Striated Regularity of Velocity for the Euler Equations

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Existence, uniqueness

$\boxed{\text{Throughout this talk we fix } \alpha \in (0, 1).}$

Theorem (Lichtenstein 1925, 1927, 1928; Gunther 1927, 1928; Wolibner 1933)

Assume that $u_0 \in C^{1,\alpha}(\mathbb{R}^d)$*,* $d \geq 2$*. There exists a unique solution,* u*, to the Euler equations with* $u \in L^\infty(0, T; C^{1,\alpha})$ *for some* $T > 0$*. When* $d = 2$*,* T *can be taken arbitrarily large.*

In fact, such well-posedness can be obtained for *striated* regularity.

Theorem (B & K 2014, 2015—roughly stated)

Assume that $\operatorname{curl} u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$*,* $d \geq 2$*. Let* $\phi_0 \in C^{1,\alpha}(\mathbb{R}^d)$ *with* $\nabla\phi_0$ *non-vanishing. If* u_0 *has* C^α *regularity along the level surfaces of* ϕ_0 *then* $u(t)$ *has* C^α *regularity along the level surfaces of* ϕ_0 *transported by the flow. Moreover, the Lagrangian velocity remains* C^α *along the level surfaces of* ϕ_0 *itself.*

Plan of the talk

1. Transport, pushforward, and frames.
2. Vector fields used to define striated regularity.
3. A little history of the problem: striated regularity of *vorticity*.
4. A more precise (but still local) expression of our main result.
5. The Lagrangian formulation.
6. The singularities in ∇u.

Transport

Let η be the flow map associated with u:

$$\partial_t \eta(t,x) = u\left(t, \eta(t,x)\right), \quad \eta(0,x) = x.$$

If ϕ is transported by the flow map then

$$\phi(t, \eta(t,x)) = \phi_0(t,x)$$

and

$$\frac{d}{dt}\phi(t,\eta(t,x)) = 0.$$

A simple calculation gives

$$\frac{d}{dt}\nabla\phi(t,\eta(t,x)) = -(\nabla u)^T \nabla\phi(t,\eta(t,x)).$$

Or, if $W := \nabla\phi$, we have

$$\frac{d}{dt}W(t,\eta(t,x)) = -((\nabla u)^T W)(t,\eta(t,x)).$$

Pushforward of a vector field

For a vector field, Y_0, define the pushforward of Y_0 by

$$Y(t, \eta(t, x)) := \nabla\eta(t, x) Y_0(x).$$

This is just the Jacobian of the diffeomorphism, $\eta(t, \cdot)$, multiplied by Y_0.

A calculation shows that

$$\frac{d}{dt} Y(t, \eta(t, x)) = (\nabla u Y)(t, \eta(t, x)).$$

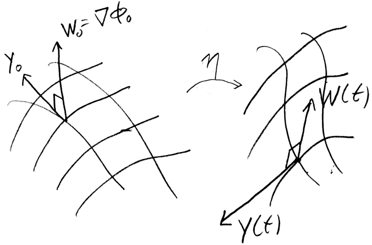
(Note $\nabla u Y = Y \cdot \nabla u$.)

Orthogonality with W conserved

Hence,

$$\frac{d}{dt}(Y \cdot W)(t,\eta(t,x)) = \left(W \cdot \frac{dY}{dt} + Y \cdot \frac{dW}{dt}\right)(t,\eta(t,x))$$
$$= \left(W \cdot (\nabla u Y) - Y \cdot ((\nabla u)^T W)\right)(t,\eta(t,x))$$
$$= \left(W^i (\nabla u)^{ij} Y^j - Y^j ((\nabla u)^T)^{ji} W^i\right)(t,\eta(t,x))$$
$$= \left(W^i \partial_j u^i Y^j - Y^j \partial_j u^i W^i\right)(t,\eta(t,x)) = 0.$$

So if $Y(0) \perp W(0)$ then $Y(t) \perp W(t)$ for all time.

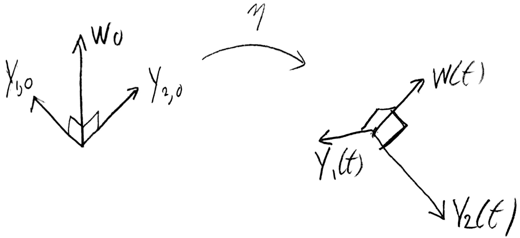


Frames for $\mathbb{R}^3$

This means that if we start with a frame, $\{Y_{1,0}, Y_{2,0}, W_0\}$, with $W_0 = \nabla\phi_0$, we can maintain a frame at time t by either:

1. Transporting ϕ_0 to ϕ, setting $W = \nabla\phi$, and building a frame around it, or
2. Pushing forward $\mathcal{Y}_0 := \{Y_{1,0}, Y_{2,0}\}$ to $\mathcal{Y}(t) := \{Y_1(t), Y_2(t)\}$, and completing the frame with $W(t)$.

In each case, we obtain (up to a multiplicative constant) the same final vector field, W, which is orthogonal to the rest of the frame. The orthogonality of the entire frame, however, is not conserved.



Transport versus pushforward

We either view $W(t) = \nabla\phi(t)$ as the direction of the singularity in the velocity gradient or $\mathcal{Y}(t)$ as the directions of higher regularity.

- Transporting ϕ_0 has the advantage that it is easier to state theorems, and transport is easier to think about geometrically. The best way to think about the propagation of regularity of the boundary of a surface, such as a vortex tube.

- Pushing forward $\mathcal{Y}_0$ makes it easier to write explicit expressions, demonstrate continuity over time, define function spaces with respect to the frame, and do estimates. It is historically the way striated regularity has been defined.

Vector fields used to define striated regularity

- For simplicity, we will present only the situation where the regularity of the velocity is prescribed by a single (ordered) set,

$$\mathcal{Y} = \{Y_1, Y_2\},$$

  explicitly in 3D, where Y_1, Y_2 are time-varying vector fields and $\mathcal{Y}(0) = \mathcal{Y}_0$. Need not be orthogonal.
- More generally, a potentially infinite collection of such sets would be used.
- For any function, f, on vector fields, define

$$f(\mathcal{Y}) := \{f(Y_1), f(Y_2)\}.$$

- For any Banach space, X,

$$\|f(\mathcal{Y})\|_X := \max\{\|f(Y_1)\|_X, \|f(Y_2)\|_X\}.$$

  When $\|f(\mathcal{Y}_0)\|_X < \infty$ we say that $f(\mathcal{Y}_0) \in X$.

Sufficient Family of Vector Fields

Define

$$I(\mathcal{Y}) := \begin{cases} \inf_{x\in\mathbb{R}^2} |Y_1(x)| & \text{in } 2D, \\ \inf_{x\in\mathbb{R}^3} \max\{|Y_1(x)|, |Y_2(x)|, |Y_1(x)\times Y_2(x)|\} & \text{in } 3D. \end{cases}$$

We say that $\mathcal{Y}_0$ is a *sufficient family of vector fields* if

- $Y_j \in C^\alpha(\mathbb{R}^2)$ with div $Y_j \in C^\alpha(\mathbb{R}^2)$, $j = 1, \ldots, d-1$,
- $I(\mathcal{Y}_0) > 0$.

A sufficient family of vector fields $\mathcal{Y}_0$ can always be obtained from $\phi_0 \in C^{1,\alpha}(\mathbb{R}^d)$ and $|\nabla\phi_0|$ bounded away from zero.

We define the vorticity in any of three different ways:

$$\begin{aligned}
d = 2: &\quad \omega = \omega(u) := \partial_1 u^2 - \partial_2 u^1, \\
d = 3: &\quad \vec{\omega} = \vec{\omega}(u) := \operatorname{curl} u, \\
d \geq 2: &\quad \Omega = \Omega(u) := \nabla u - (\nabla u)^T; \\
&\quad \Omega^j_k = \partial_k u^j - \partial_j u^k.
\end{aligned}$$

In vorticity form, the Euler equations are

$$\begin{aligned}
d = 2: &\quad \partial_t \omega + u \cdot \nabla \omega = 0, \\
d = 3: &\quad \partial_t \vec{\omega} + u \cdot \nabla \vec{\omega} = \vec{\omega} \cdot \nabla u, \\
d \geq 2: &\quad \partial_t \Omega + u \cdot \nabla \Omega + \Omega \tilde{\cdot} \nabla u = 0, \\
&\quad (\Omega \tilde{\cdot} \nabla u)^j_k := \Omega^m_j \partial_m u^k - \Omega^m_k \partial_m u^j.
\end{aligned}$$

2D: Striated initial vorticity

Theorem (Chemin 1995)

Let $\mathcal{Y}_0$ be a sufficient C^α family of vector fields in $\mathbb{R}^2$. Assume that $\omega_0 \in L^1 \cap L^\infty$ and $\operatorname{div}(\omega\mathcal{Y}) \in C^{\alpha-1}$. Then there exists a solution to the Euler equations for which $\operatorname{div}(\omega\mathcal{Y}) \in L^\infty_{loc}(\mathbb{R}; C^{\alpha-1})$. Also,

$$\|\nabla u(t)\|_{L^\infty} \le Ce^{Ct}, \quad \|\mathcal{Y}(t)\|_{C^\alpha} \le Ce^{e^{Ct}}.$$

The solution is unique for all $\omega \in L^\infty_{loc}(\mathbb{R}; L^1 \cap L^\infty)$ [Yudovich 1963].

This is a refinement of the result in Chemin's 1993 paper, which gave a detailed proof of his 1991 announcement of the persistence of striated regularity, including a proof of the persistence of regularity of a vortex patch boundary in 2D. This persistence was also proved in Bertozzi and Constantin 1993.

Serfati 1994 proved a version of persistence of striated regularity in slightly less generality than Chemin 1991 or 1993.

Negative-index Hölder spaces

For $\alpha \in (0,1)$, $C^{\alpha-1}$ is essentially the space of all functions that are sums of derivatives of functions in C^α. More precisely.

$$C^{\alpha-1}(\mathbb{R}^d) := \{f + \operatorname{div} \boldsymbol{g} \colon f, \boldsymbol{g} \in C^\alpha\},$$

$$\|h\|_{C^{\alpha-1}} := \inf_{f,\boldsymbol{g}} \max\left\{\|f\|_{C^\alpha}, \|g_1\|_{C^\alpha}, \|g_2\|_{C^\alpha}\right\}.$$

Formally,

$$\operatorname{div}(\omega Y) = Y \cdot \nabla\omega + \omega \operatorname{div} Y.$$

Since $\omega \in L^1 \cap L^\infty$ and $\operatorname{div} Y \in C^\alpha$, we *interpret* $\operatorname{div}(\omega Y) \in C^{\alpha-1}$ as meaning that ω is C^α in the direction of Y, or, when applied to a whole family of vector fields, as having *striated regularity of vorticity in C^α*.

3D: Striated initial vorticity

Theorem (Danchin 1999)

Let $\mathcal{Y}_0$ be a sufficient C^α family of vector fields in $\mathbb{R}^d$, $d \geq 2$. Assume that $\Omega_0 \in L^1 \cap L^\infty$ and $\operatorname{div}(\Omega^j_k \mathcal{Y}) \in C^{\alpha-1}$ *for all j, k. Then for some $T > 0$ there exists a solution to the Euler equations for which* $\operatorname{div}(\Omega^j_k \mathcal{Y}) \in L^\infty(0, T; C^{\alpha-1})$ *for all j, k and $\mathcal{Y} \cdot \nabla u \in L^\infty(0, T; C^\alpha)$. Also,*

$$\|\nabla u(t)\|_{L^\infty} \leq Ce^{Ct}, \quad \|\mathcal{Y}(t)\|_{C^\alpha} \leq Ce^{e^{Ct}}.$$

The solution is unique with certain conditions on u.

Earlier, Gamblin and Saint Raymond 1995 obtained a striated regularity result with less generality that encompassed 3D “vortex patches.” Fanelli in 2012 extended Danchin’s result to nonhomogeneous, incompressible fluids (homogeneous fluids being a special case).

Striated regularity of velocity

One can avoid negative Hölder spaces and prove striated regularity of *velocity* instead:

Theorem (B & K 2014, 2015)

Let $\mathcal{Y}_0$ be a sufficient C^α family of vector fields in $\mathbb{R}^d$, $d \geq 2$. For $d \geq 3$ we also assume that $\mathcal{Y}_0$ is Lipschitz. Assume that ω_0 or $\Omega_0 \in L^1 \cap L^\infty$ and $\mathcal{Y}_0 \cdot \nabla u_0 \in C^\alpha$. Then for some $T > 0$ there exists a solution to the Euler equations for which $\mathcal{Y} \cdot \nabla u \in L^\infty(0, T; C^\alpha)$. Also,

$$\|\nabla u(t)\|_{L^\infty} \leq Ce^{Ct}, \quad \|\mathcal{Y}(t)\|_{C^\alpha} \leq Ce^{e^{Ct}}.$$

In 2D, T can be arbitrarily large.

Our approach, however, was to work out the details in Serfati's 1994 paper and extend it to the striated vorticity assumptions Chemin made in his 1995 result.

Lagrangian Form

With η the flow map for u, as before, define the *Lagrangian velocity*,

$$v(t,x) := u(t,\eta(t,x)).$$

A calculation using the chain rule gives

$$Y_0(x)\cdot\nabla v(t,x) = (Y\cdot\nabla u)(t,\eta(t,x)).$$

Then,

$$\|Y_0\cdot\nabla v(t)\|_{\dot{C}^\alpha} \le \|(Y\cdot\nabla u)(t)\|_{\dot{C}^\alpha}\,\|\nabla\eta(t)\|_{L^\infty}^\alpha .$$

But $\nabla\eta(t)\in L^\infty$, so as a simple corollary we have:

Corollary (Lagrangian form)

Let $\mathcal{Y}_0$ be as in the previous theorem. Then $\mathcal{Y}_0\cdot\nabla v(t)$ remains in C^α for all time in 2D and up to time T in 3D.

Lagrangian versus Eulerian Striated Regularity

In Eulerian variables, we *pushforward* $\mathcal{Y}_0$ by the flow map giving $\mathcal{Y}(t)$, and measure regularity against this vector field.

In Lagrangian variables, we *pullback* the velocity field u by the flow map giving v, and measure regularity against the unchanging $\mathcal{Y}_0$.

One should be able to do the same thing for measures of regularity other than Hölderian. For instance, tangential Sobolev regularity (as in Coutand and Shkoller 2015) or anisotropic Gevrey regularity (as in Constantin, Kukavica, and Vicol 2015).

Dealing with higher regularity (Lagrangian or Eulerian) requires "background" regularity of vorticity, not just $\Omega_0 \in L^1 \cap L^\infty$.

Directions of the singularities in ∇u

Theorem (Serfati 1994 (2D), B & K 2015 (3D))

Let $\mathcal{Y}_0$, u, and T be as above for $d = 2, 3$. There exists a $d \times d$ matrix-valued function $A(t) \in L^\infty(0, T; C^\alpha(\mathbb{R}^d))$ such that for all $t \in [0, T]$,

$$\begin{cases} \nabla u - \omega A \in L^\infty(0, T; C^\alpha(\mathbb{R}^2)), & d = 2, \\ \nabla u - A\Omega \in L^\infty(0, T; C^\alpha(\mathbb{R}^3)), & d = 3. \end{cases}$$

The 2D version is stated in Serfati 1994.

This theorem describes how the directions of the singularities in ∇u are related to the directions of the singularities in the vorticity.

The Matrix A

2D: Letting $Y = Y_1$,

$$A := \frac{1}{|Y|^2} \begin{pmatrix} Y^1 Y^2 & -(Y^1)^2 \\ (Y^2)^2 & -Y^1 Y^2 \end{pmatrix} = -\frac{1}{|Y|^2} Y \otimes Y^\perp.$$

If instead $A = \dfrac{1}{|Y|^2} Y \otimes Y$ then $\nabla u - A\Omega \in L^\infty(0, T; C^\alpha(\mathbb{R}^3))$.

3D: First orthonormalizing Y_1, Y_2, we have

$$A = Y_1 \otimes Y_1 + Y_2 \otimes Y_2.$$

Thank you

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