2 K^1 \tilde{*} \omega_\varepsilon.$$

Thus, the p. v. integral in (6.4) is, as we know, the symmetric part of ∇u_ε . Moreover,

$$\chi(\eta_\varepsilon^{-1}) \text{p. v.} \int \nabla K(x - y)\omega_\varepsilon(y) dy = \frac{\chi(\eta_\varepsilon^{-1})}{|Y_\varepsilon|^2} \begin{pmatrix} a_\varepsilon & b_\varepsilon \\ b_\varepsilon & -a_\varepsilon \end{pmatrix},$$

where

$$\begin{aligned} a_\varepsilon &:= |Y_\varepsilon|^2 \partial_1 K^1 \tilde{*} \omega_\varepsilon = (Y_\varepsilon^1)^2 \partial_1 K^1 \tilde{*} \omega_\varepsilon + (Y_\varepsilon^2)^2 \partial_1 K^1 \tilde{*} \omega_\varepsilon \\ &= Y_\varepsilon^1 [Y_\varepsilon^1 \partial_1 K^1 \tilde{*} \omega_\varepsilon + Y_\varepsilon^2 \partial_1 K^2 \tilde{*} \omega_\varepsilon] + Y_\varepsilon^2 [Y_\varepsilon^2 \partial_1 K^1 \tilde{*} \omega_\varepsilon - Y_\varepsilon^1 \partial_1 K^2 \tilde{*} \omega_\varepsilon] \\ &= Y_\varepsilon^1 [Y_\varepsilon^1 \partial_1 K^1 \tilde{*} \omega_\varepsilon + Y_\varepsilon^2 \partial_2 K^1 \tilde{*} \omega_\varepsilon] - Y_\varepsilon^2 [Y_\varepsilon^2 \partial_2 K^2 \tilde{*} \omega_\varepsilon + Y_\varepsilon^1 \partial_1 K^2 \tilde{*} \omega_\varepsilon] \end{aligned}$$

and

$$\begin{aligned} b_\varepsilon &:= |Y_\varepsilon|^2 \partial_1 K^2 \tilde{*} \omega_\varepsilon = (Y_\varepsilon^1)^2 \partial_1 K^2 \tilde{*} \omega_\varepsilon + (Y_\varepsilon^2)^2 \partial_1 K^2 \tilde{*} \omega_\varepsilon \\ &= Y_\varepsilon^1 [Y_\varepsilon^1 \partial_1 K^2 \tilde{*} \omega_\varepsilon + Y_\varepsilon^2 \partial_2 K^2 \tilde{*} \omega_\varepsilon] + Y_\varepsilon^2 [Y_\varepsilon^2 \partial_1 K^2 \tilde{*} \omega_\varepsilon - Y_\varepsilon^1 \partial_2 K^2 \tilde{*} \omega_\varepsilon] \\ &= Y_\varepsilon^1 [Y_\varepsilon^1 \partial_1 K^2 \tilde{*} \omega_\varepsilon + Y_\varepsilon^2 \partial_2 K^2 \tilde{*} \omega_\varepsilon] + Y_\varepsilon^2 [Y_\varepsilon^2 \partial_2 K^1 \tilde{*} \omega_\varepsilon + Y_\varepsilon^1 \partial_1 K^1 \tilde{*} \omega_\varepsilon]. \end{aligned}$$

Observe that we can write

$$a_\varepsilon = Y_\varepsilon \cdot \begin{pmatrix} Y_\varepsilon \cdot (\nabla K^1 \tilde{*} \omega_\varepsilon) \\ -Y_\varepsilon \cdot (\nabla K^2 \tilde{*} \omega_\varepsilon) \end{pmatrix}, \quad b_\varepsilon = Y_\varepsilon \cdot \begin{pmatrix} Y_\varepsilon \cdot (\nabla K^2 \tilde{*} \omega_\varepsilon) \\ Y_\varepsilon \cdot (\nabla K^1 \tilde{*} \omega_\varepsilon) \end{pmatrix}.$$

Defining, for vector-valued functions v and w ,

$$v \tilde{*} w := v^j \tilde{*} w^j,$$

we have,

$$Y_\varepsilon \cdot (\nabla K^j \tilde{*} \omega_\varepsilon) = \nabla K^j \tilde{*} \cdot (\omega_\varepsilon Y_\varepsilon) + [(\nabla K^j \tilde{*} \omega_\varepsilon) \cdot Y_\varepsilon - \nabla K^j \tilde{*} \cdot (\omega_\varepsilon Y_\varepsilon)].$$

Lemma B.2 gives

$$\begin{aligned} Y_\varepsilon \cdot \nabla K^1 \tilde{*} \omega_\varepsilon &= \frac{1}{2} \omega Y^2 + K^1 \tilde{*} \operatorname{div}(\omega Y) + [(\nabla K^1 \tilde{*} \omega_\varepsilon) \cdot Y_\varepsilon - \nabla K^1 \tilde{*} \cdot (\omega_\varepsilon Y_\varepsilon)], \\ Y_\varepsilon \cdot \nabla K^2 \tilde{*} \omega_\varepsilon &= -\frac{1}{2} \omega Y^1 + K^2 \tilde{*} \operatorname{div}(\omega Y) + [(\nabla K^2 \tilde{*} \omega_\varepsilon) \cdot Y_\varepsilon - \nabla K^2 \tilde{*} \cdot (\omega_\varepsilon Y_\varepsilon)]. \end{aligned}$$

Therefore,

$$\begin{aligned} a_\varepsilon &= Y_\varepsilon^1 (Y_\varepsilon \cdot \nabla K^1 \tilde{*} \omega_\varepsilon) - Y_\varepsilon^2 (Y_\varepsilon \cdot \nabla K^2 \tilde{*} \omega_\varepsilon) \\ &= \omega_\varepsilon Y_\varepsilon^1 Y_\varepsilon^2 \\ &\quad + Y_\varepsilon^1 (K^1 \tilde{*} \operatorname{div}(\omega Y) + (\nabla K^1 \tilde{*} \omega_\varepsilon) \cdot Y_\varepsilon - \nabla K^1 \tilde{*} \cdot (\omega_\varepsilon Y_\varepsilon)) \\ &\quad - Y_\varepsilon^2 (K^2 \tilde{*} \operatorname{div}(\omega Y) + (\nabla K^2 \tilde{*} \omega_\varepsilon) \cdot Y_\varepsilon - \nabla K^2 \tilde{*} \cdot (\omega_\varepsilon Y_\varepsilon)) \\ &=: \omega_\varepsilon Y_\varepsilon^1 Y_\varepsilon^2 + \bar{a}_\varepsilon \end{aligned}$$

and

$$\begin{aligned} b_\varepsilon &= Y_\varepsilon^1 (Y_\varepsilon \cdot \nabla K^2 \tilde{*} \omega_\varepsilon) + Y_\varepsilon^2 (Y_\varepsilon \cdot \nabla K^1 \tilde{*} \omega_\varepsilon) \\ &= \frac{1}{2} \omega_\varepsilon [(Y_\varepsilon^2)^2 - (Y_\varepsilon^1)^2] \\ &\quad + Y_\varepsilon^1 (K^2 \tilde{*} \operatorname{div}(\omega Y) + (\nabla K^2 \tilde{*} \omega_\varepsilon) \cdot Y_\varepsilon - \nabla K^2 \tilde{*} \cdot (\omega_\varepsilon Y_\varepsilon)) \\ &\quad + Y_\varepsilon^2 (K^1 \tilde{*} \operatorname{div}(\omega Y) + (\nabla K^1 \tilde{*} \omega_\varepsilon) \cdot Y_\varepsilon - \nabla K^1 \tilde{*} \cdot (\omega_\varepsilon Y_\varepsilon)) \\ &=: \frac{1}{2} \omega_\varepsilon [(Y_\varepsilon^2)^2 - (Y_\varepsilon^1)^2] + \bar{b}_\varepsilon. \end{aligned}$$

This gives B_ε as stated above and

$$D_\varepsilon = \frac{\chi(\eta_\varepsilon^{-1})}{|Y_\varepsilon|^2} \begin{pmatrix} \bar{a}_\varepsilon & \bar{b}_\varepsilon \\ \bar{b}_\varepsilon & -\bar{a}_\varepsilon \end{pmatrix}.$$

Now,

$$(\nabla K^j \tilde{*} \omega_\varepsilon) \cdot Y_\varepsilon - \nabla K^j \tilde{*} \cdot (\omega_\varepsilon Y_\varepsilon) = \text{p. v.} \int \nabla K^j(x-y) \cdot [Y_\varepsilon(x) - Y_\varepsilon(y)] \omega_\varepsilon(y) dy.$$

Applying Lemma 3.3 with the kernel L_3 of Lemma 3.4 and using (5.23) and (5.24)₁ gives

$$\|(\nabla K^j \tilde{*} \omega_\varepsilon) \cdot Y_\varepsilon - \nabla K^j \tilde{*} \cdot (\omega_\varepsilon Y_\varepsilon)\|_{C^\alpha} \leq C \|Y_\varepsilon(t)\|_{C^\alpha} V_\varepsilon(t) \leq C_\alpha e^{c_2 e^{c_1 t}}.$$

This with (1.14) gives the the bound on D_ε in (6.2).

For the regularity of B_ε , first note that

$$\| |Y_\varepsilon|^{-1} \|_{\dot{C}^\alpha(\operatorname{supp} \eta^{-1})} \leq \frac{\|Y_\varepsilon\|_{\dot{C}^\alpha}}{\|Y_\varepsilon\|_{\inf(\operatorname{supp} \chi(\eta^{-1}))}^2} \leq C_\alpha e^{e^{c_\alpha t}}$$

by (1.11), (5.8) and (5.14). The bound on B_ε in (6.2) then follows from (2.4)₂ and Lemma 6.2. \square

We used the following elementary lemma above:

Lemma 6.2. *Assume that $\varphi \in C^\infty(\mathbb{R}^2)$ takes values in $[0, 1]$ and $f \in C^\alpha(\mathbb{R}^2)$. Then*

$$\|\varphi f\|_{C^\alpha(\mathbb{R}^2)} \leq \|f\|_{L^\infty(\text{supp } \varphi)} + \|\varphi\|_{C^\alpha} \|f\|_{C^\alpha(\text{supp } \varphi)}.$$

We now give a simple example where A can be explicitly calculated.

Suppose that ω_0 is radially symmetric, so that $\omega_0(x) = g(|x|)$ for some measurable function, g . Then the solution to the Euler equations is stationary, with

$$u(x) = \left(-\frac{x_2}{r^2} \int_0^r \rho g(\rho) d\rho, \frac{x_2}{r^2} \int_0^r \rho g(\rho) d\rho \right),$$

where $r = |x|$. Then

$$\begin{aligned} \nabla u(x) &= \begin{pmatrix} \frac{2x_1x_2}{r^4} \int_0^r \rho g(\rho) d\rho - \frac{x_1x_2}{r^2} g(r) & \left(\frac{2x_2^2}{r^4} - \frac{1}{r^2} \right) \int_0^r \rho g(\rho) d\rho - \frac{x_2^2}{r^2} g(r) \\ \left(-\frac{2x_1^2}{r^4} + \frac{1}{r^2} \right) \int_0^r \rho g(\rho) d\rho + \frac{x_1^2}{r^2} g(r) & -\frac{2x_1x_2}{r^4} \int_0^r \rho g(\rho) d\rho + \frac{x_1x_2}{r^2} g(r) \end{pmatrix} \\ &= \frac{1}{r^4} \int_0^r \rho g(\rho) d\rho \begin{pmatrix} 2x_1x_2 & x_2^2 - x_1^2 \\ x_2^2 - x_1^2 & -2x_1x_2 \end{pmatrix} + \frac{g(r)}{r^2} \begin{pmatrix} -x_1x_2 & -x_2^2 \\ x_1^2 & x_1x_2 \end{pmatrix}. \end{aligned}$$

For simplicity, add the assumption that $g(r) = 0$ on $B_\delta(0)$ for some $\delta > 0$. Then, choosing

$$Y = (1 - a_{\delta/4})e_\theta = (1 - a_{\delta/4}) \begin{pmatrix} -x_2 \\ x_1 \\ r \end{pmatrix},$$

with a as in Definition 2.4, and letting $\chi(|x|) = 1 - a_{\delta/2}(x)$, (6.3) gives

$$A(x) = \frac{\chi(r)}{r^2} \begin{pmatrix} -x_1x_2 & -x_2^2 \\ x_1^2 & x_1x_2 \end{pmatrix}.$$

We see, then, that

$$\begin{aligned} \nabla u(x) - \omega(x)A(x) &= \frac{1}{r^4} \int_0^r \rho g(\rho) d\rho \begin{pmatrix} 2x_1x_2 & x_2^2 - x_1^2 \\ x_2^2 - x_1^2 & -2x_1x_2 \end{pmatrix} \\ &= \frac{1}{r^2} \int_0^r \rho g(\rho) d\rho \begin{pmatrix} 2 \cos \theta \sin \theta & \sin^2 \theta - \cos^2 \theta \\ \sin^2 \theta - \cos^2 \theta & -2 \cos \theta \sin \theta \end{pmatrix} \end{aligned}$$

in polar coordinates. This is C^∞ in θ and is as smooth in r as ρ allows, but is in any case always at least Lipschitz continuous. (Across the boundary of a classical vortex patch, for instance, it is only Lipschitz continuous.)

As simple as this example is, it provides useful insight into how $\omega(t)$ and $A(t)$ combine to cancel the singularities in ∇u . In particular, it highlights how the matrix A has no direct dependence on the magnitude of ω , only upon its irregularities as described by Y . So the same pattern of irregularities in the *initial* vorticity would yield the same $A(0)$. For a nonstationary solution $A(t)$ would, of course, evolve in a way that depends upon the magnitude of ω at time zero.

7. PERSISTENCE OF REGULARITY OF A VORTEX PATCH BOUNDARY

We now prove Theorem 1.1 using Theorem 1.2, first reformulating the vortex patch problem using level sets as in [2]. We take $\phi_0 \in C^{1,\alpha}(\mathbb{R}^2)$ such that

$$\begin{aligned} \phi_0(x) &> 0 \quad \text{in } \Omega, \\ \phi_0(x) &= 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \inf_{x \in \partial\Omega} |\nabla\phi_0(x)| \geq 2c > 0. \end{aligned}$$

We will be applying (1.18) of Theorem 1.4 with the closed set $\Sigma = \partial\Omega$.

Let $Y_0 = \nabla^\perp \phi_0 \in C^\alpha$ and note that Y_0 is tangential to $\partial\Omega$ and $|Y_0| \geq c > 0$ on $\mathcal{N}_{\delta_0}(\partial\Omega)$ for some $\delta_0 > 0$, since $\partial\Omega$ is compact. Also, formally,

$$\operatorname{div}(\omega_0 Y_0) = \omega_0 \operatorname{div} Y_0 + \nabla\omega_0 \cdot Y_0 = 0 + \nabla\omega_0 \cdot Y_0 = 0.$$

More precisely, let φ be any test function in $\mathcal{D}(\mathbb{R}^2)$. Then

$$(\operatorname{div}(\omega_0 Y_0), \varphi) = -(\omega_0 Y_0, \nabla\varphi) = - \int_{\Omega} \omega_0 Y_0 \cdot \nabla\varphi = - \int_{\Omega} Y_0 \cdot \nabla\varphi = \int_{\Omega} \operatorname{div} Y_0 \varphi = 0.$$

The integration by parts is valid since Y_0 lies in $C^\alpha(\Omega) \subseteq L^2(\Omega)$ and $\operatorname{div} Y_0 = 0$ also lies in $L^2(\Omega)$ (see, for instance, Theorem I.1.2 of [25]), the boundary integral vanishing since $Y_0 \cdot \mathbf{n} = 0$.

Let $\phi(t)$ be ϕ_0 transported by the flow map, so that

$$\partial_t \phi + u \cdot \nabla \phi = 0.$$

Applying ∇^\perp to both sides of this equation gives

$$\partial_t \nabla^\perp \phi + u \cdot \nabla \nabla^\perp \phi = \nabla^\perp \phi \cdot \nabla u.$$

Comparing this to (4.4), we see that $Y(t) = \nabla^\perp \phi(t)$. Since $Y(t) \in C^\alpha$ by Theorem 1.2 applied with $\Sigma = \partial\Omega$, we have $\phi(t) \in C^{1+\alpha}$. Then, because $\partial\Omega_t$ remains a level set of $\phi(t)$, $Y(t) = \nabla^\perp \phi(t)$ is tangential to $\partial\Omega$. Hence, the boundary of the vortex patch remains in $C^{1+\alpha}$. (This argument also proves (1.18).)

8. AN EXTENSION OF SERFATI'S RESULT

It is actually slightly easier to prove a more general form of Theorem 1.2, making assumptions on the initial data using a family of vector fields, much as Chemin does in [7]. At the expense of a little extra bookkeeping, the decomposition of the vorticity into “good” and “bad” parts is no longer needed, and all the associated estimates in Section 5.4 go away.

We start with a family $Y_0 = (Y_0^{(\lambda)})_{\lambda \in \Lambda}$ of vector fields in $C^\alpha(\mathbb{R}^2)$ and define, for any $s \in \mathbb{R}$,

$$\begin{aligned} \|f(Y_0)\|_{C^s} &:= \sup_{\lambda \in \Lambda} \left\| f\left(Y_0^{(\lambda)}\right) \right\|_{C^s(\mathbb{R}^2)}, \\ I(Y_0) &:= \inf_{x \in \mathbb{R}^2} \sup_{\lambda \in \Lambda} \left| Y_0^{(\lambda)}(x) \right|. \end{aligned}$$

Here, f is any function on vector fields (such as the divergence) and we define

$$f(Y_0) = \left(f(Y_0^{(\lambda)}) \right)_{\lambda \in \Lambda}.$$

When $\|f(Y_0)\|_{C^s} < \infty$ we say that $f(Y_0) \in C^s$.

Then, in place of (1.8), we assume that

$$\begin{cases} \omega_0 \in (L^1 \cap L^\infty)(\mathbb{R}^2), \\ \|Y_0\|_{C^\alpha} < \infty, \quad I(Y_0) > 0, \\ \operatorname{div}(\omega_0 Y_0) \in C^{\alpha-1}, \\ \operatorname{div} Y_0 \in C^\alpha. \end{cases} \quad (8.1)$$

We define the pushforward of the family, Y_0 , by

$$Y(t) = (Y^{(\lambda)}(t))_{\lambda \in \Lambda}, \quad Y^{(\lambda)}(t, \eta(t, x)) := (Y_0^{(\lambda)}(x) \cdot \nabla) \eta(t, x). \quad (8.2)$$

Then the first part of Theorem 1.2 holds with these new assumptions, reproducing, as we show in the next section, the result of Chemin in [7]:

Theorem 8.1. *Suppose that ω_0 is such that the conditions in (8.1) are satisfied for some family of vector fields Y_0 . For the unique solution ω in $L^\infty(\mathbb{R}; (L^1 \cap L^\infty)(\mathbb{R}^2))$ to the Euler equations in vorticity formulation (1.2), the bound in (1.11) (with Y as in (8.2)) holds, as do (1.13) through (1.16) and (1.18).*

Proof. We outline only the changes that are needed to the proof of Theorem 1.2 given in Section 5, as well as the preliminary transport estimates in Section 4.

In place of the Y_ε , R_ε vector fields corresponding to the mollified initial vorticity, we have entire families,

$$Y_\varepsilon = (Y_\varepsilon^{(\lambda)})_{\lambda \in \Lambda}, \quad R_\varepsilon = (R_\varepsilon^{(\lambda)})_{\lambda \in \Lambda}.$$

The calculations in Section 4 now apply to each $Y_\varepsilon^{(\lambda)}$, $R_\varepsilon^{(\lambda)}$. Lemma 4.2 then gives a C^α bound on the family $R_{0,\varepsilon}$ and Lemma 4.3 bounds $\|\operatorname{div}(\omega_\varepsilon Y_\varepsilon)\|_{C^{\alpha-1}}$.

Sections 5.1 and 5.2 require no changes. The estimates in Section 5.3 are now done for each $Y_\varepsilon^{(\lambda)}$, and (5.11) become bounds on whole families. Also, (5.8) becomes a lower bound on $I(Y_\varepsilon)$.

In Section 5.4 the initial decomposition in (5.12) of the second term in V_ε is unchanged, as is the estimate of the second part. We no longer make a decomposition of the initial vorticity as in (5.15), for that is the purpose of the family of vector fields, Y_ε . What remains, then, is the estimate of the matrix B without a cutoff function; that is, with $B = \nabla[\mu_{rh} \nabla \mathcal{F}] * \omega_\varepsilon$. To estimate B at $x \in \mathbb{R}^2$, we choose arbitrarily any $\lambda \in \Lambda$ such that (5.18) holds at x with $Y_0^{(\lambda)}$ in place of Y_0 and $c = I(Y_0)$; this is always possible by (8.1)₂. This leads to the same estimate on B as in (5.20), where now Y_ε and R_ε are families of vector fields.

Sections 5.5 and 5.6 proceed with no significant changes, except that the estimates now apply to families of vector fields. \square

To obtain the second part of Theorem 1.2, (1.17), and Theorem 1.5, we must strengthen the assumptions on the vector field, Y_0 , giving more uniform-in-space control on it than the restriction that $I(Y_0) > 0$. For this purpose, we define, for any $\lambda \in \Lambda$,

$$U_\lambda = \{x \in \mathbb{R}^2: |Y_0^{(\lambda)}(x)| > c'\} \quad (8.3)$$

with $0 < c' < c := I(Y_0)$. Each U_λ is open since it is the inverse image of an open set under the continuous map, $|Y_0^{(\lambda)}|$. There always exists a countable partition of unity, $(\varphi_n)_{n \in \mathbb{N}}$, for which $\operatorname{supp} \varphi_n \subseteq U_\lambda$ for some $\lambda \in \Lambda$ (see Theorem 13.10 of [26]). We require that such a

partition of unity exist for some $c' \in (0, c)$ with the further property that

$$I'((\varphi_n)_{n \in \mathbb{N}}) := \sup_{x \in \mathbb{R}^2} \sum_{n \in \mathbb{N}} \|\varphi_n\|_{C^\alpha} \mathbb{1}_{\text{supp } \varphi_n}(x) < \infty. \quad (8.4)$$

This condition rules out families of vector fields, Y_0 , having members whose magnitude drops arbitrarily quickly from c to 0.

With this added assumption, we have Theorem 8.2.

Theorem 8.2. *With ω_0 as in Theorem 8.1 and adding the condition in (8.4), we have (1.12) and (1.17), as well as the conclusions of Theorem 1.5.*

Proof. We need only show that (1.12) holds, for the remaining facts follow from it and (1.11), which we established in Theorem 8.1.

Returning to Section 6 we employ our partition of unity to piece together A_ε . We do not make the decomposition of ∇u_ε using the cutoff function χ . Instead, we construct an $A_\varepsilon^{(n)}$ corresponding to $\varphi_n(\eta_\varepsilon^{-1})\nabla u_\varepsilon$ in the same manner that A_ε was constructed for $\chi(\eta_\varepsilon^{-1})\nabla u_\varepsilon$ in Section 6. We then let

$$A_\varepsilon = \sum_{n \in \mathbb{N}} \varphi_n(\eta_\varepsilon^{-1}) A_\varepsilon^{(n)}$$

and note that we have a doubly exponential in time bound on $\|A_\varepsilon^{(n)}\|_{C^\alpha(\eta_\varepsilon(\text{supp } \varphi_n))}$ uniform in n and ε . The regularity of A and $\nabla u - \omega A$ in (1.12) along with the bound in (1.17) then follow from an application of Lemma 8.4 with $\psi_n = \varphi_n \circ \eta_\varepsilon^{-1}$ and $f_n = A_\varepsilon^{(n)}$. We can do this since

$$I'((\psi_n)_{n \in \mathbb{N}}) \leq \|\nabla \eta_\varepsilon^{-1}\|_{L^\infty}^\alpha I'((\varphi_n)_{n \in \mathbb{N}})$$

by (2.4)₁. □

Remark 8.3. *Observe how in the proof of Theorem 8.1 we had no need of a partition of unity when treating the matrix B since the regularity of B was not at issue.*

Lemma 8.4. *Let $(\psi_n)_{n \in \mathbb{N}}$ be a partition of unity and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions in $C^\alpha(\mathbb{R}^2)$. Then*

$$\left\| \sup_{n \in \mathbb{N}} \psi_n f_n \right\|_{C^\alpha(\mathbb{R}^2)} \leq 3 \left(1 + I'((\psi_n)_{n \in \mathbb{N}})\right) \sup_{n \in \mathbb{N}} \|f_n\|_{C^\alpha(\text{supp } \psi_n)}.$$

Proof. Let

$$F = \sum_{n \in \mathbb{N}} \psi_n f_n, \quad K_n = \text{supp } \psi_n.$$

First observe that

$$\|F\|_{L^\infty} \leq \sup_{n \in \mathbb{N}} \|f_n\|_{L^\infty(K_n)},$$

so it remains only to bound the homogeneous norm.

Let $x, y \in \mathbb{R}^2$. There exists two finite sets, $N = \{n_1, \dots, n_j\}$ and $M = \{m_1, \dots, m_k\}$ such that the only elements of $(\psi_n)_{n \in \mathbb{N}}$ not vanishing at x have indices in N and the only elements of $(\psi_n)_{n \in \mathbb{N}}$ not vanishing at y have indices in M . Defining

$$\Delta_n(x, y) := \frac{|(\psi_n f_n)(x) - (\psi_n f_n)(y)|}{|x - y|^\alpha},$$

there are four cases.

If $n \in N$ and $n \in M$ then

$$\begin{aligned} \Delta_n(x, y) &\leq \frac{|\psi_n(x)(f_n(x) - f_n(y))|}{|x - y|^\alpha} + \frac{|f_n(y)(\psi_n(x) - \psi_n(y))|}{|x - y|^\alpha} \\ &\leq \psi_n(x) \|f_n\|_{\dot{C}^\alpha(K_n)} + \|\psi_n\|_{\dot{C}^\alpha} \|f_n\|_{L^\infty(K_n)}. \end{aligned} \quad (8.5)$$

If $n \notin N$ and $n \notin M$ then $\Delta_n(x, y) = 0$.

If $n \in n$ but $n \notin M$ then

$$\Delta_n(x, y) = \frac{|\psi_n(x)f_n(x)|}{|x - y|^\alpha} = \frac{|(\psi_n(x) - \psi_n(y))f_n(x)|}{|x - y|^\alpha} \leq \|\psi_n\|_{\dot{C}^\alpha} \|f_n\|_{L^\infty(K_n)}$$

and the same inequality holds for $n \notin N$ but $n \in M$.

Thus, (8.5) holds for all four cases.

Then,

$$\begin{aligned} \Delta(x, y) &:= \frac{|F(x) - F(y)|}{|x - y|^\alpha} \leq \sum_{n \in M \cup N} \Delta_n(x, y) \\ &\leq 2 \sup_{n \in \mathbb{N}} \|f_n\|_{\dot{C}^\alpha(K_n)} + 2 \sup_{n \in \mathbb{N}} \|f_n\|_{L^\infty(K_n)} I'(Y_0), \end{aligned}$$

from which the stated bound follows. \square

9. EQUIVALENCE TO CHEMIN'S VORTEX PATCH RESULT

Using our notation, Chemin in [5, 7] makes the same assumptions on the initial data as those in (8.1) except that in place of (8.1)₃ he assumes that $Y_0 \cdot \nabla \omega_0 \in C^{\alpha-1}$. We show in this section that the assumptions of Chemin and Serfati are, in fact, equivalent, and that both are equivalent to assuming that $Y_0 \cdot \nabla u_0 \in C^\alpha$. We state this more precisely as follows:

Proposition 9.1. *Assume that $\omega_0 = \text{curl } u_0$ satisfies (1.8)_{1,2,4} or (8.1)_{1,2,4}. Then*

$$Y_0 \cdot \nabla \omega_0 \in C^{\alpha-1} \iff \text{div}(\omega_0 Y_0) \in C^{\alpha-1} \iff Y_0 \cdot \nabla u_0 \in C^\alpha.$$

Proof. Since $\omega_0 \text{div } Y_0 \in L^1 \cap L^\infty$ and

$$Y_0 \cdot \nabla \omega_0 = \text{div}(\omega_0 Y_0) - \omega_0 \text{div } Y_0,$$

the first equivalence is immediate in light of Lemma 9.2, which we prove below. (This equivalence continues to hold with the weaker assumption of Remark 5.1; in fact, $\text{div } Y_0 \in L^\infty$ is sufficient for this equivalence to hold.)

We have already shown that $\text{div}(\omega_0 Y_0) \in C^{\alpha-1} \implies Y_0 \cdot \nabla u_0 \in C^\alpha$, since $Y_0 \cdot \nabla u_0 \in C^\alpha$ is (1.15) at $t = 0$. We complete the proof by showing that $Y_0 \cdot \nabla u_0 \in C^\alpha \implies \text{div}(\omega_0 Y_0) \in C^{\alpha-1}$. We will do this only in the setting of the more general assumptions in (8.1), though there is a clear analog to those in (1.8).

So assume that $Y_0 \cdot \nabla u_0 \in C^\alpha$ and that (8.1)_{1,2,4} hold. Let $Y = Y_0^{(\lambda)}$ be any element of Y_0 and let U_λ be as given in (8.3). Then on U_λ we can write ∇u_0 as

$$\nabla u_0 = \nabla u_0 \begin{pmatrix} Y & Y^\perp \end{pmatrix} \begin{pmatrix} Y & Y^\perp \end{pmatrix}^{-1} = \begin{pmatrix} Y \cdot \nabla u_0^1 & Y^\perp \cdot \nabla u_0^1 \\ Y \cdot \nabla u_0^2 & Y^\perp \cdot \nabla u_0^2 \end{pmatrix} \begin{pmatrix} Y & Y^\perp \end{pmatrix}^{-1}.$$

Now, $Y \cdot \nabla u_0^1$ and $Y \cdot \nabla u_0^2$ both lie in C^α , so using $\text{div } u_0 = 0$ and $\omega_0 = \partial_1 u_0^2 - \partial_2 u_0^1 \in L^\infty$, we have

$$\begin{aligned} \nabla u_0^1 \cdot Y^\perp &= -\partial_1 u_0^1 Y^2 + \partial_2 u_0^1 Y^1 = \partial_2 u_0^2 Y^2 + \partial_1 u_0^2 Y^1 - \omega_0 Y^1 = Y \cdot \nabla u_0^2 - \omega_0 Y^1 \in L^\infty, \\ \nabla u_0^2 \cdot Y^\perp &= -\partial_1 u_0^2 Y^2 + \partial_2 u_0^2 Y^1 = \omega_0 Y^2 - \partial_2 u_0^1 Y^2 - \partial_1 u_0^1 Y^1 = \omega_0 Y^2 - Y \cdot \nabla u_0^1 \in L^\infty. \end{aligned}$$

Also,

$$|(Y \ Y^\perp)^{-1}| = |(Y \ Y^\perp)|^{-1} \leq C|Y|^{-1}.$$

This shows that

$$\begin{aligned} \|\nabla u_0\|_{L^\infty(U_\lambda)} &\leq C|Y|^{-1} \left[\|Y \cdot \nabla u_0\|_{L^\infty(U_\lambda)} + \|\omega_0\|_{L^\infty} \|Y\|_{L^\infty(U_\lambda)} \right] \\ &\leq C(c')^{-1} \left[\|Y_0 \cdot \nabla u_0\|_{C^\alpha} + \|\omega_0\|_{L^\infty} \|Y_0\|_{C^\alpha} \right]. \end{aligned} \tag{9.1}$$

We conclude that $\nabla u_0 \in L^\infty(\mathbb{R}^2)$.

The result now follows immediately from Corollary B.3, though this corollary requires regularity of ω_0 , so an approximation argument, which we leave to the reader, is required. \square

Lemma 9.2. *For all $\beta < 0$,*

$$(L^1 \cap L^\infty)(\mathbb{R}^2) \subseteq C^\beta(\mathbb{R}^2).$$

Proof. Let $\beta \in (-1, 0)$ and $f \in (L^1 \cap L^\infty)(\mathbb{R}^2)$. From Lemma 8.1 and Proposition 8.2 of [21], $K * f$ is log-Lipschitz and so lies in $C^{1+\beta}(\mathbb{R}^2)$. But then

$$f = \text{curl}(K * f) = -\text{div}(K * f)^\perp$$

is the divergence of the $C^{1+\beta}$ -function, $-(K * f)^\perp$, and so lies in $C^\beta(\mathbb{R}^2)$. \square

In both [7] and [22], (4.4) is used to bound $\|Y_\varepsilon(t)\|_{C^\alpha}$ (or $\|X_{t,\lambda}\|_{X^\varepsilon}$ in [7]) which leads each author to bound $\|Y_\varepsilon \cdot \nabla u_\varepsilon\|_{C^\alpha}$. Serfati does this² using Corollary B.3, introducing the quantity $\text{div}(\omega_\varepsilon Y_0)$, which is transported by the flow and can be uniformly bounded in $C^{\alpha-1}$. Chemin does this on p. 101 of [7]. Since $\text{div}(\omega_\varepsilon Y_\varepsilon)$, ω_ε , and $\text{div} Y_\varepsilon$ are each transported by the flow, it follows that $Y_\varepsilon \cdot \nabla \omega_\varepsilon = \text{div}(\omega_\varepsilon Y_\varepsilon) - \omega_\varepsilon \text{div} Y_\varepsilon$ is also transported by the flow, and it turns out that it too can be uniformly bounded in $C^{\alpha-1}$. The two proofs diverge sharply in how they manage all the estimates that result, but this dichotomy of choice in what is to be transported is the origin of the difference between both their initial hypotheses and their end results.

The condition $Y_0 \cdot \nabla u_0 \in C^\alpha$ has a precise geometric interpretation: the initial velocity has $C^{1+\alpha}$ -regularity in the direction of Y_0 , and this regularity persists over time. The condition $Y_0 \cdot \nabla \omega_0 \in C^{\alpha-1}$ does not mean that ω_0 has C^α -regularity in the direction of Y_0 , except in a loose sense, and the condition $\text{div}(\omega_0 Y_0) \in C^{\alpha-1}$ or Serfati's original form of this condition that $K * \text{div}(\omega_0 Y_0) \in C^\alpha$ are hard to interpret.

Using the condition $Y_0 \cdot \nabla u_0 \in C^\alpha$ also allows one to view the result of Chemin in [7] as an extension of the well-posedness of the Euler equations for $u_0 \in C^{1+\alpha}$ (as in Chapter 4 of [7]), showing that such regularity in one direction is sufficient and will persist over time. The constants c_α and C_α of Theorem 1.2, however, do not depend only upon $\|Y_0 \cdot \nabla u_0\|_{C^\alpha}$, since if nothing else they also depend upon $\|\omega_0\|_{L^1 \cap L^\infty}$. To have well-posedness in the sense of Hadamard, then, would require a definition of the proper functions space and a closer evaluation of the manner in which c_α and C_α depend upon ω_0 and Y_0 .

²Rather, this is our interpretation of what he is doing, as the expression for $Y(x) \cdot \nabla u(x)$ in Corollary B.3 never appears in [22].

10. EXAMPLES SATISFYING THE HYPOTHESES OF SERFATI'S THEOREM

We have already seen in Section 7 that a classical vortex patch satisfies the hypotheses of Theorem 1.2 in (1.8). The following are some additional examples:

- (1) Suppose that $\omega_0 \in C^\alpha(\mathbb{R}^2)$. Then choose $\Sigma = \emptyset$ or choose Y_0 to be any nonzero constant vector on $\Sigma = \mathbb{R}^2$ with $\omega_0 Y_0 \in C^\alpha(\mathbb{R}^2)$ so $\operatorname{div}(\omega_0 Y_0) \in C^{\alpha-1}(\mathbb{R}^2)$; either way, (1.8) is satisfied.
- (2) Let $\Sigma = \partial\Omega$, where Ω is a bounded domain having a $C^{1+\alpha}$ boundary. Let $\omega_0 = f\mathbb{1}_\Omega$ for $f \in C^\alpha(\Omega)$ with $f|_{\partial\Omega} \equiv \gamma$, γ being a constant. Choose ϕ_0 and $Y_0 = \nabla^\perp \phi_0$ as for a classical vortex patch (see Section 7). Now, $\omega_0 - \gamma\mathbb{1}_\Omega$ and Y_0 both lie in C^α , so $\operatorname{div}((\omega_0 - \gamma\mathbb{1}_\Omega)Y_0) \in C^{\alpha-1}$. But,

$$\operatorname{div}((\omega_0 - \gamma\mathbb{1}_\Omega)Y_0) = \operatorname{div}(\omega_0 Y_0) - \gamma \operatorname{div}(\mathbb{1}_\Omega Y_0) = \operatorname{div}(\omega_0 Y_0),$$

since we showed that $\operatorname{div}(\mathbb{1}_\Omega Y_0) = 0$ in Section 7. Hence, (1.8) holds and $\partial\Omega_t$ will remain $C^{1+\alpha}$ for the same reason as for a classical vortex patch.

- (3) A finite sum of classical vortex patches or vorticities as in Example 2 as long as their boundaries are disjoint. The boundaries will remain $C^{1+\alpha}$.
- (4) Let $\phi_0 \in C^{1+\alpha}(\mathbb{R}^2)$ with $|\nabla\phi_0| \geq c > 0$ on all of \mathbb{R}^2 have level curves each of which crosses any given vertical line exactly once. Let $Y_0 = \nabla^\perp \phi_0$. Then $Y_0 \in C^\alpha(\mathbb{R}^2)$, $\operatorname{div} Y_0 = 0$, and its flow lines are level curves of ϕ_0 . Y_0 describes a shear flow deviating in a controlled way from horizontal. Define $f_{x_1}(x_2)$ so that the flow line that passes through $(x_1, f_{x_1}(x_2))$ also passes through $(0, x_2)$.

Let $W: \mathbb{R} \rightarrow \mathbb{R}$ be any measurable bounded function supported on some nonempty bounded interval $[c, d]$. For some fixed $L > 0$ let

$$\omega_0(x_1, x_2) = \mathbb{1}_{[-L, L]}(x_1)W(f_{x_1}(x_2))$$

and let

$$\Sigma = \{(x_1, x_2): (x_1, f_{x_1}(x_2)) \in [-L, L] \times [c, d]\}.$$

Observe that ω_0 has the same level curves as ϕ_0 , which are all in $C^{1+\alpha}$.

Now, (1.8)_{1,2} are clearly satisfied. Also, formally, $\operatorname{div}(\omega_0 Y_0) = \nabla\omega_0 \cdot Y_0 + \omega_0 \operatorname{div} Y_0 = 0$, and we can verify this as for a classical vortex patch.

Because ϕ_0 and ω_0 have the same level curves and level curves are transported by the flow, $\phi(t)$ and $\omega(t)$ have the same level curves for all time, where $\phi(t)$ is ϕ_0 transported by the flow. We conclude from (1.18) that all the level curves of ω remain $C^{1+\alpha}$, including the top and bottom boundaries of $\operatorname{supp} \omega(t)$. That is, extreme lack of regularity of ω_0 transversal to Y_0 does not disrupt the regularity of the flow lines.

- (5) Any vector field satisfying (1.8) or (8.1) plus a $C^\alpha(\mathbb{R}^2)$ vector field. Because this does not require the choice of the vector field or family of vector fields Y_0 to change, if Y_0 is divergence-free then (1.18) will continue to hold. In particular, we conclude that the initially $C^{1+\alpha}$ boundary of a classical vortex patch remains $C^{1+\alpha}$ even if the initial vorticity is perturbed by a $C^\alpha(\mathbb{R}^2)$ vector field.

APPENDIX A. PROOFS OF LEMMAS

In this section we prove the lemmas stated in Section 3.

A.1. **Proof of Lemma 3.1.** Let

$$E = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}, \quad F = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = (\det M)M^{-1},$$

so that

$$EE^T = (\det E)I, \quad MF = (\det M)I,$$

I being the 2×2 identity matrix. Therefore, B can be expressed as

$$B = \frac{EE^T BMF}{\det(ME)}. \quad (\text{A.1})$$

We now compute $E^T BM$. Let $B = (B_{ij})$. Then,

$$\begin{aligned} E^T BM &= \begin{pmatrix} a^2 B_{11} + ac B_{12} + ac B_{21} + c^2 B_{22} & ab B_{11} + ad B_{12} + bc B_{21} + cd B_{22} \\ -ac B_{11} - c^2 B_{12} + a^2 B_{21} + ac B_{22} & -bc B_{11} - cd B_{12} + ab B_{21} + ad B_{22} \end{pmatrix} \\ &=: (l_{ij}). \end{aligned}$$

Since B is symmetric, we have

$$\begin{aligned} l_{11} &= \begin{pmatrix} a \\ c \end{pmatrix}^T B \begin{pmatrix} a \\ c \end{pmatrix}, \quad l_{12} = \begin{pmatrix} b \\ d \end{pmatrix}^T B \begin{pmatrix} a \\ c \end{pmatrix}, \quad l_{21} = \begin{pmatrix} -c \\ a \end{pmatrix}^T B \begin{pmatrix} a \\ c \end{pmatrix}, \\ l_{22} &= \det M \operatorname{tr} B - \begin{pmatrix} d \\ -b \end{pmatrix}^T B \begin{pmatrix} a \\ c \end{pmatrix}. \end{aligned}$$

Therefore, we can rewrite $E^T BM$ as

$$E^T BM = \begin{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix}^T B \begin{pmatrix} a \\ c \end{pmatrix} & \begin{pmatrix} b \\ d \end{pmatrix}^T B \begin{pmatrix} a \\ c \end{pmatrix} \\ \begin{pmatrix} -c \\ a \end{pmatrix}^T B \begin{pmatrix} a \\ c \end{pmatrix} & \begin{pmatrix} d \\ -b \end{pmatrix}^T B \begin{pmatrix} a \\ c \end{pmatrix} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \det M \operatorname{tr} B \end{pmatrix}.$$

Going back to (A.1), we obtain

$$\begin{aligned} B &= \frac{E}{\det M \det E} \begin{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix}^T B \begin{pmatrix} a \\ c \end{pmatrix} & \begin{pmatrix} b \\ d \end{pmatrix}^T B \begin{pmatrix} a \\ c \end{pmatrix} \\ \begin{pmatrix} -c \\ a \end{pmatrix}^T B \begin{pmatrix} a \\ c \end{pmatrix} & \begin{pmatrix} d \\ -b \end{pmatrix}^T B \begin{pmatrix} a \\ c \end{pmatrix} \end{pmatrix} F + \frac{\operatorname{tr} B}{\det E} \begin{pmatrix} c^2 & -ac \\ -ac & a^2 \end{pmatrix} \\ &= \frac{E}{\det M (a^2 + c^2)} \begin{pmatrix} (a, c) \cdot BM_1 & (b, d) \cdot BM_1 \\ (-c, a) \cdot BM_1 & (d, -b) \cdot BM_1 \end{pmatrix} F + \frac{\operatorname{tr} B}{a^2 + c^2} \begin{pmatrix} c^2 & -ac \\ -ac & a^2 \end{pmatrix}. \end{aligned}$$

Because the matrix norm in (2.1) is sub-multiplicative,

$$|B| \leq \frac{|E|}{\det M (a^2 + c^2)} \left| \begin{pmatrix} (a, c) \cdot BM_1 & (b, d) \cdot BM_1 \\ (-c, a) \cdot BM_1 & (d, -b) \cdot BM_1 \end{pmatrix} \right| |F| + \frac{|\operatorname{tr} B| (a^2 + c^2)}{a^2 + c^2}.$$

But, as we can see from (2.2),

$$|E| = (a^2 + c^2)^{1/2}, \quad |F| < (a^2 + b^2 + c^2 + d^2)^{1/2},$$

and, also applying the Cauchy-Schwarz inequality,

$$\begin{aligned} \left| \begin{pmatrix} (a, c) \cdot BM_1 & (b, d) \cdot BM_1 \\ (-c, a) \cdot BM_1 & (d, -b) \cdot BM_1 \end{pmatrix} \right| &\leq \sqrt{2} \left(|(a, c)|^2 |BM_1|^2 + |(b, d)|^2 |BM_1|^2 \right)^{1/2} \\ &\leq \sqrt{2} (a^2 + b^2 + c^2 + d^2)^{1/2} |BM_1|. \end{aligned}$$

Hence, applying (2.2) one final time,

$$\begin{aligned} |B| &< \sqrt{2} \frac{(a^2 + b^2 + c^2 + d^2) |M|}{|\det M| (a^2 + c^2)^{1/2}} |BM_1| + |\operatorname{tr} B| \\ &\leq \sqrt{2} \frac{(a^2 + b^2 + c^2 + d^2)^{1/2} |M|}{|\det M|} |BM_1| + |\operatorname{tr} B| \leq 2 \frac{|M|}{|\det M|} |BM_1| + |\operatorname{tr} B|. \end{aligned}$$

□

A.2. Proof of Lemma 3.2. We need to show that

$$\left| \int L(x, z) [f(z) - f(x)] dz - \int L(y, z) [f(z) - f(y)] dz \right| \leq C \|L\|_* \|f\|_{C^\alpha} |x - y|^\alpha.$$

We set $h = |x - y|$ and write

$$\begin{aligned} &\int L(x, z) [f(z) - f(x)] dz - \int L(y, z) [f(z) - f(y)] dz \\ &= \int_{|x-z| \leq 2h} L(x, z) [f(z) - f(x)] dz - \int_{|x-z| \leq 2h} L(y, z) [f(z) - f(y)] dz \\ &+ \left[\int_{|x-z| > 2h} L(x, z) [f(z) - f(x)] dz - \int_{|x-z| > 2h} L(y, z) [f(z) - f(y)] dz \right] \\ &=: \text{I} + \text{II} + \text{III}. \end{aligned}$$

We first estimate I by

$$\begin{aligned} |\text{I}| &\leq \int_{|x-z| \leq 2h} |L(x, z)| |x - z|^\alpha \frac{|f(z) - f(x)|}{|x - z|^\alpha} dz \\ &\leq \|f\|_{C^\alpha} \int_{|x-z| \leq 2h} (|L(x, z)| |x - z|^2) |x - z|^{\alpha-2} dz \\ &\leq C \|f\|_{\dot{C}^\alpha} \|L\|_* \int_{|x-z| \leq 2h} |x - z|^{\alpha-2} dz \leq C \alpha^{-1} \|L\|_* \|f\|_{\dot{C}^\alpha} h^\alpha. \end{aligned}$$

If $|x - z| \leq 2h$ and $|x - y| = h$, we have $|y - z| \leq 3h$ and thus

$$|\text{II}| \leq C \left| \int_{|y-z| \leq 3h} L(y, z) [f(z) - f(y)] dz \right| \leq C \alpha^{-1} \|L\|_* \|f\|_{\dot{C}^\alpha} h^\alpha.$$

To estimate III, we decompose it further (using (3.1)) into

$$\begin{aligned} \text{III} &= [f(y) - f(x)] \int_{|x-z| > 2h} L(x, z) dz + \int_{|x-z| > 2h} [f(z) - f(y)] (L(x, z) - L(y, z)) dz \\ &=: \text{III}_1 + \text{III}_2. \end{aligned}$$

We immediately have that

$$|\text{III}_1| \leq \|f\|_{\dot{C}^\alpha} \int_{|x-z| > 2h} |x - z|^{\alpha-2} dz \leq C \alpha^{-1} \|L\|_* \|f\|_{C^\alpha} h^\alpha.$$

We finally estimate III₂. Since

$$|L(x, z) - L(y, z)| \leq |\nabla_x L(\tilde{x}, z)| |x - y|, \quad \tilde{x} = ty + (1 - t)x, \text{ for some } t \in [0, 1],$$

we have

$$\begin{aligned}
|\text{III}_2| &\leq \int_{|x-z|>2h} |[f(z) - f(y)](L(x, z) - L(y, z))| dz \\
&\leq \int_{|x-z|>2h} |f(z) - f(y)| |\nabla_x L(\tilde{x}, z)| |x - y| dz \\
&= h \int_{|x-z|>2h} \frac{|f(z) - f(y)|}{|y - z|^\alpha} (|\nabla_x L(\tilde{x}, z)| |\tilde{x} - z|^3 \frac{|y - z|^\alpha}{|\tilde{x} - z|^3}) dz \\
&\leq \|L\|_* \|f\|_{C^\alpha} h \int_{|x-z|>2h} \frac{|y - z|^\alpha}{|\tilde{x} - z|^3} dz \leq C \|L\|_* \|f\|_{C^\alpha} h \int_{|\tilde{x}-z|>h} \frac{1}{|\tilde{x} - z|^{3-\alpha}} dz \\
&\leq C(1 - \alpha)^{-1} \|L\|_* \|f\|_{C^\alpha} h^\alpha,
\end{aligned}$$

where we used the inequalities $|\tilde{x} - y| \leq (1-t)|x - y| \leq h$ and $|\tilde{x} - z| \geq |x - z| - t|x - y| \geq h$ to obtain $|y - z| \leq |\tilde{x} - y| + |\tilde{x} - z| \leq 2|\tilde{x} - z|$. Collecting all terms, we have (3.2).

Finally, if (3.3) holds then $\int_{\mathbb{R}^2} L(\cdot, z) f(\cdot) dz = 0$, and (3.4) follows from (3.2). \square

A.3. Proof of Lemma 3.3. In light of Lemma 3.2, we need only bound the corresponding L^∞ norms. We have,

$$\begin{aligned}
&\left\| \text{p. v.} \int_{\mathbb{R}^2} L(\cdot, z) [f(z) - f(\cdot)] dz \right\|_{L^\infty} \\
&\leq \|f\|_{\dot{C}^\alpha} \left\| \lim_{h \rightarrow 0} \int_{B_h(x)^c \cap B_1(x)} |L(x, z)| |x - z|^\alpha dz \right\|_{L^\infty} + 2 \|f\|_{L^\infty} \sup_{x \in \mathbb{R}^2} \|L(x, \cdot)\|_{L^1(B_1(x)^c)} \\
&\leq \|L\|_* \|f\|_{\dot{C}^\alpha} \left\| \lim_{h \rightarrow 0} \int_{B_h(x)^c \cap B_1(x)} |x - z|^{\alpha-2} dz \right\|_{L^\infty} + 2 \|L\|_{**} \|f\|_{L^\infty} \\
&\leq C\alpha^{-1} \|L\|_{**} \|f\|_{C^\alpha}.
\end{aligned}$$

A.4. Proof of Lemma 3.5. Suppose that $\text{div } Z \in C^{\alpha-1}(\mathbb{R}^2)$ with $Z \in L^\infty(\mathbb{R}^2)$. We have,

$$\nabla \mathcal{F} * \text{div } Z = m(D) \text{div } Z = n_i(D) Z^i,$$

where m and n_i , $i = 1, 2$, are the Fourier-multipliers,

$$m(\xi) = \frac{\xi}{|\xi|^2}, \quad n_i(\xi) = \frac{\xi^i \xi}{|\xi|^2},$$

up to unimportant multiplicative constants. We can thus write $\nabla \mathcal{F} * \text{div } Z$ using a Littlewood-Paley decomposition in the form,

$$\nabla \mathcal{F} * \text{div } Z = \sum_{j \geq -1} \Delta_j m(D) \text{div } Z = \Delta_{-1} n_i(D) Z^i + \sum_{j \geq 0} \Delta_j m(D) \text{div } Z, \quad (\text{A.2})$$

where Δ_j are the nonhomogeneous Littlewood-Paley operators (dyadic blocks). We use the notation of [1] and refer the reader to Section 2.2 of that text for more details. The sum in (A.2) will converge in the space $\mathcal{S}'(\mathbb{R}^2)$ of Schwartz-class distributions as long as $\text{div } Z \in \mathcal{S}'(\mathbb{R}^2)$.

Now, for any noninteger $r \in [-1, \infty)$,

$$\sup_{j \geq -1} 2^{jr} \|\Delta_j f\|_{L^\infty}$$

is equivalent to the C^r norm of f (see Propositions 6.3 and 6.4 in Chapter II of [8]). Also,

$$\|\Delta_j m(D)f\|_{L^\infty} \leq C2^{-j} \|\Delta_j f\|_{L^\infty}, \quad \|\Delta_{-1} n_i(D)f\|_{L^\infty} \leq C \|f\|_{L^\infty}$$

for all $j \geq 0$ and $i = 1, 2$. The first inequality follows from Lemma 2.2 of [1] because m is homogeneous of degree -1 . The second inequality follows by a direct calculation, using only that n_i is bounded.

Hence,

$$\begin{aligned} \|\nabla \mathcal{F} * \operatorname{div} Z\|_{C^\alpha} &\leq \|\Delta_{-1} n_i(D)Z^i\|_{L^\infty} + \sup_{j \geq 0} 2^{j\alpha} \|\Delta_j m(D) \operatorname{div} Z\|_{L^\infty} \\ &\leq C \|Z\|_{L^\infty} + \sup_{j \geq 0} 2^{j(\alpha-1)} \|\Delta_j \operatorname{div} Z\|_{L^\infty} \\ &\leq C \|Z\|_{L^\infty} + C \|\operatorname{div} Z\|_{C^{\alpha-1}}, \end{aligned}$$

which gives the second inequality in (3.6).

Conversely, assume that $v := \nabla \mathcal{F} * \operatorname{div} Z \in C^\alpha(\mathbb{R}^2)$. Then,

$$\operatorname{div} v = \Delta \mathcal{F} * \operatorname{div} Z = \operatorname{div} Z.$$

Therefore, we conclude that $\operatorname{div} Z \in C^{\alpha-1}(\mathbb{R}^2)$ and obtain the first inequality in (3.6). \square

APPENDIX B. CALCULATIONS INVOLVING ∇u

Recall from Section 2 that $\nabla u = Du$, the Jacobian matrix of u (rather than its transpose, as it is sometimes defined).

Lemma B.1 is a standard way of expressing ∇u ; it is, in fact, the decomposition into its antisymmetric and symmetric parts. It follows, for instance, from Proposition 2.17 of [21]. In Lemma B.2, we inject the C^α -vector field Y into the formula given in Lemma B.1; the expression that results lies at the heart of the proof of Theorem 1.2, via Corollary B.3. Finally, Lemma B.4 justifies switching between two ways of calculating principal value integrals. We leave the proofs of Lemmas B.2 and B.4 to the reader.

Our applications of these results are to our approximate solutions, which lie in $L^1 \cap C^\infty$.

Lemma B.1. *Let u be a divergence-free vector field in $(L^1 \cap C^\infty)(\mathbb{R}^2)$ with vorticity ω . Then*

$$\nabla u(x) = \frac{\omega(x)}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \text{p. v.} \int \nabla K(x-y) \omega(y) dy,$$

where $K = \nabla^\perp \mathcal{F}$ is the Biot-Savart kernel. The first term is the antisymmetric, the second term the symmetric part of $\nabla u(x)$.

Lemma B.2. *Let $\omega \in (L^1 \cap C^\infty)(\mathbb{R}^2)$ and let Y be a vector field in $C^\alpha(\mathbb{R}^2)$. Then*

$$\text{p. v.} \int \nabla K(x-y) Y(y) \omega(y) dy = -\frac{\omega(x)}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y(x) + [K * \operatorname{div}(\omega Y)](x),$$

where $K = \nabla^\perp \mathcal{F}$ is the Biot-Savart kernel. Alternately, if for $j = 1, 2$ we let $i = 2, 1$ then

$$\text{p. v.} \int \nabla K^j(x-y) Y(y) \omega(y) dy = \frac{(-1)^i}{2} \omega_\varepsilon(x) Y_\varepsilon^i(x) + [K^j * \operatorname{div}(\omega_\varepsilon Y_\varepsilon)](x).$$

The following is a corollary of Lemmas B.1 and B.2.

Corollary B.3. *Let $\omega \in (L^1 \cap C^\infty)(\mathbb{R}^2)$ and let Y be a vector field in $C^\alpha(\mathbb{R}^2)$. Then*

$$Y(x) \cdot \nabla u(x) = \text{p. v.} \int \nabla K(x-y) [Y(x) - Y(y)] \omega(y) dy + [K * \text{div}(\omega Y)](x),$$

where $K = \nabla^\perp \mathcal{F}$ is the Biot-Savart kernel. Moreover,

$$\left\| \text{p. v.} \int \nabla K(x-y) [Y(x) - Y(y)] \omega(y) dy \right\|_{C^\alpha} \leq CV(\omega) \|Y\|_{C^\alpha},$$

$V(\omega)$ being given in (3.5).

Proof. The expression for $Y(x) \cdot \nabla u(x)$ follows from comparing the expressions in Lemmas B.1 and B.2. The C^α -bound follows from applying Lemma 3.3 with the kernel L_3 of Lemma 3.4. \square

Lemma B.4. *Let $f \in C^\beta(\mathbb{R}^2)$ for $\beta > 0$ be such that $\nabla(a_r K) \tilde{*} f(x)$ is defined for some $r > 0$, $x \in \mathbb{R}^2$. Then*

$$\nabla(a_r K) \tilde{*} f(x) = \lim_{h \rightarrow 0} \nabla(\mu_{rh} K) \tilde{*} f(x).$$

APPENDIX C. ON TRANSPORT EQUATION ESTIMATES

Together, Lemmas C.1 and C.2 justify our use of strong transport equations in obtaining estimates in the C^α -norm of the transported and pushed-forward quantities. First, the initial data is mollified using a mollification parameter δ independent of ε , the strong transport equation estimates are made, then δ is taken to zero. This is all while ε is held fixed. Lemma C.1 is used to obtain the C^α -bound on $\text{div} Y_\varepsilon(t)$ (leading to (1.13)), while Lemma C.2 is used to obtain the C^α -bounds on the vector fields, $Y_\varepsilon(t)$, $R_\varepsilon(t)$, and $Y_\varepsilon \cdot \nabla u_\varepsilon(t)$.

The proofs of Lemmas C.1 and C.2, which are left to the reader, employ only (2.4)_{1,2}, the boundedness of $\nabla \eta_\varepsilon^{-1}(t)$ in L^∞ over time (for fixed ε), and the convergence in C^α of a mollified function to the function itself.

Lemma C.1. *For $f_0 \in C^\alpha$, let*

$$\begin{aligned} f(t, x) &:= f_0(\eta_\varepsilon^{-1}(t, x)), \\ f^{(\delta)}(t, x) &= (\rho_\delta * f_0)(\eta_\varepsilon^{-1}(t, x)) \end{aligned}$$

for $\delta > 0$. Then

$$\|f^{(\delta)} - f\|_{L^\infty([0, T]; C^\alpha)} \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

Lemma C.2. *Let Y_ε be as in (4.3), so that*

$$Y_\varepsilon(t, \eta_\varepsilon(t, x)) = Y_0(x) \cdot \nabla \eta_\varepsilon(t, x).$$

Define $Y_\varepsilon^{(\delta)}$ by

$$Y_\varepsilon^{(\delta)}(t, \eta_\varepsilon(t, x)) = (\rho_\delta * Y_0)(x) \cdot \nabla \eta_\varepsilon(t, x).$$

Then

$$\|Y_\varepsilon^{(\delta)} - Y_\varepsilon\|_{L^\infty([0, T]; C^\alpha)} \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

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