THE STRONG VANISHING VISCOSITY LIMIT WITH DIRICHLET BOUNDARY CONDITIONS: FACTS, SPECULATIONS, AND CONJECTURES

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Abstract. We employ the simple corrector used by Tosio Kato in his seminal 1983 paper to establish necessary and sufficient conditions for the solutions to the Navier-Stokes equations to converge to a solution to the Euler equations in the presence of a boundary as the viscosity is taken to zero. We extend conditions developed by various authors for no-slip boundary conditions to allow non-homogeneous Dirichlet boundary conditions, establishing a few new conditions along the way. Finally, we make a few speculations and conjectures on the strong vanishing viscosity limit.

Compiled on Saturday 30 March 2019 at 14:52

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \ge 2$. We consider solutions to

$$(NS_g) \begin{cases} \partial_t u_g + u_g \cdot \nabla u_g + \nabla p_g = \nu \Delta u_g & \text{on } \Omega, \\ \operatorname{div} u_g = 0 & \text{on } \Omega, \\ u_g(0) = u^0 & \text{on } \Omega, \\ u_g = g & \text{on } \partial\Omega. \end{cases}$$

Here, $\nu > 0$ is the constant viscosity and u^0 is the divergence-free initial velocity with $u^0 \cdot \boldsymbol{n} = 0$ on the boundary, $\partial\Omega$, where $\boldsymbol{n}$ is the outward unit normal vector. The function g is defined on $\partial\Omega$, with $g \cdot \boldsymbol{n} = 0$.

Minimal regularity requirements are not a topic of this paper, so for simplicity of presentation, we assume that $u^0 \in C^\infty(\overline{\Omega})$, $\partial\Omega$ is C^∞, and $g \in (C^\infty([0, \infty) \times \partial\Omega))^d$.

The vector field g induces a type of boundary forcing that influences the solution near the boundary, its effects spreading over time through the body of the fluid. A simple example would be a constant-magnitude g that describes the rotation of a circular boundary, as analyzed in [9, 10]. Our use of g, however, will be as a tool to try to better understand the key special case in which $g \equiv 0$.[1] This yields the Navier-Stokes equations with their classical, no-slip boundary conditions, $u = 0$ on the boundary:

$$(NS) \begin{cases} \partial_t u_0 + u_0 \cdot \nabla u_0 + \nabla p_0 = \nu \Delta u_0 & \text{on } \Omega, \\ \operatorname{div} u_0 = 0 & \text{on } \Omega, \\ u_0(0) = u^0 & \text{on } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that u_0, like u_g, depends upon ν, though we suppress ν in our notation.

When $\nu = 0$, (NS_g), for any g, formally reduces to the Euler equations with no-penetration boundary conditions:

$$(E) \begin{cases} \partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{p} = 0 & \text{on } \Omega, \\ \operatorname{div} \overline{u} = 0 & \text{on } \Omega, \\ \overline{u}(0) = u^0 & \text{on } \Omega, \\ \overline{u} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

A longstanding open question in incompressible fluid mechanics is whether u_0 converges to $\overline{u}$ as $\nu \to 0$ and, if so, in what manner. That u_0 has some weak limit in $L^2(0, T; L^2(\Omega))$ is assured by the uniform-in-ν bound in the space of weak solutions (as in (1.4)). Recently, the work of Constanin and Vicol in [6] has brought renewed interest in weak convergence to weak solutions. We say a few words on this in Section 8, but in this paper, we

[1] Most of the literature that follows in the tradition of Kato assumes $g \equiv 0$. A notable exception is Xiaoming Wang's [36], whose setting is similar to the one we have here.

will be concerned with the question of whether or not what we will call the *strong vanishing viscosity* limit,

$$\left\|u_g(t) - \overline{u}(t)\right\|^2 + \nu \int_0^t \left\|\nabla(u_g(s) - \overline{u}(s))\right\|^2 \, ds \to 0 \text{ as } \nu \to 0, \tag{1.1}$$

holds for all $t \in [0, T]$ for some fixed $T > 0$. Here and throughout,

$$\|\cdot\| := \|\cdot\|_{L^2(\Omega)} .$$

We are most interested in (1.1) in the special case of no-slip boundary condition, in which $g \equiv 0$. It was shown by Tosio Kato in [17] that when $\overline{u}$ is sufficiently regular, (1.1) is implied for $g \equiv 0$ by the weaker condition,

$$u_0 \to \overline{u} \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ as } \nu \to 0, \tag{1.2}$$

which is often referred to as the *classical vanishing viscosity limit*. This equivalence comes from the observation that if (1.2) holds it necessarily follows that

$$\limsup_{\nu \to 0} \, \nu \int_0^t \left\|\nabla u_0\right\|^2 = 0. \tag{1.3}$$

(If the limsup is positive, we say the sequence $(u_0)_{\nu > 0}$ has an *energy defect*.)

That (1.2) implies (1.3), and hence implies (when $\overline{u}$ is sufficiently regular) (1.1), when $g \equiv 0$ is clear: if (1.2) is to hold, then the energy for u_0 must converge to the energy for $\overline{u}$, which is conserved over time. By the classical energy equality for (NS) ((1.4), below) this can only happen if (1.3) holds. The situation for $g \not\equiv 0$ is more complicated, as we will see, because of the more complicated energy bound in (1.5).

We require that the initial velocities be the same for all solutions, so that the vanishing viscosity limit has some chance to hold. (It is also possible to allow $u_g(0) \to u^0$ as $\nu \to 0$.) As a consequence, unless $u^0|_{\partial\Omega} = g(0)$, a condition we do not impose, u_g has an initial boundary layer in that there is an immediate discrepancy in boundary values after the initial time.

We restrict our arguments to dimension $d = 2$, which yields four related simplifications. First, because we assume smooth data, all of our weak solutions (for all $\nu \geq 0$) will actually be smooth, globally in time (being 2D), which makes it easy to justify all of our energy arguments. Second, the various energy equalities that we obtain would only be energy inequalities in higher dimension, which would require additional work to properly treat. Third, for $d \geq 3$, weak solutions would have only a type of weak continuity to time zero. Fourth, the vorticity, $\omega_g = \operatorname{curl}(u_g) := \partial_1 u_g^2 - \partial_2 u_g^1$, is a scalar in 2D, which simplifies the form of certain expressions. We do not use the vorticity formulation of the equations, however, so this simplification is more cosmetic than fundamental, as vortex stretching would never be (directly) encountered.

Nonetheless, most of our analyses and results would apply to all $d \geq 3$ up to the time of existence of smooth solutions to the Euler equations, with only minor, technical adaptations.

The well-posedness of (NS) and (E) are classical (uniqueness being for short time in dimensions 3 and higher). We have, in particular, that $\overline{u} \in C^1([0,T]; C^\infty(\overline{\Omega}))$ for some $T > 0$, with any $T < \infty$ in 2D. We have, as well, the basic energy equalities,

$$\begin{aligned} &\|u_0(t)\|^2 + \nu \int_0^t \|\nabla u_0\|^2 = \|u^0\|^2, \\ &\|\overline{u}(t)\| = \|u^0\|. \end{aligned} \tag{1.4}$$

These equalities hold for weak solutions to (NS) and (E) in 2D and strong solutions to (E) for any dimension (up to the time of existence, of course). In 3D and higher, the equality for (NS) becomes an inequality ($\le$).

For (NS_g), we have well-posedess as stated in Proposition 1.2. Its proof is standard, but we include it in Section 9 because of the specific form of the energy inequality that we use. To obtain it, we need to extend g as in Lemma 1.1 (also proved in Section 9).

Lemma 1.1. *There exists a divergence-free extension of g to $C^\infty([0,\infty)\times\overline{\Omega})$ (which we continue to call g). If $u^0|_{\partial\Omega} = g(0)$ then we can have $g(0) = u^0$.*

Proposition 1.2. *There exists a (unique) smooth solution to (NS_g) on some time interval, $[0,T)$ for some time $T > 0$. In 2D, $T = \infty$. Moreover, extending g as in Lemma 1.1, we have*

$$\begin{aligned} \|u_g(t)\|^2 + 2\nu \int_0^t \|\nabla u_g\|^2 &\le 2\left(\|g(t)\|^2 + 2\nu \int_0^t \|\nabla g\|^2\right) \\ &\quad + 2\left(2\|u^0\|^2 + C(\nu,t)\right) e^{t + 2\int_0^t (\|\nabla g\|_{L^\infty})}, \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} C(\nu,t) &:= 2\|g(0)\|^2 + \int_0^t \|F_g\|^2, \\ F_g &:= \nu\Delta g - \partial_t g - g\cdot\nabla g. \end{aligned}$$

Because g is independent of ν, both (1.4) and (1.5) yield an energy bound that is independent of the viscosity. When $g \equiv 0$, the energy inequality in (1.5) reduces to the inequality arising from (1.4) with an additional factor of $4e^t$. Hence, the bound is not optimal in terms of g, an issue that we will find closely connected to the strong vanishing viscosity limit itself (see the discussion at the end of Section 10).

We will not make use of the special case of compatible initial data, $u^0|_{\partial\Omega} = g(0)$, except in the extreme case where we set $g \equiv u^0|_{\partial\Omega}$, as in Theorem 10.1.

Now let us consider the geometry of Ω. Since $\partial\Omega$ is C^∞ there exists a tubular neighborhood (in Ω) of width $\overline{\delta} > 0$.

Definition 1.3. *For any δ we define*

$$\Gamma_\delta = \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) < \delta\}.$$

Remark 1.4. *Throughout this paper, we always assume without comment that $\delta \in (0, \min\{\overline{\delta}/2, 1\})$.*

Let $(\boldsymbol{n}, \boldsymbol{\tau})$ be the unit normal, tangent vectors $\partial\Omega$ chosen so that $(\boldsymbol{n}, \boldsymbol{\tau})$ is in the standard orientation of $(\boldsymbol{e}_1, \boldsymbol{e}_2)$. Each component of $\partial\Omega$ has its own component of Γ_δ. We define coordinates on $\Gamma_{\overline{\delta}}$, and hence on each Γ_δ, component-by-component.

Fix an arbitrary point b in a given component of $\partial\Omega$ and let a be any point in the corresponding component of $\Gamma_{\overline{\delta}}$. Then let a' be the closest point to a on $\partial\Omega$. We define coordinates (x_1, x_2) for the point a by

$$x_1 = \text{the arc length along } \partial\Omega \text{ from } b \text{ to } a' \text{ in the } \boldsymbol{\tau} \text{ direction},$$
$$x_2 = |a - a'|.$$

Another way of expressing this is that (x_1, x_2) are coordinate values in the $(\boldsymbol{\tau}, -\boldsymbol{n})$ coordinate frame with $(\boldsymbol{\tau}, -\boldsymbol{n})$ extended from $\partial\Omega$ to $\Gamma_{\overline{\delta}}$ in the natural way.

Although we focus on a bounded domain in 2D, our results apply as well to a half-plane, $\{(x_1, x_2) \colon x_2 > 0\}$, or a channel periodic in the x_1-direction. (In particular, note that our only use of Poincaré's inequality is through Lemma 4.1, which remains valid in these settings.)

The basic theme of this paper is that, as regards the strong vanishing viscosity limit for no-slip boundary conditions having no special symmetries, everything that has been learned about it fits neatly into Kato's original approach using his original corrector. There have been refinements, most notably those of Xioaming Wang in [36] building on his work with Roger Temam in [35] (these two papers seem to have revived interest in [17]). See also [5, 3, 19, 20, 21, 23]. Nonetheless, none of these works can be said to push the envelope of Kato's fundamental result very far—unsurprising, given the difficulty of this problem without added simplifications.

The main concrete thing we accomplish in this paper is to turn Kato's energy argument using his original corrector—specifically in 2D with smooth data, where lessened technicalities make the structure of the argument clearer—into a tool, Theorem 3.3, that can be applied to obtain the various existing conditions for the strong vanishing viscosity limit to hold. In this, we have been greatly aided by a surprisingly recently discovered decomposition in [3] of one of the key terms appearing in Kato's energy argument: this decomposition simplifies a number of arguments considerably. We then apply this tool to rederive the existing conditions in [36, 19, 20], and a few other conditions along the way. Note that only [36] works with what we are calling u_g, so in re-deriving the conditions from [19, 20], we are also extending them to non-homogeneous boundary conditions. (The conditions in [3] can also be simply derived using Theorem 3.3.)

Kato's insight was to clearly identify the balance of the two, uncontrollable terms appearing in his energy argument and to understand that the only feasible thing to do was to create from them a single necessary and sufficient condition to control them both. Yet at its core, Kato's argument is a simple energy argument that almost anyone exploring the vanishing viscosity limit for the first time would attempt. Hence, one cannot say that the use of energy arguments in the vanishing viscosity limit or related singular limits, natural as they are, necessarily means that the author is following in the tradition of Kato. But there is by now a fairly sizeable literature going beyond the study of the strong vanishing viscosity limit, the topic of this paper, that is clearly very influenced by his approach, adapting his argument and philosophy to a greater or lesser extent.

This literature includes, to this author's knowledge, papers where the boundary condition is (directly or indirectly) changed [37, 31], the domain is expanded to the whole space or shrunk to a point or points [24, 13], there is some special symmetry to the geometry and initial data [29, 22], or the argument is applied to slightly different equations with sometimes different boundary conditions [27, 32, 38, 1, 25, 26, 33, 39].

Organization of this paper. We start by defining Kato's corrector in Section 2, using this corrector in Section 3 to develop a tool we use throughout most of the remainder of the paper to develop necessary and sufficient conditions for the strong vanishing viscosity limit to hold. In sections Sections 4 to 6 we apply this tool using Kato's original layer and then using the infinitesimally wider layer to reproduce the result of Xiaoming Wang's in [36]. We explain in Section 7 how the result from [20] on the formation of a vortex sheet on the boundary continues to hold for non-homogeneous boundary conditions. We say a few words in Section 8 on the recent result of Constantin and Vicol on weaker convergence in the vanishing viscosity limit. We give the proof of Lemma 1.1 and Proposition 1.2 in Section 9. Finally, in Section 10 we make a few speculations and conjectures on the strong vanishing viscosity limit.

Appendix A proves the estimates on Kato's corrector stated in Section 2, and Appendix B contains two utility lemmas used earlier in the paper.

2. Kato's corrector

We will find that the very simple corrector defined by Kato in [17] will be sufficient for all of our results. We describe it here and state the estimates, leaving the detailed derivation of these estimates to Appendix A.

Definition 2.1. *Define* $\varphi\colon [0,\infty) \to [0,1]$ *to be a* C^∞ *function with* $\varphi \equiv 1$ *on* $[0,1/2]$ *and* $\varphi \equiv 0$ *on* $[1,\infty]$. *Define* $\varphi_\delta(\cdot) = \varphi(\cdot/\delta)$.

Let g be as in Lemma 1.1. We define the corrector separately in each component of $\Gamma_{\overline{\delta}}$. Let ψ be the stream function for

$$v := g - \overline{u}, \tag{2.1}$$

so that $v = \nabla^\perp \psi$ and chosen so that $\psi = 0$ on the given component of $\Gamma_{\overline{\delta}}$. Define

$$z(x_1, x_2) = z_\delta(x_1, x_2) := \nabla^\perp(\varphi_\delta(x_2)\psi(x_1, x_2)). \tag{2.2}$$

Kato defined his corrector to have a width δ that was constant in time, shrinking only in viscosity. We will also allow δ to vary with time. For clarity, we make an explicit definition:

Definition 2.2. *Assume that either:*

(1) $\delta = \delta(\nu)$ is continuous at $\nu = 0$ with $\delta(0) = 0$ or
(2) $\delta = \delta(t, \nu)$ is continuous at $\nu = 0$ with $\delta(t, 0) = 0$ and δ increasing in ν.

Remark 2.3. *Definition 2.2 (2) is a generalization of (1), though only when we assume that $\delta(0, \nu) = 0$ does it extend (1) in a meaningful way. Also, we cannot assume in (2) any regularity of δ beyond continuity at $\nu = 0$, because we will find the need to construct a δ in a manner for which we cannot insure regularity, only monotonicity (see (6.8)). This will be sufficient to take time derivatives of δ, however, as we note in the derivation of (2.5), below. Although in practice one would typically choose δ to be increasing in ν, this is not strictly needed.*

Remark 2.4. *As mentioned near the end of Section 1, we always assume that $\delta(\nu)$ or $\delta(t, \nu)$ lies in $(0, \min\{\overline{\delta}/2, 1\}$ without explicitly commenting on that fact. In practice, this means that ν must be sufficiently small, how small depending upon the choice of the δ function.*

Theorem 2.5. *Assume that δ is independent of time (though it may depend upon viscosity, for instance, as in Definition 2.2 (1)). We have the following estimates for the Kato corrector as defined in (2.2):*

$$\left\|\partial_1^j \partial_2^k \partial_t^m z^1\right\|_{L^p(\Omega)} \le C\delta^{\frac{1}{p}-k}, \quad \left\|\partial_1^j \partial_2^k \partial_t^m z^2\right\|_{L^p(\Omega)} \le C\delta^{\frac{1}{p}+1-k} \tag{2.3}$$

for any $p \in [1, \infty]$, $j, k \ge 0$, $m = 0, 1$, any $t \in [0, T]$. The constants are independent of p and depend only upon the initial data, T, j, k, and m.

Let δ be as in Definition 2.2 (2). The estimates in (2.3) for $m = 0$ (no time derivative) continue to hold. We also have, for all $p \in [1, \infty]$ and $t \in [0, T]$,

$$\begin{aligned} \left\|\partial_t z^1\right\|_{L^p(\Omega)} &\le C\delta^{\frac{1}{p}} + C\partial_t\delta\, \delta^{\frac{1}{p}-1}, \\ \left\|\partial_t z^2\right\|_{L^p(\Omega)} &\le C\delta^{\frac{1}{p}+1} + C\partial_t\delta\, \delta^{\frac{1}{p}}, \\ \left\|\partial_t z\right\|_{L^p(\Omega)} &\le C\delta^{\frac{1}{p}-1}(\delta + \partial_t\delta). \end{aligned} \tag{2.4}$$

Proof. We defer the proof to Appendix A. □

A few observations regarding Kato's corrector are in order, as they will help guide our strategy in employing it:

(1) Because z is supported on a set of Lebesgue measure $C\delta$, the bounds in L^p for $p < \infty$ follow from the bounds in L^∞.
(2) Because z^2 vanishes on the boundary and grows linearly away from it, it is small compared to z^1, which is merely bounded.

(3) Derivatives in x_1 (tangential direction) are totally benign, having no effect on the estimates, while each derivative in x_2 (normal direction) increases the bound by a factor of δ^{-1}.
(4) Time derivatives have no effect when δ is independent of time, and even when δ varies, they are benign as long as we integrate the estimates in time.

As an application of observation (4), the final bound in (2.4) gives

$$\begin{aligned}\int_0^t \|\partial_s z(s,\nu)\, ds\| &\le C\int_0^t \delta(s,\nu)^{\frac{1}{2}}\, ds + C\int_0^t \partial_s(\delta(s,\nu)^{\frac{1}{2}})\, ds \\ &\le Ct\delta(t,\nu)^{\frac{1}{2}} + C\left[\delta(t,\nu)^{\frac{1}{2}} - \delta(0,\nu)^{\frac{1}{2}}\right] \le C(1+t)\delta(t,\nu)^{\frac{1}{2}},\end{aligned} \tag{2.5}$$

where we used that $\delta(\cdot,\nu)$ is increasing. We also used that for any increasing function, $f\colon [a,b]\to\mathbb{R}$, $f'\ge 0$ exists almost everywhere, and

$$\int_a^b f'(s)\, ds \le f(b) - f(a).$$

The bound in (2.5), which we will apply in (3.6), is the only bound on $\partial_t z$ that we will need.

3. Kato's energy argument

From now until Section 10, we will drop the g subscript on u_g, writing

$$u = u_g.$$

We will return to writing u_g in Section 10, where we will be treating u_g and u_0 at the same time.

The starting point for almost all of our analysis will be the energy inequality we obtain in Proposition 3.1 for

$$w := u - \overline{u}.$$

Proposition 3.1. *Let δ be as in Definition 2.2 and let z be the Kato corrector as in (2.2). Then*

$$\frac{1}{2}\|w(t)\|^2 + \frac{\nu}{2}\int_0^t \|\nabla w\|^2 = A(t,\nu) + B(t,\nu) + C\int_0^t \|w\|^2, \tag{3.1}$$

where

$$A(t,\nu) := -\int_0^t (u^1u^2, \partial_2 z^1) + \nu\int_0^t (\nabla u, \nabla z) \tag{3.2}$$

and

$$B(t,\nu) \le C(1+t)\delta^{\frac{1}{2}}.$$

The constants C depend upon T, u^0, and g, though not ν.

Proof. Recalling Remark 2.3, we will assume that $\delta = \delta(t, \nu)$ is time varying as in (2) of Definition 2.2.

Let

$$\widetilde{w} := w - z = u - \overline{u} - z,$$

and note that $\widetilde{w} = 0$ on $\partial\Omega$. Observe that from (1.5) and Theorem 2.5, we know up front that at least

$$\|\widetilde{w}(t)\|, \|w(t)\| \le C(T)$$

for all $t \in [0, T]$.

Subtracting the Euler equations from the Navier-Stokes equations gives

$$\partial_t w + \nabla(p - \overline{p}) = \nu\Delta u - u \cdot \nabla w - w \cdot \nabla\overline{u}.$$

Pairing with $\widetilde{w}$ and using

$$\begin{aligned}
(\partial_t w, \widetilde{w}) &= \frac{1}{2}\frac{d}{dt}\|w\|^2 - (\partial_t w, z), \\
\nu(\Delta u, \widetilde{w}) &= -\nu(\nabla u, \nabla\widetilde{w}) = -\nu(\nabla u, \nabla w) + \nu(\nabla u, \nabla z) \\
&= -\nu(\nabla w, \nabla w) - \nu(\nabla\overline{u}, \nabla w) + \nu(\nabla u, \nabla z) \\
&\le -\nu\|\nabla w\|^2 + \nu\|\nabla\overline{u}\|^2 + \frac{\nu}{2}\|\nabla w\|^2 + \nu(\nabla u, \nabla z) \\
&\le C\nu - \frac{\nu}{2}\|\nabla w\|^2 + \nu(\nabla u, \nabla z), \\
(\nabla(p - \overline{p}), \widetilde{w}) &= 0, \\
-(u \cdot \nabla w, \widetilde{w}) &= -(u \cdot \nabla w, w) + (u \cdot \nabla w, z) = (u \cdot \nabla w, z) \\
&= (u \cdot \nabla u, z) - (u \cdot \nabla\overline{u}, z) \\
&= -(u \cdot \nabla z, u) - (u \cdot \nabla\overline{u}, z) \\
&\le -(u \cdot \nabla z, u) + \|\nabla\overline{u}\|_{L^\infty}\|u\|\|z\| \\
&\le -(u \cdot \nabla z, u) + C\|z\| \le -(u \cdot \nabla z, u) + C\delta^{\frac{1}{2}}, \\
-(w \cdot \nabla\overline{u}, \widetilde{w}) &= -(w \cdot \nabla\overline{u}, w) + (w \cdot \nabla\overline{u}, z) \\
&\le \|\nabla\overline{u}\|_{L^\infty}\left(\|w\|^2 + \|w\|\|z\|\right) \le C\|w\|^2 + C\delta^{\frac{1}{2}},
\end{aligned}$$

we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|^2 + \frac{\nu}{2}\|\nabla w\|^2 \le (\partial_t w, z) &+ C\nu + C\delta^{\frac{1}{2}} + C\|w\|^2 \\
&- (u \cdot \nabla z, u) + \nu(\nabla u, \nabla z).
\end{aligned} \tag{3.3}$$

We divide $-(u \cdot \nabla z, u)$ into parts as in [3], writing

$$\begin{aligned}
-(u \cdot \nabla z, u) &= -(u^i\partial_i z^j, u^j) \\
&= -(u^1u^2, \partial_1 z^2) + ((u^2)^2 - (u^1)^2, \partial_1 z^1) - (u^1u^2, \partial_2 z^1).
\end{aligned} \tag{3.4}$$

Here, we used that $\operatorname{div} z = 0$ so that $\partial_2 z^2 = -\partial_1 z^1$.

Now,

$$-(u^1 u^2, \partial_1 z^2) \le \left\|\partial_1 z^2\right\|_{L^\infty} \|u\|^2 \le C\|u^0\|^2 \delta \le C\delta,$$

since $\left\|\partial_1 z^2\right\|_{L^\infty} \le C\delta$ by Theorem 2.5. Also,

$$\begin{aligned} |((u^j)^2, \partial_1 z^1)| &= |((u^j - \overline{u}^j)^2, \partial_1 z^1) + 2(u^j \overline{u}^j, \partial_1 z^1) - ((\overline{u}^j)^2, \partial_1 z^1)| \\ &\le \|\partial_1 z^1\|_{L^\infty} \|w\|^2 + 2\|\overline{u}\|_{L^\infty} \left\|\partial_1 z^1\right\| \|u\| + \|\overline{u}\|_{L^\infty}^2 \|\partial_1 z^1\|_{L^1} \\ &\le C\|w\|^2 + C\delta^{\frac{1}{2}} + C\delta \le C\delta^{\frac{1}{2}} + C\|w\|^2. \end{aligned}$$

Hence,

$$((u^2)^2 - (u^1)^2, \partial_1 z^1) \le C\delta^{\frac{1}{2}} + C\|w\|^2,$$

so that

$$-(u \cdot \nabla z, u) \le C\delta^{\frac{1}{2}} + C\|w\|^2 - (u^1 u^2, \partial_2 z^1). \tag{3.5}$$

Returning to (3.3), then, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2 + \frac{\nu}{2}\|\nabla w\|^2 &\le (\partial_t w, z) + C\nu + C\delta^{\frac{1}{2}} + C\|w\|^2 \\ &\quad - (u^1 u^2, \partial_2 z^1) + \nu(\nabla u, \nabla z). \end{aligned}$$

Integrating in time and using (2.5), we have

$$\begin{aligned} \int_0^t (\partial_t w, z) &= \int_\Omega \int_0^t \partial_t w \cdot z = \int_\Omega \left[w(t) \cdot z(t) - \int_0^t w \partial_t z \right] \\ &\le \|w(t)\| \|z(t)\| + \int_0^t \|w\| \|\partial_t z\| \le C\|z(t)\| + C\int_0^t \|\partial_t z\| \le C\delta^{\frac{1}{2}}. \end{aligned} \tag{3.6}$$

Then,

$$\begin{aligned} \frac{1}{2}\|w(t)\|^2 + \frac{\nu}{2}\int_0^t \|\nabla w\|^2 &\le C(1+t)\delta^{\frac{1}{2}} + C\nu t \\ &\quad - \int_0^t (u^1 u^2, \partial_2 z^1) + \nu \int_0^t (\nabla u, \nabla z) + C\int_0^t \|w\|^2, \end{aligned} \tag{3.7}$$

which can be re-expressed in the form of (3.1). Note that we used here that

$$\int_0^t \delta^{\frac{1}{2}}(s, \nu)\, ds \le \delta^{\frac{1}{2}}(t, \nu) t = \delta^{\frac{1}{2}} t,$$

since $\delta(s, \nu)$ is increasing in s. □

Proposition 3.1 leads to Theorem 3.3, which gives general necessary and sufficient criteria for the vanishing viscosity limit to hold. But we will need first the following lemma, also useful in its own right:

Lemma 3.2. *Assume that* $g \equiv 0$. *If (1.2) holds then (1.1) holds—and hence,* $\nu \int_0^T \|\nabla u\|^2, \nu \int_0^T \|\nabla w\|^2 \to 0$ *as* $\nu \to 0$.

Proof. This is proved in [17] using only the energy *inequality* for the Navier-Stokes equations. The argument in 2D, where the energy equality holds is slightly simpler: We have, from (1.4),

$$\|u(t)\|^2 - \|\overline{u}(t)\|^2 + 2\nu \int_0^t \|\nabla u\|^2 = 0.$$

If (1.2) then $\|u(t)\|^2 - \|\overline{u}(t)\|^2 \to 0$, hence, $\nu \int_0^t \|\nabla u\|^2 \to 0$. But also $\nu \int_0^t \|\nabla \overline{u}\|^2 \to 0$, and we conclude that $\nu \int_0^t \|\nabla w\|^2 \to 0$. □

Theorem 3.3. *If there exists some δ as in Definition 2.2 (1) or (2) for which $A(\cdot, \nu) \to 0$ in $L^\infty([0, T])$ as $\nu \to 0$, with A as defined in (3.2), then the strong vanishing viscosity limit as in (1.1) holds.*

Conversely, if (1.1) holds (when $g \equiv 0$ we only require (1.2)) then $A(\cdot, \nu) \to 0$ in $L^\infty([0, T])$ as $\nu \to 0$ for any δ as in Definition 2.2 (1) or (2).

Furthermore, we can equivalently define $A = A_1 + A_2$, where A_1 is either

$$A_1^1 := -\int_0^t (u^1 u^2, \partial_2 z^1) \text{ or } A_1^2 := -\int_0^t (u \cdot \nabla z, u)$$

and A_2 is either

$$A_2^1 := \nu \int_0^t (\nabla u, \nabla z) \text{ or } A_2^2 := \nu \int_0^t (\operatorname{curl} u, \operatorname{curl} z).$$

Also equivalently, we can add to A either

$$a_1 \nu \int_0^t \|\nabla u\|^2 + a_2 \nu \|w\|^2 \text{ or } a_1 \nu \int_0^t \|\nabla w\|^2 + a_2 \nu \|w\|^2 \tag{3.8}$$

for any $a_1 < \frac{1}{2}$ and any $a_2 \in \mathbb{R}$ without affecting the conclusions of the theorem.

Remark 3.4. *The function δ appears implicitly in this theorem through A, which contains the δ-dependent corrector, z.*

Proof of Theorem 3.3. Assume that $A(\cdot, \nu) \to 0$ in $L^\infty([0, T])$ as $\nu \to 0$, with A as defined in (3.2), for some choice of δ as in Definition 2.2. Applying Gronwall's inequality to (3.7), we conclude that

$$\begin{aligned} &\frac{1}{2} \|w(t)\|^2 + \frac{\nu}{2} \int_0^t \|\nabla w\|^2 \\ &\quad \le \left[\left\| A(\cdot, \nu)_{L^\infty([0,T])} \right\| + C(1 + t) t \delta^{\frac{1}{2}} + C \nu t^2 \right] e^{Ct}, \end{aligned}$$

which vanishes as $\nu \to 0$ since $\delta(\nu) \to 0$ or $\delta(t, \nu) \to 0$ as $\nu \to 0$. This gives (1.1).

Either of the terms in (3.8) can be added to A since they can be absorbed in the energy inequality in (3.7).

Conversely, assume that the vanishing viscosity limit holds. Then by Lemma 3.2 when $g \equiv 0$ or otherwise by assumption, $t \mapsto \nu \int_0^t \|\nabla w\|^2 \to 0$ in $L^\infty([0, T])$. For *any* δ as in Definition 2.2, $B(\cdot, \nu) \to 0$ in $L^\infty([0, T])$, with

B as in Proposition 3.1, since $\delta(\nu) \to 0$ or $\delta(t, \nu) \to 0$ as $\nu \to 0$. This leaves only the term $A(\cdot, \nu)$ in (3.1), which therefore must vanish as $\nu \to 0$ as well.

Note also that the terms in (3.8) also vanish if (1.2) holds by Lemma 3.2.

The equivalence of A_1^1 and A_1^2 follows from (3.5). For the equivalence of A_2^1 and A_2^2, we apply Lemma B.1, which gives

$$\nu(\nabla u, \nabla z) = \nu(\operatorname{curl} u, \operatorname{curl} z) + \nu \int_{\partial\Omega} (\operatorname{curl}(z_2)(z \cdot \boldsymbol{\tau}) - \kappa z \cdot u).$$

Then,

$$\nu \int_{\partial\Omega} (\operatorname{curl}(z_2)(z \cdot \boldsymbol{\tau}) - \kappa z \cdot u) = \nu \int_{\partial\Omega} (\operatorname{curl}(z_2)((g - \overline{u}) \cdot \boldsymbol{\tau}) - \kappa(g - \overline{u}) \cdot g),$$

which is bounded by $C\nu$, since z_2, g, and $\overline{u}$ are each bounded independently of ν on the boundary. Hence, A_2^1 and A_2^2 are interchangeable. □

Remark 3.5. *Since the converse in Theorem 3.3 holds for any δ it follows that so, too, does the forward direction of the theorem in the sense that if $A(\cdot, \nu) \to 0$ in $L^\infty([0, T])$ for one choice of δ then A vanishes in the same manner for any other choice of δ. (All δ's must be as in Definition 2.2, of course.) A priori, however, the forward direction is stronger with "there exists δ" rather than "for all δ."*

In applying Theorem 3.3, the key is the control of the two terms A_1 and A_2 in A, regardless of which form is used. The term A_1 originates in the convective terms in the Navier-Stokes and Euler equations, A_2 from the effect of the boundary on the viscous term in the Navier-Stokes equations. Either term can be controlled individually: Without the convective term we have the Stokes equation (the Euler equations becoming stationary) and the vanishing viscosity limit holds as shown in [12]. Without the boundary, the vanishing viscosity limit holds as shown in many contexts ([34, 15, 16, 4, 28], for instance). Ideally, one could handle the combined effect of these terms, but no such technique is currently available. We have little choice, then, but to handle the two terms separately.

Thus, if we wish to establish sufficient condition for the vanishing viscosity limit to hold, we require that

$$\int_0^t (u^1 u^2, \partial_2 z^1) \to 0 \text{ as } \nu \to 0 \tag{3.9}$$

and

$$\nu \int_0^t (\nabla u, \nabla z) \to 0 \text{ as } \nu \to 0. \tag{3.10}$$

In his seminal paper [17], Tosio Kato chose to set (with $g \equiv 0$)

$$\delta = \nu.$$

In this case, (3.9) and (3.10) are both critical in the sense that they can be shown to be bounded by the basic energy inequality for the Navier-Stokes equations, but the energy inequality is insufficient to show that these integrals vanish with viscosity. Kato shows that both of these conditions (though not making the nice division of $(u \cdot \nabla z, u)$ into parts as in (3.4) that originated in [3], as we did above) can be replaced by

$$\nu \int_0^t \|\nabla u\|^2_{L^2(\Gamma_\nu)} \to 0 \text{ as } \nu \to 0.$$

Following in this same spirit, [19] gives two other ways to find a common condition that applies to (3.9) and (3.10) (also not making the nice division of $(u \cdot \nabla z, u)$ into parts). These are the conditions in (4.2) and (4.3) that we discuss in Section 4, along with an improvement that comes from dividing $(u \cdot \nabla z, u)$ as in [3].

Definition 3.6. *The boundary layer, Γ_ν, is called the Kato (boundary) layer and ν is called the Kato width.*

Kato actually used $\Gamma_{c\nu}$, but there is no loss of generality in setting $c = 1$: only the rates of convergence change.

Alternately, we can allow δ to be infinitesimally larger than ν, though still vanishing as $\nu \to 0$. This approach, in the full generality in which we will use it (except for being time-independent), was first taken by Xiaoming Wang in [36] (see [35] for an earlier, less general version of this idea). We define it as follows:

Definition 3.7. *Let δ be as in Definition 2.2 (2) with the additional property that*

$$\int_0^T \frac{\nu}{\delta(s, \nu)}\, ds \to 0 \text{ as } \nu \to 0. \tag{3.11}$$

The resulting boundary layer, Γ_δ, we call a Wang (boundary) layer and such a δ we call a Wang width.

With a Wang layer, (3.10) follows very easily (see the proof of Theorem 6.1). This is because the factor of ν in (3.10) came from the diffusion term in the Navier-Stokes equations, while the bounds on ∇z improves as δ increases. This leaves only the condition in (3.9) or an equivalent condition to be treated.

Although we use the Wang layer, we will employ it with the Kato corrector rather than the corrector employed by Wang in [36]. We do this in Section 6, but first we explore the Kato layer in Section 4.

4. Using the Kato layer

The use of the Kato layer of width proportional to ν leads naturally to Theorem 4.3, the result for (4.2) and (4.3) appearing in [19]. In its proof, and multiple times in later sections, we will use (through its corollary) the following version of Poincaré's inequality, which is essentially the form that applies to a domain of given width vanishing on one component of the boundary:

Lemma 4.1. *Fix $p \in [1, \infty]$ and assume that $f \in W^{1,p}(\Gamma_\delta)$ with $f = 0$ on $\partial\Omega$. Then*

$$\|f\|_{L^p(\Gamma_\delta)} \le C\delta \|\partial_2 f\|_{L^p(\Gamma_\delta)},$$

where the constant $C = C(\Omega)$ is independent of p and δ (recall Remark 1.4).

Corollary 4.2. *For all $p \in [1, \infty]$,*

$$\begin{aligned} \|u^1\|_{L^p(\Gamma_\delta)} &\le C\delta \left\|\partial_2 u^1\right\|_{L^p(\Gamma_\delta)} + C'\delta^{\frac{1}{p}}, \\ \|u^2\|_{L^p(\Gamma_\delta)} &\le C\delta \left\|\partial_2 u^2\right\|_{L^p(\Gamma_\delta)}, \end{aligned} \tag{4.1}$$

where the constant C is as in Lemma 4.1 and $C' = \|g\|_{W^{1,\infty}(\Omega)}$ is independent of p and δ.

Proof. Since $u^2 = -g \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, the inequality for $\|u^2\|_{L^p(U)}$ follows directly from Lemma 4.1. For the other inequality, we have

$$\begin{aligned} \|u^1\|_{L^p(\Gamma_\delta)} &\le \|u^1 - g^1\|_{L^p(\Gamma_\delta)} + \|g^1\|_{L^p(\Gamma_\delta)} \\ &\le C\delta \|\partial_2(u^1 - g^1)\|_{L^p(\Gamma_\delta)} + C\delta^{\frac{1}{p}}\|g^1\|_{L^\infty(\Omega)} \\ &\le C\delta\|\partial_2 u^1\|_{L^p(\Gamma_\delta)} + C\delta^{1+\frac{1}{p}}\|\partial_2 g^1\|_{L^\infty(\Gamma_\delta)} + C\delta^{\frac{1}{p}}\|g^1\|_{L^\infty(\Omega)} \\ &\le C\delta \left\|\partial_2 u^1\right\|_{L^p(\Gamma_\delta)} + C\|g\|_{W^{1,\infty}}\delta^{\frac{1}{p}}, \end{aligned}$$

where we again applied Lemma 4.1, and used that Ω has finite measure. □

Theorem 4.3. *The strong vanishing viscosity limit in (1.1) holds if*

$$\nu \int_0^t \|\operatorname{curl} u\|^2_{L^2(\Gamma_\nu)} \to 0 \text{ as } \nu \to 0 \tag{4.2}$$

or

$$\frac{1}{\nu}\int_0^t \|u\|^2_{L^2(\Gamma_\nu)} \to 0 \text{ as } \nu \to 0 \tag{4.3}$$

or

$$\frac{1}{\nu}\int_0^t \int_{\Gamma_\nu} ((u^1)^2 + |u^1 u^2|) \to 0 \text{ as } \nu \to 0. \tag{4.4}$$

Conversely, if (1.1) holds (or simply (1.2) when $g \equiv 0$) then (4.2) through (4.4) hold.

Proof. We prove first the converse—the necessity of (4.2) through (4.4). Because

$$\begin{aligned} \frac{1}{\nu}\int_0^t \int_{\Gamma_\nu} ((u^1)^2 + |u^1 u^2|) &\le \frac{1}{\nu}\int_0^t \|u\|^2_{L^2(\Gamma_\nu)} \le \frac{1}{\nu}\int_0^t C\nu^2 \|\partial_2 u\|^2_{L^2(\Gamma_\nu)} \\ &\le C\nu \int_0^t \|\nabla u\|^2_{L^2(\Gamma_\nu)}, \\ \nu \int_0^t \|\operatorname{curl} u\|^2_{L^2(\Gamma_\nu)} &\le C\nu \int_0^t \|\nabla u\|^2_{L^2(\Gamma_\nu)}, \end{aligned}$$

the necessity of all three conditions follows from Lemma 3.2 when $g \equiv 0$ or, when $g \not\equiv 0$, by the assumption of (1.1), which implies (1.3). The first bound also shows that the sufficiency of (4.4) implies the sufficiency of (4.3). It remains, then, to show the sufficiency of (4.2) and (4.4).

First assume (4.2). In place of (3.5), we bound $A_1 = A_1^2$ by

$$\begin{aligned}
\Big|\int_0^t (u \cdot \nabla z, u)\Big| &= \Big|\int_0^t (u \cdot \nabla u, z)\Big| = \Big|\int_0^t (u^\perp \operatorname{curl} u, z)\Big| \\
&\le \|z\|_{L^\infty([0,T]\times\Omega)} \int_0^t \|u\|_{L^2(\Gamma_\nu)} \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)} \\
&\le C\nu \int_0^t \|\nabla u\|_{L^2(\Gamma_\nu)} \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)} + C\nu^{\frac{1}{2}} \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)}.
\end{aligned}$$

In the second equality we applied Lemma B.2 to exchange ∇u for $\operatorname{curl} u$, and in the last inequality we applied Corollary 4.2.

For the first term,

$$\begin{aligned}
&C\nu \int_0^t \|\nabla u\|_{L^2(\Gamma_\nu)} \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)} \\
&\qquad \le C \left(\nu \int_0^t \|\nabla u\|_{L^2(\Omega)}^2 \, ds\right)^{\frac{1}{2}} \left(\nu \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)}^2 \, ds\right)^{\frac{1}{2}} \\
&\qquad \le C(T) \left(\nu \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)}^2 \, ds\right)^{\frac{1}{2}}.
\end{aligned}$$

In the last inequality we applied the energy inequality in (1.5). Also,

$$C\nu^{\frac{1}{2}} \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)} \le t^{\frac{1}{2}} \left(\nu \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)}^2 \, ds\right)^{\frac{1}{2}},$$

so

$$\Big|\int_0^t (u \cdot \nabla z, u)\Big| \le C(T) \left(\nu \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)}^2 \, ds\right)^{\frac{1}{2}}.$$

We then bound A_2^2 by

$$\begin{aligned}
&\nu\Big|\int_0^t (\operatorname{curl} u, \operatorname{curl} z)\Big| \le \nu \int_0^t \|\nabla z\| \, \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)} \\
&\qquad \le C\nu^{\frac{1}{2}} \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)} \le Ct^{\frac{1}{2}} \left(\nu \int_0^t \|\operatorname{curl} u\|_{L^2(\Gamma_\nu)}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Then (1.1) follows from Theorem 3.3.

Assume now (4.4). We have,

$$(\nabla u, \nabla z) = -(u, \Delta z) = -(u, \partial_1^2 z) - (u^1, \partial_2^2 z^1) - (u^2, \partial_2^2 z^2).$$

Using Theorem 2.5, we have

$$\begin{aligned}\nu\left|(u, \partial_1^2 z)\right| &\le \nu \left\|u\right\| \left\|\partial_1^2 z\right\| \le C\nu\nu^{\frac{1}{2}} = C\nu^{\frac{3}{2}},\\ \nu\left|(u^2, \partial_2^2 z^2)\right| &\le \nu \left\|u\right\| \left\|\partial_2^2 z^2\right\| \le C\nu\nu^{-\frac{1}{2}} = C\nu^{\frac{1}{2}}.\end{aligned}$$

Therefore, we can write

$$A(t, \nu) = f(t, \nu) - \int_0^t \left((u^1 u^2, \partial_2 z^1) + \nu(u^1, \partial_2^2 z^1)\right),$$

where $f(\cdot, \nu) \to 0$ in $L^\infty(0, T; L^2(\Omega))$ as $\nu \to 0$. But, applying

$$-(u^1 u^2, \partial_2 z^1) \le \int_{\Gamma_\delta} \left\|\partial_2 z^1\right\|_{L^\infty} |u^1 u^2| \le \int_{\Gamma_\delta} \frac{C}{\delta} |u^1 u^2| \tag{4.5}$$

with $\delta = \nu$, and again using Theorem 2.5,

$$\begin{aligned}\nu\left|\int_0^t (u^1, \partial_2^2 z^1)\right| &\le C\nu \int_0^t \left\|u^1\right\|_{L^2(\Gamma_\nu)} \left\|\partial_2^2 z^1\right\| \le \frac{C}{\sqrt{\nu}} \int_0^t \left\|u^1\right\|_{L^2(\Gamma_\nu)}\\ &\le C \left(\int_0^t 1\right)^{\frac{1}{2}} \left(\frac{1}{\nu}\int_0^t \left\|u^1\right\|^2_{L^2(\Gamma_\nu)}\right)^{\frac{1}{2}}.\end{aligned}$$

Then (1.1) follows from Theorem 3.3. □

We might hope to extend Kato's conditions and the Kato-like conditions in Theorem 4.3 to use a layer of width νt. We should expect the effect of the initial layer of vorticity forming at the boundary to take some time to move into the fluid, so the width of the layer should increase with time. The heat equation solution depends only upon νt with simple geometries for instance (though its weak boundary layer is of "width" $\sqrt{\nu t}$), so such a scaling would seem reasonable. It is not, however, possible.

To see this, let us consider the condition,

$$\nu \int_0^t \left\|\operatorname{curl} u\right\|^2_{L^2(\Gamma_{\nu s})} \, ds \to 0 \text{ as } \nu \to 0 \tag{4.6}$$

in place of (4.2). Certainly this is a necessary condition, being weaker than the condition in (4.2). To adapt the proof of sufficiency of (4.2) above, we need only change the width of the layer. Note that this brings powers of the time into the time integrals. For bounding the convective term in A, we find (including only the key steps) that

$$\begin{aligned}\left|\int_0^t (u \cdot \nabla z, u)\right| &\le \int_0^t \left\|u\right\|_{L^2(\Gamma_{\nu s})} \left\|\operatorname{curl} u\right\|_{L^2(\Gamma_{\nu s})} \left\|z\right\|_{L^\infty} \, ds\\ &\le C \int_0^t \nu s \left\|\nabla u\right\|_{L^2(\Gamma_{\nu s})} \left\|\operatorname{curl} u\right\|_{L^2(\Gamma_{\nu s})} \, ds + C\nu^{\frac{1}{2}} \int_0^t s^{\frac{1}{2}} \left\|\operatorname{curl} u\right\|_{L^2(\Gamma_\nu)}\\ &\le Ct \left(\nu \int_0^t \left\|\operatorname{curl} u\right\|^2_{L^2([0,T]; L^2(\Gamma_{\nu s}))} \, ds\right)^{\frac{1}{2}}.\end{aligned}$$

Here, Poincare's inequality via Corollary 4.2 brings an additional factor of s into the integral, which we bound above by t and bring outside the integral. The end result is a harmless additional factor of t.

The boundary term, however, has a significant problem. To see this, let us treat this term for a general δ as in Definition 2.2, a bound we will find useful later in the proof of Theorem 6.1. We have, using A_2^2,

$$\begin{aligned}\nu\left|\int_0^t (\operatorname{curl} u, \operatorname{curl} z)\right| &\le \nu \int_0^t \|\operatorname{curl} z\| \, \|\operatorname{curl} u\|_{L^2(\Gamma_{\nu s})} \, ds \\ &\le C\nu \int_0^t \delta(s,\nu)^{-\frac{1}{2}} \|\operatorname{curl} u\|_{L^2(\Gamma_{\nu s})} \, ds \\ &\le C \left(\int_0^t \frac{\nu}{\delta(s,\nu)} \, ds\right)^{\frac{1}{2}} \left(\nu \int_0^t \|\operatorname{curl} u\|^2_{L^2(\Gamma_{\nu t})}\right)^{\frac{1}{2}}.\end{aligned} \tag{4.7}$$

So the first time integral above must at least be finite for $A(t, \nu)$ to have a chance to vanish with ν. When $\delta(s, \nu) = \nu s$, however, the integral is infinite.

In estimating the convective term, we integrated by parts in the first step, removing the gradient on $z = z_\delta$ ($\delta = \nu$ or νs, here). The estimate for $\|z_\delta\|_{L^\infty}$ is independent of δ, so this simply leads to an additional factor of t in the estimate. There appears to be no way to avoid leaving at least part of the derivative on z in estimating the boundary term, however; in particular, $\partial_1 z^2$, which dominates ∇z, seems unavoidable.

It is clear from these estimates that we could use a boundary layer of width νt^α for any $\alpha \in [0, 1)$ in (4.2). For (4.3) and (4.4), however, the boundary term estimate scales in time even more severely, as we must integrate by parts to get $\nu \int_0^t (u, \Delta z)$, in order to make the estimates on the velocity rather than the vorticity.

5. A little more with Kato's layer

In [36], Wang gives necessary and sufficient conditions for the vanishing viscosity limit to hold based upon the magnitude of the tangential derivatives of either the tangential components of the velocity or the of the normal component of the velocity. The penalty is that the boundary layer considered must be infinitesimally larger than that of Kato (as in (3.11)).

We discuss [36] in detail in Section 6, but first we derive in a more simple manner a result using Kato's original boundary layer. The conditions required are stronger than that of [36] in that they each involve a derivative normal to the boundary. They apply, however, to the thinner boundary layer of Kato.

Theorem 5.1. *If*

(1) $\nu \int_0^T \|\partial_2 u\|^2_{L^2(\Gamma_\nu)} = \nu \int_0^T \|\partial_2 u^1\|^2_{L^2(\Gamma_\nu)} + \|\partial_2 u^2\|^2_{L^2(\Gamma_\nu)} \to 0$ *as* $\nu \to 0$

or

(2) $\nu \int_0^T \|\nabla u^1\|^2_{L^2(\Gamma_\nu)} = \nu \int_0^T \|\partial_1 u^1\|^2_{L^2(\Gamma_\nu)} + \|\partial_2 u^1\|^2_{L^2(\Gamma_\nu)} \to 0$ *as* $\nu \to 0$

then the strong vanishing viscosity limit in (1.1) holds. Conversely, if (1.1) holds (or simply (1.2) when $g \equiv 0$*) then* (1) *and* (2) *hold.*

Proof. First observe that (1) and (2) are equivalent since u is divergence-free.

That (1.2) when $g \equiv 0 \implies (1), (2)$ follows from Lemma 3.2; when $g \not\equiv 0$ they follows directly from the stronger assumption in (1.1).

For the forward implications, assume (1). We will apply Theorem 3.3 to A as in (3.2).

Setting $\delta = \nu$, we have,

$$\begin{aligned}
(u^1 u^2, \partial_2 z^1) &\le \left\|\partial_2 z^1\right\|_{L^\infty} \left\|u^1\right\|_{L^2(\Gamma_\nu)} \left\|u^2\right\|_{L^2(\Gamma_\nu)} \\
&\le \frac{C}{\nu}\left(\nu \left\|\partial_2 u^1\right\|_{L^2(\Gamma_\nu)} + \nu^{\frac{1}{2}}\right) \nu \left\|\partial_2 u^2\right\|_{L^2(\Gamma_\nu)} \\
&= C\nu \left\|\partial_2 u^1\right\|_{L^2(\Gamma_\nu)} \left\|\partial_2 u^2\right\|_{L^2(\Gamma_\nu)} + C\nu^{\frac{1}{2}} \left\|\partial_2 u^2\right\|_{L^2(\Gamma_\nu)} \\
&\le C\nu \left(\left\|\partial_2 u^1\right\|^2_{L^2(\Gamma_\nu)} + \left\|\partial_2 u^2\right\|^2_{L^2(\Gamma_\nu)}\right) + C\nu^{\frac{1}{2}} \left\|\partial_2 u^2\right\|_{L^2(\Gamma_\nu)},
\end{aligned} \tag{5.1}$$

where we used Corollary 4.2.

Next,

$$\begin{aligned}
-\nu(\nabla u, \nabla z) &= -\nu \partial_i z^j \partial_i u^j \\
&\le \nu \sum_{(i,j) \ne (2,1)} \left\|\partial_i z^j\right\| \left\|\partial_i u^j\right\|_{L^2(\Gamma_\nu)} + \nu \left\|\partial_2 z^1\right\| \left\|\partial_2 u^1\right\|_{L^2(\Gamma_\nu)} \\
&\le \nu \nu^{\frac{1}{2}} \|\nabla u\| + \nu \nu^{-\frac{1}{2}} \left\|\partial_2 u^1\right\|_{L^2(\Gamma_\nu)} \\
&\le \frac{\nu}{2} + \frac{\nu^2}{2} \|\nabla u\|^2 + \nu^{\frac{1}{2}} \left\|\partial_2 u^1\right\|_{L^2(\Gamma_\nu)}.
\end{aligned}$$

Integrating in time, we have

$$\begin{aligned}
A(t, \nu) &\le C\nu \int_0^t \left(\left\|\partial_2 u^1\right\|^2_{L^2(\Gamma_\nu)} + \left\|\partial_2 u^2\right\|^2_{L^2(\Gamma_\nu)}\right) + C\nu^{\frac{1}{2}} \int_0^t \left\|\partial_2 u^2\right\|_{L^2(\Gamma_\nu)} \\
&\quad + \frac{\nu}{2} t + \frac{\nu}{2}\left(\nu \int_0^t \|\nabla u\|^2\right) + \nu^{\frac{1}{2}} \int_0^t \left\|\partial_2 u^1\right\|_{L^2(\Gamma_\nu)} \\
&\le C\nu \int_0^t \sum_{j=1}^2 \left\|\partial_2 u^j\right\|^2_{L^2(\Gamma_\nu)} + C(T)\nu + \sum_{j=1}^2 t^{\frac{1}{2}} \left(\nu \int_0^t \left\|\partial_2 u^j\right\|^2_{L^2(\Gamma_\nu)}\right)^{\frac{1}{2}},
\end{aligned}$$

where we used (1.5). The assumption (1) insures that $A(t, \nu) \to 0$ as $\nu \to 0$, which gives (1.1) by Theorem 3.3. □

6. Using a Wang layer

Theorem 3.3 applied to a Wang layer easily yields sufficient conditions for the vanishing viscosity limit to hold for such a layer, leading to Theorem 6.1.

Theorem 6.1. *Let δ be a Wang width as in Definition 3.7. If*

$$\int_0^t \int_{\Gamma_\delta} \frac{1}{\delta} |u^1 u^2| \to 0 \text{ or } \int_0^t (u^1 u^2, \partial_2 z^1) \to 0 \text{ as } \nu \to 0 \tag{6.1}$$

then (1.1) holds. Conversely, if (1.1) holds (or simply (1.2) when $g \equiv 0$) (6.1) holds for any Wang width.

Proof. The two conditions in (6.1) are equivalent by the argument that led to (4.5). So assume that (6.1) holds along with (3.11). Then, $\nu \int_0^t |(\nabla u, \nabla z)|$ vanishes as $\nu \to 0$ by (4.7). (Note that since $\delta(\cdot, \nu)$ is increasing, $\delta(\cdot, \nu) \to 0$ in $L^\infty(0, T)$.) The result then follows by Theorem 3.3. □

A simple and direct use of a Wang layer yields Theorem 6.2.

Theorem 6.2. *Let δ be a Wang width as in Definition 3.7. If*

$$\frac{1}{\nu} \int_0^t \left\|u^1\right\|_{L^2(\Gamma_\delta)}^2 \to 0 \text{ or } \frac{1}{\nu} \int_0^t \left\|u^2\right\|_{L^2(\Gamma_\delta)}^2 \to 0 \text{ as } \nu \to 0 \tag{6.2}$$

then (1.1) holds.

Proof. We have,

$$\begin{aligned}
|(u^1 u^2, \partial_2 z^1)| &\le \left\|\partial_2 z^1\right\|_{L^\infty} \left\|u^1 u^2\right\|_{L^1(\Gamma_\delta)} \le \frac{C}{\delta} \left\|u^1 u^2\right\|_{L^1(\Gamma_\delta)} \\
&\le \frac{C}{\delta} \left\|u^1\right\|_{L^2(\Gamma_\delta)} \left\|u^2\right\|_{L^2(\Gamma_\delta)} \le \frac{C}{\delta} \left\|u^1\right\|_{L^2(\Gamma_\delta)} C\delta \left\|\partial_2 u^2\right\|_{L^2(\Gamma_\delta)} \\
&= C \left\|u^1\right\|_{L^2(\Gamma_\delta)} \left\|\partial_1 u^1\right\|_{L^2(\Gamma_\delta)},
\end{aligned}$$

where we used $(4.1)_2$ of Corollary 4.2. Hence,

$$\begin{aligned}
\int_0^t (u^1 u^2, \partial_2 z^1) &\le C \int_0^t \left\|u^1\right\|_{L^2(\Gamma_\delta)} \left\|\partial_1 u^1\right\|_{L^2(\Gamma_\delta)} \\
&\le C \left(\int_0^t \left\|u^1\right\|_{L^2(\Gamma_\delta)}^2 \right)^{\frac{1}{2}} \left(\int_0^t \left\|\partial_1 u^1\right\|_{L^2(\Gamma_\delta)}^2 \right)^{\frac{1}{2}} \\
&= C \left(\nu^{-1} \int_0^t \left\|u^1\right\|_{L^2(\Gamma_\delta)}^2 \right)^{\frac{1}{2}} \left(\nu \int_0^t \left\|\partial_1 u^1\right\|_{L^2(\Gamma_\delta)}^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

The second factor on the right-hand side is bounded by (1.4) or (1.5). The result for the first condition in (6.2) thus follows from Theorem 6.1, while the result for the second condition follows by replacing u^1 by u^2 in the argument above (noting that the key use of Poincaré's inequality is unchanged). □

A more subtle use of the infinitesimally thicker boundary layer leads to the result of Xiaoming Wang in Theorem 6.4. (Theorem 6.4 extends Wang's result to allow a time-varying boundary layer.) Its proof is based upon the following estimates:

Lemma 6.3. *Let δ as in Definition 2.2 be a the width of a boundary layer. Then*

$$|(u^1u^2, \partial_2 z^1)| \le \frac{\nu}{4} \|\nabla u\|_{L^2(\Gamma_\delta)}^2 + \frac{C\nu}{\delta} + C\left(\frac{\delta}{\nu}\right)^2 \left(\nu \|\partial_1 u^1\|_{L^2(\Gamma_\delta)}^2\right) \tag{6.3}$$

and

$$\begin{aligned} |(u^1u^2, \partial_2 z^1)| \le \frac{\nu}{4} \|\nabla u\|_{L^2(\Gamma_\delta)}^2 + C\|w\|^2 + C\delta^{\frac{1}{2}} + \left(\frac{\delta}{\nu^{\frac{1}{4}}}\right)^{\frac{4}{3}} \\ + C\left(\frac{\delta}{\nu}\right)^4 \left(\nu \|\partial_1 u^2\|_{L^2(\Gamma_\delta)}^2\right), \end{aligned} \tag{6.4}$$

Proof. To prove (6.3), we start with (4.5):

$$\begin{aligned} |(u^1u^2, \partial_2 z^1)| &\le \frac{C}{\delta}\int_{\Gamma_\delta} |u^1u^2| \le C\delta^{-1}\|u^1\|_{L^2(\Gamma_\delta)}\|u^2\|_{L^2(\Gamma_\delta)} \\ &\le \frac{C}{\delta}\left(\delta \left\|\partial_2 u^1\right\|_{L^2(\Gamma_\delta)} + \delta^{\frac{1}{2}}\right)\delta \left\|\partial_2 u^2\right\|_{L^2(\Gamma_\delta)} \\ &= C\nu^{\frac{1}{2}} \left\|\partial_2 u^1\right\|_{L^2(\Gamma_\delta)} \frac{\delta}{\nu^{\frac{1}{2}}} \left\|\partial_1 u^1\right\|_{L^2(\Gamma_\delta)} + C\frac{\delta}{\nu^{\frac{1}{2}}}\left(\frac{\nu}{\delta}\right)^{\frac{1}{2}} \left\|\partial_1 u^1\right\|_{L^2(\Gamma_\delta)} \\ &\le \frac{\nu}{4}\left\|\partial_2 u^1\right\|_{L^2(\Gamma_\delta)}^2 + C\frac{\delta^2}{\nu}\left\|\partial_1 u^1\right\|_{L^2(\Gamma_\delta)}^2 + C\frac{\nu}{\delta}, \end{aligned}$$

where we paralleled the argument in (5.1), but using $\partial_2 u^2 = -\partial_1 u^1$ and applying Young's inequality asymmetrically.

The proof of (6.4) is more involved. We first make the decomposition,

$$-(u^1u^2, \partial_2 z^1) = (u^1 \partial_2 u^2, z^1) + (\partial_2 u^1 u^2, z^1),$$

where we integrated by parts, using that $u^2 = 0$ on $\partial\Omega$. For the first term in $-(u^1u^2, \partial_2 z^1)$, we use that $\operatorname{div} u = 0$ to obtain

$$\begin{aligned} (u^1\partial_2 u^2, z^1) &= -(u^1\partial_1 u^1, z^1) = -\frac{1}{2}(\partial_1 (u^1)^2, z^1) = \frac{1}{2}((u^1)^2, \partial_1 z^1) \\ &= \frac{1}{2}((w^1)^2, \partial_1 z^1) + (u^1\overline{u}^1, \partial_1 z^1) - \frac{1}{2}((\overline{u}^1)^2, \partial_1 z^1), \end{aligned}$$

where, since we integrated by parts in the tangential variable, we needed no boundary condition. Hence,

$$\begin{aligned} |(u^1\partial_2 u^2, z^1)| &\le \frac{1}{2}\|w\|^2 \left\|\partial_1 z^1\right\|_{L^\infty} + \|u\| \|\overline{u}\|_{L^\infty} \left\|\partial_1 z^1\right\| + \frac{1}{2}\|\overline{u}\|_{L^\infty}\|\overline{u}\| \left\|\partial_1 z^1\right\| \\ &\le C\|w\|^2 + C\delta^{\frac{1}{2}}. \end{aligned}$$

For the second term in $-(u^1u^2, \partial_2 z^1)$, we have

$$|(\partial_2 u^1 u^2, z^1)| \le \|\partial_2 u^1\|_{L^2(\Gamma_\delta)}\|u^2 z^1\|.$$

Defining β by

$$\beta(t, x_1, x_2) := -\int_{x_2}^{\delta(t,\nu)} (z^1(t, x_1, y))^2\, dy,$$

we see that

$$\partial_2\beta = (z^1)^2$$

and

$$\|\beta\|_{L^\infty(\Gamma_\delta)} \le \delta\|z^1\|^2_{L^\infty} \le C\delta,$$
$$\|\partial_1\beta\|_{L^\infty(\Gamma_\delta)} \le \delta\|\partial_1 z^1\|^2_{L^\infty} \le C\delta.$$

Then,

$$\begin{aligned}
\|u^2 z^1\|^2 &= \int_{\Gamma_\delta} (u^2)^2(z^1)^2 = \int_{\partial\Omega}\int_0^\delta (u^2(t,x_1,x_2))^2 \partial_{x_2}\beta(t,x_1,y)\,dx_2\,dx_1 \\
&= -\int_{\partial\Omega}\int_0^\delta \partial_{x_2}(u^2(t,x_1,x_2))^2\beta(t,x_1,x_2)\,dx_2\,dx_1 \\
&= -\int_{\Gamma_\delta}\partial_2(u^2)^2\beta = -2\int_{\Gamma_\delta} u^2\partial_2 u^2\beta = 2\int_{\Gamma_\delta} u^2\partial_1 u^1\beta \\
&= -2\int_{\Gamma_\delta} u^1\partial_1(u^2\beta) = -2\int_{\Gamma_\delta} u^1\partial_1 u^2\beta - 2\int_{\Gamma_\delta} u^1 u^2\partial_1\beta.
\end{aligned}$$

In both integrations by parts, we used that $u^2 = 0$ on $\partial\Omega$, the outer component of $\partial\Gamma_\delta$, while $\beta = 0$ on the inner component of $\partial\Gamma_\delta$.

Proceeding,

$$-2\int_{\Gamma_\delta} u^1u^2\partial_1\beta \le 2\|u^1\|\|u^2\|\,\|\partial_1\beta\|_{L^\infty(\Gamma_\delta)} \le C\delta^2\|\partial_2 u^2\|_{L^2(\Gamma_\delta)},$$

$$-2\int_{\Gamma_\delta} u^1\partial_1 u^2\beta \le 2\|u^1\|\|\partial_1 u^2\|_{L^2(\Gamma_\delta)}\|\beta\|_{L^\infty(\Gamma_\delta)} \le C\delta^2\|\partial_2 u^1\|_{L^2(\Gamma_\delta)}\left\|\partial_1 u^2\right\|_{L^2(\Gamma_\delta)}.$$

Thus,

$$\|u^2z^1\| \le C\delta\,\|\nabla u\|^{\frac12}_{L^2(\Gamma_\delta)}\left(\left\|\partial_1 u^2\right\|^{\frac12}_{L^2(\Gamma_\delta)}+1\right) \quad \le C\delta\,\|\nabla u\|^{\frac12}_{L^2(\Gamma_\delta)}\left(\left\|\partial_1 u^2\right\|^{\frac12}_{L^2(\Gamma_\delta)}+1\right)$$

and therefore,

$$\begin{aligned}
|(\partial_2 u^1 u^2, z^1)| &\le \|\partial_2 u^1\|_{L^2(\Gamma_\delta)}\|u^2z^1\| \\
&\le C\delta\,\|\nabla u\|^{\frac32}_{L^2(\Gamma_\delta)}\left(\left\|\partial_1 u^2\right\|^{\frac12}_{L^2(\Gamma_\delta)}+1\right) \\
&= C\delta\,\|\nabla u\|^{\frac32}_{L^2(\Gamma_\delta)}\left\|\partial_1 u^2\right\|^{\frac12}_{L^2(\Gamma_\delta)} + C\delta\,\|\nabla u\|^{\frac32}_{L^2(\Gamma_\delta)}.
\end{aligned}$$

Applying Young's inequality,

$$\begin{aligned}
C\delta\,\|\nabla u\|^{\frac32}_{L^2(\Gamma_\delta)}\left\|\partial_1 u^2\right\|^{\frac12}_{L^2(\Gamma_\delta)} &= C\left(\nu^{\frac34}\,\|\nabla u\|^{\frac32}_{L^2(\Gamma_\delta)}\right)\left(\frac{\delta}{\nu}\nu^{\frac14}\left\|\partial_1 u^2\right\|^{\frac12}_{L^2(\Gamma_\delta)}\right) \\
&\le \frac{\nu}{8}\|\nabla u\|^2_{L^2(\Gamma_\delta)} + \frac{\delta^4}{\nu^4}\left(\nu\left\|\partial_1 u^2\right\|^2_{L^2(\Gamma_\delta)}\right)
\end{aligned}$$

and

$$C\delta \left\|\nabla u\right\|_{L^2(\Gamma_\delta)}^{\frac{3}{2}} = C\frac{\delta}{\nu^{\frac{1}{4}}}\nu^{\frac{1}{4}} \left\|\nabla u\right\|_{L^2(\Gamma_\delta)}^{\frac{3}{2}} \le \frac{\nu}{8} \left\|\nabla u\right\|_{L^2(\Gamma_\delta)}^2 + C\left(\frac{\delta}{\nu^{\frac{1}{4}}}\right)^{\frac{4}{3}}.$$

Collecting these bounds gives (6.4). □

Theorem 6.4. *[Wang [36]] Let δ be a Wang width as in Definition 3.7 with*

$$\nu \int_0^T \|\partial_1 u^1\|_{L^2(\Gamma_\delta(s,\nu))}^2 \, ds \to 0 \text{ as } \nu \to 0 \tag{6.5}$$

or

$$\nu \int_0^T \|\partial_1 u^2\|_{L^2(\Gamma_\delta(s,\nu))}^2 \, ds \to 0 \text{ as } \nu \to 0. \tag{6.6}$$

Then the strong vanishing viscosity limit in (1.1) holds. Conversely, if (1.1) holds (or simply (1.2) when $g \equiv 0$) then (6.5) and (6.6) hold for any Wang width.

Proof. For each of (6.5) and (6.6), the converse follows immediately from Lemma 3.2 or the assumption in (1.1), which implies (1.3).

For the forward direction, we know that (3.10) holds simply because δ is a Wang width (see the comment following Definition 3.7). It remains to show that (3.9) holds, for it will follow that $A \to 0$ as in Theorem 3.3.

Assume, first, that (6.5) holds. Integrating (6.3) over time gives

$$\int_0^T |(\partial_2 u^1 u^2, z^1)| \le \frac{\nu}{4} \int_0^t \|\nabla u\|_{L^2(\Gamma_\delta)}^2 + C\nu \int_0^t \frac{ds}{\delta(s,\nu)} + C\frac{\delta^2}{\nu^2} F_\nu(\delta) \tag{6.7}$$

(we used here that $\delta(s,\nu) \le \delta(t,\nu)$), where

$$F_\nu(t,\delta) := \nu \int_0^t \|\partial_1 u^1\|_{L^2(\Gamma_\delta)}^2.$$

Note that even in 2D, we cannot say that $F_\nu(t,\delta)$ is increasing in ν even for *fixed* δ; we would be hard pressed even to show that it is continuous.

Let us agree to call the function δ for which the condition in (6.5) is assumed to hold, δ_0; this means that we are given that $F_\nu(t, \delta_0(t,\nu)) \to 0$ as $\nu \to 0$. We will show that there exists a possibly smaller Wang width, which we will relabel δ, for which $\frac{\delta^2(t,\nu)}{\nu^2} F_\nu(t, \delta(t,\nu)) \to 0$ as $\nu \to 0$.

As long as $\delta \le \delta_0$ (as functions of ν), we will have

$$\frac{\delta^2}{\nu^2} F_\nu(t,\delta) \le \frac{\delta^2}{\nu^2} F_\nu(t,\delta_0).$$

So let

$$\delta(t,\nu) = \min\left\{\delta_0(t,\nu), \inf_{s\in[t,T]} \frac{\nu}{F_\nu(s,\delta_0(s,\nu))^{\frac{1}{4}}}\right\}, \tag{6.8}$$

which we note is continuous at $\nu = 0$ with $\delta(t, 0) = 0$, and is increasing in t. Then,

$$\frac{\nu}{\delta(t,\nu)} \le \max\left\{\frac{\nu}{\delta_0(t,\nu)}, F_\nu(t, \delta_0(t,\nu))^{\frac{1}{4}}\right\} \to 0,$$

$$\begin{aligned}\frac{\delta(t,\nu)^2}{\nu^2} F_\nu(t, \delta(t,\nu)) \le \frac{\delta(t,\nu)^2}{\nu^2} F_\nu(t, \delta_0(t,\nu)) \le \frac{\frac{\nu^2}{\sqrt{F_\nu(t,\delta_0(t,\nu))}}}{\nu^2} F_\nu(t, \delta_0(t,\nu))\\ = \sqrt{F_\nu(t, \delta_0(\nu))} \to 0\end{aligned}$$

as $\nu \to 0$, and the convergence is uniform in time. Also,

$$\int_0^T \frac{\nu}{\delta(t,\nu)}\, dt \le \max\left\{\int_0^T \frac{\nu}{\delta_0(t,\nu)}\, dt, \int_0^T F_\nu(\delta_0(t,\nu))^{\frac{1}{4}}\, dt\right\}.$$

As $\nu \to 0$, the first integral on the right-hand side vanishes because δ_0 is a Wang width, while the second integral vanishes because $F_\nu(\delta_0(t,\nu)) \le F_\nu(\delta_0(T,\nu)) \to 0$. Hence, we see that δ is a Wang width, so we can apply Theorem 3.3 to the bound in (6.7) using (3.8) to conclude that (1.1) holds.

Now assume that (6.6) holds. Integrating (6.4) over time, we have

$$\begin{aligned}\int_0^T |(\partial_2 u^1 u^2, z^1)| \le \frac{\nu}{4}\int_0^T \|\nabla u\|^2_{L^2(\Gamma_\delta)} + C\left(\frac{\delta}{\nu^{\frac{1}{4}}}\right)^{\frac{4}{3}} T\\ + C\frac{\delta^4}{\nu^4}\int_0^T \left(\nu \left\|\partial_1 u^2\right\|^2_{L^2(\Gamma_\delta)}\right).\end{aligned}$$

We can absorb the first term above by virtue of (3.8), and, if needed, we can always decrease δ to be less than $\nu^{\frac{1}{4}}$ while still keeping the conditions in (3.11) and in Definition 2.2 (2), insuring that the second term above vanishes with ν. The final term we treat in the same manner as we treated the final term in (6.7), writing it in the form, $C\frac{\delta^4}{\nu^4}F_\nu(\delta)$, where now

$$F_\nu(\delta) := \nu\int_0^T \|\partial_1 u^2\|^2_{L^2(\Gamma_\delta)}.$$

Applying Theorem 3.3 using (3.8) to conclude that (1.1) holds, the proof of sufficiency of (6.6) is complete. □

Remark 6.5. *The construction in (6.8) is a little easier to understand when δ is time-independent. We set*

$$\delta(\nu) = \min\left\{\delta_0(\nu), \frac{\nu}{F_\nu(T, \delta_0(\nu))^{\frac{1}{4}}}\right\}.$$

Then δ is continuous at zero with $\delta(0) = 0$. Then since $F_\nu(T, \delta_0(\nu)) \to 0$ by assumption, $\delta(\nu)$ is a Wang width, and

$$\frac{\delta(\nu)^2}{\nu^2} F_\nu(t, \delta(\nu)) \le \frac{\delta(\nu)^2}{\nu^2} F_\nu(t, \delta_0(\nu)) \le \frac{\frac{\nu^2}{\sqrt{F_\nu(t,\delta_0(\nu))}}}{\nu^2} F_\nu(t, \delta_0(\nu))$$
$$= \sqrt{F_\nu(t, \delta_0(\nu))} \le \sqrt{F_\nu(T, \delta_0(\nu))} \to 0.$$

Remark 6.6. *In [36], Wang uses an energy argument that starts with the equation for what we are calling $\widetilde{w}$ (rather than w, as we did) then multiplies by $\widetilde{w}$ and integrates over time and space. The introduction of F_ν and the use of β, which are at the heart of the proof, are adopted from [36]. Also, Wang uses a different corrector, though all the necessary estimates hold for the Kato corrector we are using.*

7. Vortex sheet on the boundary

Let $\mathcal{M}(\Omega)$ be the space of finite Borel signed measures on $\overline{\Omega}$—$\mathcal{M}(\Omega)$ is the dual space of $C(\overline{\Omega})$. Let μ in $\mathcal{M}(\overline{\Omega})$ be the measure supported on Γ for which $\mu|_\Gamma$ corresponds to Lebesgue measure on Γ (arc length, since $d = 2$). Then μ is also a member of $H^1(\Omega)'$.

The proof of Theorem 7.1 for $g \equiv 0$ is given in [20]. Its proof for a general g requires only the trivial replacement of $\overline{u}$ by $\overline{u} - g$ in the arguments in [20]. Note that the presence or absence of an energy defect as in (1.3) does not affect the arguments in [20]. In some sense, this is because a corrector is not employed in [20].

Theorem 7.1. *The following conditions are equivalent, when Ω is simply connected:*

(1) (1.1) holds,
(2) $\omega \to \overline{\omega} + ((g - \overline{u}) \cdot \boldsymbol{\tau})\mu$ in $(H^1(\Omega))'$ uniformly on $[0, T]$,
(3) $\omega \to \overline{\omega}$ in $H^{-1}(\Omega)$ uniformly on $[0, T]$.

8. Weaker convergence

The question of whether the vanishing viscosity limit in the sense of (1.2) holds has been a long open problem in mathematical fluid mechanics, and a satisfactory answer to it appears as out or reach now as it did 100 years ago. Recently, in [6], Constantin and Vicol initiated a program to consider whether perhaps a weaker type of convergence to a weaker type of solution to the Euler equations might hold. As relates to 2D Euler solutions, the authors of [6] consider the condition:

> There exists a positive sequence (ν_n) converging to zero such that for any compact $K \subseteq \Omega$ there exists $\mathcal{E}_K \ge 0$ such that
>
> $$\|\omega_{\nu_n}\|^2_{L^\infty(0,T;L^2(K))} \le \mathcal{E}_K \text{ for all } n. \tag{8.1}$$

Here, $\mathcal{E}_K$ can depend on u^0, T, and K, but nothing else (in particular, no dependence on ν is allowed). By virtue of u_0 having energy bounded uniformly over $\nu > 0$, there always exist a weak limit of $(u_0)_{\nu>0}$ in $L^2(0, T; L^2(\Omega))$. They show that if (8.1) holds then any such weak limit is a very weak solution to the Euler equations. These solutions satisfy no initial or boundary conditions and are possibly dissipative, though the energy is bounded. The condition in (8.1) is sufficient for weak convergence, but not, or at least not shown to be, necessary.

An assumption on the smoothness of u^0 is not needed, and so not made, in [6], and, indeed, one might expect that if the type of weak convergence in [6] holds it has nothing to do with the smoothness or lack thereof of the initial data. Nonetheless, it is interesting that when the initial data is smooth, (8.1) implies no Kato-like condition, and no Kato-like condition implies (8.1).

Finally, we note that in [2], Dongho Chae establishes that, in fact, the vanishing viscosity limit holds for no-slip boundary conditions; the convergence being, however, in a weak sense to a measure-valued solution to the Euler equations. Measure-valued solutions, a concept developed in [7, 8], are even weaker than those of [6].

9. Well-posedness of (NS_g)

We now give the proof of Lemma 1.1 and use it to prove the existence of solutions to (NS_g), Proposition 1.2.

Proof of Lemma 1.1. Let $\overline{g}$ solve the stationary Stokes problem,

$$\begin{cases} \nabla q = \Delta \overline{g} & \text{in } \Omega, \\ \operatorname{div} \overline{g} = 0 & \text{in } \Omega, \\ \overline{g} = g & \text{on } \partial\Omega. \end{cases}$$

It follows that $\overline{g} \in C^\infty(\overline{\Omega})$ (see, for instance, Theorem IV.7.1 of [11].) We see also that $\partial_t \overline{g}$ satisfies the stationary Stokes problem, $\nabla \partial_t q = \Delta \partial_t \overline{g}$, $\operatorname{div} \partial_t \overline{g} = 0$ in Ω, $\partial_t \overline{g} = \partial_t g$ on $\partial\Omega$, so $\overline{g} \in C^\infty([0, \infty) \times \overline{\Omega})$, is divergence-free, and equals g on $\partial\Omega$.

If, in addition, $u^0|_{\partial\Omega} = g(0)$, then $\overline{g} + u^0 - \overline{g}(0) \in C^\infty([0, \infty) \times \overline{\Omega})$, is divergence-free, equals g on $\partial\Omega$ and equals u^0 at time zero.

Relabeling by setting $g = \overline{g}$ or $g = \overline{g} + u^0 - \overline{g}(0)$ completes the proof. □

Proof of Proposition 1.2. With g as in Lemma 1.1, we can rewrite (NS_g) as

$$\partial_t r + \partial_t g + r \cdot \nabla r + r \cdot \nabla g + g \cdot \nabla r + g \cdot \nabla g + \nabla p_g = \nu \Delta r + \nu \Delta g, \quad (9.1)$$

where $r := u_g - g$, noting that $r = 0$ on $\partial\Omega$. Hence, we look for a weak solution to

$$\begin{cases} \partial_t r + r \cdot \nabla r + r \cdot \nabla g + g \cdot \nabla r + \nabla p_g = \nu \Delta r + F_g & \text{on } \Omega, \\ \operatorname{div} r = 0 & \text{on } \Omega, \\ r(0) = u^0 - g(0) & \text{on } \Omega, \\ r = 0 & \text{on } \partial\Omega. \end{cases}$$

This is a linear perturbation of the Navier-Stokes equations with the forcing term, F_g. Existence and, in 2D, uniqueness, is standard (see, for instance, [14], where a similar perturbation is worked out in detail).

The energy inequality that results we can derive formally by multiplying the equation for r by r and integrating over Ω:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|r\|^2 + \nu\|\nabla r\|^2 &= -(r \cdot \nabla g, r) + (F_g, r) \\ &\le \|\nabla g\|_{L^\infty}\|r\|^2 + \|F_g\|\|r\| \le \frac{\|F_g\|^2}{2} + \left(\|\nabla g\|_{L^\infty} + \frac{1}{2}\right)\|r\|^2 \end{aligned}$$

so that

$$\frac{d}{dt}\|r\|^2 + 2\nu\|\nabla r\|^2 \le \|F_g\|^2 + (2\|\nabla g\|_{L^\infty} + 1)\|r\|^2.$$

Integrating in time, we see that

$$\begin{aligned} &\|r(t)\|^2 + 2\nu\int_0^t \|\nabla r\|^2 \\ &\qquad \le \|r(0)\|^2 + \int_0^t \|F_g\|^2 + \int_0^t (2\|\nabla g\|_{L^\infty} + 1)\|r\|^2. \end{aligned}$$

Applying Gronwall's lemma gives

$$\|r(t)\|^2 + 2\nu\int_0^t \|\nabla r\|^2 \le \left(\|r(0)\|^2 + \int_0^t \|F_g\|^2\right) e^{\int_0^t (2\|\nabla g\|_{L^\infty} + 1)}. \tag{9.2}$$

Using (9.2) with $\|r(0)\|^2 \le 2\|u^0\|^2 + 2\|g_0\|^2$ and

$$\|u_g(t)\|^2 + 2\nu\int_0^t \|\nabla u_g\|^2 \le 2\left(\|r(t)\|^2 + 2\nu\int_0^t \|\nabla r\|^2 + \|g(t)\|^2 + 2\nu\int_0^t \|\nabla g\|^2\right)$$

yields the bound in (1.5). Moreover, the continuity in time and higher regularity properties of r follow from (9.2) in the standard way, which yield the corresponding properties for $u_g = r + g$. □

10. How might convergence happen?

We return to writing u_g rather than simply u, as we did in Sections 3 to 7.

If we choose to set $g = \overline{u}|_{\partial\Omega}$, we see that $u_g = u_{\overline{u}} = \overline{u}$ on $\partial\Omega$. This eliminates the boundary term in the basic energy argument, giving $u_{\overline{u}} \to \overline{u}$ as in the boundary-free case (though only in the $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$-norm, not in higher norms, since there is still no control of vorticity production of $u_{\overline{u}}$ on the boundary). Thus, we easily obtain Theorem 10.1.

Theorem 10.1. *We have*

$$u_{\overline{u}} \to \overline{u} \text{ in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$$

with

$$\|u_{\overline{u}}(t) - \overline{u}(t)\| \le C\nu e^{Ct}, \quad \int_0^t \|\nabla(u_{\overline{u}}(s) - \overline{u}(s))\|^2 \, ds \le C\nu t^{\frac{1}{2}} e^{Ct}.$$

Proof. Let $w = u_{\overline{u}} - \overline{u}$. Then

$$\begin{aligned}(\partial_t w, w) + (w \cdot \nabla \overline{u}, w) + (u_{\overline{u}} \cdot \nabla w, w) + (\nabla(q - \overline{p}), w) &= \nu(\Delta u_{\overline{u}}, w) \\ &= \nu(\Delta w, w) + \nu(\Delta \overline{u}, w).\end{aligned}$$

The third and fourth terms on the left-hand side vanish after integrating by parts. Also, $w = 0$ on $\partial\Omega$ so we can integrate the first term on the right-hand side by parts to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2 + \nu\|\nabla w\|^2 &= -(w \cdot \nabla \overline{u}, w) + \nu(\Delta \overline{u}, w) \\ &\le \frac{\nu^2}{2}\|\Delta \overline{u}\|_{L^2}^2 + \frac{1}{2}\|w\|^2 + \|\nabla \overline{u}\|_{L^\infty}\|w\|^2.\end{aligned}$$

It follows from Gronwall's inequality that

$$\|w(t)\|^2 + 2\nu \int_0^t \|\nabla w\|^2 \le C\nu^2 t e^{Ct}$$

from which the convergence with the stated rates follow. □

(It is also possible to show that if $g = g(\nu) \to \overline{u}$ in, say, $L^\infty([0,T] \times \partial\Omega)$, then $u_g \to \overline{u}$ in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ as $\nu \to 0$. The idea is that the bounds on the z^1 component of Kato's corrector decrease in proportion to $\|g - \overline{u}\|_{L^\infty}$; incorporating these bounds into the energy argument in Proposition 3.1 gives the convergence.)

A simple corollary of Theorem 10.1 is the following:

Corollary 10.2. *We have,*

$$u_g \to \overline{u} \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ as } \nu \to 0$$

if and only if

$$u_g - u_{\overline{u}} \to 0 \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ as } \nu \to 0.$$

Proof. By the triangle inequality,

$$\begin{aligned}\|u_g - \overline{u}\| &\le \|u_g - u_{\overline{u}}\| + \|u_{\overline{u}} - \overline{u}\|, \\ \|u_g - u_{\overline{u}}\| &\le \|u_g - \overline{u}\| + \|u_{\overline{u}} - \overline{u}\|,\end{aligned}$$

and the result follows from Theorem 10.1. □

The key to the proof of Corollary 10.2 was that (1.1) holds for $u_{\overline{u}}$. We showed in Section 7 that if (1.1) holds for u_g then a vortex sheet forms on the boundary with strength proportional to $\overline{u} - g$. Hence, we know that no vortex sheet forms in the limit as $\nu \to 0$ for $u_{\overline{u}}$, which is what distinguishes it among all possible u_g.

Now consider the issue of the convergence of $u_g - u_{\overline{u}}$ to 0. Let $w = u_g - u_{\overline{u}}$. Then

$$(\partial_t w, w) + (w \cdot \nabla u_{\overline{u}}, w) + (u_g \cdot \nabla w, w) + (\nabla(p - q), w) = \nu(\Delta w, w).$$

The third and fourth terms on the left-hand side vanish after integrating by parts. We integrate the right-hand side by parts to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|w\right\|^2 + \nu \int_0^T \left\|\nabla w\right\|^2 &= -(w \cdot \nabla u_{\overline{u}}, w) + \nu \int_{\partial\Omega} (\nabla w \cdot \boldsymbol{n}) \cdot w \\ &= -(w \cdot \nabla u_{\overline{u}}, w) - \nu \int_{\partial\Omega} (\nabla w \cdot \boldsymbol{n}) \cdot (g - \overline{u}).\end{aligned}$$

Now, to obtain convergence we need control both on $\nabla u_{\overline{u}}$ in something close to $L^1(0, T; L^\infty)$, as well as control on the boundary term. So proving $u_g - u_{\overline{u}} \to 0$ appears to be even more difficult than proving $u_g \to \overline{u}$.

Moving into the realm of speculation, consider the following opposed possibilities:

- *Positive*: (1.1) holds for all smooth u_0 and smooth g.
- *Negative*: (1.1) fails to hold for generic u_0 and generic g.

The qualification "generic," is not meant in any precise technical way, but is to rule out, for instance, initial data for which $\overline{u}$ vanishes on the boundary or which has some degree of analyticity.

Whether one or the other of these possibilities holds (they are not exhaustive, so neither may hold) is related to the question, "Is the solution to (NS) at low viscosity indifferent to the boundary value g, or is it sensitive to it?" Indifference would support the positive possibility, sensitivity would support the negative (or at least non-positive) possibility. We can give some support for each position:

> *Indifferent*: As $\nu \to 0$, the imposition of $u = g$ on the boundary should become less important, since as the fluid becomes less viscous, the boundary forcing should have less effect on it, so less vorticity should be shed off the boundary and transported into the bulk of the fluid. Nonetheless, there is enough shedding of vorticity for a vortex sheet to form at the boundary. Indeed, this is shown to be the case for radially symmetric solutions in [9, 10], and is likely the case for other scenarios in which the non-linearity is weakened or eliminated (though such examples do not seem to have been worked out explicitly in the literature, since $g = 0$ is generally assumed).
>
> *Sensitive*: When $g = \overline{u}|_{\partial\Omega}$, the shedding of vorticity off the boundary is shut down (at least at the level seen by the energy of the fluid) and the vanishing viscosity limit holds.

Weaker than either of the two positions is the following conjecture:

Conjecture 1. *Generically, (1.1) holds for* u_0 *if and only if (1.1) holds for any* $g \in (C^\infty([0, T] \times \partial\Omega))^d$ *with* $g \cdot \boldsymbol{n} = 0$ *on* $\partial\Omega$.

This conjecture is saying, in effect, that except in very special circumstances, the vanishing viscosity limit can hold only if the indifferent position is correct, though it takes no position on whether the vanishing viscosity limit holds generically at all. A motivation for this conjecture is that, as we have seen, the form of the Kato and Kato-like conditions are all indifferent to the choice of g.

Consider the special case where $u^0 \equiv 0$, so $\overline{u} \equiv 0$ is a (stationary) solution to the Euler equations. There is an incompatibility in the boundary conditions for (NS_g) at time zero when $g \not\equiv 0$, so the solution to the Navier-Stokes equations does not vanish. This leads to a special case of the vanishing viscosity limit not included in the classical setting (where $g \equiv 0$ would trivialize to $u_0 \equiv \overline{u} \equiv 0$). There are only two possibilities:

- *Positive*: $u_g \to 0$ as $\nu \to 0$ for all smooth g.
- *Negative*: there exists smooth g such that $u_g \not\to 0$ as $\nu \to 0$.

A route to a positive answer would be to find a more optimum bound on the energy of u_g than that in (1.5), one that would lead to $\|u_g(t)\| \to 0$ as $\nu \to 0$. But this is entirely equivalent, as we can see from Theorem 3.3, to obtaining a bound on $A(t, \nu)$ that insures it vanishes with ν. Even in simple geometries such as a disk with constant $g \cdot \boldsymbol{\tau}$, then, even this simplified form of the vanishing viscosity limit question seems out of reach.

To gain a little insight, though, let us consider a linearized version of (NS_g) in which we drop the term $u_g \cdot \nabla u_g$ in (NS_g): that is, the time-dependent Stokes problem, $\partial_t u_g + \nabla p_g = \nu \Delta u_g$. We will assume, however, that g is time-independent. We begin by making the same energy argument as in the proof above of Proposition 1.2, but instead of using g itself, we use a "corrector," z. We define z as in Section 2, using $v = g$ in place of (2.1), and with δ to be chosen below. (Hence, the corrector is "correcting" only the boundary value of g.) We can see from Lemma 1.1 and Theorem 2.5 that

$$\|z\| \le C\delta^{\frac{1}{2}}, \quad \nu \|\nabla z\|^2 \le C\frac{\nu}{\delta}.$$

Set $r = u_g - z$ and choose $\delta = \nu^{1/2}$. Because there are no nonlinear terms and $\partial_t z$ vanishes, in place of (9.1) we have

$$\partial_t r + \nabla p_g = \nu \Delta r + \nu \Delta z.$$

Multiplying by r and integrating over the domain, we have

$$\frac{1}{2}\frac{d}{dt}\|r\|^2 + \nu \|\nabla r\|^2 = \nu(\nabla z, \nabla r) \le \frac{\nu}{2}\|\nabla z\|^2 + \frac{\nu}{2}\|\nabla r\|^2.$$

We conclude that

$$\frac{d}{dt}\|r\|^2 + \nu \|\nabla r\|^2 \le \nu \|\nabla z\|^2 \le C\frac{\nu}{\delta}.$$

Integrating in time, we see that

$$\|r(t)\|^2 + \nu \int_0^t \|\nabla r\|^2 \le \|r(0)\|^2 + C\frac{\nu}{\delta}t = \|z\|^2 + C\frac{\nu}{\delta}t \le C\delta + C\frac{\nu}{\delta}t$$
$$\le C(1+t)\nu^{\frac{1}{2}},$$

where in the last step we chose $\delta = \nu^{\frac{1}{2}}$ to balance the ν-dependence of the two terms.

Hence, for the linearized problem, at least in the special case in which the boundary data is constant in time, we obtain the positive possibility. Of course, this linear situation should not dominate our intuition: the question is whether the nonlinear, convective term disrupts this linear behavior sufficiently to obtain a negative answer.

Appendix A. Proof of corrector estimates

Proof of Theorem 2.5. Working on a single component of $\Gamma_{\overline{\delta}}$, we have,

$$z(x_1, x_2) = -\left(\varphi_\delta'(x_2)\psi(x_1, x_2), 0\right) + \varphi_\delta(x_2)v(x_1, x_2).$$

Hence,

$$\begin{aligned}
\partial_1 z^1 &= -\varphi_\delta'(x_2)\partial_1\psi(x_1, x_2) + \varphi_\delta(x_2)\partial_1 v^1(x_1, x_2) \\
&= -\varphi_\delta'(x_2)v^2(x_1, x_2) + \varphi_\delta(x_2)\partial_1 v^1(x_1, x_2), \\
\partial_2 z^1 &= -\varphi_\delta'(x_2)\partial_2\psi(x_1, x_2) - \varphi_\delta''(x_2)\psi(x_1, x_2) \\
&\qquad\qquad + \varphi_\delta'(x_2)v^1(x_1, x_2) + \varphi_\delta(x_2)\partial_2 v^1(x_1, x_2) \\
&= 2\varphi_\delta'(x_2)v^1 - \varphi_\delta''(x_2)\psi(x_1, x_2) + \varphi_\delta(x_2)\partial_2 v^1(x_1, x_2), \\
\partial_1 z^2 &= \varphi_\delta(x_2)\partial_1 v^2(x_1, x_2), \\
\partial_2 z^2 &= -\partial_1 z^1.
\end{aligned}$$

Now,

$$\begin{aligned}
|\psi(x_1, x_2)| &\le \|v\|_{L^\infty} x_2 = Cx_2, \\
|v^2(x_1, x_2)| &\le \left\|\partial_2 v^2\right\|_{L^\infty} x_2 \le Cx_2, \\
|\partial_1 v^2(x_1, x_2)| &\le \left\|\partial_2\partial_1 v^2\right\|_{L^\infty} x_2 \le Cx_2, \\
|\varphi_\delta'(x_2)x_2| &\le C, \qquad |\varphi_\delta''(x_2)x_2| \le C\delta^{-1},
\end{aligned}$$

so we have the pointwise bounds (for all $\delta \le \delta_0$, for some fixed $\delta_0 > 0$),

$$\begin{array}{ll}
|z^1(x_1, x_2)| \le C, & |z^2(x_1, x_2)| \le Cx_2, \\
|\partial_1 z^1(x_1, x_2)| \le C, & |\partial_2 z^1(x_1, x_2)| \le C\delta^{-1}, \\
|\partial_1 z^2(x_1, x_2)| \le Cx_2, & |\partial_2 z^2(x_1, x_2)| \le C
\end{array} \tag{A.1}$$

with all quantities supported in Γ_δ. These bounds lead directly to the bounds in Theorem 2.5 given in (2.3). Because

$$\partial_t z(x_1, x_2) = \nabla^\perp(\varphi_\delta(x_2)\partial_t\psi(x_1, x_2))$$

and $\partial_t \psi$ is bounded in the same manner as ψ (just with different constants), the estimates in (A.1) and so in (2.3) hold as well for $\partial_t z$ in place of z.

This establishes (2.3) for $j, k = 0$; $j = 1, k = 0$; $j = 0, k = 1$. Because additional derivatives in x_1 of z^1 or z^2 affect only ψ and v, which are C^∞, we also obtain the result for any value of j. Each additional derivative of z^1 or z^2 in x_2 has the same effect on ψ and v, but also adds one additional derivative on φ_δ, introducing an additional factor of δ. This leads to an additional factor of δ^{-k} for ∂_2^k. Since, however, $\partial_2 z^2 = -\partial_1 z^1$, there is one less factor of δ^{-1} for $\partial_2^k z^2$ than there is for $\partial_2^k z^1$. Similar considerations apply to $\partial_1^j \partial_2^k$, completing the proof of (2.3).

We now turn to the proof of (2.4). The estimates in (2.3) continue to hold unchanged when $m = 0$. If δ also varies with time, however, the cutoff function, φ_δ, has an additional dependence on time thorough δ, so that

$$\partial_t \varphi_\delta(x_2) = \partial_t \varphi\left(\frac{x_2}{\delta}\right) = \varphi'\left(\frac{x_2}{\delta}\right) \frac{\partial}{\partial t} \frac{x_2}{\delta} = -x_2 \frac{\partial_t \delta}{\delta^2} \varphi'\left(\frac{x_2}{\delta}\right).$$

Hence,

$$\begin{aligned}\partial_t z(x_1, x_2) &= \nabla^\perp(\varphi_\delta(x_2) \partial_t \psi(x_1, x_2)) - \nabla^\perp \left(x_2 \frac{\partial_t \delta}{\delta^2} \varphi'\left(\frac{x_2}{\delta}\right) \psi(x_1, x_2)\right) \\ &=: v_1 + v_2.\end{aligned}$$

To obtain the estimates in (A.1) for $\partial_t z$ in place of z, v_1 is bounded as before, so that, in particular,

$$\begin{aligned}\left\|v_1^1(x_1, x_2)\right\|_{L^p(\Omega)} &\le C\delta^{\frac{1}{p}}, \\ \left\|v_1^2(x_1, x_2)\right\|_{L^p(\Omega)} &\le C\delta^{\frac{1}{p}+1}.\end{aligned}$$

In bounding v_2, $-\varphi'(x_2/\delta)$ plays the role that $\varphi_\delta(x_2)$ played in bounding z, and is bounded in the same manner (the vanishing of φ' in a neighborhood of the boundary does not improve any estimates), but there is an additional factor of $x_2 \frac{\partial_t \delta}{\delta^2}$ that is included in each of the corresponding bounds in (2.3) for v_2. We need only the first two bounds,

$$\begin{aligned}|v_2^1(x_1, x_2)| &\le C x_2 \frac{\partial_t \delta}{\delta^2}, \\ |v_2^2(x_1, x_2)| &\le C x_2^2 \frac{\partial_t \delta}{\delta^2}.\end{aligned} \tag{A.2}$$

Hence (assuming that $\partial_t \delta > 0$),

$$\begin{aligned}\left\|v_2^1\right\|_{L^p(\Omega)} &\le C \frac{\partial_t \delta}{\delta^2} \left(\int_0^\delta x_2^p\right)^{\frac{1}{p}} \le C \frac{\partial_t \delta}{\delta^2} \delta^{1+\frac{1}{p}}, \\ \left\|v_2^2\right\|_{L^p(\Omega)} &\le C \frac{\partial_t \delta}{\delta^2} \left(\int_0^\delta x_2^{2p}\right)^{\frac{1}{p}} \le C \frac{\partial_t \delta}{\delta^2} \delta^{2+\frac{1}{p}}.\end{aligned}$$

From this, $(2.4)_{1,2}$ follow directly. Then

$$\begin{aligned}\|\partial_t z\|_{L^p(\Omega)} &\le C\delta^{\frac{1}{p}}(1+\delta) + C\partial_t\delta\,\delta^{\frac{1}{p}-1}(1+\delta)\\ &\le C\delta^{\frac{1}{p}-1}(\delta+\partial_t\delta)(1+\delta) \le C\delta^{\frac{1}{p}-1}(\delta+\partial_t\delta)\end{aligned}$$

for δ less than any fixed $\delta_0 > 0$, which is $(2.4)_3$. □

Appendix B. Some lemmas

Lemma B.1. *Let $w_1, w_2 \in H \cap H^2$ and set $\omega_j = \operatorname{curl} w_j$, $j = 1, 2$. Then,*

$$(\nabla w_1, \nabla w_2) = (\omega_1, \omega_2) + \int_{\partial\Omega} (\omega_2(w_1 \cdot \boldsymbol{\tau}) - \kappa w_1 \cdot w_2),$$

where κ is the curvature of the boundary.

Proof. We have,

$$\begin{aligned}(\nabla w_1, \nabla w_2) &= -(w_1, \Delta w_2) + \int_{\partial\Omega} (\nabla w_2 \cdot \boldsymbol{n}) \cdot w_1\\ &= -(w_1, \nabla^\perp \omega^2) + \int_{\partial\Omega} (\omega_2(w_1 \cdot \boldsymbol{\tau}) - \kappa w_1 \cdot w_2),\end{aligned}$$

where we used Lemma 4.1 of [18] for the boundary integrand. But,

$$-(w_1, \nabla^\perp \omega^2) = (w_1^\perp, \nabla\omega^2) = -(\operatorname{div} w_1^\perp, \omega^2) = (\omega_1, \omega_2).$$

□

The following is adapted from Lemma A.4 of [19]:

Lemma B.2. *For all vector fields, $u \in H^1(\Omega)$, $v \in H$,*

$$(u \cdot \nabla u, v) = (u^\perp \operatorname{curl} u, v).$$

Proof. We have,

$$(u \cdot \nabla u, v) = (u \cdot (\nabla u - (\nabla u)^T), v) + (u \cdot (\nabla u)^T, v).$$

But,

$$(u \cdot (\nabla u)^T) \cdot v = (u^i \partial_j u^i, v^j) = \frac{1}{2}(v, \nabla |u|^2) = 0,$$

so

$$\begin{aligned}(v, u \cdot \nabla u) &= (u^i(\partial_i u^j - \partial_j u^i), v^j)\\ &= (u^1(\partial_1 u^2 - \partial_2 u^1), v^2) + (u^2(\partial_2 u^1 - \partial_1 u^2), v^1)\\ &= \int_\Omega (u^1 v^2 - u^2 v^1) \operatorname{curl} u = (u^\perp \operatorname{curl} u, v).\end{aligned}$$

□

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