STRIATED REGULARITY FOR THE EULER EQUATIONS

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ABSTRACT. In 1991, Chemin proved that vorticity possessing negative Hölder regularity in directions given by a sufficient family of vector fields (striated regularity) maintains such regularity for all time when measured against the push-forward of those vector fields under the flow map for a solution to the 2D Euler equations. We give an alternative proof, in 2D and 3D, largely following an approach of Ph. Serfati 1994, and establish the propagation of striated regularity of the Lagrangian velocity.

CONTENTS

1. INTRODUCTION

The Euler equations in velocity form (without forcing) on $\mathbb{R}^d$, $d \geq 2$, can be written,

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p &= 0, \\ \operatorname{div} u &= 0, \\ u(0) &= u_0, \end{cases} \tag{1.1}$$

where u is the velocity field, p is the pressure, and u_0 is the divergence-free initial velocity. These equations model the flow of an incompressible inviscid fluid.

Throughout this paper we fix $\alpha \in (0,1)$.

The fundamental well-posedness (though not in the sense of Hadamard) result in Hölder spaces is given in the following theorem:

Theorem (Lichtenstein 1925, 1927, 1928; Gunther 1927, 1928; Wolibner 1933)**.** *Assume that $u_0 \in C^{1,\alpha}(\mathbb{R}^d)$, $d = 3$. There exists a unique solution to the Euler equations with $u \in L^\infty(0,T;C^{1,\alpha})$ for some $T > 0$. When $d = 2$, T can be taken arbitrarily large.*

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The 3D result goes back to papers of Lichtenstein and Gunther [15, 16, 17, 18, 9, 10, 11], the 2D result is due to Wolibner [24]. We mention also Chemin's proof in [4].

In this paper, we will show that, in fact, such well-posedness can be obtained assuming $C^{1,\alpha}$ regularity of the velocity only in directions given by a sufficient family of vector fields. To describe this result, we first need to review the vorticity formulation of the Euler equations, introduce the flow map associated to the Eulerian velocity along with the pushforward of a velocity field by the flow map, and define some function spaces on families of vector fields.

We define the vorticity in any of three different ways as follows:

$$\begin{aligned} d = 2: &\quad \omega = \omega(u) := \partial_1 u^2 - \partial_2 u^1, \\ d = 3: &\quad \vec{\omega} = \vec{\omega}(u) := \operatorname{curl} u, \\ d \geq 2: &\quad \Omega = \Omega(u) := \nabla u - (\nabla u)^T; \\ &\quad \Omega^j_k = \partial_k u^j - \partial_j u^k. \end{aligned} \tag{1.2}$$

When working exclusively in 2D, it is always most convenient to use the first definition. Even specialized to 3D, most of our computations are more easily accomplished using the third definition than the second. When we express results or give proofs that apply to all dimensions $d \geq 2$ we will use the third form; when specializing to 2D we will use the first. A similar comment applies to the expressions that appear below in (1.3) and (1.4), Propositions 4.1 and 4.2, and Corollary 4.3.

Taking the vorticity of $(1.1)_1$, we obtain the vorticity equations,

$$\begin{aligned} d = 2: &\quad \partial_t \omega + u \cdot \nabla \omega = 0, \\ d = 3: &\quad \partial_t \vec{\omega} + u \cdot \nabla \vec{\omega} = \vec{\omega} \cdot \nabla u, \\ d \geq 2: &\quad \partial_t \Omega + u \cdot \nabla \Omega + \Omega \cdot \nabla u = 0. \end{aligned} \tag{1.3}$$

To turn (1.3) into a vorticity formulation, the velocity is recovered from the vorticity using the Biot-Savart law. Letting $\mathcal{F}_d$ be the fundamental solution of the Laplacian in $\mathbb{R}^d$ ($\Delta \mathcal{F}_d = \delta$), we can write this as

$$\begin{aligned} d = 2: &\quad u = K * \omega, \qquad K := \nabla^\perp \mathcal{F}_2 := (-\partial_2 \mathcal{F}_2, \partial_1 \mathcal{F}_2), \\ d \geq 2: &\quad u^j = K^k_d * \Omega^j_k, \quad K_d := \nabla \mathcal{F}_d, \end{aligned} \tag{1.4}$$

where here and in all that follows we implicitly sum over repeated indices. For $d = 2, 3$,

$$\begin{aligned} &\mathcal{F}_2(x) = \frac{1}{2\pi} \log |x|, \quad K_2(x) = \frac{1}{2\pi} \frac{x}{|x|^2}, \quad K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \\ &\mathcal{F}_3(x) = -\frac{1}{4\pi |x|}, \qquad K_3(x) = \frac{1}{4\pi} \frac{x}{|x|^3}, \end{aligned} \tag{1.5}$$

where $x^\perp := (-x_2, x_1)$.

Suppose that u is sufficiently regular that it has a unique associated flow map η,

$$\partial_t \eta(t, x) = u(t, \eta(t, x)), \quad \eta(0, x) = x. \tag{1.6}$$

Let Y_0 be a vector field on $\mathbb{R}^d$ and define the pushforward of Y_0 by

$$Y(t, \eta(t, x)) := (Y_0(x) \cdot \nabla) \eta(t, x). \tag{1.7}$$

This is just the Jacobian of the diffeomorphism $\eta(t, \cdot)$ multiplied by Y_0. Equivalently,

$$Y(t, x) = \eta(t)_* Y_0(t, x) := (Y_0(\eta^{-1}(t, x)) \cdot \nabla) \eta(t, \eta^{-1}(t, x)).$$

For a $d \times d$ matrix M, let $\operatorname{cofac} M$ be its cofactor matrix; thus, $(\operatorname{cofac} M)^i_j = (-1)^{i+j}$ times the (i, j)-minor of M (the determinant of the $(d-1)\times(d-1)$ matrix formed by removing the i-th row and j-th column). For any $Y_1, \cdots, Y_{d-1}$ in $\mathbb{R}^d$, we define $\wedge_{i<d} Y_i$ to be the vector, Z, appearing in the last column of the cofactor matrix,

$$\operatorname{cofac}\begin{pmatrix} Y^1 & Y^2 & \cdots & Y^{d-1} & Z \end{pmatrix}.$$

Note that the last column of this cofactor matrix depends only upon $Y^1, \dots, Y^{d-1}$, so this uniquely defines Z. (There are various equivalent ways to define $\wedge_{i<d} Y_i$, as, for instance, in [6].) Specifically, in 2D and 3D,

$$\begin{aligned} &\wedge_{i<2} Y_i = Y_1^\perp := (-Y_1^2, Y_1^1), && d = 2, \\ &\wedge_{i<3} Y_i = Y_1 \times Y_2, && d = 3. \end{aligned}$$

Let $\mathcal{Y} = (Y^{(\lambda)})_{\lambda\in\Lambda}$ be a family of vector fields on $\mathbb{R}^d$ indexed over the set Λ. For any function f on vector fields (such as div), define

$$f(\mathcal{Y}) := \left(f(Y^{(\lambda)})\right)_{\lambda\in\Lambda}.$$

For example, if Ω is as in (1.2), then for $j, k = 1, \dots, d$,

$$\operatorname{div}(\Omega^j_k \mathcal{Y}) = \left(\operatorname{div}\left(\Omega^j_k(Y^{(\lambda)})\right)\right)_{\lambda\in\Lambda}.$$

For any Banach space, X, define

$$\|f(\mathcal{Y})\|_X := \sup_{\lambda\in\Lambda}\left\|f\left(Y^{(\lambda)}\right)\right\|_X.$$

When $\|f(\mathcal{Y})\|_X < \infty$ we say that $f(\mathcal{Y}) \in X$. Define,

$$\begin{aligned} &d = 2: \quad I(\mathcal{Y}) := \inf_{x\in\mathbb{R}^2}\sup_{\lambda\in\Lambda}\left|Y^{(\lambda)}(x)\right|, \\ &d \ge 2: \quad I(\mathcal{Y}) := \min\left\{\inf_{x\in\mathbb{R}^d}\sup_{\lambda\in\Lambda}\left|Y^{(\lambda)}(x)\right|, \inf_{x\in\mathbb{R}^d}\sup_{\lambda_1,\dots,\lambda_{d-1}\in\Lambda}\left|\wedge_{j<d} Y^{(\lambda_j)}(x)\right|\right\}. \end{aligned} \tag{1.8}$$

We define the pushforward, $\mathcal{Y}$, of the family $\mathcal{Y}_0$ by

$$\mathcal{Y}(t) = (Y^{(\lambda)}(t))_{\lambda\in\Lambda}, \quad Y^{(\lambda)}(t, \eta(t, x)) := (Y_0^{(\lambda)}(x)\cdot\nabla)\eta(t, x). \tag{1.9}$$

We call $\mathcal{Y}_0$ a *sufficient* C^α family of vector fields if

$$\mathcal{Y}_0 \in C^\alpha, \ \operatorname{div}\mathcal{Y}_0 \in C^\alpha, \quad \text{and } I(\mathcal{Y}_0) > 0.$$

We will see that the pushforward, $\mathcal{Y}(t)$, of $\mathcal{Y}_0$ will remain a sufficient family for all time for $d = 2$ and for short time for $d \ge 3$, though the bound on $I(\mathcal{Y}(t))$ will increase with time.

We can now state our main results, Theorems 1.1 to 1.3. We note that Theorem 1.1 precisely states the well-posedness of the Euler equations assuming $C^{1,\alpha}$ regularity of the velocity only in directions given by a sufficient family of vector fields.

Theorem 1.1. *Let $\mathcal{Y}_0$ be a sufficient C^α family of vector fields in $\mathbb{R}^d$, $d \ge 2$. Assume that $\Omega(u_0) \in L^1 \cap L^\infty(\mathbb{R}^d)$ and $\mathcal{Y}_0 \cdot \nabla u_0 \in C^\alpha$. Then for some $T > 0$, T being arbitrarily large when $d = 2$, there exists a unique (see Remark 1.8) solution to the Euler equations, with $\mathcal{Y}\cdot\nabla u \in L^\infty(0, T; C^\alpha)$. Moreover, we have the following estimates:*

$$\|\nabla u(t)\|_{L^\infty} \le c_2 e^{c_1 t}, \tag{1.10}$$

$$\|\mathcal{Y}(t)\|_{C^\alpha} \le c_3 e^{c_1 e^{c_1 t}}, \tag{1.11}$$

$$\|\operatorname{div} \mathcal{Y}(t)\|_{C^\alpha} \le \|\operatorname{div} \mathcal{Y}_0\|_{C^\alpha} e^{e^{c_1 t}}, \tag{1.12}$$

$$\|\operatorname{div}(\Omega^j_k \mathcal{Y})(t)\|_{C^{\alpha-1}} \le c_3 e^{c_1 e^{c_1 t}} \; \forall\, j, k, \tag{1.13}$$

$$\|\mathcal{Y} \cdot \nabla u(t)\|_{C^\alpha} \le c_4 e^{c_1 e^{c_1 t}}, \tag{1.14}$$

$$\|\nabla \eta(t)\|_{L^\infty}, \|\nabla \eta^{-1}(t)\|_{L^\infty} \le e^{c_1 e^{c_1 t}}, \tag{1.15}$$

$$I(\mathcal{Y})(t) \ge I(\mathcal{Y}_0) e^{-c_1 e^{c_1 t}}. \tag{1.16}$$

Here,

$$c_1 := \frac{C}{\alpha}, \quad c_2 := \frac{C}{\alpha^2}, \quad c_3 := \frac{C}{\alpha(1-\alpha)}, \quad c_4 := \frac{C}{\alpha^2(1-\alpha)}.$$

The constant $C = C(u_0, \mathcal{Y}_0)$ depends on u_0 and $\mathcal{Y}_0$; specifically, on $\|\Omega_0\|_{L^1 \cap L^\infty}$, $\|\mathcal{Y}_0 \cdot \nabla u_0\|_{C^\alpha}$, $\|\mathcal{Y}_0\|_{C^\alpha}$, $\|\operatorname{div} \mathcal{Y}_0\|_{C^\alpha}$, $I(\mathcal{Y}_0)^{-1}$. In each case, C increases with each of these quantities. In 3D, c_1 through c_4 also have an additional dependence on T.

Theorem 1.2. *[Serfati [22] (in 2D)] Let u be the solution given by Theorem 1.1 for $d = 2$. There exists a matrix $A(t) \in C^\alpha(\mathbb{R}^2)$ such that for all $t \ge 0$,*

$$\|A(t)\|_{C^\alpha}, \|\nabla u(t) - \omega(t) A(t)\|_{C^\alpha} \le c_5 e^{c_1 e^{c_1 t}}, \tag{1.17}$$

where c_1 is in Theorem 1.1 and

$$c_5 := \frac{C(u_0, \mathcal{Y}_0)}{\alpha^4 (1-\alpha)^4}.$$

When $d = 3$, the same result holds, though now we have $A(t)\Omega(t)$ in place of $\omega(t)A(t)$. In 3D, c_5 also has an additional dependence on T.

Theorem 1.3. *Let $\mathcal{Y}$ be a sufficient family of C^α vector fields and $\Omega \in L^1 \cap L^\infty(\mathbb{R}^d)$. Then*

$$\begin{aligned} &\mathcal{Y} \cdot \nabla u \in C^\alpha \iff \operatorname{div}(\omega \mathcal{Y}) \in C^{\alpha-1} && d = 2, \\ &\mathcal{Y} \cdot \nabla u \in C^\alpha \iff \operatorname{div}(\Omega^j_k \mathcal{Y}) \in C^{\alpha-1} \, \forall\, j, k, && d \ge 2. \end{aligned} \tag{1.18}$$

Theorem 1.3 is very close to Lemma 4.6 in Fanelli's [7], and indeed follows from that lemma combined (for the forward implications) with Proposition 6.2, below—see Section 6. The backward implications in (1.18) are implicit in the proofs in [2, 3, 4, 22, 6] (see Remark 1.7).

Theorem 1.3 allows us to rephrase our striated regularity results in Lagrangian form. Define the *Lagrangian velocity*,

$$v(t, x) := u(t, \eta(t, x)).$$

A classical calculation using the chain rule gives, for any $Y_0 \in \mathcal{Y}_0$,

$$Y_0(x) \cdot \nabla v(t, x) = (Y \cdot \nabla u)(t, \eta(t, x)).$$

Thus (see (2.5)),

$$\|Y_0 \cdot \nabla v(t)\|_{C^\alpha} \le \|(Y \cdot \nabla u)(t)\|_{C^\alpha} \|\nabla \eta(t)\|^\alpha_{L^\infty}.$$

As a simple corollary of Theorem 1.1, then, we see that the striated regularity of the Lagrangian velocity is propagated over time:

Corollary 1.4. *Making the assumptions in Theorem 1.1, $\mathcal{Y}_0 \cdot \nabla v(t)$ remains in C^α for all time in 2D and up to time T for $d \ge 3$.*

The equivalence of striated regularity of the initial vorticity and velocity in Theorem 1.3 yields an immediate proof of Theorem 1.1 when combined with the following two existing results for the propagation of regularity of striated vorticity.

Theorem 1.5. *[Chemin [4]] Let $\mathcal{Y}_0$ be a sufficient C^α family of vector fields in $\mathbb{R}^2$. Assume that $\omega(u_0) \in L^1 \cap L^\infty(\mathbb{R}^2)$ and $\operatorname{div}(\omega_0 \mathcal{Y}_0) \in C^{\alpha-1}$. (The negative Hölder space, $C^{\alpha-1}$, is defined in Section 2.) Then there exists a unique global solution to the Euler equations, with $\operatorname{div}(\omega(t)\mathcal{Y}(t)) \in L^\infty_{loc}([0,\infty); C^{\alpha-1})$.*

Theorem 1.6. *[Danchin [6]] Let $\mathcal{Y}_0$ be a sufficient C^α family of vector fields in $\mathbb{R}^d$, $d \geq 3$. Assume that $\Omega(u_0) \in L^1 \cap L^\infty(\mathbb{R}^d)$ and $\operatorname{div}(\Omega^j_k(u_0)\mathcal{Y}_0) \in C^{\alpha-1}(\mathbb{R}^d)$ for all j, k. Then for some $T > 0$ there exists a unique solution to the Euler equations, with $\operatorname{div}(\Omega^j_k(u(t))\mathcal{Y}(t)) \in L^\infty(0,T; C^{\alpha-1})$ for all j, k.*

Remark 1.7. *As part of the proofs of Theorems 1.5 and 1.6 in [4, 6], it is shown that $\mathcal{Y} \cdot \nabla u \in L^\infty(0,T; C^\alpha)$. Some form of all the estimates stated in (1.10) through (1.16) are also obtained, some implicitly, though the specific dependence on α is not noted.*

Remark 1.8. *For uniqueness in Theorems 1.1 and 1.6 for $d = 2$ and in Theorem 1.5, the condition that $\omega \in L^\infty(0,T; L^1 \cap L^\infty)$ suffices, by Yudovich [25]. For higher dimension, a uniqueness condition that suffices for Theorems 1.1 and 1.6 is that $u \in L^\infty(0,T; Lip) \cap C(0,T; H^1)$, as established in [6]. The family $\mathcal{Y}$ itself clearly cannot enter into any uniqueness criterion.*

The 2D striated regularity result was first proved by Chemin in [2, 3]. Serfati in [22] also obtained the equivalent of Theorem 1.5, using a different approach, and gave the 2D version of Theorem 1.2. In each of these works, the initial vorticity was assumed to have striated regularity in a portion of the plane as described by a single non-vanishing vector field, and to have C^α regularity elsewhere. The use of a family of vector fields to characterize striated regularity throughout the plane was a later refinenement of Chemin: we give in Theorem 1.5 the result as it appears in [4]. (The use of a family of vector fields adds some bookkeeping to the proofs, but the heart of the matter was already addressed in [2].)

In dimensions 2 and higher, Theorem 1.6 is as stated (essentially) in [6] (recently extended to nonhomogeneous incompressible fluids by Fanelli in [7]). See also [8, 21, 12].

We give proofs of Theorems 1.5 and 1.6 following Serfati's approach to 2D Euler in [22]. The proof is self-contained in 2D, though in 3D we use an estimate on vortex stretching in dimensions 3 and higher from [6]. We present the full details in 2D, but just outline what is different in the higher-dimensional argument.

We close this introduction by observing a simple consequence of Theorem 1.2: the local propagation in 2D of Hölder regularity stated in Theorem 1.9.

Theorem 1.9. *[2D] Let ω_0, Y_0 be as in Theorem 1.2. If $\omega_0 \in C^\alpha(U)$ for some open subset U of $\mathbb{R}^2$ and $\alpha \in [0,1)$ then $\omega(t) \in C^\alpha(U_t)$ for all t, with*

$$\|\omega(t)\|_{C^\alpha(U_t)} \leq \|\omega_0\|_{C^\alpha(U)} e^{\alpha c_1 e^{c_1 t}}, \tag{1.19}$$

where $U_t = \eta(t, U)$. Further,

$$\|\nabla u(t)\|_{C^\alpha(U_t)} \leq c_5 \left(1 + \|\omega_0\|_{C^\alpha(U)}\right) e^{c_1 e^{c_1 t}}. \tag{1.20}$$

The constants c_1 and c_5 are as in Theorem 1.2.

Proof. For any $x, y \in U_t$,

$$\frac{|\omega(t,x) - \omega(t,y)|}{|x-y|^\alpha} = \frac{\left|\omega_0(\eta^{-1}(t,x)) - \omega_0(\eta^{-1}(t,y))\right|}{|\eta^{-1}(t,x) - \eta^{-1}(t,y)|^\alpha} \left(\frac{\left|\eta^{-1}(t,x) - \eta^{-1}(t,y)\right|}{|x-y|} \right)^\alpha .$$

Together with (1.15) this gives (1.19) (a bound that would hold for any Lipschitz velocity field). The bound in (1.20) then follows from (1.17). ☐

Theorem 1.9 improves, for initial data having striated regularity, existing estimates of local propagation of Hölder regularity for bounded initial vorticity. For instance, Proposition 8.3 of [19] would only give $\nabla u(t) \in C^\alpha_{loc}(U_t)$.

This paper is organized as follows. In Section 2, we fix some notation and make a few definitions. We develop the basic estimates we need on singular integrals in Section 3. Section 4 includes a number of lemmas centered around ∇u, these lemmas being central to the proofs of all of our results. Our proofs of Theorems 1.3, 1.5, and 1.6 all rely upon a linear algebra lemma of Serfati's to obtain a refined estimate on ∇u in L^∞. We present this lemma in Section 5. The proof of Theorem 1.3, giving the equivalence of striated regularity of velocity and vorticity, is presented in Section 6. In Section 7, we give the proof of Theorem 1.2 in 2D, giving the 3D proof in Section 8.

With Section 8, we have a complete proof of our main results: we directly proved Theorems 1.2 and 1.3, and Theorem 1.1 follows from Theorem 1.3 applied to Theorems 1.5 and 1.6 of [2, 3, 4, 6]. In Section 9 we begin a direct proof of Theorem 1.5, following [22]. From this we derive, as well, the specific estimates stated in (1.10) through (1.16).

The subject of Section 9 is the transport equations of a vector field $Y_0 \in \mathcal{Y}_0$ as well as the propagation of regularity of $\operatorname{div}(\omega Y)$. Section 10 contains the body of the proof of Theorem 1.5. In Section 11 we outline the changes to the proof of Theorem 1.5 needed to obtain Theorem 1.6 for $d \geq 3$.

Finally, in Appendix A, we discuss our use of weak transport equations.

2. Notation, conventions, and definitions

We define ∇u, the Jacobian matrix of u, as the $d \times d$ matrix with

$$(\nabla u)^i_j = \partial_j u^i$$

and define the gradient of other vector fields in the same manner. We follow the common convention that the gradient and divergence operators apply only to the spatial variables.

We write $C(p_1, \dots, p_n)$ to mean a constant that depends only upon the parameters $p_1, \dots, p_n$. We follow the convention that such constants can vary from expression to expression and even between two occurrences within the same expression. We will make frequent use of constants of the form,

$$c_\alpha := C(\omega_0, \mathcal{Y}_0)\alpha^{-1}, \quad C_\alpha := C(\omega_0, \mathcal{Y}_0)\alpha^{-1}(1-\alpha)^{-1}, \tag{2.1}$$

where $C(\omega_0, \mathcal{Y}_0)$ is a constant that depends upon only ω_0 and $\mathcal{Y}_0$.

We define

$$
\begin{aligned}
&M_{m\times n}(\mathbb{R}) = \text{ the space of all } m\times n \text{ real matrices},\\
&M^i_j = \text{the element at the } i\text{-th row, } j\text{-th column of } M\in M_{d\times d}(\mathbb{R}),\\
&M_j = \text{ the } j\text{-th column of } M\in M_{d\times d}(\mathbb{R}),\\
&M\cdot N = \sum_{i,j} M^i_j N^i_j = \sum_j M_j\cdot N_j \text{ for all } M,N\in M_{m\times n}(\mathbb{R}).
\end{aligned}
$$

Repeated indices appearing in upper/lower index pairs are summed over, but no summation occurs if the indices are both upper or both lower.

We write $|v|$ for the Euclidean norm of $v=(v^1,v^2,\cdots,v^d)$, $|v|^2=(v^1)^2+(v^2)^2+\cdots(v^d)^2$. For $M\in M_{d\times d}(\mathbb{R})$, we use the operator norm,

$$
|M| := \max_{|v|=1}|Mv|. \tag{2.2}
$$

Of course, all norms on finite-dimensional spaces are equivalent, so the choice of matrix norm just affects the values of constants. Our choice has the convenient properties, however, that it is sub-multiplicative, gives the identity matrix norm 1, and

$$
|M| = \sqrt{\max \text{ eigenvalue of } MM^*} \le \left(\sum_{i,j=1}^d (M^i_j)^2\right)^{\frac12} \le \sqrt{d}\,|M|, \tag{2.3}
$$

the first inequality being strict when M is nonsingular. If X is a function space, we define

$$
\|v\|_X := \||v|\|_X, \quad \|M\|_X := \||M|\|_X.
$$

Definition 2.1 (Hölder and Lipschitz spaces)**.** *Let $\alpha\in(0,1)$ and $U\subseteq\mathbb{R}^d$, $d\ge 1$, be open. Then $C^\alpha(U)$ is the space of all measurable functions for which*

$$
\|f\|_{C^\alpha(U)} := \|f\|_{L^\infty(U)} + \|f\|_{\dot{C}^\alpha(U)} < \infty, \quad \|f\|_{\dot{C}^\alpha(U)} := \sup_{\substack{x,y\in U\\ x\ne y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.
$$

For $\alpha=1$, we obtain the Lipschitz space, which is not called C^1 but rather $Lip(U)$. We also define $lip(U)$ for the homogeneous space. Explicitly, then,

$$
\|f\|_{Lip(U)} := \|f\|_{L^\infty(U)} + \|f\|_{lip(U)}, \quad \|f\|_{lip(U)} := \sup_{\substack{x,y\in U\\ x\ne y}} \frac{|f(x)-f(y)|}{|x-y|}.
$$

For any positive integer k, $C^{k+\alpha}(U)$ is the space of k-times continuously differentiable functions on U for which

$$
\|f\|_{C^{k+\alpha}(U)} := \sum_{|\beta|\le k}\|D^\beta f\|_{L^\infty(U)} + \sum_{|\beta|=k}\|D^\beta f\|_{C^\alpha(U)} < \infty.
$$

We define the negative Hölder space, $C^{\alpha-1}(U)$, by

$$
\begin{aligned}
C^{\alpha-1}(U) &= \{f+\operatorname{div} v\colon f,v\in C^\alpha(U)\},\\
\|h\|_{C^{\alpha-1}(U)} &= \inf\{\|f\|_{C^\alpha(U)}+\|v\|_{C^\alpha(U)} : h=f+\operatorname{div} v;\ f,\ v\in C^\alpha(U)\}.
\end{aligned}
$$

It follows immediately from the definition of $C^{\alpha-1}$ that

$$
\|\operatorname{div} v\|_{C^{\alpha-1}} \le \|v\|_{C^\alpha}. \tag{2.4}
$$

We also have the elementary inequalities,

$$\begin{aligned} \|f \circ g\|_{\dot{C}^\alpha} &\le \|f\|_{\dot{C}^\alpha} \|\nabla g\|_{L^\infty}^\alpha, \\ \|fg\|_{C^\alpha} &\le \|f\|_{C^\alpha} \|g\|_{C^\alpha}, \\ \|1/f\|_{\dot{C}^\alpha} &\le \frac{\|f\|_{\dot{C}^\alpha}}{(\inf |f|)^2}. \end{aligned} \tag{2.5}$$

Definition 2.2 (Radial cutoff functions)**.** *We make an arbitrary, but fixed, choice of a radially symmetric function $a \in C_C^\infty(\mathbb{R}^d)$ taking values in $[0,1]$ with $a = 1$ on $B_1(0)$ and $a = 0$ on $B_2(0)^C$. For $r > 0$, we define the rescaled cutoff function, $a_r(x) = a(x/r)$, and for $r, h > 0$ we define*

$$\mu_{rh} = a_r(1 - a_h).$$

Remark 2.3. *When using the cutoff function μ_{rh} we will be fixing r while taking $h \to 0$, in which case we can safely assume that h is sufficiently smaller than r so that μ_{rh} vanishes outside of $(h, 2r)$ and equals 1 identically on $(2h, r)$. It will then follow that*

$$\begin{cases} |\nabla \mu_{rh}(x)| \le Ch^{-1} \le C\,|x|^{-1} & \text{for } |x| \in (h, 2h), \\ |\nabla \mu_{rh}(x)| \le Cr^{-1} \le C\,|x|^{-1} & \text{for } |x| \in (r, 2r), \\ \nabla \mu_{rh} \equiv 0 & \text{elsewhere.} \end{cases}$$

Hence, also, $|\nabla \mu_{rh}(x)| \le C\,|x|^{-1}$ everywhere.

Definition 2.4 (Mollifier)**.** *Let $\rho \in C_C^\infty(\mathbb{R}^d)$ with $\rho \ge 0$ have $\|\rho\|_{L^1} = 1$ and be radially symmetric. For $\varepsilon > 0$, define $\rho_\varepsilon(\cdot) = (\varepsilon^{-d})\rho(\cdot/\varepsilon)$.*

Definition 2.5 (Principal value integral)**.** *For any measurable integral kernel, $L\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, and any measurable function, $f\colon \mathbb{R}^d \to \mathbb{R}$, define the integral transform $L[f]$ by*

$$L[f](x) := \text{p. v.} \int_{\mathbb{R}^d} L(x,y) f(y)\,dy := \lim_{h \to 0^+} \int_{|x-y|>h} L(x,y)\,f(y)\,dy,$$

whenever the limit exists.

Finally, we give the form of Gronwall's lemma that we will need.

Lemma 2.6 (Gronwall's lemma and reverse Gronwall's lemma)**.** *Suppose $h \ge 0$ is a continuous nondecreasing or nonincreasing function on $[0,T]$, $g \ge 0$ is an integrable function on $[0,T]$, and*

$$f(t) \le h(t) + \int_0^t g(s) f(s)\,ds \quad \text{or} \quad f(t) \ge h(t) - \int_0^t g(s) f(s)\,ds$$

for all $t \in [0,T]$. Then

$$f(t) \le h(t) \exp \int_0^t g(s)\,ds \quad \text{or} \quad f(t) \ge h(t) \exp\left(-\int_0^t g(s)\,ds\right),$$

respectively, for all $t \in [0,T]$.

3. Estimates on singular integrals

Because ∇u, via the Biot-Savart law (1.4), involves a singular integral, estimates on such integrals are central to all of our results. In this section, we give the basic estimates we will need for such integrals.

Lemma 3.1 is a fairly standard result on singular integral operators (so we suppress its proof). We do not apply it directly, but rather indirectly through its corollary, Lemma 3.2. Lemma 3.3 gives explicit estimates on the kernels to which we apply Lemma 3.2. We note that one of these kernels is not derived from the Biot-Savart kernel.

Lemma 3.1. *Let $L\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be an integral kernel for which*

$$\|L\|_* := \sup_{x,y \in \mathbb{R}^d} \left\{ |x-y|^d \, |L(x,y)| + |x-y|^{d+1} \, |\nabla_x L(x,y)| \right\} < \infty$$

and for which

$$\left| \text{p. v.} \int_{\mathbb{R}^d} L(x,y) \, dy \right| < \infty \text{ for all } x \in \mathbb{R}^d. \tag{3.1}$$

Let $L[f]$ be as in Definition 2.5. Then

$$\left\| \text{p. v.} \int_{\mathbb{R}^d} L(x,y) \, [f(y) - f(x)] \, dy \right\|_{\dot{C}^\alpha_x} \le C\alpha^{-1}(1-\alpha)^{-1} \, \|L\|_* \, \|f\|_{\dot{C}^\alpha}. \tag{3.2}$$

If

$$\text{p. v.} \int_{\mathbb{R}^d} L(\cdot, y) \, dy \equiv 0 \tag{3.3}$$

then

$$\|L[f]\|_{\dot{C}^\alpha} \le C\alpha^{-1}(1-\alpha)^{-1} \, \|L\|_* \, \|f\|_{\dot{C}^\alpha}. \tag{3.4}$$

The inequality in (3.4) is a classical result relating a Dini modulus of continuity of f to a singular integral operator applied to f in the special case where the modulus of continuity is $r \mapsto Cr^\alpha$. (See, for instance, the lemma in [13], and note that applying that lemma to a C^α function gives the same factor of $\alpha^{-1}(1-\alpha)^{-1}$ that appears in Lemma 3.1. This reflects the fact that the integral transform in (3.2) applied to a C^1-function gives only a log-Lipschitz function, and applied to a C^0-function yields no modulus of continuity.)

Lemma 3.2 allows us to bound the full C^α norm.

Lemma 3.2. *Let L be as in Lemma 3.1 and suppose further that*

$$\|L\|_{**} := \|L\|_* + \sup_{x \in \mathbb{R}^d} \|L(x, \cdot)\|_{L^1(B_1(x)^C)} < \infty.$$

Then the conclusions of Lemma 3.1 hold with each $\dot{C}^\alpha$ replaced by C^α and $\|L\|_*$ replaced by $\|L\|_{**}$.

Proof. In light of Lemma 3.1, we only need to bound the corresponding L^∞ norms. We have,

$$
\begin{aligned}
&\left\| \text{p. v.} \int_{\mathbb{R}^d} L(\cdot, z)\, [f(z) - f(\cdot)]\, dz \right\|_{L^\infty} \\
&\quad \le \|f\|_{\dot{C}^\alpha} \left\| \lim_{h\to 0} \int_{B_h(x)^C \cap B_1(x)} |L(x,z)|\, |x-z|^\alpha\, dz \right\|_{L^\infty_x} + 2\, \|f\|_{L^\infty} \sup_{x\in\mathbb{R}^2} \|L(x,\cdot)\|_{L^1(B^1(x)^C)} \\
&\quad \le \|L\|_* \|f\|_{\dot{C}^\alpha} \left\| \lim_{h\to 0} \int_{B_h(x)^C \cap B_1(x)} |x-z|^{\alpha-d}\, dz \right\|_{L^\infty_x} + 2\, \|L\|_{**} \|f\|_{L^\infty} \\
&\quad \le C\alpha^{-1} \|L\|_{**} \|f\|_{C^\alpha}.
\end{aligned}
$$

□

We shall apply Lemma 3.2 to the kernels of Lemma 3.3. Note that for L_2, we are actually applying Lemma 3.1 to each of its components. Also, for no $\varepsilon > 0$ is L_1 singular, but it becomes singular in the limit as $\varepsilon \to 0$.

Lemma 3.3. *Assume that $\Omega \in L^1 \cap L^\infty(\mathbb{R}^d)$ and define the kernels,*

(1) $L_1(x,y) = \rho_\varepsilon(x-y)\Omega(y)$;
(2) $L_2(x,y) = \Omega(y)\nabla K_d(x-y)$.

Here, $K_d = \nabla \mathcal{F}_d$ is the Biot-Savart kernel of (1.4) (in 2D, we can use K). Then $\|L_1\|_{**} \le C\|\Omega_0\|_{L^\infty}$ for C independent of ε and $\|L_2\|_{**} \le CV(\Omega)$ with

$$
V(\Omega) := \|\Omega\|_{L^\infty} + \left\| \text{p. v.} \int \Omega(y)\nabla K_d(x-y)\, dy \right\|_{L^\infty}. \tag{3.5}
$$

Proof. The bounds on the $*$-norms of L_1 and L_2 are easily verified, the key points being their L^1-bound uniform in x, the decay of $K_d(x-y)$ and $\nabla_x K_d(x-y)$, and the scaling of $\rho_\varepsilon(x-y)$ and $\nabla_x \rho_\varepsilon(x-y)$ in terms of ε. The p. v. integral in (3.5) comes from the final term in $\|L\|_{**}$. □

Lemma 3.4. *Let $r \in (0,1]$. For all $f \in \dot{C}^\alpha(\mathbb{R}^d)$, $g \in L^\infty(\mathbb{R}^d)$, we have*

$$
\left| \int \nabla[\mu_{rh}\nabla\mathcal{F}_d](x-y)(f(x)-f(y))g(y)\, dy \right| \le C\alpha^{-1} \|f\|_{\dot{C}^\alpha} \|g\|_{L^\infty} r^\alpha. \tag{3.6}
$$

For all $f \in L^\infty(\mathbb{R}^d)$, we have

$$
\left| \int (\mu_{rh}\nabla\mathcal{F}_d)(x-y) f(y)\, dy \right| \le C\alpha^{-1} \|f\|_{C^{\alpha-1}} r^\alpha. \tag{3.7}
$$

Proof. First observe that the integrals in (3.6) and (3.7) are well-defined because in both cases, f is bounded on any compact subset of $\mathbb{R}^d$ and the kernels are locally integrable.

For (3.6), we have $|\nabla[\mu_{rh}\nabla\mathcal{F}_d](x-y)| \le Cs^{-d}$ by Remark 2.3 and $|f(x)-f(y)| \le \|f\|_{\dot{C}^\alpha} s^\alpha$, where $s = |x-y|$. Hence,

$$
\begin{aligned}
\left| \int \nabla[\mu_{rh}\nabla\mathcal{F}_d](x-y)(f(x)-f(y))g(y)\, dy \right| &\le C\|f\|_{\dot{C}^\alpha} \|g\|_{L^\infty} \int_h^r s^{-d} s^\alpha s^{d-1}\, ds \\
&\le C\alpha^{-1} \|f\|_{\dot{C}^\alpha} \|g\|_{L^\infty} r^\alpha.
\end{aligned}
$$

We now show (3.7). Since $f \in L^\infty(\mathbb{R}^d) \subseteq C^{\alpha-1}(\mathbb{R}^d)$, we see from Definition 2.1 that there exist $f_0, f_1 \in C^\alpha$ with $f = f_0 + \operatorname{div} f_1$ such that

$$
\|f_0\|_{C^\alpha}, \|f_1\|_{C^\alpha} \le 2\|f\|_{C^{\alpha-1}}. \tag{3.8}
$$

(The 2 could be any value greater than 1 by definition of the infimum.)

For f_0, we have,

$$\begin{aligned}\left|\int(\mu_{rh}\nabla\mathcal{F}_d)(x-y)f_0(y)\,dy\right| &= \left|\int(\mu_{rh}\nabla\mathcal{F}_d)(x-y)(f_0(x)-f_0(y))\,dy\right| \\ &\le C\,\|f_0\|_{\dot{C}^\alpha}\int_h^r s^{-(d-1)}s^\alpha s^{d-1}\,ds \le C\,\|f_0\|_{\dot{C}^\alpha}\,r^{\alpha+1} \le C\,\|f_0\|_{C^\alpha}\,r^{\alpha+1} \\ &\le C\,\|f\|_{C^{\alpha-1}}\,r^{\alpha+1} \le C\,\|f\|_{C^{\alpha-1}}\,r^\alpha.\end{aligned}$$

The equality in the first step holds because the mean value of the kernel is zero. In the final inequality, we used (3.8). Observe that $\operatorname{div} f_1 \in L^\infty$, since $f, f_0 \in L^\infty$. This gives us sufficient regularity to integrate by parts and use (3.6) to obtain

$$\begin{aligned}&\left|\int(\mu_{rh}\nabla\mathcal{F}_d)(x-y)\operatorname{div}_y f_1(y))\,dy\right| \\ &= \left|\int(\mu_{rh}\nabla\mathcal{F}_d)(x-y)(\operatorname{div}_y(f_1(y)-f_1(x))\,dy\right| \\ &= \left|\int\nabla[\mu_{rh}\nabla\mathcal{F}_d](x-y)(f_1(x)-f_1(y))\,dy\right| \le C\alpha^{-1}\,\|f_1\|_{\dot{C}^\alpha}\,r^\alpha \\ &\le C\alpha^{-1}\,\|f_1\|_{C^\alpha}\,r^\alpha \le C\alpha^{-1}\,\|f\|_{C^{\alpha-1}}\,r^\alpha,\end{aligned}$$

where we use (3.8) at the end. Since

$$\left|\int(\mu_{rh}\nabla\mathcal{F}_d)(x-y)f(y)\,dy\right| \le \left|\int(\mu_{rh}\nabla\mathcal{F}_d)(x-y)f_0(y)\,dy\right| + \left|\int(\mu_{rh}\nabla\mathcal{F}_d)(x-y)\operatorname{div} f_1(y)\,dy\right|,$$

adding the bounds for these two integrals yields (3.7). □

4. Lemmas involving the velocity gradient

In this section we give the lemmas involving ∇u that we will need.

Proposition 4.1 is a standard way of expressing ∇u; it is, in fact, the decomposition of ∇u into its antisymmetric and symmetric parts. It follows, for instance, from Proposition 2.17 of [19].

In Proposition 4.2, we inject the C^α-vector field Y into the formula given in Proposition 4.1; the expression that results lies at the heart of the proofs of Theorems 1.3, 1.5, and 1.6, via Corollary 4.3, and the proof of Theorem 1.2, via Corollary 4.6. Proposition 4.7 justifies switching between two ways of calculating principal value integrals. Proposition 4.5 and Lemma 4.4 are used in the proofs of these results; Proposition 4.5 is also used directly in the proof of Theorem 1.5. We leave the proofs of Propositions 4.2 and 4.7 to the reader.

Recall the definitions of K and K_d in (1.4). We note that ∇K_d is a symmetric matrix.

Proposition 4.1. *Let u be a divergence-free vector field vanishing at infinity with vorticity $\Omega \in L^1 \cap L^\infty(\mathbb{R}^d)$. Then*

$$\begin{aligned} d=2:\quad &\nabla u(x) = \frac{\omega(x)}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} + \text{p. v.}\int\nabla K(x-y)\omega(y)\,dy,\\ d\ge 2:\quad &\nabla u(x) = \partial_j u^i(x) = \frac{\Omega(x)}{2} + \text{p. v.}\int\Omega(y)\nabla K_d(x-y)\,dy;\\ &(\nabla u)^i_j(x) = \partial_j u^i(x) = \frac{\Omega^i_j(x)}{2} + \text{p. v.}\int\partial_i\partial_k\mathcal{F}_d(x-y)\Omega^j_k(y)\,dy.\end{aligned}$$

The first term is the antisymmetric, the second term the symmetric part of $\nabla u(x)$.

In Proposition 4.1, the principal value integral is a singular integral operator, which is well-defined as a map from L^p to L^p for any $p \in (1, \infty)$. (See, for instance, Theorem 2 Chapter 2 of [23].)

Proposition 4.2. *Let* $\Omega \in L^1 \cap L^\infty(\mathbb{R}^d)$ *and let* Y *be a vector field in* $C^\alpha(\mathbb{R}^d)$. *Then*

$$d = 2: \quad \text{p. v.} \int \nabla K(x-y) Y(y)\, \omega(y)\, dy = -\frac{\omega(x)}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y(x) + [K * \operatorname{div}(\omega Y)](x),$$

$$d \geq 2: \quad \left[\text{p. v.} \int \Omega(y) \nabla K_d(x-y) Y(y)\, dy\right]^j = -\frac{(\Omega(x)Y(x))^j}{2} + \left[K_d^k * \operatorname{div}(\Omega_k^j Y)\right](x).$$

Corollary 4.3. *Let* $\Omega \in L^1 \cap L^\infty(\mathbb{R}^d)$ *and let* Y *be a vector field in* $C^\alpha(\mathbb{R}^d)$. *Then*

$$d = 2: \quad Y(x) \cdot \nabla u(x) = \text{p. v.} \int \nabla K(x-y)\, [Y(x) - Y(y)]\, \omega(y)\, dy + [K * \operatorname{div}(\omega Y)](x),$$

$$\begin{aligned} d \geq 2: \quad [Y(x) \cdot \nabla u(x)]^j = &\left[\text{p. v.} \int \Omega(y) \nabla K_d(x-y)\, [Y(x) - Y(y)]\, dy\right]^j \\ &+ \left[K_d^k * \operatorname{div}(\Omega_k^j Y)\right](x). \end{aligned}$$

Moreover, for $d = 2$,

$$\left\| \text{p. v.} \int \nabla K(x-y)\, [Y(x) - Y(y)]\, \omega(y)\, dy \right\|_{C^\alpha} \leq C V(\omega) \|Y\|_{C^\alpha},$$

$V(\omega)$ *being given in (3.5). The analogous bound holds for* $d \geq 3$.

Proof. The expression for $Y(x) \cdot \nabla u(x)$ follows from comparing the expressions in Propositions 4.1 and 4.2. The C^α-bound follows from applying Lemma 3.1 and Lemma 3.2 with the kernel L_2 of Lemma 3.3. □

Lemma 4.4. *Let* Z *be a vector field in* $L^1 \cap L^\infty(\mathbb{R}^2)$. *Then*

$$K * \operatorname{div} Z = Z^\perp - (K * \operatorname{curl} Z)^\perp.$$

Proof. A direct calculation shows that as tempered distributions, the divergence of each side is zero, while the curl of each side is $\operatorname{div} Z$. Since each side decays at infinity, it follows that the two sides are equal (see, for instance, Proposition 1.3.1 of [4]). □

Proposition 4.5. *If* $Z \in (L^1 \cap L^\infty)(\mathbb{R}^d)$ *and* $\operatorname{div} Z \in C^{\alpha-1}(\mathbb{R}^d)$, *then* $\nabla \mathcal{F}_d * \operatorname{div} Z \in C^\alpha(\mathbb{R}^d)$ *(equivalently,* $K * \operatorname{div} Z \in C^\alpha(\mathbb{R}^2)$, *for* $d = 2$*). Moreover,*

$$\|\nabla \mathcal{F}_d * \operatorname{div} Z\|_{C^\alpha} \leq C \left(\|Z\|_{L^1 \cap L^\infty} + \|\operatorname{div} Z\|_{C^{\alpha-1}}\right). \tag{4.1}$$

We also have $\operatorname{div} Z \in C^{\alpha-1}(\mathbb{R}^d)$ *if* $\nabla \mathcal{F}_d * \operatorname{div} Z \in C^\alpha(\mathbb{R}^d)$ *(equivalently,* $K * \operatorname{div} Z \in C^\alpha(\mathbb{R}^2)$, *for* $d = 2$*) and*

$$\|\operatorname{div} Z\|_{C^{\alpha-1}} \leq \|\nabla \mathcal{F}_d * \operatorname{div} Z\|_{C^\alpha}. \tag{4.2}$$

Proof. Suppose that $Z \in (L^1 \cap L^\infty)(\mathbb{R}^d)$ with $\operatorname{div} Z \in C^{\alpha-1}(\mathbb{R}^d)$. We have,

$$\nabla \mathcal{F}_d * \operatorname{div} Z = m(D) \operatorname{div} Z = n_i(D) Z^i,$$

where m and n_i, $i = 1, 2, \cdots, d$, are the Fourier-multipliers,

$$m(\xi) = \frac{\xi}{|\xi|^2}, \quad n_i(\xi) = \frac{\xi^i \xi}{|\xi|^2},$$

up to unimportant multiplicative constants. We can thus write $\nabla \mathcal{F}_d * \operatorname{div} Z$ using a Littlewood-Paley decomposition in the form,

$$\nabla \mathcal{F}_d * \operatorname{div} Z = \sum_{j \geq -1} \Delta_j m(D) \operatorname{div} Z = \Delta_{-1} n_i(D) Z^i + \sum_{j \geq 0} \Delta_j m(D) \operatorname{div} Z, \tag{4.3}$$

where Δ_j are the nonhomogeneous Littlewood-Paley operators (dyadic blocks). We use the notation of [1] and refer the reader to Section 2.2 of that text for more details. The sum in (4.3) will converge in the space $\mathcal{S}'(\mathbb{R}^d)$ of Schwartz-class distributions as long as $\operatorname{div} Z \in \mathcal{S}'(\mathbb{R}^d)$.

Now, for any noninteger $r \in [-1, \infty)$,

$$\sup_{j \geq -1} 2^{jr} \|\Delta_j f\|_{L^\infty}$$

is equivalent to the C^r norm of f (see Propositions 6.3 and 6.4 in Chapter II of [5], which apply to all $d \geq 2$). Also,

$$\|\Delta_{-1} n_i(D) f\|_{L^\infty} \leq C \|f\|_{L^2} \leq C \|f\|_{L^1 \cap L^\infty}, \quad \|\Delta_j m(D) f\|_{L^\infty} \leq C 2^{-j} \|\Delta_j f\|_{L^\infty}$$

for $j \geq 0$ and $i = 1, 2$. These inequalities follow from Lemma 2.1 and Lemma 2.2 of [1]. Hence,

$$\begin{aligned} \|\nabla \mathcal{F}_d * \operatorname{div} Z\|_{C^\alpha} &\leq \left\|\Delta_{-1} n_i(D) Z^i\right\|_{L^\infty} + \sup_{j \geq 0} 2^{j\alpha} \|\Delta_j m(D) \operatorname{div} Z\|_{L^\infty} \\ &\leq C\|Z\|_{L^2} + \sup_{j \geq 0} 2^{j(\alpha-1)} \|\Delta_j \operatorname{div} Z\|_{L^\infty} \leq C\|Z\|_{L^1 \cap L^\infty} + C \|\operatorname{div} Z\|_{C^{\alpha-1}}, \end{aligned}$$

which gives the inequality in (4.1).

Conversely, assume that $v := \nabla \mathcal{F}_d * \operatorname{div} Z \in C^\alpha(\mathbb{R}^d)$. Then,

$$\operatorname{div} v = \Delta \mathcal{F}_d * \operatorname{div} Z = \operatorname{div} Z.$$

Therefore, we conclude that $\operatorname{div} Z \in C^{\alpha-1}(\mathbb{R}^d)$ and obtain the inequality in (4.2). □

Corollary 4.6. *[2D] Let $\omega \in L^1 \cap L^\infty(\mathbb{R}^2)$ and let Y be a vector field in $C^\alpha(\mathbb{R}^2)$ with $\operatorname{div}(\omega Y) \in C^{\alpha-1}$. Then*

$$Y^\perp(x) \cdot \nabla u(x) = \text{p. v.} \int \nabla K(x-y) \left[Y^\perp(x) - Y^\perp(y)\right] \omega(y)\, dy + \left[K * \operatorname{div}(\omega Y)\right]^\perp(x) - \omega Y(x).$$

Moreover,

$$\|Y^\perp \cdot \nabla u + \omega Y\|_{C^\alpha} \leq C V(\omega) \|Y\|_{C^\alpha} + C \|\operatorname{div}(\omega Y)\|_{C^{\alpha-1}}.$$

Proof. Applying Lemma 4.4 with $Z = \omega Y^\perp$ gives

$$K * \operatorname{div}(\omega Y^\perp) = (\omega Y^\perp)^\perp - (K * \operatorname{curl}(\omega Y^\perp))^\perp = -\omega Y + (K * \operatorname{div}(\omega Y))^\perp.$$

Applying Corollary 4.3 with $Y^\perp$ in place of Y then gives the expression for $Y^\perp \cdot \nabla u$, and the C^α bound on $Y^\perp \cdot \nabla u + \omega Y$ follows as in the proof of Corollary 4.3, and using Proposition 4.5. □

Proposition 4.7. *Let $f \in C^\beta(\mathbb{R}^d)$ for $\beta > 0$ be. Then for all $r > 0$.*

$$\text{p. v.} \int (a_r K)(x-y) f(y)\, dy = \lim_{h \to 0} \nabla(\mu_{rh} K) * f(x),$$

where a_r and μ_{rh} are defined in Definition 2.2.

5. Serfati's linear algebra lemma

In this section we state and prove a simple linear algebra lemma due to Serfati. (The authors are not aware of earlier versions of similar lemmas. This lemma is key in Serfati's approach; it does not appear, for instance, in Chemin's approach in [2, 3].) This lemma will be used both in establishing the equivalence of striated vorticity and velocity in Section 6 and in proving the propagation of striated vorticity in Sections 10 and 11.

The $2D$ version of Lemma 5.1 appeared, in slightly different form, in [22]. A version for $d \geq 2$ appeared in Serfati's doctoral thesis, [20], and in [21].

In our application of Lemma 5.1, the vectors $M_1, \dots, M_{d-1}$ will represent the $d-1$ directions in which we have some regularity of the vorticity. This will give us control of BM_i for $i < d$. Separate control on $\operatorname{tr} B$ will then allow us to bound the full matrix, B.

Lemma 5.1. *For any symmetric $B \in M_{d\times d}(\mathbb{R})$, $d \geq 1$, we have*

$$|B| \leq \frac{P(M_1, \dots, M_{d-1})}{|\wedge_{i<d} M_i|^4} \sum_{i=1}^{d-1} |BM_i| + d\,|\operatorname{tr} B|\,, \tag{5.1}$$

where $M_1, \dots, M_{d-1}$ are any linearly independent vectors in $\mathbb{R}^d$ and P is a polynomial of degree $3d-2$.

Proof. Let

$$Z = \bigwedge_{i<d} M_i$$

and form the matrix M by column as

$$M := \begin{bmatrix} M_1 & \cdots & M_{d-1} & Z \end{bmatrix}.$$

Note that Z is the last column of $\underline{M}$, the cofactor matrix of M (as in our definition of the wedge product itself in Section 1, this is not a circular definition). Expanding about the last column of M, we see that

$$|Z|^2 = \det M \neq 0,$$

because we assumed that $M_1, \dots, M_{d-1}$ are linearly independent.

Now,

$$M\underline{M}^T = \underline{M}M^T = \det M\, I,$$

from which it follows that

$$B = \frac{\underline{M}}{(\det M)^2} D \underline{M}^T, \quad D := M^T BM. \tag{5.2}$$

Then, noting that $D^i_j = M_i \cdot BM_j$, we can write,

$$D = \begin{pmatrix} M_1 \cdot BM_1 & \cdots & M_1 \cdot BM_d \\ & \vdots & \\ M_{d-1} \cdot BM_1 & \cdots & M_{d-1} \cdot BM_d \\ M_d \cdot BM_1 & \cdots & M_d \cdot BM_d \end{pmatrix}.$$

Because B is symmetric, so is D. Hence, we can replace $M_j \cdot BM_d$ in the final column with $M_d \cdot BM_j$, eliminating BM_d from all but $D^d_d = M_d \cdot BM_d$. For D^d_d, we calculate,

$$M_d \cdot BM_d = \underline{M}_d \cdot BM_d = \sum_{i=1}^{d} \underline{M}_i \cdot BM_i - \sum_{i=1}^{d-1} \underline{M}_i \cdot BM_i = \sum_{i=1}^{d} \underline{M}_i \cdot BM_i - \sum_{i=1}^{d-1} M_i \cdot BM_i,$$

where we used that $\underline{M}_d = M_d (= Z)$. But,

$$\sum_{i=1}^{d} \underline{M}_i \cdot BM_i = \sum_{i=1}^{d} (\underline{M}^T BM)^i_i = \operatorname{tr}(\underline{M}^T BM) = \operatorname{tr}(M\underline{M}^T B) = d \det M \operatorname{tr} B,$$

since $\operatorname{tr}(DE) = \operatorname{tr}(ED)$ for any D, E in $M_{d\times d}(\mathbb{R})$. So,

$$M_d \cdot BM_d = d \det M \operatorname{tr} B - \sum_{i=1}^{d-1} M_i \cdot BM_i,$$

We conclude that

$$D = D_1 + d \det M \operatorname{tr} BD_2,$$

where

$$D_1 := \begin{pmatrix} M_1 \cdot BM_1 & \cdots & M_d \cdot BM_1 \\ & \vdots & \\ M_{d-1} \cdot BM_1 & \cdots & M_d \cdot BM_{d-1} \\ M_d \cdot BM_1 & \cdots & -\sum_{i=1}^{d-1} M_i \cdot BM_i \end{pmatrix}, \quad D_2 := \begin{pmatrix} 0 & \cdots & 0 \\ & \vdots & \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{pmatrix}.$$

For any $E, F \in M_{d\times d}(\mathbb{R})$,

$$(ED_2F)^i_j = E^i_k (D_2)^k_\ell F^\ell_j = E^i_d F^d_j = E_d \cdot F^d,$$

we see that

$$\frac{M}{(\det M)^2} (d \det M \operatorname{tr} BD_2) \underline{M}^T = d \frac{M_d (\underline{M}^T)^d}{\det M} \operatorname{tr} B = d \frac{|\underline{M}_d|^2}{\det M} \operatorname{tr} B = d \operatorname{tr} B,$$

where we used that $|\underline{M}_d|^2 = |Z|^2 = \det M$.

This gives the form of the bound on $|B|$ in (5.1). Noting that $\underline{M}_d$ is of degree $d-1$ while M_j is of degree 1 for $j < d$, we see that the highest-order terms in $P(M)$ are of degree $3d-2$. This completes the proof. □

6. Equivalence of striated vorticity and velocity

To prove Theorem 1.3, we first show that $\nabla u \in L^\infty$. This can be done via a direct calculation, simple in 2D, but substantially more involved in higher dimensions. The idea behind this bound is that ∇u is bounded by assumption in the $d-1$ directions determined at any point by elements of $\mathcal{Y}$, while the divergence-free condition on u along with the boundedness of Ω are sufficient to control ∇u in L^∞ in the remaining direction.

The proof we give, however, will rely instead on Lemma 5.1. This will allow us to obtain the bound on $\nabla u \in L^\infty$ very easily in a manner that works for all dimensions 2 and higher. (We will use Lemma 5.1 again in the proofs of Theorems 1.5 and 1.6.)

Remark 6.1. *Observe that $Y \cdot \nabla u = \nabla u Y$. We write $Y \cdot \nabla u$ when we wish to emphasize the role of $Y \cdot \nabla$ as a directional derivative (as we do in all sections but this one). We write $\nabla u Y$ when primarily performing linear algebra manipulations.*

Proposition 6.2. *Assume that $\mathcal{Y} \cdot \nabla u \in L^\infty(\mathbb{R}^d)$, $\Omega \in L^\infty(\mathbb{R}^d)$, and $\mathcal{Y} \in L^\infty(\mathbb{R}^d)$. Then $\nabla u \in L^\infty$.*

Proof. Fix $x \in \mathbb{R}^d$ and let $Y_1, \dots, Y_{d-1} \in \mathcal{Y}$ have lengths of at least $I(\mathcal{Y})$ and be such that $|\wedge_{i<d} Y_i| \geq I(\mathcal{Y})$ as well. This is always possible by the definition of $I(\mathcal{Y})$.

Now,

$$|\nabla u(x)| \leq \frac{1}{2} |B| + \frac{1}{2} \|\Omega(u)\|_{L^\infty},$$

where $B = \nabla u(x) + (\nabla u(x))^T$. Since B is symmetric, we can apply Lemma 5.1 to bound it.

Since $\operatorname{tr} B = 2 \operatorname{div} u = 0$, Lemma 5.1 gives

$$|B| \leq \frac{P(Y_1, \dots, Y_{d-1})}{|\wedge_{i<d} Y_i|^4} \sum_{i=1}^{d-1} |BY_i| \leq C \frac{\|\mathcal{Y}\|_{L^\infty(\mathbb{R}^d)}^{3d-2}}{I(\mathcal{Y})^4} \sum_{i=1}^{d-1} |BY_i| \leq C(\mathcal{Y}) \sum_{i=1}^{d-1} |BY_i|.$$

But, writing $B = 2\nabla u - (\nabla u - (\nabla u)^T) = 2\nabla u - \Omega(u)$, we see that

$$|BY_i| \leq 2 \|\mathcal{Y} \cdot \nabla u\|_{L^\infty(\mathbb{R}^d)} + \|\Omega(u)\|_{L^\infty(\mathbb{R}^d)} \|\mathcal{Y}\|_{L^\infty(\mathbb{R}^d)},$$

which completes the proof. □

The forward implications in Theorem 1.3 follow from Lemma 4.6 of [7] combined with Proposition 6.2, which in turn relies upon a key paraproduct estimate of Danchin's in Section 2 of [6] . We give a self-contained, elementary proof below that uses, however, the additional assumption in $d \geq 3$ that $\nabla \mathcal{Y} \in L^\infty(\mathbb{R}^d)$. (Because we apply Theorem 1.3 only at the initial time, the restriction that $\nabla \mathcal{Y} \in L^\infty$ would need only be imposed on the initial data—the pushforward of the sufficient family need not be Lipschitz, nor should we expect it to be.)

Proof of Theorem 1.3. That $\operatorname{div}(\omega \mathcal{Y}) \in C^{\alpha-1} \implies \mathcal{Y} \cdot \nabla u \in C^\alpha$ in 2D and that $\operatorname{div}(\Omega^j_k \mathcal{Y}) \in C^{\alpha-1} \, \forall j, k \implies \mathcal{Y} \cdot \nabla u \in C^\alpha$ in higher dimensions follow by applying Remark 1.7 at $t = 0$. It remains to prove the forward implications in (1.18).

So assume that $\mathcal{Y} \cdot \nabla u \in C^\alpha$, imposing the additional assumption that $\nabla \mathcal{Y} \in L^\infty(\mathbb{R}^d)$, as explained above.

If $d = 2$ then $\nabla u \in L^\infty$ by Proposition 6.2, and $\operatorname{div}(\omega \mathcal{Y}) \in C^{\alpha-1}$ follows immediately from Corollary 4.3 and Proposition 4.5.

Now assume that $d \geq 3$ and that $\mathcal{Y}$ is Lipschitz. We have, for any i, k,

$$\partial_k (Y \cdot \nabla u)^i - \partial_i (Y \cdot \nabla u)^k \in C^{\alpha-1}$$

by Lemma 6.4, below. But,

$$\begin{aligned}
\partial_k (Y \cdot \nabla u)^i - \partial_i (Y \cdot \nabla u)^k &= \partial_k (Y^j \partial_j u^i) - \partial_i (Y^j \partial_j u^k) \\
&= Y^j \partial_j (\partial_k u^i - \partial_i u^k) + \partial_k Y^j \partial_j u^i - \partial_i Y^j \partial_j u^k \\
&= Y \cdot \nabla \Omega^i_k + \left[\nabla Y (\nabla u)^T - \nabla u (\nabla Y)^T \right]^i_k.
\end{aligned}$$

Fix $p \in (1, \infty)$. Then $\nabla u \in L^p \cap L^\infty$ for any $p \in (1, 2)$, because of Proposition 6.2 and because $\|\nabla u\|_{L^p} \leq C(p) \|\Omega\|_{L^p}$ (a form of the Calderon-Zygmun inequality). Hence, $\left[\nabla Y (\nabla u)^T - \nabla u (\nabla Y)^T \right] \in L^p \cap L^\infty \subseteq C^{\alpha-1}$, so that then $Y \cdot \nabla \Omega^i_k \in C^{\alpha-1}$ for all i, k. But,

$$Y \cdot \nabla \Omega^i_k = \operatorname{div}(Y \Omega^i_k) - (\operatorname{div} Y) \Omega^i_k$$

and $\operatorname{div}(Y)\Omega^i_k \in L^1 \cap L^\infty \subseteq C^{\alpha-1}$. Hence, $\operatorname{div}(Y\Omega^i_k) \in C^{\alpha-1}$. □

Remark 6.3. *It follows from the proof of Theorem 1.3, Lemma 4.6 of [7], and Proposition 6.2 that if $\mathcal{Y} \cdot \nabla u \in C^\alpha$ then it must be that $\nabla Y(\nabla u)^T - \nabla u(\nabla Y)^T \in C^{\alpha-1}$.*

We used the following simple lemma above:

Lemma 6.4. *If $f \in C^\alpha$ then $\partial_j f \in C^{\alpha-1}$ with $\|\partial_j f\|_{C^{\alpha-1}} \le \|f\|_{C^\alpha}$.*

Proof. We have $\partial_j f = \operatorname{div}(f\boldsymbol{e}_j)$, where $f\boldsymbol{e}_j \in C^\alpha$. □

7. Higher regularity of corrected velocity gradient in 2D

To obtain Theorem 1.2, we need to construct a partition of unity associated to the sufficient family of C^α vector fields, $\mathcal{Y}$, as in the following proposition:

Proposition 7.1. *Let $\mathcal{Y} = (Y^{(\lambda)})_{\lambda\in\Lambda}$ be a sufficient family of C^α vector fields. There exists an $R > 0$, $M_0 = C(\mathcal{Y}, \alpha) > 0$, and a partition of unity, $(\varphi_n)_{n\in\mathbb{N}}$, with the property that for all $n \in \mathbb{N}$,*

$$\begin{aligned}
&\|\varphi_n\|_{C^\alpha} \le M_0,\\
&\exists\, Y \in \mathcal{Y} \text{ such that } |Y| > I(\mathcal{Y})/2 \text{ on } \operatorname{supp}\varphi_n,\\
&\#\{k \in \mathbb{N} \colon \operatorname{supp}\varphi_n \cap \operatorname{supp}\varphi_k \ne \emptyset\} \le 2.
\end{aligned} \tag{7.1}$$

Proof. Because $\mathcal{Y}$ is C^α, there is a modulus of continuity that applies uniformly to all elements of $\mathcal{Y}$. It follows that there exists some $R > 0$ such that for any $x \in \mathbb{R}^2$ there exists some $Y \in \mathcal{Y}$ such that $|Y| > I(\mathcal{Y})/2$ on $B_R(x)$.

Now let $f \in C_0^\infty((0,1))$ taking values in $[0,1]$ with $f \equiv 1$ on $(1/2, 3/4)$. Then extend f to be periodic on all of $\mathbb{R}$. For any $i, j \in \mathbb{Z}$ define $f_{ij}, g_{ij} \in C_0^\infty(\mathbb{R}^2)$ by

$$\begin{aligned}
&f_{ij}(x_1, x_2) = f(x_1)f(x_2) \text{ on } [i, i+1] \times [j, j+1],\\
&g_{ij}(x_1, x_2) = 1 - f(x_1)f(x_2) \text{ on } [i + \frac{1}{2}, i + \frac{3}{2}] \times [j + \frac{1}{2}, j + \frac{3}{2}],\\
&f_{ij}, g_{ij} = 0 \text{ elsewhere in } \mathbb{R}^2.
\end{aligned}$$

Let $(\varphi_n)_{n\in\mathbb{N}}$ consist of the collection of all the $f_{ij}(\cdot/R)$ and $g_{ij}(\cdot/R)$ functions indexed in an arbitrary manner. It is easy to see that all the properties in (7.1) hold. □

From Proposition 7.1, with (1.15) and (2.5), Lemmas 7.2 and 7.3 follow easily. (Note that $(7.1)_3$ is critical to obtaining these bounds, though the 2 could be any finite number.)

Lemma 7.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions with $f_n \in C^\alpha(\operatorname{supp}(\varphi_n \circ \eta^{-1}))$ for all n. Then*

$$\left\|\sum_{n\in\mathbb{N}} \varphi_n(\eta^{-1}) f_n\right\|_{C^\alpha(\mathbb{R}^2)} \le C e^{e^{C(\omega_0, \mathcal{Y}_0)t}} \sup_{n\in\mathbb{N}} \|f_n\|_{C^\alpha(\operatorname{supp}(\varphi_n\circ\eta^{-1}))}.$$

Lemma 7.3. *Assume that $\varphi \in C_C^\infty(\mathbb{R}^2)$ takes values in $[0,1]$ and let $f \in C^\alpha(\mathbb{R}^2)$. Then*

$$\|\varphi f\|_{C^\alpha(\mathbb{R}^2)} \le \|f\|_{L^\infty(\operatorname{supp}\varphi)} + \|\varphi\|_{\dot{C}^\alpha} \|f\|_{C^\alpha(\operatorname{supp}\varphi)}.$$

We now have the machinery we need to prove Theorem 1.2 in 2D.

***Proof of Theorem 1.2** in 2D.* For any $n \in \mathbb{Z}$ let $Y_n^0 \in \mathcal{Y}_0$ be such that $\left|Y_n^0\right| > I(\mathcal{Y})/2$ on $\operatorname{supp} \varphi_n$, and let Y_n be the pushforward of Y_n^0 under the flow map, η. Define for all $t \geq 0$,

$$A_n := \frac{1}{|Y_n|^2} \begin{pmatrix} Y_n^1 Y_n^2 & -(Y_n^1)^2 \\ (Y_n^2)^2 & -Y_n^1 Y_n^2 \end{pmatrix}, \qquad A := \sum_n \varphi_n(\eta^{-1}) A_n, \tag{7.2}$$

setting $A_n = 0$ outside of $\operatorname{supp} \varphi_n$. A simple calculation shows that

$$A_n Y_n = 0, \quad A_n Y_n^\perp = -Y_n. \tag{7.3}$$

Let $V_n = \operatorname{supp} \varphi_n(\eta^{-1})$ and note that $|Y_n(t)| > I(\mathcal{Y}(t))/2$ on V_n for all n. Using (2.5),

$$\|A_n(t)\|_{C^\alpha(V_n)} \leq \|Y_n(t)\|^4_{C^\alpha(V_n)} / I(\mathcal{Y}(t))^2.$$

The bound on $\|A\|_{C^\alpha}$ in (1.17) follows, then, from Lemmas 7.2 and 7.3, (1.11), and (1.16).

By (7.3), $(\nabla u - \omega A_n) Y_n = \nabla u Y_n \in C^\alpha(V_n)$ with norm bounded uniformly over n by Theorem 1.1. Also,

$$(\nabla u - \omega A_n) Y_n^\perp = \nabla u Y_n^\perp + \omega Y_n \in C^\alpha(V_n)$$

with norm bounded uniformly over n by Corollary 4.6 and Theorem 1.1. Since in the (orthogonal) basis, $\{Y_n, Y_n^\perp\}$, the matrix $\nabla u - \omega A_n$ is

$$\begin{pmatrix} (\nabla u - \omega A_n) Y_n \\ (\nabla u - \omega A_n) Y_n^\perp \end{pmatrix}^T,$$

and $Y_n \in C^\alpha$ with $\|Y_n\|_{C^\alpha(V_n)}$ uniformly bounded, it follows that $\nabla u - \omega A_n \in C^\alpha(V_n)$ with norm bounded uniformly over n. Hence, $\nabla u - \omega A \in C^\alpha$ with the bound in (1.17). □

8. Higher regularity of corrected velocity gradient in 3D

As in Section 7, we need a partition of unity, as provided by Proposition 8.1, the 3D analog of Proposition 7.1.

Proposition 8.1. *Let $\mathcal{Y} = (Y^{(\lambda)})_{\lambda \in \Lambda}$ be a 3D sufficient family of C^α vector fields. There exists an $R > 0$, $M_0 = C(Y, \alpha) > 0$, and a partition of unity, $(\varphi_n)_{n \in \mathbb{N}}$, with the property that for all $n \in \mathbb{N}$,*

$$\begin{aligned} &\|\varphi_n\|_{C^\alpha} \leq M_0, \\ &\exists\, Y_1, Y_2 \in \mathcal{Y} \text{ such that } |Y_1|, |Y_2|, |Y_1 \times Y_2| > I(\mathcal{Y})/2 \text{ on } \operatorname{supp} \varphi_n, \\ &\#\{k \in \mathbb{N} \colon \operatorname{supp} \varphi_n \cap \operatorname{supp} \varphi_k \neq \emptyset\} \leq 2. \end{aligned}$$

Proof. A minor variant of that of Proposition 7.1. □

For the remainder of this section, we give only the local argument, dealing with one pair of vector fields Y_1, $Y_2 \in \mathcal{Y}$ satisfying $|Y_1|, |Y_2|, |Y_1 \times Y_2| \geq I(\mathcal{Y})/2$ on some open set, $U = \operatorname{supp} \varphi_k$. This yields locally a matrix field which we will call, A. Piecing these matrices together to form a single matrix field is done just as in Section 7, so we suppress the details.

Gram-Schmidt orthonormalization yields a C^α map, $\mathcal{G}$, from $\{Y_1, Y_2\}$ to $\{Y_1', Y_2'\}$ that makes $\{Y_1', Y_2', Y_1' \times Y_2'\}$ an orthonormal frame on U in the standard orientation and is such that $\|\mathcal{G}\|_{C^\alpha} \leq C \|\mathcal{Y}\|_{C^\alpha}$. We suppress this map and simply relabel Y_1', Y_2' as Y_1, Y_2, so that $\{Y_1, Y_2, Y_1 \times Y_2\}$ is an orthonormal frame.

We can decompose $\vec{\omega}$ using our orthonormal frame as

$$\vec{\omega} = a_1 Y_1 + a_2 Y_2 + a_3 Y_1 \times Y_2, \tag{8.1}$$

where each a_j is a function of space.

Proposition 8.2. *Writing $\vec{\omega}$ as (8.1), we have $a_3 \in C^\alpha$ with $\|a_3\|_{C^\alpha} \le 2\|Y\|_{C^\alpha}\|Y\cdot\nabla u\|_{C^\alpha}$.*

Proof. Fix a point $x \in U$ and let $Z_1 = Y_1(x)$, $Z_2 = Y_2(x)$. (In effect, we are freezing the frame as give at the point, x, and calculating the third component of the curl in an orthonormal frame in the standard orientation.) Since $\{Z_1, Z_2, Z_1 \times Z_2\}$ is orthonormal, we have,

$$\begin{aligned}
a_3(x) &= \partial_{Z_1}(u\cdot Z_2) - \partial_{Z_2}(u\cdot Z_1) = Z_1\cdot\nabla(u\cdot Z_2) - Z_2\cdot\nabla(u\cdot Z_1)\\
&= (Z_1\cdot\nabla u)\cdot Z_2 - (Z_2\cdot\nabla u)\cdot Z_1 + (Z_1\cdot\nabla Z_2 - Z_2\cdot\nabla Z_1)\cdot u\\
&= (Y_1(x)\cdot\nabla u)\cdot Y_2(x) - (Y_2(x)\cdot\nabla u)\cdot Y_1(x).
\end{aligned}$$

The last equality holds because Z_1, Z_2 are constant throughout space. We conclude that

$$a_3 = (Y_1\cdot\nabla u)\cdot Y_2 - (Y_2\cdot\nabla u)\cdot Y_1.$$

But, $(Y_1\cdot\nabla u)\cdot Y_2 \in C^\alpha$ since $Y_1\cdot\nabla u \in C^\alpha$, $Y_2 \in C^\alpha$ by assumption and C^α is an algebra. Similarly, $(Y_2\cdot\nabla u)\cdot Y_1 \in C^\alpha$. Hence, $a_3 \in C^\alpha$. □

To determine what form the matrix A might take, let us return for a moment to the 2D result of Section 7. There, we found that the irregularities in the velocity gradient could be corrected by subtracting from it a matrix-multiple of the scalar vorticity; that is, $\nabla u - \omega A \in C^\alpha$, where $A \in C^\alpha$ is given by (7.2). There is no correction in the tangential direction, since $\omega AY = 0$, and a correction tangential to the boundary in the normal direction. Also, $\omega AY^\perp = -\omega Y$, so the discontinuity in ∇u in the normal direction is in the tangential direction.

To extend this result to 3D, it will be more convenient to use (mostly) the vorticity in the form of an antisymmetric matrix as opposed to a three-vector. Toward this end, observe that in 2D, a simple calculation shows that

$$\omega A = \sum_n \frac{\varphi_n(\eta^{-1})}{|Y_n|^2}\begin{pmatrix}(Y_n^1)^2 & Y_n^1Y_n^2\\ Y_n^2Y_n^1 & (Y_n^2)^2\end{pmatrix}\Omega = \left[\sum_n \frac{\varphi_n(\eta^{-1})}{|Y_n|^2} Y_n\otimes Y_n\right]\Omega. \tag{8.2}$$

So if we had instead defined A to be equal to the expression in brackets on the right-hand side we would have expressed our result in the form $A\Omega$ rather than ωA, and this form makes sense in any number of dimensions.

The analog of the relations $\omega AY = 0$, $\omega AY^\perp = -\omega Y$ in 3D are that

$$\begin{aligned}
&A\Omega(Y_1\times Y_2) = \Omega(Y_1\times Y_2),\\
&AP_{\mathrm{span}\{Y_1,Y_2\}}\Omega Y_1 = a_3 Y_2,\\
&AP_{\mathrm{span}\{Y_1,Y_2\}}\Omega Y_2 = -a_3 Y_1,
\end{aligned} \tag{8.3}$$

where P_V is projection into the subspace V. We derive such a matrix A in Proposition 8.4, below, but first we show in Proposition 8.3 that (8.3) gives, in fact, the required properties.

To prove (8.3), we will find it useful to have a way to translate between the three-vector and antisymmetric forms of the vorticity by defining, for any three-vector, $\varphi = \langle\varphi^1, \varphi^2, \varphi^3\rangle$,

$$Q(\varphi) = \begin{pmatrix}0 & -\varphi^3 & \varphi^2\\ \varphi^3 & 0 & -\varphi^1\\ -\varphi^2 & \varphi^1 & 0\end{pmatrix}.$$

Then Q is a bijection from the space of 3-vectors to the space of antisymmetric 3×3 matrices. A direct calculation shows that

$$Q(\varphi)v = \varphi\times v \tag{8.4}$$

for any three-vectors, φ, v.

If $V \subseteq \mathbb{R}^3$ is a subspace, we define

$$P_V \Omega := Q(\mathrm{proj}_V \vec{\omega}).$$

Proposition 8.3. *Suppose that $A \in C^\alpha$ satisfies (8.3). Then*

$$\nabla u - A\Omega \in C^\alpha.$$

Proof. Let $V = \mathrm{span}\{Y_1, Y_2\}$ so that $V^\perp = \mathrm{span}\{Y_1 \times Y_2\}$. Then,

$$\begin{aligned}(\nabla u - A\Omega)Y_1 &= (\nabla u - AP_V\Omega)Y_1 - AP_{V^\perp}\Omega Y_1 = \nabla u Y_1 - a_3 Y_2 - AP_{V^\perp}\Omega Y_1,\\ (\nabla u - A\Omega)Y_2 &= (\nabla u - AP_V\Omega)Y_2 - AP_{V^\perp}\Omega Y_2 = \nabla u Y_2 + a_3 Y_1 - AP_{V^\perp}\Omega Y_2\end{aligned}$$

by (8.3). But $\nabla u Y_j - AP_{V^\perp}\Omega Y_j \in C^\alpha$ since $\nabla u Y_j, A, Y_j \in C^\alpha$ by assumption and $a_3 \in C^\alpha$ and $P_{V^\perp}\Omega \in C^\alpha$ by Proposition 8.2.

Also,

$$(\nabla u - A\Omega)(Y_1 \times Y_2) = (\nabla u - \Omega)(Y_1 \times Y_2) = (\nabla u)^T(Y_1 \times Y_2) \in C^\alpha$$

by Lemma 8.7.

Because $(\nabla u - A\Omega)Y_1$, $(\nabla u - A\Omega)Y_2$, and $(\nabla u - A\Omega)(Y_1 \times Y_2)$ are C^α and the Gram-Schmidt orthonormalization map, $\mathcal{G}$, is C^α, it follows that $\nabla u - A\Omega \in C^\alpha$. □

Proposition 8.4. *Define the matrix A (locally) by*

$$A = A_1 + A_2, \quad A_j := Y_j \otimes Y_j. \tag{8.5}$$

Then $A \in C^\alpha$ and satisfies (8.3).

Remark 8.5. *This form of A only applies when $\{Y_1, Y_2, Y_1 \times Y_2\}$ form an orthonormal frame in the standard orientation. An expression for A in terms of more general Y_1, Y_2 would need to incorporate the map, $\mathcal{G}$—as (8.2) does for 2D.*

Proof of Proposition 8.4. What we must show is that A as given in (8.5) satisfies (8.3). For any vorticity, $\vec{\omega} = a_1 Y_1 + a_2 Y_2 + a_3 Y_1 \times Y_2$, we can write

$$\Omega = a_1\Omega_1 + a_2\Omega_2 + a_3\Omega_3,$$

where $\Omega_1 = Q(Y_1)$, $\Omega_2 = Q(Y_2)$, $\Omega_3 = Q(Y_1 \times Y_2)$. It follows immediately from (8.4) that

$$\Omega_1 Y_1 = \Omega_2 Y_2 = \Omega_3(Y_1 \times Y_2) = 0, \quad \Omega_1 Y_2 = -\Omega_2 Y_1. \tag{8.6}$$

We first prove $(8.3)_1$. Writing, $\vec{\omega} = a_1 Y_1 + a_2 Y_2 + a_3 Y_1 \times Y_2$, we have,

$$\Omega(Y_1 \times Y_2) = a_1\Omega_1(Y_1 \times Y_2) + a_2\Omega_2(Y_1 \times Y_2),$$

where the a_3 term disappeared by (8.6).

Now,

$$\begin{aligned}\Omega_1(Y_1 \times Y_2) &= \begin{pmatrix} 0 & Y_1^3 & -Y_1^2 \\ -Y_1^3 & 0 & Y_1^1 \\ Y_1^2 & -Y_1^1 & 0 \end{pmatrix} \begin{pmatrix} Y_1^2 Y_2^3 - Y_2^2 Y_1^3 \\ Y_2^1 Y_1^3 - Y_1^1 Y_2^3 \\ Y_1^1 Y_2^2 - Y_2^1 Y_1^2 \end{pmatrix}\\ &= \begin{pmatrix} (Y_1^3)^2 Y_2^1 - Y_1^3 Y_1^1 Y_2^3 - Y_1^2 Y_1^1 Y_2^2 + (Y_1^2)^2 Y_2^1 \\ -Y_1^3 Y_1^2 Y_2^3 + (Y_1^1)^2 Y_2^2 + (Y_1^1)^2 Y_2^2 - Y_1^1 Y_2^2 Y_1^2 \\ (Y_1^2)^2 Y_2^3 - Y_1^2 Y_2^2 Y_1^3 - Y_1^1 Y_2^1 Y_1^3 + (Y_1^1)^2 Y_2^3 \end{pmatrix}.\end{aligned}$$

Each of these components simplifies. We have

$$[\Omega_1(Y_1 \times Y_2)]^1 = \left[|Y_1|^2 - (Y_1^1)^2\right] Y_2^1 - Y_1^1(Y_1^3 Y_2^3 + Y_1^2 Y_2^2) = Y_2^1.$$

Similarly,

$$[\Omega_1(Y_1 \times Y_2)]^2 = Y_2^2, \quad [\Omega_1(Y_1 \times Y_2)]^3 = Y_2^3.$$

We conclude that

$$\Omega_1(Y_1 \times Y_2) = Y_2, \quad \Omega_2(Y_1 \times Y_2) = -Y_1, \tag{8.7}$$

the latter following from symmetry by transposing Y_1 and Y_2 and using $Y_2 \times Y_1 = -Y_1 \times Y_2$. Thus, by linearity,

$$\Omega(Y_1 \times Y_2) = a_1 Y_2 - a_2 Y_1, \quad A\Omega(Y_1 \times Y_2) = a_1 A Y_2 - a_2 A Y_1.$$

Noting that we can also write A_j in the form,

$$A_j = \begin{pmatrix} Y_j^1 \, Y_j \\ Y_j^2 \, Y_j \\ Y_j^3 \, Y_j \end{pmatrix}, \tag{8.8}$$

each row of A_j being a row vector, we see that

$$A_j Y_k = \begin{pmatrix} Y_j^1 Y_j \cdot Y_k \\ \\ Y_j^2 Y_j \cdot Y_k \\ \\ Y_j^3 Y_j \cdot Y_k \end{pmatrix} = (Y_j \cdot Y_k) Y_j.$$

Hence,

$$A_1 Y_1 = Y_1, \quad A_1 Y_2 = 0, \quad A_2 Y_1 = 0, \quad A_2 Y_2 = Y_2,$$

so that

$$AY_1 = Y_1, \quad AY_2 = Y_2, \tag{8.9}$$

and hence,

$$A\Omega(Y_1 \times Y_2) = a_1 Y_2 - a_2 Y_1 = \Omega(Y_1 \times Y_2). \tag{8.10}$$

This establishes $(8.3)_1$.

We next prove $(8.3)_2$ and $(8.3)_3$. By (8.6) and (8.7),

$$\begin{aligned} P_{\text{span}\{Y_1, Y_2\}} \Omega Y_1 &= P_{\text{span}\{Y_1, Y_2\}} \left(a_2 Y_2 \times Y_1 + a_3 Y_2\right) = a_3 Y_2, \\ P_{\text{span}\{Y_1, Y_2\}} \Omega Y_2 &= P_{\text{span}\{Y_1, Y_2\}} \left(a_1 Y_1 \times Y_2 - a_3 Y_1\right) = -a_3 Y_1. \end{aligned}$$

By (8.9), we obtain $(8.3)_2$ and $(8.3)_3$. □

Proof of Theorem 1.2 *in 3D.* The result follows locally in space from Propositions 8.3 and 8.4. To obtain the result in all space, we apply a partition of unity as in the 2D proof in Section 7. □

Remark 8.6. *This same approach could be used to prove the 2D result, though it would be longer than our approach in Section 7, which employed Corollary 4.6. The proof in this section, however, emphasizes that Theorem 1.2 is almost purely geometric in nature.*

We used the following lemma above:

Lemma 8.7. *For $d = 3$, $(\nabla u)^T(Y_1 \times Y_2) \in C^\alpha$ with*

$$\left\|(\nabla u)^T(Y_1 \times Y_2)\right\|_{C^\alpha} \leq \max_{j=1,2} \|Y_j \cdot \nabla u\|_{C^\alpha} \max_{j=1,2} \|Y_j\|_{C^\alpha}.$$

Proof. We have,

$$(\nabla u)^T(Y_1 \times Y_2) = \begin{pmatrix} \partial_1 u^1 & \partial_1 u^2 & \partial_1 u^3 \\ \partial_2 u^1 & \partial_2 u^2 & \partial_2 u^3 \\ \partial_3 u^1 & \partial_3 u^2 & \partial_3 u^3 \end{pmatrix} \begin{pmatrix} Y_1^2 Y_2^3 - Y_1^3 Y_2^2 \\ Y_1^3 Y_2^1 - Y_1^1 Y_2^3 \\ Y_1^1 Y_2^2 - Y_1^2 Y_2^1 \end{pmatrix}.$$

We will write out only first component in detail, the other two components being very similar. Multiplying, we have

$$\begin{aligned}
&[(\nabla u)^T(Y_1 \times Y_2)]^1 \\
&\quad = \partial_1 u^1 (Y_1^2 Y_2^3 - Y_1^3 Y_2^2) + \partial_1 u^2 (Y_1^3 Y_2^1 - Y_1^1 Y_2^3) + \partial_1 u^3 (Y_1^1 Y_2^2 - Y_1^2 Y_2^1) \\
&\quad = Y_2^1(\partial_1 u^2 Y_1^3 - \partial_1 u^3 Y_1^2) + Y_2^2(-\partial_1 u^1 Y_1^3 + \partial_1 u^3 Y_1^1) + Y_2^3(\partial_1 u^1 Y_1^2 - \partial_1 u^2 Y_1^1) \\
&\quad = Y_2^1(\partial_1 u^2 Y_1^3 - \partial_1 u^3 Y_1^2) + Y_2^2((\partial_2 u^2 + \partial_3 u^3) Y_1^3 + \partial_1 u^3 Y_1^1) \\
&\qquad\qquad + Y_2^3((-\partial_2 u^2 - \partial_3 u^3) Y_1^2 - \partial_1 u^2 Y_1^1) \\
&\quad = Y_2^1(\partial_1 u^2 Y_1^3 - \partial_1 u^3 Y_1^2) + Y_2^2((\partial_1 u^3 Y_1^1 + \partial_2 u^3 Y_1^2 + \partial_3 u^3 Y_1^3) - \partial_2 u^3 Y_1^2 + \partial_2 u^2 Y_1^3) \\
&\qquad\qquad + Y_2^3(-(\partial_1 u^2 Y_1^1 + \partial_2 u^2 Y_1^2 + \partial_3 u^2 Y_1^3) + \partial_2 u^2 Y_1^3 - \partial_3 u^3 Y_1^2) \\
&\quad = Y_2^1(\partial_1 u^2 Y_1^3 - \partial_1 u^3 Y_1^2) + Y_2^2(\nabla u^3 Y_1 - \partial_2 u^3 Y_1^2 + \partial_2 u^2 Y_1^3) \\
&\qquad\qquad + Y_2^3(-\nabla u^2 Y_1 + \partial_2 u^2 Y_1^3 - \partial_3 u^3 Y_1^2) \\
&\quad = Y_2^2 \nabla u^3 Y_1 - Y_2^3 \nabla u^2 Y_1 + \partial_1 u^2 Y_1^3 Y_2^1 - \partial_1 u^3 Y_1^2 Y_2^1 - \partial_2 u^3 Y_1^2 Y_2^2 + \partial_2 u^2 Y_1^3 Y_2^2 \\
&\qquad\qquad + \partial_2 u^2 Y_1^3 Y_2^3 - \partial_3 u^3 Y_1^2 Y_2^3 \\
&\quad = Y_2^2 \nabla u^3 Y_1 - Y_2^3 \nabla u^2 Y_1 + Y_1^3(\partial_1 u^2 Y_2^1 + \partial_2 u^2 Y_2^2 + \partial_2 u^2 Y_2^3) \\
&\qquad\qquad - Y_1^2(\partial_1 u^3 Y_2^1 + \partial_2 u^3 Y_2^2 + \partial_3 u^3 Y_2^3) \\
&\quad = Y_2^2 \nabla u^3 Y_1 - Y_2^3 \nabla u^2 Y_1 + Y_1^3 \nabla u^2 Y_2 - Y_1^2 \nabla u^3 Y_2 \\
&\quad = Y_2^2 (Y_1 \cdot \nabla u)^3 - Y_2^3 (Y_1 \cdot \nabla u)^2 + Y_1^3 (Y_2 \cdot \nabla u)^2 - Y_1^2 (Y_2 \cdot \nabla u)^3 \in C^\alpha.
\end{aligned}$$

□

Remark 8.8. *Theorem 1.2 has a clear extension to all dimensions $d \geq 2$. It is the computation of the analogous bound to that in Lemma 8.7 that complicates the general-dimensional proof.*

9. Approximate solutions and transport equations

Having established Theorem 1.3, Theorem 1.1 follows immediately from Theorems 1.5 and 1.6. We now, however, begin the presentation of a self-contained proof of Theorem 1.5 following Serfati's [22]. (We outline the changes to this proof needed to obtain Theorem 1.6 in Section 11.)

We start in this section with a mollification of the initial data so we can work with smooth solutions, and then discuss the various transport equations that enter into the proof.

We regularize the initial data by setting $u_{0,\varepsilon} = \rho_\varepsilon * u_0$, where ρ_ε is the standard mollifier of Definition 2.4, letting ε range over values in $(0, 1]$. It follows that $\omega_{0,\varepsilon} = \rho_\varepsilon * \omega_0$. Then there exists a smooth solution, $\omega_\varepsilon(t) \in C^\infty(\mathbb{R}^2)$, to the Euler equations in vorticity form, (1.3), (1.4), for all time with C^∞ velocity field, u_ε ([14, 24] or see Theorem 4.2.4 of [4]). These solutions converge to a weak solution $\omega(t)$ of (1.3), (1.4). (We say more about convergence in Section 10.5.)

The flow map, η_ε, is given in (1.6) with u_ε in place of u. Moreover, all the L^p-norms of ω_ε are conserved over time with

$$\|\omega_\varepsilon(t)\|_{L^p} = \|\omega_{\varepsilon,0}\|_{L^p} \le \|\omega_0\|_{L^p} \le \|\omega_0\|_{L^1\cap L^\infty} =: \|\omega_0\|_{L^1} + \|\omega_0\|_{L^\infty} \tag{9.1}$$

for all $p \in [1, \infty]$. Also,

$$\|u_\varepsilon(t)\|_{L^\infty} \le C \|\omega_0\|_{L^1\cap L^\infty} \tag{9.2}$$

(see Proposition 8.2 of [19]) so $\|u_\varepsilon\|_{L^\infty(\mathbb{R}\times\mathbb{R}^2)}$ is uniformly bounded in ε.

For most of the proof we will use these smooth solutions, passing to the limit as $\varepsilon \to 0$ in the final steps in Section 10.5.

Let $Y_0 \in C^\alpha$ with $\operatorname{div} Y_0 \in C^\alpha$. We let

$$Y_\varepsilon(t, \eta_\varepsilon(t, x)) = Y_0(x) \cdot \nabla \eta_\varepsilon(t, x) \tag{9.3}$$

be the pushforward of Y_0 under the flow map η_ε, as in (1.7). (Because Y_0 has all the regularity we need, it would be counterproductive to mollify it, as we do the initial data.) Similarly, we define the pushforward of the family $\mathcal{Y}_0$ of Theorem 1.1 as in (1.9), by

$$\mathcal{Y}_\varepsilon(t) = (Y_\varepsilon^{(\lambda)}(t))_{\lambda\in\Lambda}, \quad Y_\varepsilon^{(\lambda)}(t, \eta(t, x)) := (Y_0^{(\lambda)}(x) \cdot \nabla)\eta_\varepsilon(t, x). \tag{9.4}$$

(Note the slight notational collision between Y_ε and Y_0, $\mathcal{Y}_\varepsilon$ and $\mathcal{Y}_0$, and ω_ε and ω_0; this should not, however, cause any confusion.)

For the remainder of this section we focus on one element, $Y_0 \in \mathcal{Y}_0$.

Standard calculations show that

$$\partial_t Y_\varepsilon + u_\varepsilon \cdot \nabla Y_\varepsilon = Y_\varepsilon \cdot \nabla u_\varepsilon \tag{9.5}$$

and that

$$\begin{aligned} \partial_t \operatorname{div} Y_\varepsilon + u_\varepsilon \cdot \nabla \operatorname{div} Y_\varepsilon &= 0, \\ \partial_t \operatorname{div}(\omega_\varepsilon Y_\varepsilon) + u_\varepsilon \cdot \nabla \operatorname{div}(\omega_\varepsilon Y_\varepsilon) &= 0, \end{aligned} \tag{9.6}$$

the latter equality using that the vorticity is transported by the flow map. Hence,

$$\begin{aligned} \operatorname{div} Y_\varepsilon(t, x) &= \operatorname{div} Y_0(\eta_\varepsilon^{-1}(t, x)), \\ \operatorname{div}(\omega_\varepsilon Y_\varepsilon)(t, x) &= \operatorname{div}(\omega_{0,\varepsilon} Y_0)(\eta_\varepsilon^{-1}(t, x)). \end{aligned} \tag{9.7}$$

Remark 9.1. *Actually, the transport equations in (9.5) and (9.6), and others we will state later, are satisfied in a weak sense. We refer to Definition 3.13 of [1] for the notion of weak transport. With the exception of the use of Theorem 3.19 of [1] in the proof of Lemma 9.2, we will treat all transport equations as though they are satisfied in a strong sense, however, justifying such use in Appendix A.*

We can also write (9.5) and (9.6) as

$$\begin{aligned} &\frac{d}{dt} Y_\varepsilon(t, \eta_\varepsilon(t, x)) = (Y_\varepsilon \cdot \nabla u_\varepsilon)(t, \eta_\varepsilon(t, x)), \\ &\frac{d}{dt} \operatorname{div}(\omega_\varepsilon Y_\varepsilon)(t, \eta_\varepsilon(t, x)) = 0. \end{aligned} \tag{9.8}$$

Finally, we prove the propagation of regularity of $\operatorname{div}(\omega_\varepsilon Y_\varepsilon)$.

Lemma 9.2. *We have* $\operatorname{div}(\omega_\varepsilon Y_\varepsilon)(t) \in C^{\alpha-1}(\mathbb{R}^2)$ *with*

$$\|\operatorname{div}(\omega_\varepsilon Y_\varepsilon)(t)\|_{C^{\alpha-1}} \le C_\alpha \exp \int_0^t \|\nabla u_\varepsilon(s)\|_{L^\infty} \, ds.$$

Proof. We first obtain the following bound:

$$\|\operatorname{div}(\omega_\varepsilon Y_\varepsilon)(t)\|_{C^{\alpha-1}} \le C \|\operatorname{div}(\omega_{0,\varepsilon} Y_0)\|_{C^{\alpha-1}} \exp \int_0^t \|\nabla u_\varepsilon(s)\|_{L^\infty} \, ds.$$

The details of obtaining this bound is presented in the appendix (see Proposition A.3).

We must still, however, bound $\|\operatorname{div}(\omega_{0,\varepsilon} Y_0)\|_{C^{\alpha-1}}$ uniformly in ε. From the triangle inequality,

$$\|\operatorname{div}(\omega_{0,\varepsilon} Y_0)\|_{C^{\alpha-1}} \le \|\operatorname{div}(\omega_{0,\varepsilon} Y_0) - \rho_\varepsilon * \operatorname{div}(\omega_0 Y_0)\|_{C^{\alpha-1}} + \|\rho_\varepsilon * \operatorname{div}(\omega_{0,\varepsilon} Y_0)\|_{C^{\alpha-1}}.$$

Now,

$$\begin{aligned}
&\|\operatorname{div}(\omega_{0,\varepsilon} Y_0) - \rho_\varepsilon * \operatorname{div}(\omega_0 Y_0)\|_{C^{\alpha-1}} \le \|\omega_{0,\varepsilon} Y_0 - \rho_\varepsilon * (\omega_0 Y_0)\|_{C^\alpha} \\
&\quad = \|(\rho_\varepsilon * \omega_0) Y_0 - \rho_\varepsilon * (\omega_0 Y_0)\|_{C^\alpha} = \left\| \int_{\mathbb{R}^2} \rho_\varepsilon(x-y)\omega_0(y) \left[Y_0(x) - Y_0(y)\right] dy \right\|_{C^\alpha} \\
&\quad \le C(\omega_0, Y_0) \left(\alpha^{-1}(1-\alpha)^{-1}\right) = C_\alpha.
\end{aligned}$$

The first inequality followed from (2.4); the second followed from Lemma 3.2 with the kernel L_1 of Lemma 3.3. Also,

$$\begin{aligned}
\|\rho_\varepsilon * \operatorname{div}(\omega_0 Y_0)\|_{C^{\alpha-1}} &\le C \|\nabla \mathcal{F}_2 * (\rho_\varepsilon * \operatorname{div}(\omega_0 Y_0))\|_{C^\alpha} = C \|\rho_\varepsilon * (\nabla \mathcal{F}_2 * \operatorname{div}(\omega_0 Y_0))\|_{C^\alpha} \\
&\le C \|\nabla \mathcal{F}_2 * \operatorname{div}(\omega_0 Y_0)\|_{C^\alpha} \le C \left(\|\omega_0 Y_0\|_{L^1 \cap L^\infty} + \|\operatorname{div}(\omega_0 Y_0)\|_{C^{\alpha-1}}\right).
\end{aligned}$$

For the first inequality we applied Proposition 4.5, for the second inequality we used $\|\rho_\varepsilon * f\|_{C^\alpha} \le \|f\|_{C^\alpha}$, and for the third we applied Proposition 4.5 once more. Hence,

$$\|\operatorname{div}(\omega_{0,\varepsilon} Y_0)\|_{C^{\alpha-1}} \le C_\alpha + \left(\|\omega_0 Y_0\|_{L^1 \cap L^\infty} + \|\operatorname{div}(\omega_0 Y_0)\|_{C^{\alpha-1}}\right) \le C_\alpha.$$

□

Remark 9.3. *It is easy to see that the estimates in Lemma 9.2 apply equally well to the whole family,* $\mathcal{Y}$.

10. Propagation of striated regularity of vorticity in 2D

We first present the overall strategy of the proof of Theorem 1.5.

We start in Section 10.1 by bounding $\|\nabla u_\varepsilon(t)\|_{L^\infty}$ above by the quantity,

$$V_\varepsilon(t) := \|\omega_0\|_{L^\infty} + \left\| \text{p. v.} \int \nabla K(\cdot - y)\omega_\varepsilon(t, y) \, dy \right\|_{L^\infty}. \tag{10.1}$$

We also bound the gradients of the flow map and inverse flow map in terms of $V_\varepsilon(t)$. These estimates are entirely classical and do not involve $\mathcal{Y}_\varepsilon$.

In Section 10.2, we bound $\|Y_\varepsilon\|_{C^\alpha}$ in terms of $V_\varepsilon(t)$ and $\|K * \operatorname{div}(\omega_\varepsilon Y_\varepsilon)\|_{C^\alpha}$. This gives us a bound on $\|Y_\varepsilon\|_{C^\alpha}$ in terms of $V_\varepsilon(t)$ alone. We also develop a pointwise bound from below of $|Y_\varepsilon|(t, x)$ in terms of $V_\varepsilon(t)$.

In Section 10.3, we bound $V_\varepsilon(t)$ in terms of $\|Y_\varepsilon\|_{C^\alpha}$. Here, we make use of Lemma 5.1. We also need the pointwise bound from below of $|Y_\varepsilon|(t, x)$ developed in Section 10.2, for $|Y_\varepsilon|$ appears in the denominator in our estimates. The end result is a bound on $V_\varepsilon(t)$ in terms of itself that will allow us to close the estimates and so apply Gronwall's lemma to bound $V_\varepsilon(t)$.

The bound on $\|Y_\varepsilon\|_{C^\alpha}$ in terms of $V_\varepsilon(t)$ in Section 10.3 also involves $\|K * \operatorname{div}(\omega_\varepsilon Y_\varepsilon)\|_{C^\alpha}$, but this is bounded in terms of $V_\varepsilon(t)$ easily by Lemma 9.2 and Proposition 4.5. This, in turn, yields the bounds on all the other quantities, as in (1.10) through (1.16).

It remains, however, to show that the sequence of approximate solutions converge to a solution in a manner such that (1.10) through (1.16) hold. Such an argument is given in [4]; we restrict ourselves to describing in Section 10.5 the role that assuming $\operatorname{div} \mathcal{Y}_0 \in C^\alpha$ plays in the convergence argument, for this is a somewhat subtle point.

10.1. **Preliminary estimate of $\|\nabla u_\varepsilon(t)\|_{L^\infty}$, $\|\nabla \eta_\varepsilon(t)\|_{L^\infty}$, and $\left\|\nabla \eta_\varepsilon^{-1}(t)\right\|_{L^\infty}$.** By the expression for ∇u_ε in Proposition 4.1, and using (9.1), we have,

$$\|\nabla u_\varepsilon(t)\|_{L^\infty} \le V_\varepsilon(t).$$

As in (1.6), the defining equation for η_ε is

$$\partial_t \eta_\varepsilon(t,x) = u_\varepsilon(t, \eta_\varepsilon(t,x)), \quad \eta_\varepsilon(0,x) = x, \tag{10.2}$$

or, in integral form,

$$\eta_\varepsilon(t,x) = x + \int_0^t u_\varepsilon(s, \eta_\varepsilon(s,x))\, ds. \tag{10.3}$$

This immediately implies that

$$\|\nabla \eta_\varepsilon(t)\|_{L^\infty} \le \exp \int_0^t V_\varepsilon(s)\, ds. \tag{10.4}$$

Similarly,

$$\left\|\nabla \eta_\varepsilon^{-1}(t)\right\|_{L^\infty} \le \exp \int_0^t V_\varepsilon(s)\, ds. \tag{10.5}$$

The bound in (10.5) does not follow as immediately as that in (10.4) because the flow is not autonomous. For the details, see, for instance, the proof of Lemma 8.2 p. 318-319 of [19].

10.2. **Estimate of Y_ε.** Taking the inner product of $(9.8)_1$ with $Y_\varepsilon(t, \eta_\varepsilon(t,x))$ gives

$$\frac{d}{dt} Y_\varepsilon(t, \eta_\varepsilon(t,x)) \cdot Y_\varepsilon(t, \eta_\varepsilon(t,x)) = (Y_\varepsilon \cdot \nabla u_\varepsilon)(t, \eta_\varepsilon(t,x)) \cdot Y_\varepsilon(t, \eta_\varepsilon(t,x)).$$

The left-hand side equals

$$\frac{1}{2} \frac{d}{dt} \left| Y_\varepsilon(t, \eta_\varepsilon(t,x)) \right|^2$$

so

$$\begin{aligned} \left| \frac{d}{dt} \left| Y_\varepsilon(t, \eta_\varepsilon(t,x)) \right|^2 \right| &\le 2 \left\| \nabla u_\varepsilon(t, \eta_\varepsilon(t,\cdot)) \right\|_{L^\infty} \left| Y_\varepsilon(t, \eta_\varepsilon(t,x)) \right|^2 \\ &= 2 \left\| \nabla u_\varepsilon(t) \right\|_{L^\infty} \left| Y_\varepsilon(t, \eta_\varepsilon(t,x)) \right|^2 \le 2 V_\varepsilon(t) \left| Y_\varepsilon(t, \eta_\varepsilon(t,x)) \right|^2. \end{aligned}$$

It follows that

$$\frac{d}{dt} \left| Y_\varepsilon(t, \eta_\varepsilon(t,x)) \right|^2 \le 2 V_\varepsilon(t) \left| Y_\varepsilon(t, \eta_\varepsilon(t,x)) \right|^2.$$

Similarly,

$$\frac{d}{dt} \left| Y_\varepsilon(t, \eta_\varepsilon(t,x)) \right|^2 \ge -2 V_\varepsilon(t) \left| Y_\varepsilon(t, \eta_\varepsilon(t,x)) \right|^2.$$

Integrating in time and applying Lemma 2.6 gives

$$|Y_0(x)|\, e^{-\int_0^t \|\nabla u_\varepsilon(s)\|_{L^\infty}\, ds} \le |Y_\varepsilon(t, \eta_\varepsilon(t,x))| \le |Y_0(x)|\, e^{\int_0^t \|\nabla u_\varepsilon(s)\|_{L^\infty}\, ds}.$$

We conclude that

$$|Y_\varepsilon(t, \eta_\varepsilon(t,x))| \ge |Y_0(x)|\, e^{-\int_0^t V_\varepsilon(s)\, ds} \tag{10.6}$$

and taking the L^∞ norm in x that

$$\|Y_\varepsilon(t)\|_{L^\infty} \le \|Y_0\|_{L^\infty}\, e^{\int_0^t V_\varepsilon(s)\, ds}. \tag{10.7}$$

Integrating $(9.8)_1$ in time and substituting $\eta_\varepsilon^{-1}(t,x)$ for x yields

$$Y_\varepsilon(t,x) = Y_0(\eta_\varepsilon^{-1}(t,x)) + \int_0^t (Y_\varepsilon \cdot \nabla u_\varepsilon)(s, \eta_\varepsilon(s, \eta_\varepsilon^{-1}(t,x)))\, ds. \tag{10.8}$$

Taking the $\dot{C}^\alpha$ norm and applying $(2.5)_1$, we have

$$\|Y_\varepsilon(t)\|_{\dot{C}^\alpha} \le \|Y_0\|_{\dot{C}^\alpha} \left\|\nabla \eta_\varepsilon^{-1}(t)\right\|_{L^\infty}^\alpha + \int_0^t \|(Y_\varepsilon \cdot \nabla u_\varepsilon)(s)\|_{\dot{C}^\alpha} \left\|\nabla(\eta_\varepsilon(s, \eta_\varepsilon^{-1}(t,x)))\right\|_{L^\infty}^\alpha\, ds.$$

Now, by Corollary 4.3, we have

$$\begin{aligned} Y_\varepsilon \cdot \nabla u_\varepsilon(s,x) = \text{p. v.} \int \nabla K(x-y) \omega_\varepsilon(s,y)\, [Y_\varepsilon(s,x) - Y_\varepsilon(s,y)]\, dy \\ + K * \operatorname{div}(\omega_\varepsilon Y_\varepsilon)(s,x) =: \mathrm{I} + \mathrm{II} \end{aligned}$$

with

$$\|\mathrm{I}\|_{C^\alpha} \le C\, \|Y_\varepsilon(s)\|_{C^\alpha}\, V_\varepsilon(s).$$

By Proposition 4.5 and Lemma 9.2, we have

$$\|\mathrm{II}\|_{C^\alpha} \le C_\alpha \exp \int_0^s V_\varepsilon(\tau)\, d\tau.$$

It follows that

$$\|Y_\varepsilon \cdot \nabla u_\varepsilon(t)\|_{C^\alpha} \le \|Y_\varepsilon(t)\|_{C^\alpha}\, V_\varepsilon(t) + C_\alpha \exp \int_0^t V_\varepsilon(\tau)\, d\tau. \tag{10.9}$$

To estimate $\|\nabla(\eta_\varepsilon(s, \eta_\varepsilon^{-1}(t,x)))\|_{L^\infty}$, we start with

$$\partial_\tau \eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t,x)) = u_\varepsilon(\tau, \eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t,x))),$$

which follows from (10.2). Applying the spatial gradient and the chain rule gives

$$\partial_\tau \nabla \left(\eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t,x))\right) = \nabla u_\varepsilon(\tau, \eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t,x))) \nabla(\eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t,x))).$$

Integrating in time and using $\nabla(\eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t,x)))|_{\tau=t} = I^{2\times 2}$, the identity matrix, we have

$$\nabla \left(\eta_\varepsilon(s, \eta_\varepsilon^{-1}(t,x))\right) = I^{2\times 2} - \int_s^t \nabla u_\varepsilon(\tau, \eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t,x))) \nabla(\eta_\varepsilon(\tau, \eta_\varepsilon^{-1}(t,x)))\, d\tau.$$

By Lemma 2.6, then,

$$\left\|\nabla(\eta_\varepsilon(s, \eta_\varepsilon^{-1}(t,x)))\right\|_{L^\infty} \le \exp \int_s^t \|\nabla u_\varepsilon(\tau)\|_{L^\infty}\, d\tau \le \exp \int_s^t V_\varepsilon(\tau)\, d\tau.$$

These bounds with (10.5), and accounting for (10.7), give

$$\begin{aligned}\|Y_\varepsilon(t)\|_{C^\alpha} &\le \|Y_0\|_{C^\alpha} \exp\left(\alpha \int_0^t V_\varepsilon(s)\, ds\right) \\ &\qquad + \int_0^t \left[\|Y_\varepsilon(s)\|_{C^\alpha} V_\varepsilon(s) + C_\alpha \exp \int_0^s V_\varepsilon(\tau)\, d\tau\right] \exp\left(\alpha \int_s^t V_\varepsilon(\tau)\, d\tau\right) ds \\ &\le (\|Y_0\|_{C^\alpha} + C_\alpha t) \exp \int_0^t V_\varepsilon(s)\, ds + \int_0^t \|Y_\varepsilon(s)\|_{C^\alpha} V_\varepsilon(s) \left[\exp \int_s^t V_\varepsilon(\tau)\, d\tau\right] ds.\end{aligned}$$

Letting

$$y_\varepsilon(t) = \|Y_\varepsilon(t)\|_{C^\alpha} \exp\left[-\int_0^t V_\varepsilon(s)\, ds\right]$$

it follows that y_ε satisfies the inequality,

$$y_\varepsilon(t) \le \|Y_0\|_{C^\alpha} + C_\alpha t + \int_0^t V_\varepsilon(s) y_\varepsilon(s)\, ds.$$

Therefore, by Lemma 2.6, we obtain

$$y_\varepsilon(t) \le (\|Y_0\|_{C^\alpha} + C_\alpha t) \exp\left(\int_0^t V_\varepsilon(s)\, ds\right) \le C_\alpha(1+t) \exp\left(\int_0^t V_\varepsilon(s)\, ds\right)$$

and thus,

$$\|Y_\varepsilon(t)\|_{C^\alpha} \le C_\alpha(1+t) \exp\left(2\int_0^t V_\varepsilon(s)\, ds\right). \tag{10.10}$$

10.3. **Estimate of V_ε.** In Proposition 6.2 we bounded ∇u in L^∞ using a bound on $Y \cdot \nabla u$ in L^∞. Given (10.9) and (10.10), we could do the same now for bounding ∇u_ε in L^∞. The resulting bound, however, would too weak to close the estimates. We instead employ Lemma 5.1 to obtain a more refined estimate of ∇u_ε in L^∞.

Until the very end of this section, we will estimate quantities at a fixed point, $(t, x) \in (\mathbb{R} \times \mathbb{R}^2)$, though we will generally suppress these arguments for simplicity of notation.

We start by splitting the second term in V_ε in (10.1) into two parts, as

$$\begin{aligned}&\text{p. v.} \int \nabla K(x-y)\omega_\varepsilon(t,y)\, dy \\ &\quad = \text{p. v.} \int \nabla((a_r K))(x-y)\omega_\varepsilon(t,y)\, dy + \text{p. v.} \int \nabla((1-a_r)K)(x-y)\omega_\varepsilon(t,y)\, dy.\end{aligned} \tag{10.11}$$

where $r \in (0, 1]$ will be chosen later (in (10.19)).

On the support of $\nabla(1 - a_r) = -\nabla a_r$, $|x - y| \le 2r$, so

$$|\nabla((1-a_r)K)| \le |(1-a_r)\nabla K| + |\nabla a_r \otimes K| \le C\, |x-y|^{-2}. \tag{10.12}$$

Hence, one term in (10.11) is easily bounded by

$$\begin{aligned}&\left|\text{p. v.} \int \nabla((1-a_r)K)(x-y)\omega_\varepsilon(t,y)\, dy\right| \le C \int_{B_r^C(x)} |x-y|^{-2} |\omega_\varepsilon(t,y)|\, dy \\ &\quad \le C \int_r^1 \frac{\|\omega_\varepsilon\|_{L^\infty}}{\rho^2} \rho\, d\rho + C\| \, |x-\cdot|^{-2} \, \|_{L^\infty(B_1^C(x))} \|\omega_{\varepsilon,0}\|_{L^1} \\ &\quad \le -C \log r \, \|\omega_0\|_{L^\infty} + C\, \|\omega_0\|_{L^1} \le C(-\log r + 1)\, \|\omega_0\|_{L^1 \cap L^\infty}.\end{aligned} \tag{10.13}$$

For the other term in (10.11), choose any $Y_0 \in \mathcal{Y}_0$ such that

$$|Y_0| \left(\eta_\varepsilon^{-1}(t,x)\right) \geq I(\mathcal{Y}_0). \tag{10.14}$$

Letting μ_{rh} be as in Definition 2.2, by virtue of Proposition 4.7, we can write

$$\left| \text{p. v.} \int \nabla((a_r K))(x-y)\omega_\varepsilon(t,y)\, dy \right| = \left| \lim_{h\to 0} \nabla(\mu_{hr} K) * \omega_\varepsilon(t,x) \right| = \lim_{h\to 0} |B|,$$

where

$$B = B(t,x) := \nabla \left[\mu_{rh} \nabla \mathcal{F}_2\right] * \omega_\varepsilon.$$

Because $\nabla \left[\mu_{rh} \nabla \mathcal{F}_2\right]$ is not in L^1 uniformly in $h > 0$, we cannot estimate $|B|$ directly. Instead, we will apply Lemma 5.1 with $M_1 = Y_\varepsilon$ (so $|\wedge_{i<2}| = \left|Y_\varepsilon^\perp\right| = |Y_\varepsilon|$), giving

$$|B| \leq C \frac{P(Y_\varepsilon)}{|Y_\varepsilon|^4} |BY_\varepsilon| + 2 \left|\operatorname{tr} B\right|.$$

We now compute $\operatorname{tr} B$. We have,

$$\begin{aligned}
\operatorname{tr} B &= [\partial_1 \mu_{rh} \partial_1 \mathcal{F}_2] * \omega_\varepsilon + [\partial_2 \mu_{rh} \partial_2 \mathcal{F}_2] * \omega_\varepsilon + [\mu_{rh} \Delta \mathcal{F}_2] * \omega_\varepsilon \\
&= [\partial_1 \mu_{rh} \partial_1 \mathcal{F}_2] * \omega_\varepsilon + [\partial_2 \mu_{rh} \partial_2 \mathcal{F}_2] * \omega_\varepsilon,
\end{aligned}$$

using $\Delta \mathcal{F}_2 = \delta_0$ and $\mu_{rh}(0) = 0$ to remove the last term.

But, referring to Remark 2.3, for $j = 1, 2$, we have

$$\begin{aligned}
|[\partial_j \mu_{rh} \partial_j \mathcal{F}_2] * \omega_\varepsilon| &\leq \frac{C}{r} \int_{r<|x-y|<2r} \frac{|\omega_\varepsilon(t,y)|}{|x-y|}\, dy + \frac{C}{h} \int_{h<|x-y|<2h} \frac{|\omega_\varepsilon(t,y)|}{|x-y|}\, dy \\
&\leq \frac{C}{r} \int_r^{2r} \frac{\|\omega_\varepsilon(t)\|_{L^\infty}}{\rho} \rho\, d\rho + \frac{C}{h} \int_h^{2h} \frac{\|\omega_\varepsilon(t)\|_{L^\infty}}{\rho} \rho\, d\rho \\
&= C \left\|\omega_\varepsilon(t)\right\|_{L^\infty}
\end{aligned}$$

so that

$$\lim_{h\to 0} |\operatorname{tr} B| \leq C \left\|\omega_0\right\|_{L^\infty}.$$

We next estimate $|BY_\varepsilon|$. Because

$$B = \begin{pmatrix} \partial_1 \left[\mu_{rh} \partial_1 \mathcal{F}_2\right] * \omega_\varepsilon & \partial_2 \left[\mu_{rh} \partial_1 \mathcal{F}_2\right] * \omega_\varepsilon \\ \partial_1 \left[\mu_{rh} \partial_2 \mathcal{F}_2\right] * \omega_\varepsilon & \partial_2 \left[\mu_{rh} \partial_2 \mathcal{F}_2\right] * \omega_\varepsilon \end{pmatrix}$$

we have

$$BY_\varepsilon = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} := \begin{pmatrix} (\partial_1 \left[\mu_{rh} \partial_1 \mathcal{F}_2\right] * \omega_\varepsilon) Y_\varepsilon^1 + (\partial_2 \left[\mu_{rh} \partial_1 \mathcal{F}_2\right] * \omega_\varepsilon) Y_\varepsilon^2 \\ (\partial_1 \left[\mu_{rh} \partial_2 \mathcal{F}_2\right] * \omega_\varepsilon) Y_\varepsilon^1 + (\partial_2 \left[\mu_{rh} \partial_2 \mathcal{F}_2\right] * \omega_\varepsilon) Y_\varepsilon^2 \end{pmatrix}.$$

We now decompose F_1 and F_2 into two parts as $F_k = d_k + e_k$, where

$$\begin{aligned}
d_k &= \sum_{j=1}^{2} (\partial_j \left[\mu_{rh} \partial_k \mathcal{F}_2\right] * \omega_\varepsilon) Y_\varepsilon^j - \partial_j \left[\mu_{rh} \partial_k \mathcal{F}_2\right] * (\omega_\varepsilon Y_\varepsilon^j), \\
e_k &= \partial_1 \left[\mu_{rh} \partial_k \mathcal{F}_2\right] * (\omega_\varepsilon Y_\varepsilon^1) + \partial_2 \left[\mu_{rh} \partial_k \mathcal{F}_2\right] * (\omega_\varepsilon Y_\varepsilon^2) = \operatorname{div} \Big(\mu_{rh} \partial_k \mathcal{F}_2 * (\omega_\varepsilon Y_\varepsilon)\Big).
\end{aligned}$$

By Lemma 3.4 (noting that $\operatorname{div}(\omega_\varepsilon Y_\varepsilon) = \omega_\varepsilon \operatorname{div} Y_\varepsilon + Y_\varepsilon \cdot \nabla \omega_\varepsilon \in C^\alpha$),

$$\begin{aligned}\sum_{k=1,2} \left| \lim_{h\to 0} d_k \right| &\le 2 \left| \lim_{h\to 0} \int_{\mathbb{R}^2} \nabla \left[\mu_{rh} \nabla \mathcal{F}_2\right](x-y)(Y_\varepsilon(x) - Y_\varepsilon(y))\omega_\varepsilon(y)\, dy \right| \\ &\le C\alpha^{-1} \|Y_\varepsilon(t)\|_{C^\alpha} \|\omega_\varepsilon(t)\|_{L^\infty} r^\alpha \le C\alpha^{-1} \|Y_\varepsilon(t)\|_{C^\alpha} \|\omega_0\|_{L^\infty} r^\alpha\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1,2} \left| \lim_{h\to 0} e_k \right| &\le 2 \left| \lim_{h\to 0} \int_{\mathbb{R}^2} \left[\mu_{rh} \nabla \mathcal{F}_2\right](x-y) \operatorname{div}(\omega_\varepsilon Y_\varepsilon)(y)\, dy \right| \\ &\le C\alpha^{-1} \|\operatorname{div}(\omega_\varepsilon Y_\varepsilon)(t)\|_{C^{\alpha-1}} r^\alpha.\end{aligned} \tag{10.15}$$

Thus,

$$\lim_{h\to 0} |B| \le C\alpha^{-1} \frac{P(Y_\varepsilon)}{|Y_\varepsilon|^4} \left(\|Y_\varepsilon\|_{C^\alpha} \|\omega_0\|_{L^\infty} + \|\operatorname{div}(\omega_\varepsilon Y_\varepsilon)\|_{C^{\alpha-1}}\right) r^\alpha + C \|\omega_0\|_{L^\infty}. \tag{10.16}$$

Both sides of the inequality above are functions of t and x. By (10.6) and (10.14),

$$|Y_\varepsilon(t,x)| \ge I(\mathcal{Y}_0) e^{-\int_0^t V_\varepsilon(s)\, ds}.$$

From this, combined with (10.7), we conclude that

$$\begin{aligned}&\sup_{x\in\mathbb{R}^2} \lim_{h\to 0} |B(t,x)| \\ &\quad \le C\alpha^{-1} \|Y_0\|_{L^\infty} e^{a_0 \int_0^t V_\varepsilon(s)\, ds} \left(\|Y_\varepsilon\|_{C^\alpha} \|\omega_0\|_{L^\infty} + \|\operatorname{div}(\omega_\varepsilon Y_\varepsilon)\|_{C^{\alpha-1}}\right) r^\alpha + \|\omega_0\|_{L^\infty},\end{aligned} \tag{10.17}$$

where $a_0 = 8$, since P is of degree 4 by Lemma 5.1.

From the estimates in (10.10), (10.13), (10.17), and Lemma 9.2, which apply uniformly over all elements of $\mathcal{Y}_0$, we conclude that

$$\begin{aligned}V_\varepsilon(t) &\le C(1 - \log r) \|\omega_0\|_{L^1\cap L^\infty} + \sup_{Y_0\in\mathcal{Y}_0} \sup_{x\in\mathbb{R}^2} \lim_{h\to 0} |B(t,x)| \\ &\le C(\omega_0)(1 - \log r) + \frac{C_\alpha}{\alpha}(1+t) e^{(a_0+2)\int_0^t V_\varepsilon(s)\, ds} r^\alpha + C_\alpha,\end{aligned} \tag{10.18}$$

where C_α is defined in (2.1).

Remark 10.1. *Observe how, in contrast to the proof of Theorem 1.2 in Section 7, we had no need of a partition of unity when bounding ∇u, since the regularity of ∇u was not at issue, only a bound on the value of $|\nabla u(t,x)|$.*

10.4. **Closing the estimates using Gronwall's lemma.** Now choose

$$r = \exp\left(-C' \int_0^t V_\varepsilon(s)\, ds\right), \tag{10.19}$$

delaying the choice of C' for the moment. Then,

$$1 - \log r \le 1 + C' \int_0^t V_\varepsilon(s)\, ds, \quad r^\alpha \le \exp\left(-C'\alpha \int_0^t V_\varepsilon(s)\, ds\right).$$

Returning to (10.18), then, these bounds on $1 - \log r$ and r^α yield the estimate,

$$\begin{aligned}V_\varepsilon(t) &\le C(\omega_0) + C'C(\omega_0) \int_0^t V_\varepsilon(s)\, ds + \frac{C_\alpha}{\alpha}(1+t) \exp\left((a_0 + 2 - \alpha C') \int_0^t V_\varepsilon(s)\, ds\right) \\ &\le \frac{C_\alpha}{\alpha}(1+t) + \frac{C(\omega_0)}{\alpha} \int_0^t V_\varepsilon(s)\, ds\end{aligned}$$

as long as we set $C' = (a_0 + 2)/\alpha$

By Lemma 2.6, we conclude that

$$\|\nabla u_\varepsilon(t)\|_{L^\infty} \le V_\varepsilon(t) \le \frac{C_\alpha}{\alpha}(1+t)e^{C(\omega_0)\alpha^{-1}t}.$$

If $\alpha > 1/2$, we can apply the above bound with $1/2$ in place of α, eliminating the factor of $(1-\alpha)^{-1}$ that appear in C_α. This gives

$$\|\nabla u_\varepsilon(t)\|_{L^\infty} \le V_\varepsilon(t) \le \frac{c_\alpha}{\alpha}(1+t)e^{C(\omega_0)\alpha^{-1}t} \le \frac{c_\alpha}{\alpha}e^{C(\omega_0)\alpha^{-1}t}. \tag{10.20}$$

The final inequality is obtained by increasing the value of the constant in the exponent (in a manner that is independent of α.) We do this again, below.

Then

$$\int_0^t V_\varepsilon(s)\,ds < \frac{c_\alpha}{C(\omega_0)}e^{C(\omega_0)\alpha^{-1}t} = c_\alpha e^{C(\omega_0)\alpha^{-1}t}$$

so by virtue of (10.10),

$$\|\mathcal{Y}_\varepsilon(t)\|_{C^\alpha} \le C_\alpha \exp\left(c_\alpha e^{C(\omega_0)\alpha^{-1}t}\right). \tag{10.21}$$

It follows from (10.9) that

$$\|\mathcal{Y}_\varepsilon \cdot \nabla u_\varepsilon(t)\|_{C^\alpha} \le C_\alpha \alpha^{-1} \exp\left(c_\alpha e^{C(\omega_0)\alpha^{-1}t}\right).$$

This gives, once we take $\varepsilon \to 0$ in the next subsection, the estimates in (1.10), (1.11) and (1.14). Similarly, (1.13) follows from Lemma 9.2; (1.15) follows from (10.4) and (10.5); and (1.16) follows from (10.6).

Finally, (1.12) follows from $(2.5)_1$ applied to $(9.7)_1$. Here, though, we can absorb the constant $\alpha c_\alpha = C(\omega_0, \mathcal{Y}_0)$ into the exponent without introducing an additional dependence of the constants on α.

10.5. **Convergence of approximate solutions.** That the approximate solutions (u_ε) converge to the solution u for bounded initial vorticity is by now classical (see Section 8.2 of [19], for instance). It remains to show, however, that in the limit as $\varepsilon \to 0$, $Y_\varepsilon \to Y$ in such a way that all the estimates in (1.10) through (1.16) hold. This is done by Chemin on pages 105-106 of [4]; we highlight here, only the role that assuming $\operatorname{div} \mathcal{Y}_0 \in C^\alpha$ plays in the convergence argument.

Chemin first establishes that the sequence of flow maps (and inverse flow maps) converge in the sense that $\eta_\varepsilon - \eta \to 0$ in $L^\infty([0,T] \times \mathbb{R}^2)$ and, similarly, that $\eta_\varepsilon^{-1} - \eta^{-1} \to 0$ in $L^\infty([0,T] \times \mathbb{R}^2)$. Hence, by interpolation, $\eta_\varepsilon - \eta \to 0$ in $L^\infty(0,T; C^\beta(\mathbb{R}^2))$ for all $\beta < 1$ because $\eta_\varepsilon \in L^\infty(0,T; Lip(\mathbb{R}^d))$ uniformly in ε.

We can write (9.3) as

$$Y_0 \cdot \nabla \eta_\varepsilon = Y_\varepsilon \circ \eta_\varepsilon.$$

By $(2.5)_1$ and (10.21), then, $Y_0 \cdot \nabla \eta_\varepsilon$ is uniformly bounded in $L^\infty(0,T; C^\alpha(\mathbb{R}^2))$. But $C^\alpha(\mathbb{R}^2)$ is compactly embedded in $C^\beta(\mathbb{R}^2)$ for all $\beta < \alpha$ so a subsequence of $(Y_0 \cdot \nabla \eta_\varepsilon)$ converges in $L^\infty(0,T; C^\beta(\mathbb{R}^2))$ to some f for all $\beta < \alpha$, and it is easy to see that $f \in L^\infty(0,T; C^\alpha(\mathbb{R}^2))$.

To show that $f = Y_0 \cdot \nabla \eta$, we need only show convergence of $Y_0 \cdot \nabla \eta_\varepsilon \to Y_0 \cdot \nabla \eta$ in some weaker sense. To do this, observe that

$$(Y_0 \cdot \nabla \eta_\varepsilon)^j = Y_0 \cdot \nabla \eta_\varepsilon^j = \operatorname{div}(\eta_\varepsilon^j Y_0) - \eta_\varepsilon^j \operatorname{div} Y_0.$$

But $\eta_\varepsilon - \eta \to 0$ in $L^\infty(0,T;C^\beta(\mathbb{R}^2))$ for all $\beta < 1$ as we showed above so $\eta_\varepsilon^j Y_0 - \eta^j Y_0 \to 0$ in $L^\infty(0,T;C^\alpha(\mathbb{R}^2))$ and $\eta_\varepsilon^j \operatorname{div} Y_0 - \eta^j \operatorname{div} Y_0 \to 0$ in $L^\infty(0,T;C^\alpha(\mathbb{R}^2))$. (Here, we used $\operatorname{div} Y_0 \in C^\alpha$.) By the definition of negative Hölder spaces in Definition 2.1 it follows that $Y_0 \cdot \nabla\eta_\varepsilon \to Y_0 \cdot \nabla\eta$ in $L^\infty(0,T;C^{\alpha-1}(\mathbb{R}^2))$. Hence, $f = Y_0 \cdot \nabla\eta$, so we can conclude that $Y_0 \cdot \nabla\eta \in L^\infty(0,T;C^\alpha(\mathbb{R}^2))$ and $Y_0 \cdot \nabla\eta_\varepsilon \to Y_0 \cdot \nabla\eta$ in $L^\infty(0,T;C^\beta(\mathbb{R}^2))$ for all $\beta < \alpha$.

Then, since $Y_\varepsilon = (Y_0 \cdot \nabla\eta_\varepsilon) \circ \eta_\varepsilon^{-1}$ and $Y = (Y_0 \cdot \nabla\eta) \circ \eta^{-1}$ (see (1.7) and (9.3)), we have,

$$\begin{aligned}
\|Y_\varepsilon - Y\|_{L^\infty} &\le \left\|(Y_0 \cdot \nabla\eta_\varepsilon) \circ \eta_\varepsilon^{-1} - (Y_0 \cdot \nabla\eta_\varepsilon) \circ \eta^{-1}\right\|_{L^\infty} \\
&\qquad + \left\|(Y_0 \cdot \nabla\eta_\varepsilon) \circ \eta^{-1} - (Y_0 \cdot \nabla\eta) \circ \eta^{-1}\right\|_{L^\infty} \\
&\le \|Y_0 \cdot \nabla\eta_\varepsilon\|_{C^\alpha} \|\eta_\varepsilon^{-1} - \eta^{-1}\|_{L^\infty}^\alpha + \|Y_0 \cdot \nabla\eta_\varepsilon - Y_0 \cdot \nabla\eta\|_{L^\infty} \\
&\to 0 \text{ as } \varepsilon \to 0,
\end{aligned}$$

where we used $(2.5)_1$. Here the L^∞ norms are bounded over $[0,T] \times \mathbb{R}^2$ for any fixed $T > 0$. Arguing as for $Y_0 \cdot \nabla\eta$, it also follows that $Y \in L^\infty(0,T;C^\alpha(\mathbb{R}^2))$ and that the bound on $Y(t)$ in (1.11) holds. Then (1.15) follows from (1.10) as in (10.4) and (10.5).

Also,

$$(Y_\varepsilon \cdot \nabla u_\varepsilon)^j = \operatorname{div}(u_\varepsilon^j Y_\varepsilon) - u_\varepsilon^j \operatorname{div} Y_\varepsilon,$$

and given that we now know that $Y_\varepsilon \to Y$ in $C^\beta(\mathbb{R}^2)$ for all $\beta < \alpha$ with $Y \in C^\alpha(\mathbb{R}^2)$, (1.14) can be proved much the way we proved the convergence of $Y_0 \cdot \nabla\eta_\varepsilon \to Y_0 \cdot \nabla\eta$, above (taking advantage of (1.12), and again using $\operatorname{div} Y_0 \in C^\alpha$).

The proofs of the other bounds in (1.10) through (1.16), which we suppress, follow much the same course as the bounds above. This completes the proof of Theorem 1.5 by Serfati's approach.

11. Propagation of striated regularity of vorticity in higher dimensions

We outline the changes that are needed to the proof of Theorem 1.5 to obtain Theorem 1.6.

Section 9: The transport equations involving vorticity are dimension-dependent. Vorticity will remain in L^∞ only for short time because of vortex stretching, which will ultimately limit us to a short-time result. Also, we use the transport of $\operatorname{div}(\Omega_k^j Y)$ for all j, k, in place of $\operatorname{div}(\omega_\varepsilon Y_\varepsilon)$, though this also will apply only for short time. This is done as in [6].

Section 10.1: We define

$$V_\varepsilon(t) := \|\Omega_\varepsilon(t)\|_{L^\infty} + \max_{1\le i,j,k\le d} \left\| \text{p. v.} \int (\partial_i K_d^k)(x-y)\Omega_k^j(y)\, dy \right\|_{L^\infty} \tag{11.1}$$

to control $\|\nabla u_\varepsilon(t)\|_{L^\infty}$. (We suppress the ε subscript that should appear on Ω_k^j to avoid notational clutter; also notice that there is no sum over k.) The estimates of $\|\nabla\eta_\varepsilon(t)\|_{L^\infty}$ and $\left\|\nabla\eta_\varepsilon^{-1}(t)\right\|_{L^\infty}$ in (10.4) and (10.5) are unchanged.

Section 10.2: The estimates of $\|Y_\varepsilon\|_{L^\infty}$ in (10.7) and the bound from below on $|Y(t,x)|$ in (10.6) are unchanged. The bound on $\|Y_\varepsilon\|_{C^\alpha}$ is derived as in 2D, though now the vortex stretching term in (1.3) complicates matters. The resolution of this issue is involved, but is handled as in [8, 6, 7]). See, in particular, Sections 4.2.4 and 4.3 of [7], the vortex stretching term being bounded as in (47) of [7]. (Note that Fanelli is bounding, in effect, $Y_\varepsilon \cdot \nabla\Omega_k^j$ in $C^{\alpha-1}$ for all j, k rather than $\operatorname{div}(\Omega_k^j Y_\varepsilon)$, but the two are related by his Lemma 4.5.) This yields bounds of the form,

$$\|Y_\varepsilon(t)\|_{C^\alpha} \le C_\alpha(1 + F(t)) \exp\left(2 \int_0^t V_\varepsilon(s)\, ds\right). \tag{11.2}$$

Here, $F(t)$ is a factor, due to the vortex stretching term, that increases in time in a manner that ultimately prevents Gronwall's inequality from being applied globally in time. (See (49) of [7].)

Section 10.3: Fix t, x. Let $Y_0^{(1)}, \dots, Y_0^{(d-1)} \in \mathcal{Y}_0$ be such that

$$|Y_0^{(1)}(x)|, \dots, |Y_0^{(d-1)}(x)|, \quad \left|\wedge_{i<d} Y_0^{(i)}(x)\right| > I(\mathcal{Y}_0).$$

Let $Y_\varepsilon^{(1)}(t), \dots, Y_\varepsilon^{(d-1)}(t)$ be the pushforwards of $Y_0^{(1)}, \dots, Y_0^{(d-1)}$. Let $W_\varepsilon = \wedge_{i<d} Y^{(i)}$. From the proof of Proposition 4.1 of [6], we have

$$\partial_t W_\varepsilon + u \cdot \nabla W_\varepsilon = -(\nabla u)^T W_\varepsilon.$$

Examining the estimate that led to (10.6), we see that that argument works just as well for estimating W_ε from below. This gives

$$|W_\varepsilon(t, \eta_\varepsilon(t,x))| \ge |W_0(x)|\, e^{-\int_0^t V_\varepsilon(s)\, ds}.$$

The application of Lemma 5.1 in dimension $d \ge 3$ is little different from than for $d = 2$. We apply it using $M_1 = Y_\varepsilon^{(1)}, \dots, M_{d-1} = Y_\varepsilon^{(d-1)}$. Then the estimates in (11.2) allow us to bound $|BM_1|, \dots, |BM_{d-1}|$ just as we did $|BM_1|$ in 2D. The value of the constant a_0 in (10.17) becomes $4d+1$, because P_1 is of degree $4d-3$, but this does not affect the argument.

Section 10.4: The presence of $F(t)$ in (11.2) means that the bound on $V_\varepsilon(t)$ can only be closed for finite time.

Section 10.5: Unlike in 2D, where the existence of a unique solution is assured merely by ω_0 lying in $L^1 \cap L^\infty$ (by Yudovich [25]), existence has to be established using the sequence of approximate solutions. This can be done as in [8, 6, 7]. The proofs of the bounds in (1.10) through (1.16) are unchanged, however, once we have convergence of the flow map and inverse flow map.

Appendix A. On transport equation estimates

Together, Lemmas A.1 and A.2 justify our use of strong transport equations in obtaining estimates in the C^α-norm of the transported and pushed-forward quantities. First, the initial data is mollified using a mollification parameter δ independent of ε, the strong transport equation estimates are made, then δ is taken to zero. This is all while ε is held fixed. Lemma A.1 is used to obtain the C^α-bound on $\operatorname{div} Y_\varepsilon(t)$ (leading to (1.12)), while Lemma A.2 is used to obtain the C^α-bounds on the vector fields, $Y_\varepsilon(t)$ and $Y_\varepsilon \cdot \nabla u_\varepsilon(t)$.

The proofs of Lemmas A.1 and A.2, which are left to the reader, employ only $(2.5)_{1,2}$, the boundedness of $\nabla \eta_\varepsilon^{-1}(t)$ in L^∞ over time (for fixed ε), and the convergence in C^α of a mollified function to the function itself.

Lemma A.1. *For $f_0 \in C^\alpha$ and $\eta_\varepsilon^{-1} \in L^\infty(0, T; Lip(\mathbb{R}^d))$, let*

$$f(t,x) := f_0(\eta_\varepsilon^{-1}(t,x)),$$
$$f^{(\delta)}(t,x) = (\rho_\delta * f_0)(\eta_\varepsilon^{-1}(t,x))$$

for $\delta > 0$. *Then*

$$\|f^{(\delta)} - f\|_{L^\infty([0,T];C^\alpha)} \to 0 \text{ as } \delta \to 0.$$

Lemma A.2. *Let* Y_ε *be as in (9.3), so that*

$$Y_\varepsilon(t, \eta_\varepsilon(t,x)) = Y_0(x) \cdot \nabla \eta_\varepsilon(t,x).$$

Define $Y_\varepsilon^{(\delta)}$ *by*

$$Y_\varepsilon^{(\delta)}(t, \eta_\varepsilon(t,x)) = (\rho_\delta * Y_0)(x) \cdot \nabla \eta_\varepsilon(t,x).$$

Then

$$\|Y_\varepsilon^{(\delta)} - Y_\varepsilon\|_{L^\infty([0,T];C^\alpha)} \to 0 \text{ as } \delta \to 0.$$

Proposition A.3 provides the Hölder bound employed in the proof of Lemma 9.2.

Proposition A.3. *Let* $f_0 \in C^{\alpha-1}(\mathbb{R}^d)$, $d \ge 1$, $\alpha \in (0,1)$ *and suppose that* f *is* f_0 *transported by the flow map,* η*, for the divergence-free velocity field,* v*, which we assume is Lipschitz continuous with a Lipschitz constant that is uniform over the time interval* $[0,T]$. *Then*

$$\|f(t)\|_{C^{\alpha-1}} \le C e^{\int_0^t \|\nabla v(s)\|_{L^\infty}\, ds} \|f_0\|_{C^{\alpha-1}}.$$

Proof. We first note that $\eta, \eta^{-1} \in L^\infty(0,T; Lip(\mathbb{R}^d))$ and, moreover, that

$$e^{-\int_0^t \|\nabla v(s)\|_{L^\infty}\, ds} \le \|\nabla \eta(t)\|_{L^\infty}, \left\|\nabla \eta^{-1}(t)\right\|_{L^\infty} \le e^{\int_0^t \|\nabla v(s)\|_{L^\infty}\, ds}. \tag{A.1}$$

Lemma A.4, below, gives

$$\|f(t)\|_{C^{\alpha-1}} = \|f(t)\|_{B^{\alpha-1}_{\infty,\infty}} \le C \sup_{\phi \in Q^{1-\alpha}_{1,1}} \langle f(t), \phi \rangle,$$

where $Q^{1-\alpha}_{1,1}$ is the set of all Schwartz class functions lying in $B^{1-\alpha}_{1,1}$ having norm ≤ 1.

By the density of $C^\infty(\mathbb{R}^d)$ in $C^{\alpha-1}(\mathbb{R}^d)$ it is sufficient to assume that $f_0 \in C^\infty(\mathbb{R}^d)$.

Let $\phi \in Q^{1-\alpha}_{1,1}$ be such that $\|f(t)\|_{C^{\alpha-1}} \le C \langle f(t), \phi \rangle$, as guaranteed by Lemma A.4. Then

$$\begin{aligned}
\|f(t)\|_{C^{\alpha-1}} &\le C \langle f(t), \phi \rangle = C \left\langle f_0 \circ \eta^{-1}(t), \phi \right\rangle = C \int_{\mathbb{R}^d} f_0(\eta^{-1}(t,x)) \phi(x)\, dx \\
&= C \int_{\mathbb{R}^d} f_0(y) \phi(\eta(t,y)) \left|\det \nabla \eta(t,y)\right| dy = C \left\langle f_0, \phi \circ \eta(t) \left|\det \nabla \eta(t)\right| \right\rangle \\
&= C \langle f_0, \phi \circ \eta(t) \rangle \le C \|f_0\|_{C^{\alpha-1}} \|\phi \circ \eta(t)\|_{B^{1-\alpha}_{1,1}}
\end{aligned} \tag{A.2}$$

($\det \nabla \eta(t,y) = 1$ since v is divergence-free). Here, we again used Lemma A.4.

Using Theorem 2.36 of [1], we have

$$\begin{aligned}
\|\phi \circ \eta(t)\|_{B^{1-\alpha}_{1,1}} &\le C \left\| \frac{\|\tau_{-y}(\phi \circ \eta(t)) - \phi \circ \eta(t)\|_{L^1(\mathbb{R}^d)}}{|y|^{1-\alpha}} \right\|_{L^1(\mathbb{R}^d; \frac{dy}{|y|^d})} \\
&= C \int_{\mathbb{R}^d} \frac{1}{|y|^{d+1-\alpha}} \int_{\mathbb{R}^d} \left|\phi \circ \eta(t, x+y) - \phi \circ \eta(t,x)\right| dx\, dy \\
&= C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\phi(\eta_t(x+y)) - \phi(\eta_t(x))|}{|y|^{d+1-\alpha}}\, dy\, dx.
\end{aligned}$$

Here we switched to the notation $\eta_t(x)$ for $\eta(t,x)$. Making the change of variables, $\eta_t(x+y) = \eta_t(x)+z$, which we note induces a C^1-diffeomorphism of $\mathbb{R}^d$ with Jacobian $|\det \nabla\eta_t(x+t)| = 1$, we have

$$\begin{aligned}
\|\phi\circ\eta(t)\|_{B^{1-\alpha}_{1,1}} &\le C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\phi(\eta_t(x)+z)-\phi(\eta_t(x))|}{\left|\eta_t^{-1}(\eta_t(x)+z)-x\right|^{d+1-\alpha}}\,dz\,dx\\
&\le Ce^{(d+1-\alpha)\int_0^t\|\nabla v(s)\|_{L^\infty}\,ds}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\phi(\eta_t(x)+z)-\phi(\eta_t(x))|}{|z|^{d+1-\alpha}}\,dz\,dx\\
&\le Ce^{\int_0^t\|\nabla v(s)\|_{L^\infty}\,ds}\int_{\mathbb{R}^d}\frac{1}{|z|^{d+1-\alpha}}\int_{\mathbb{R}^d}|\phi(\eta_t(x)+z)-\phi(\eta_t(x))|\,dx\,dz,
\end{aligned}$$

where we used (A.1). Making another change of variables, $w=\eta_t(x)$, which also has Jacobian of 1, we have

$$\begin{aligned}
\|\phi\circ\eta(t)\|_{B^{1-\alpha}_{1,1}} &\le Ce^{\int_0^t\|\nabla v(s)\|_{L^\infty}\,ds}\int_{\mathbb{R}^d}\frac{1}{|z|^{d+1-\alpha}}\int_{\mathbb{R}^d}|\phi(w+z)-\phi(w)|\,dw\,dz\\
&= Ce^{\int_0^t\|\nabla v(s)\|_{L^\infty}\,ds}\int_{\mathbb{R}^d}\frac{1}{|z|^{1-\alpha}}\int_{\mathbb{R}^d}|\phi(w+z)-\phi(w)|\,dw\,\frac{dz}{|z|^d}\\
&= Ce^{\int_0^t\|\nabla v(s)\|_{L^\infty}\,ds}\|\phi\|_{B^{1-\alpha}_{1,1}} = Ce^{\int_0^t\|\nabla v(s)\|_{L^\infty}\,ds}.
\end{aligned}$$

We conclude that

$$\|f(t)\|_{C^{\alpha-1}} \le Ce^{\int_0^t\|\nabla v(s)\|_{L^\infty}\,ds}\|f_0\|_{C^{\alpha-1}}. \tag{A.3}$$

□

Lemma A.4. *Let $p,q\in[1,\infty]$ and $s\in\mathbb{R}$ and let p', q' be the Hölder conjugates of p, q. Define $Q^{-s}_{p',q'}$ to be the set of all Schwartz-class functions lying in $B^{-s}_{p',q'}$ having norm ≤ 1. Then for all $u\in B^s_{p,q}$,*

$$\|u\|_{B^s_{p,q}} \le C\sup_{\phi\in Q^{-s}_{p',q'}}\langle u,\phi\rangle \tag{A.4}$$

and

$$|\langle u,\phi\rangle| \le C\|u\|_{B^s_{p,q}}. \tag{A.5}$$

(Note that $\langle u,\phi\rangle$ is the pairing between u and ϕ in the duality between $\mathcal{S}$ and $\mathcal{S}'$, $\mathcal{S}$ being the class of Schwartz-class functions.)

Proof. The inequality in (A.4) is part of Proposition 2.76 of [1]. For (A.5), we have, using the notation of [1],

$$\begin{aligned}
|\langle u,\phi\rangle| &= \left|\left\langle\sum_j\Delta_j u,\sum_{j'}\Delta_{j'}\phi\right\rangle\right| = \left|\sum_j\sum_{j'}\left\langle\Delta_j u,\Delta_{j'}\phi\right\rangle\right|\\
&= \left|\sum_{|j-j'|\le 1}\left\langle\Delta_j u,\Delta_{j'}\phi\right\rangle\right| = |L(u,\phi)|,
\end{aligned}$$

since the supports of the Fourier transforms of $\Delta_j u$ and $\Delta_{j'}\phi$ are disjoint when $|j-j'|\ge 2$. Here, L is the continuous bilinear functional on $B^{-s}_{p',q'}\times B^s_{p,q}$ defined in Proposition 2.76 of [1], and (A.5) follows from the continuity of L. □

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