Operational Algebraic Cobordism

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Notation

- Let k be a field of characteristic zero. Let G be a linear algebraic group over k.
- Sch_k = Category of finite type separated schemes over k.
- $G-\mathrm{Sch}_k$ = Subcategory of Sch_k consisting of G-schemes that admit an ample G-linearizable line bundle; Morphisms are G-equivariant.
- $\mathcal{C}$ = Either Sch_k or $G-\mathrm{Sch}_k$; $\mathcal{C}'$ = Category with same objects as $\mathcal{C}$ but only projective morphisms.
- Ab_* = Category of graded abelian groups.

Min. Requirem.: Refined Oriented Borel-Moore pre-Homology Theory

(D$_1$) Functor $H_* : \mathcal{C}' \to \mathrm{Ab}_*$ s.t. $\bigoplus_{i=1}^r H_*(X_i) \xrightarrow{\sim} H_*(\coprod_{i=1}^r X_i)$. $(f_* := H_* f)$

(D$_2$) $\forall\, g : Y \to X$ smooth rel. codim. d, $g^* : H_*(X) \to H_{*-d}(Y)$.

(D$_3$) An element $1 \in H_0(\mathrm{Spec}\, k)$ and for each pair (X, Y) of schemes in $\mathcal{C}$,

$$\times : H_*(X) \times H_*(Y) \to H_*(X \times Y)$$

(D$_4$) $\forall\, f : Z \to X$ reg. embed. rel. codim. d and $\forall\, g : Y \to X$

$$\begin{tikzcd} Y \times_X Z \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ Z \arrow[r, "f"] & X \end{tikzcd} \qquad f_g^! : H_*(Y) \to H_{*-d}(Y \times_X Z)$$

All compatible!!

Examples: Chow theory A_*, K-theory $G_0[\beta, \beta^{-1}]$, algebraic cobordism Ω_*.

The Axioms of Homotopy, Localization and Gillet's Sequence

(H) Let $p : E \to X$ be a vector bundle of rank r over X. Then $p^* : H_*(X) \to H_{*+r}(E)$ is an isomorphism.

(L) For any closed immersion $i : Z \to X$ with open complement $j : U = X \setminus Z \to X$ the following sequence is exact:

$$H_*(Z) \xrightarrow{i_*} H_*(X) \xrightarrow{j^*} H_*(U) \to 0.$$

(G) For any envelope $\pi : \tilde{X} \to X$ (i.e. π is proper and $\forall Z \subset X$ subv. $\exists Z' \subset X'$ subv. s.t. $\pi| : Z' \to Z$ is birational), with π projective, the following sequence is exact $H_*(\tilde{X} \times_X \tilde{X}) \xrightarrow{p_{1*} - p_{2*}} H_*(\tilde{X}) \xrightarrow{\pi_*} H_*(X) \longrightarrow 0.$

Equivariant Version of an ROBM pre-Homology Theory

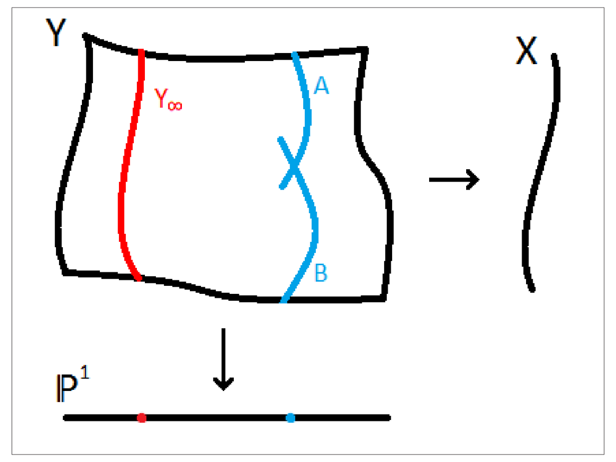
- **Definition:** Assume that the ROBM pre-homology theory H_* on Sch_k satisfies (H) and (L). For any scheme $X \in G-\mathrm{Sch}_k$, choose a good system of representations $\{(V_i, U_i)\}$ of G and define $H_*^G(X) = \bigoplus_{n \in \mathbf{Z}} H_n^G(X)$, where
$$H_n^G(X) := \varprojlim_i H_{n+\dim U_i - \dim G}(X \times^G U_i).$$
- This definition is independent of good system of representations, and H_*^G is an ROBM pre-homology theory on $G-\mathrm{Sch}_k$.

(Cf. Equivariant Chow Theory A_*^G (Totaro 90's, Edidin-Graham 96') & Equivariant Algebraic Cobordism Ω_*^G (Krishna 10', Heller-Malagón·López 10'))

Algebraic Cobordism

(Levine-Morel 01', Levine-Pandharipande 08')

$$\Omega_*(X) = \frac{\text{Free ab. on iso. classes of } [f{:}\,Y \to X] : f \text{ proj., } Y \text{ smooth quasiproj. variety}}{\langle\, [Y_\infty \to X] - [A \to X] - [B \to X] + [\mathbb{P}_{A\cap B} \to X]\,\rangle}$$



Y smooth quasiproj. var., $Y \to X \times \mathbb{P}^1$ proj., Y_∞ smooth; A and B smooth and intersect transversally.

$$\mathbb{P}_{A\cap B} := \mathbb{P}(N_{A\cap B}A \oplus \mathcal{O}_{A\cap B})$$

- Ω_* is the universal OBM homology theory on Sch_k.
- E.g. $A_*(X) = \Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z}$

Computing Algebraic Cobordism as Envelope Cobordism

THEOREM For any envelope $\pi : \tilde{X} \to X$ in Sch_k, with π projective,

$$\Omega_*(X) = \Omega_*^{\pi}(X) := \frac{\text{Span of cycles pushed forward from } \tilde{X}}{\text{Span of relations pushed forward from } \tilde{X}}$$

Operational Bivariant Algebraic Cobordism

(Cf. Fulton-MacPherson 81' operational A_* case)

Definition: Let $f : X \to Y$ be any morphism. For each morphism $g : Y' \to Y$, form the fiber square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

A *bivariant class* c in $\Omega^p(X \xrightarrow{f} Y)$ is a collection of homomorphisms

$$c_g^{(k)} : \Omega_k Y' \to \Omega_{k-p} X'$$

for all $g : Y' \to Y$, and all $k \in \mathbf{Z}$, compatible with projective push-forwards, smooth pull-backs, intersection products and exterior products. I.e:

$$\begin{tikzcd} X'' \arrow[r, "f''"] \arrow[d, "h'"] & Y'' \arrow[d, "h"] \\ X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} X'' \arrow[r, "f''"] \arrow[d, "i''"] & Y'' \arrow[r, "h'"] \arrow[d, "i'"] & Z'' \arrow[d, "i"] \\ X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[r, "h"] \arrow[d, "g"] & Z' \\ X \arrow[r, "f"] & Y & \end{tikzcd} \qquad \begin{tikzcd} X' \times Z \arrow[r, "f''"] \arrow[d, "h'"] & Y' \times Z \arrow[d, "h"] \\ X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Operational Algebraic Cobordism

(Cf. Fulton-MacPherson 81' operational A_* case)

- **Remark:** We can similarly define the operational bivariant theories:
$$\Omega_G^* \qquad H^* \qquad H_G^*$$
- **Definition:** The *Operational Algebraic Cobordism* of $X \in \mathrm{Sch}_k$ is
$$\Omega^*(X) = \Omega^*(X \to X)$$
- **Definition:** The *Operational Equivariant Algebraic Cobordism* of $X \in G-\mathrm{Sch}_k$ is
$$\Omega_G^* X = \Omega_G^*(X \to X)$$

Homology and Cohomology

(Cf. Fulton-MacPherson 81' A_* case)

THEOREM For any X: $\varphi : \Omega^{-*}(X \to \mathrm{Spec}\, k) \xrightarrow{\sim} \Omega_*(X)$

For X smooth: $\psi : \Omega^*(X \to X) \xrightarrow{\sim} \Omega_{\dim X - *}(X)$

The same holds for Ω_*^G, H_* and H_*^G.

Operational Equivariant vs. Equivariant Operational

(Cf. Edidin-Graham 97' A_* case)

THEOREM If the ROBM pre-homology theory H_* satisfies properties (H), (L) and (G), then for any X in $G-\mathrm{Sch}_k$ there are isomorphisms

$$H_G^*(X) \xrightarrow{\sim} \varprojlim_i H^*(X \times^G U_i) \quad \text{and in part.} \quad \Omega_G^*(X) \xrightarrow{\sim} \varprojlim_i \Omega^*(X \times^G U_i).$$

Gillet & Kimura Sequences for Algebraic Cobordism

(Cf. Gillet 82', Kimura 92' A_* case)

THEOREM For any envelope $\pi : \tilde{X} \to X$, with π projective, the following sequences are exact:

$$\Omega_*(\tilde{X} \times_X \tilde{X}) \xrightarrow{p_{1*} - p_{2*}} \Omega_*(\tilde{X}) \xrightarrow{\pi_*} \Omega_*(X) \longrightarrow 0.$$

$$0 \to \Omega^*(Y \to X) \xrightarrow{\pi^*} \Omega^*(\tilde{Y} \to \tilde{X}) \xrightarrow{p_1{}^* - p_2{}^*} \Omega^*(\tilde{Y} \times_Y \tilde{Y} \to \tilde{X} \times_X \tilde{X}).$$

$$0 \to \Omega_G^*(Y \to X) \xrightarrow{\pi^*} \Omega_G^*(\tilde{Y} \to \tilde{X}) \xrightarrow{p_1{}^* - p_2{}^*} \Omega_G^*(\tilde{Y} \times_Y \tilde{Y} \to \tilde{X} \times_X \tilde{X}).$$

Inductive Computation of Operational Cobordism

(Cf. Kimura 92' A_* case)

THEOREM Assume that the envelope $\pi : \tilde{X} \to X$ is birational and projective, $\pi : \pi^{-1}(U) \xrightarrow{\sim} U$ for some open nonempty $U \subset X$. Let $S_i \subset X$ be closed subschemes, such that $X \setminus U = \cup S_i$. Let $E_i = \pi^{-1}(S_i)$ and let $\pi_i : E_i \to S_i$ be the induced morphism. Then for a class $\tilde{c} \in \Omega^*(\tilde{Y} \to \tilde{X})$ the following are equivalent:

- $\tilde{c} = \pi^*(c)$ for some $c \in \Omega^*(Y \to X)$.
- For all i, $\tilde{c}|_{E_i} = \pi_i^*(c_i)$ for some $c_i \in \Omega^*(Y \times_X S_i \to S_i)$.

Operational Equivariant Cobordism of Toric Varieties

(Cf. Krishna-Uma 10' Ω_*^T of smooth TV case)

Let X_Δ be a toric variety.

- Brion+Payne: The operational T-equivariant Chow ring of X_Δ is isomorphic to the algebra $PP(\Delta)$ of piecewise integral polynomial functions on the fan Δ.
- Brion-Vergne+Payne-Anderson: The operational T-equivariant K-theory ring of X_Δ is isomorphic to the algebra $PLP(\Delta)$ of piecewise integral exponential functions on the fan Δ.
- **THEOREM** Let X_Δ be a toric variety. Then the operational T-equivariant cobordism $\Omega_T^*(X_\Delta)$ is isomorphic to the algebra $PPS(\Delta)$ of piecewise graded power series on the fan Δ with coefficients in the Lazard ring $\mathbb{L} = \Omega_*(\mathrm{Spec}\, k)$.

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