Scattering and Complete Integrability in Four Dimensions

John C. Baez

Department of Mathematics
Wellesley College
Wellesley, Massachusetts 02181
(on leave from the University of California at Riverside)

August 21, 1991

Abstract

The method of 'linearization via the wave operator' establishes a close connection between scattering theory and complete integrability for nonlinear wave equations in four-dimensional Minkowski space. We review the proofs of complete integrability for the massive ϕ^4 theory on the space of solutions of finite energy, and for the massless ϕ^4 theory on the space of solutions all of whose images under conformal transformations have finite energy. We show that the complete integrability is associated to the infinite-dimensional symmetry group $C(S^3, U(1))$ in the massive case, and a subgroup of $C^1(S^3, U(1))$ in the massless case. We review the construction of gauge-invariant conserved quantities for suitably regular solutions of the Yang-Mills equations in terms of asymptotic 'in' or 'out' fields, and discuss the prospects for complete integrability in this case.

Introduction

A great deal of work has been done on completely integrable nonlinear wave equations in two-dimensional spacetime, uncovering a rich complex of mathematical structures. In comparison, the situation in four dimensions is largely unexplored. There has been a detailed investigation of the self-dual Yang-Mills equations in four dimensions [19, 25, 35], but it should be recalled that there is no such thing as a self-dual solution on Minkowski space of the Yang-Mills equations with compact semisimple gauge group, since in this case the square of the Hodge star operator on 2-forms is -1, while the Lie algebra of a compact semisimple group admits no invariant complex structure. In this paper we review some of what *is* known about complete integrability for nonlinear wave equations in four dimensions. Our focus is on relativistically invariant equations, and primarily three basic cases: the massive ϕ^4 theory, the massless ϕ^4 theory, and the Yang-Mills equations.

We wish to emphasize that by 'completely integrable' we do not mean 'exactly solvable,' which is a rather vague notion in any event. Instead, we mean the existence of a complete set of conserved quantities with vanishing Poisson brackets, arising from the action of a commutative Lie group of symmetries. Strictly speaking, this is a property not of an equation, but of a one-parameter group of canonical transformations of phase space. There are various ways of formulating this property precisely, especially when the phase space is infinite-dimensional, so we begin with some definitions. In all that follows we assume that the phase space M is a (smooth) Banach manifold. We say that a function from M to another Banach manifold is 'smooth' if it is infinitely Frechét-differentiable. We say that (M, ω) is a 'symplectic manifold' if ω is a smooth closed 2-form on M that is weakly nondegenerate, that is, for any nonzero $u \in T_xM$ there exists $v \in T_xM$ with $\omega(u, v) \neq 0$. Given a symplectic manifold (M, ω), we say that a subspace $V \subset T_xM$ is 'isotropic' if $\omega|V = 0$, and 'Lagrangian' if it is a maximal isotropic subspace. We say that a diffeomorphism $f\colon M \to M$ of a symplectic manifold is a 'symplectomorphism' (in the language of physics, a canonical transformation) if $f^*\omega = \omega$.

Let (M, ω) be a symplectic manifold. We say a function $f \in C^\infty(M)$ is 'nice' if it generates a smooth vector field ζ_f on M in the following sense:

$$df = \omega(\zeta_f, \cdot).$$

Since the map $v \mapsto \omega(v, \cdot)$ need not define an isomorphism of T_xM and T_x^*M, not every smooth function need be nice, but ζ_f is unique if it exists. Given nice functions f_1, f_2 generating vector fields ζ_1, ζ_2, the function

$$\{f_1, f_2\} = -\omega(\zeta_1, \zeta_2)$$

generates the vector field $[v_1, v_2]$. In other words, the nice functions form a Lie algebra under the Poisson bracket $\{\cdot, \cdot\}$, and the map $f \mapsto \zeta_f$ is a Lie algebra homomorphism. Let G be a Banach Lie group and $\rho\colon G \times M \to M$ a smooth action of G on M as symplectomorphisms. The action ρ defines a homomorphism $d\rho$ from the Lie algebra $\mathfrak{g}$ of G to the Lie algebra of smooth vector fields on M. We say that the action ρ is 'Hamiltonian' if for each $v \in \mathfrak{g}$ there is a nice function f_v generating the vector field $d\rho(v)$, and the map $v \mapsto f_v$ is a Lie algebra homomorphism.

Now let $V(t)$ be a one-parameter group of symplectomorphisms of M. We say that $V(t)$ is 'completely integrable' if there is an abelian Banach Lie group G and a Hamiltonian action ρ of G on M such that: 1) for all $g \in G$ and $t \in \mathbb{R}$, $\rho(g)V(t) = V(t)\rho(g)$; 2) except for x in a set of first category in M, the closed span of the vectors $d\rho(v)$ for $v \in \mathfrak{g}$ is a Lagrangian subspace of T_xM. In this situation the functions f_v are conserved quantities:

$$V(t)^* f_v = f_v,$$

and the Poisson brackets $\{f_u, f_v\}$ vanish for all $u, v \in \mathfrak{g}$.

Returning to nonlinear wave equations, it should be clear that the question of complete integrability depends not only on the equation, but on a choice of a particular symplectic manifold of solutions. For example, for the massive ϕ^4 theory we shall work with solutions having finite energy, which seems natural on physical grounds. On the other hand, the finite-energy space is not well adapted to the conformal invariance of the massless ϕ^4 theory, since the transform of a finite-energy solution by an element of the conformal group may not have finite energy. Thus we work with solutions all of whose conformal transforms have finite energy.

While there has not been much work explicitly concerned with the complete integrability of nonlinear wave equations in four dimensions, there has been a detailed study of scattering theory for such equations. We consider an approach to complete integrability that could be called 'linearization via the wave operator'. This idea goes back at least to the work of Calogero [10] on n-body systems with repulsive interactions. For such systems, the particles' motions are essentially free in the distant future and past, and their asymptotic momenta define a complete set of conserved quantities with vanishing Poisson brackets. Similarly, for the ϕ^4 theory the solutions we consider approach solutions of the linear Klein-Gordon equation in the distant future and past. This permits the construction of 'wave operators,' symplectomorphisms intertwining the actions of time evolution on the phase spaces of the free and interacting theories. Using these one may immediately deduce the complete integrability of the interacting theory from that of the free theory. The difficulty lies solely in proving the existence of wave operators with the desired properties. This is plausible on the grounds of deformation theory, as shown by Flato and Simon [17, 18], but for the massive ϕ^4 theory the rigorous proof for finite-energy solutions is extremely technical. For the massless ϕ^4 theory the use of conformal invariance permits a more conceptual proof.

The complete integrability of the Yang-Mills equations on a suitable space of solutions is a fascinating open question, and the study of the massless ϕ^4 theory may be regarded as a 'warm-up' for this case. Like the massless ϕ^4 theory, the Yang-Mills equations are conformally invariant, but their gauge invariance introduces additional complications. In particular, since the physical phase space of the Yang-Mills equations is not a vector space, one cannot expect a true 'linearization' of time evolution. Nonetheless, there exists a very simple description of time evolution for the solutions we consider in terms of 'in' and 'out' fields, certain limits as time approaches $\pm\infty$. This permits the construction of infinitely many gauge-invariant conserved quantities.

In our brief (and incomplete) histories of various results, dates always refer to dates of publication. The author wishes to thank Irving Segal and Zhengfang Zhou for many conversations on scattering theory for nonlinear wave equations. In particular, the proof of Theorem 3 was developed in conversations with Zhou. The author is also indebted to Walter Strauss for reading a draft of this paper and suggesting some improvements.

The Massive ϕ^4 Theory

Let $\mathbf{M}_0$ denote Minkowski space, which we identify with $\mathbb{R}^4$ with the coordinates $x = (x_0, x_1, x_2, x_3) = (x_0, \vec{x})$. The massive ϕ^4 theory is given by the equation

$$(\Box + m^2)\phi + \lambda\phi^3 = 0, \qquad m > 0,\ \lambda \geq 0, \tag{1}$$

where ϕ is a real function on $\mathbf{M}_0$. The Hamiltonian of this theory is given by

$$H = \int_{x_0=0} \left(\frac{1}{2}((\nabla\phi)^2 + m^2\phi^2 + \dot{\phi}^2) + \frac{\lambda}{4}\phi^4 \right) d^3x. \tag{2}$$

For the free theory ($\lambda = 0$), the space of Cauchy data $v = (v_1, v_2)$ for which the energy is finite forms a real Hilbert space $\mathbf{H}$ with norm given by

$$\|v\|^2 = \frac{1}{2}\int_{\mathbf{R}^3} (\nabla v_1)^2 + m^2 v_1^2 + v_2^2.$$

In fact, the Sobolev inequalities imply that

$$\int \phi^4\, d^3x \leq c\left(\int ((\nabla\phi)^2 + m^2\phi^2) d^3x\right)^2,$$

so even for the interacting theory ($\lambda > 0$), the energy H is finite if $(\phi, \dot{\phi})|_{x_0=0}$ lies in $\mathbf{H}$.

The basic global existence theorem for the massive ϕ^4 theory is that for any $v \in \mathbf{H}$ there is a unique solution ϕ of equation (1) with $(\phi, \dot{\phi})|_{x_0=0} = v$. This was proved by Jörgens [23] and Segal [40] in the early 1960's. Note that part of proving this theorem consists of defining precisely in what sense ϕ is a solution. Segal's proof makes use of the technique of nonlinear semigroups, in which equation (1) is transformed into an integral equation for the Cauchy data $v(t) = (\phi, \dot{\phi})|_{x_0=t}$ as a function of t, and it is shown that for any $v \in \mathbf{H}$ there is a unique continuous solution $v(t)$ with $v(0) = v$. This approach underlies most of the work we will describe, but we will not emphasize it here, referring the reader instead to the expositions by Reed and Segal [37, 47]. Other technical aspects which we will downplay, such as *a priori* estimates, have been reviewed by Strauss [52].

The global existence theorem allows us to identify finite-energy solutions of the massive ϕ^4 theory with their time-zero Cauchy data, which we do consistently henceforth, treating $\mathbf{H}$ as the phase space of the theory. The standard symplectic structure ω on $\mathbf{H}$ is given by

$$\omega(u, v) = \int_{\mathbf{R}^3} u_1 v_2 - v_1 u_2,$$

where we identify the tangent vectors $u, v \in T_x\mathbf{H}$ with elements of $\mathbf{H}$ by means of parallel translation. Of course, for the interacting theory what matters is not the vector space structure of $\mathbf{H}$ *per se*, but its structure as a symplectic manifold.

Let $\mathbf{P}_0$ denote the connected component of the Poincaré group. Given $g \in \mathbf{P}_0$ and a finite-energy solution ϕ of (1), define $g\phi$ by

$$g\phi(x) = \phi(g^{-1}x).$$

Then $g\phi$ is also a finite-energy solution of (1), and the map $(g, \phi) \mapsto g\phi$ defines an action of $\mathbf{P}_0$ on $\mathbf{H}$, which we denote by V_λ. When $\lambda = 0$, the maps $V_\lambda(g)$ are linear. As it turns out, even for $\lambda > 0$ the maps $V_\lambda(g)$ are not only smooth but real-analytic, that is, they have Taylor series with a nonzero radius of convergence about each point. Thus the phase space $\mathbf{H}$ has canonically the structure of an analytic manifold:

Theorem 1. *For any $g \in \mathbf{P}_0$, $V_\lambda(g) \colon \mathbf{H} \to \mathbf{H}$ is an analytic symplectomorphism. For all $v \in \mathbf{H}$, $V_\lambda(g)v$ is continuous as a function of $g \in \mathbf{P}_0$.*

Proof - The smoothness of the time evolution maps $v \mapsto v(t)$ on $\mathbf{H}$ was first shown by Segal [40], who also showed that $\mathbf{P}_0$ acts as symplectomorphisms of a dense subspace of $\mathbf{H}$ [41]. Analyticity of time evolution follows from the results of Baez and Zhou on analytic nonlinear semigroups [7]. A proof that the whole Poincaré group acts continuously and as analytic symplectomorphisms does not appear in the literature, but Theorems 2 and 3 below reduce the problem to the free case, which is straightforward. $\square$

We write simply t for the element of $\mathbf{P}_0$ corresponding to translation forwards in time by $t \in \mathbb{R}$, so that

$$V_\lambda(t)\left((\phi, \dot{\phi})|_{x_0=0}\right) = (\phi, \dot{\phi})|_{x_0=t}.$$

Finite-energy solutions of the interacting massive ϕ^4 theory are asymptotic to solutions of the free theory in the following sense:

Theorem 2. *For any $u \in \mathbf{H}$, there exist $u_+, u_- \in \mathbf{H}$ such that*

$$\lim_{t \to \pm\infty} \|V_\lambda(t)u - V_0(t)u_\pm\| = 0.$$

There exist 'wave operators,' analytic diffeomorphisms $W_\pm \colon \mathbf{H} \to \mathbf{H}$, such that $u = W_\pm(u_\pm)$ for all $u \in \mathbf{H}$.

Proof - The study of this question was initiated by Segal [42] in 1966. The wave operators were constructed as diffeomorphisms of a dense subspace $\mathcal{F} \subset \mathbf{H}$ by Morawetz and Strauss [30, 31]. The analyticity properties of these wave operators were studied by Raczka and Strauss [36]. Strauss [50, 51] later proved the existence of wave operators on all of $\mathbf{H}$, and inverted them at low energy, i.e., in a neighborhood of the origin of $\mathbf{H}$. In 1985, Brenner [9] constructed inverses for the wave operators throughout $\mathbf{H}$. Baez and Zhou proved that the wave operators are homeomorphisms, and analytic diffeomorphisms at low energy [7, 8]. In a paper to be published, Kumlin [24] has proved that the wave operators are analytic diffeomorphisms throughout $\mathbf{H}$. All this work relies primarily on hard analysis, and particularly on sharp decay estimates for solutions of the free theory. The technique for obtaining such estimates was found by Strichartz [53], and developed by Marshall, Strauss and Wainger [26, 27]. $\square$

The wave operators have the following basic properties:

Theorem 3. *The wave operators* $W_\pm\colon \mathbf{H} \to \mathbf{H}$ *are symplectomorphisms. Moreover, they intertwine the free and interacting actions of the Poincaré group, i.e.,*

$$V_\lambda(g) = W_\pm V_0(g) W_\pm^{-1}$$

for all $g \in \mathbf{P}_0$.

Proof - Morawetz and Strauss [31] proved the symplectic and intertwining properties of the wave operators on the space $\mathcal{F} \subset \mathbf{H}$. Baez and Zhou proved the symplectic property at low energy [8], and the proof extends to all of $\mathbf{H}$ using Kumlin's global analyticity result. The proof of the intertwining property on all of $\mathbf{H}$ is straightforward for spatial translations and rotations, so it suffices to treat Lorentz boosts $g \in \mathbf{P}_0$. By the result of Morawetz and Strauss, $V_\lambda(g)$ equals $W_\pm V_0(g) W_\pm^{-1}$ on the dense subspace $\mathcal{F} \subset \mathbf{H}$. Since $W_\pm$ is an analytic diffeomorphism and $V_0(g)$ is continuous and linear, it suffices to show that $V_\lambda(g)$ is continuous to conclude the result for all $v \in \mathbf{H}$. Writing $V_\lambda(g)v$ explicitly in terms of v in terms of an integral equation, continuity follows from the result of Kumlin [24] that the map from v to the solution ϕ is continuous from $\mathbf{H}$ to $L^3(\mathbb{R}, L^6(\mathbb{R}^3))$. $\square$

As a corollary, the scattering operator $S\colon \mathbf{H} \to \mathbf{H}$, given by $S = W_+^{-1} W_-$, is an analytic symplectomorphism commuting with the free action of $\mathbf{P}_0$:

$$S V_0(g) = V_0(g) S.$$

This implies that scattering may be computed at low energy using an explicit Taylor series. Another corollary is the complete integrability of the interacting massive ϕ^4 theory on the finite-energy space:

Corollary 1. *The one-parameter group* $V_\lambda(t)$ *of symplectomorphisms of* $\mathbf{H}$ *is completely integrable, with analytic conserved quantities.*

Proof - This follows from Theorems 2 and 3 and the complete integrability of the free theory. The results of Baez and Zhou [8] together with those of Kumlin [24] give a set of analytic conserved quantities on $\mathbf{H}$ with vanishing Poisson brackets, generating vector fields spanning a Lagrangian subspace of $T_x\mathbf{H}$ except for $x \in \mathbf{H}$ in a set of first category. Here we construct a Hamiltonian group action of the abelian Banach

Lie group $G = C(S^3, U(1))$ on $\mathbf{H}$, commuting with the action of time evolution, with $\{f_\alpha\}$ as associated conserved quantities. By Theorem 3 it suffices to consider the free theory. In this case any finite-energy solution ϕ is determined by the restriction of its Fourier transform to the 'mass hyperboloid'

$$\{k \in \mathbf{M}_0^* : k_\mu k^\mu = m^2,\ k_0 > 0\}.$$

Time evolution acts on these Fourier transforms as multiplication by a phase:

$$(V_0(t)\phi)\widehat{\ }(k) = e^{itk_0}\widehat{\phi}(k).$$

We identify elements of G with continuous functions $g\colon \mathbb{R}^3 \to U(1)$ approaching a constant at infinity, and identify the Lie algebra $\mathfrak{g}$ of G with the abelian Lie algebra of continuous functions $v\colon \mathbb{R}^3 \to \mathbb{R}$ approaching a constant at infinity. There is a representation $\rho\colon G \times \mathbf{H} \to \mathbf{H}$ given by

$$(\rho(g)\phi)\widehat{\ }(k) = g(\vec{k})\widehat{\phi}(k).$$

It is easy to check that ρ is smooth and commutes with $V_0(t)$ for all $t \in \mathbb{R}$. Given $v \in \mathfrak{g}$, let f_v be the function on $\mathbf{H}$ mapping the finite-energy solution ϕ to

$$\int v(\vec{k})\, |\widehat{\phi}(k)|^2\, d\mu(k),$$

where μ is the properly normalized Lorentz-invariant measure on the mass hyperboloid. It can be shown [8] that f_v generates the vector field $d\rho(v)$. The conserved quantities constructed by Baez and Zhou are all of this form. $\Box$

For the n-component generalization of the massive ϕ^4 theory there is a similar action of the group $C(S^3, U(n))$ as symplectomorphisms of the finite-energy space commuting with time evolution, and one again has complete integrability on the finite-energy space. The n-component ϕ^4 theory can be thought of as an approximation to any n-component wave equation of the form

$$\Box\phi_i + \frac{\partial F}{\partial x_i}(\phi) = 0, \qquad 1 \le i \le n,$$

where $F\colon \mathbb{R}^n \to \mathbb{R}$ is smooth, for solutions in the vicinity of a nondegenerate local minimum of F. Thus one should expect complete integrability and an action of $C(S^3, U(n))$ on the solutions of such an equation that lie near enough to a nondegenerate local minimum of F. It would be an interesting project to make this statement precise and prove it. The work of Simon and Taflin [48] takes a step in this direction. It is also tempting to speculate that groups of the form $C(S^3, G)$ play a role in four dimensions somewhat analogous to the role of loop groups in two dimensions.

The Massless ϕ^4 Theory

The massless ϕ^4 theory is the equation

$$\Box\phi + \lambda\phi^3 = 0, \qquad \lambda \ge 0, \tag{3}$$

where ϕ is a real function on Minkowski space $\mathbf{M}_0$. The Hamiltonian of this theory is

$$H = \int_{x_0=0} \left(\frac{1}{2}((\nabla\phi)^2 + \dot{\phi}^2) + \frac{\lambda}{4}\phi^4 \right) d^3x. \tag{4}$$

Taking Cauchy data such that $\dot{\phi} = 0$ and ϕ is a smooth bump function dilated so as to have support of radius r, one easily easily estimates that $\int (\nabla\phi)^2 \, d^3x \sim r$, while $\int \phi^4 \, d^3x \sim r^3$. Thus in contrast to the massive theory, one cannot bound the interaction Hamiltonian by the free Hamiltonian. This is the simplest of the 'infrared problems' plaguing the massless theory. To take another example, solutions of the free massless equation with C_0^∞ Cauchy data have

$$\sup_{\vec{x}} |\phi(t, \vec{x})| = O(t^{-1})$$

rather than $O(t^{-3/2})$ as in the massive case. The slower decay in the massless case has so far prevented the formulation of a scattering theory on the finite-energy space, despite interesting partial results [21, 49].

Luckily, all these infrared problems are directly linked to additional symmetries of the massless ϕ^4 theory. It is easy to see at a formal level that this theory is invariant, not only under the Poincaré group $\mathbf{P}_0$, but under dilations:

$$\phi(x) \mapsto c\phi(cx),$$

where $c > 0$. In fact, the massless ϕ^4 theory is conformally invariant, and one can develop a very elegant scattering theory for it by taking full advantage of this fact.

Let $\mathbf{P}$ denote the 11-dimensional group generated by $\mathbf{P}_0$ together with dilations. Every orientation-preserving conformal diffeomorphism of $\mathbf{M}_0$ is an element of $\mathbf{P}$, but there is a 15-dimensional group of conformal transformations that are defined only on open dense subsets of $\mathbf{M}_0$. This group, the 'conformal group' $\mathbf{G} = SO(2,4)$, is the identity component of the group generated by $\mathbf{P}$ and conformal inversion:

$$x \mapsto \frac{4x}{x_\mu x^\mu}.$$

To take advantage of conformal symmetry it is best to embed Minkowski space in a larger spacetime on which $\widetilde{\mathbf{G}}$ acts as diffeomorphisms. We briefly review this part of conformal geometry, referring the reader to Paneitz and Segal [33] and Penrose and Rindler [35] for the proofs and explicit formulae.

Minkowski space has a conformal compactification $\mathbf{M}$, that is, a compact spacetime into which it may be conformally embedded as a dense open submanifold. As a homogeneous space, $\mathbf{M}$ is just $\mathbf{G}/\mathbf{P}$. As a manifold with conformal structure, $\mathbf{M}$ may be identified with $S^1 \times S^3/\mathbb{Z}_2$ (where $\mathbb{Z}_2$ acts as the product of the antipodal maps on S^1 and S^3), equipped with the metric pushed down from the metric $d\theta^2 - ds^2$ on $S^1 \times S^3$, where $d\theta^2$ and ds^2 are the standard metrics on S^1 and S^3. The group $\mathbf{G}$ acts on $\mathbf{M}$ as conformal diffeomorphisms.

Since $\mathbf{M}$ has closed timelike loops, it is unsatisfactory for the study of nonlinear wave equations. It is better to work instead with the universal cover $\widetilde{\mathbf{M}}$, which may be identified with the 'Einstein universe,' $\mathbb{R} \times S^3$ with the metric $d\tau^2 - ds^2$. The coordinate $\tau \in \mathbb{R}$ is called the 'Einstein time.'

Let $\iota\colon \mathbf{M}_0 \to \widetilde{\mathbf{M}}$ be a conformal embedding lifting that of $\mathbf{M}_0$ in $\widetilde{\mathbf{M}}$. Minkowski space as embedded in the Einstein universe is given by the open set

$$\iota(\mathbf{M}_0) = \{(\tau, u) \in \mathbb{R} \times S^3 \colon\ |\tau| + |\rho| < \pi\},$$

where ρ is the angle of the point $u \in S^3$ from a fixed point, for example $(1, 0, 0, 0) \in S^3$. We will identify Minkowski space with its image in $\widetilde{\mathbf{M}}$. The boundary of $\mathbf{M}_0$ in $\widetilde{\mathbf{M}}$ is the union of two lightcones

$$C_\pm = \{(\tau, u) \in \mathbb{R} \times S^3 \colon\ \pm\tau = \pi - \rho\},$$

which we call the lightcones at 'future (resp. past) infinity,' since points in $C_\pm$ may be regarded as limits of points in $\mathbf{M}_0$ as $t \to \pm\infty$. The action of $\mathbf{G}$ on $\mathbf{M}$ lifts to an action of the universal cover $\widetilde{\mathbf{G}}$ as conformal transformations of $\widetilde{\mathbf{M}}$, and the subgroup of $\widetilde{\mathbf{G}}$ preserving $\mathbf{M}_0$ is just the universal cover $\widetilde{\mathbf{P}}$ of the group $\mathbf{P}$. It follows that $\widetilde{\mathbf{P}}$ also preserves each of the lightcones $C_\pm$.

To take advantage of the conformal invariance of the massive ϕ^4 theory, it is crucial to use the fact that it is equivalent to a very simple equation on $\widetilde{\mathbf{M}}$. Let $p\colon \mathbf{M}_0 \to \mathbb{R}$ be the conformal factor relating the Einstein and Minkowski metrics on $\mathbf{M}_0$:

$$\iota^*(d\tau^2 - ds^2) = p^2(dx_0^2 - d\vec{x}^2).$$

Let $\widetilde{\Box}$ denote the Laplace-Beltrami operator associated to the Einstein metric on $\widetilde{\mathbf{M}}$. If $\widetilde{\phi}$ is a function on $\widetilde{\mathbf{M}}$ satisfying

$$(\widetilde{\Box} + 1)\widetilde{\phi} + \lambda\widetilde{\phi}^3 = 0, \qquad \lambda \geq 0, \tag{5}$$

then the function ϕ on $\mathbf{M}_0$ given by

$$\phi(x) = p(x)\, \widetilde{\phi}(\iota(x))$$

satisfies the massless ϕ^4 theory equation (3). Conversely, any solution ϕ of (3) on $\mathbf{M}_0$ defines a solution $\widetilde{\phi}$ of (5) at least on $\iota(\mathbf{M}_0) \subset \widetilde{\mathbf{M}}$. As we shall see, $\widetilde{\phi}$ extends to a solution on all of $\widetilde{\mathbf{M}}$ if ϕ has 'finite Einstein energy.'

While the Einstein time τ is not simply a function of the Minkowski time x_0, the surface $x_0 = 0$ is contained in the surface $\tau = 0$. Thus we may study the Cauchy problem on Minkowski space in terms of the Cauchy problem on $\widetilde{\mathbf{M}}$. For equation (5), Einstein time evolution is associated with the Hamiltonian

$$H_e = \int_{\tau=0} \left(\frac{1}{2}((\widetilde{\nabla}\widetilde{\phi})^2 + \widetilde{\phi}^2 + (\partial_\tau \widetilde{\phi})^2) + \frac{\lambda}{4}\widetilde{\phi}^4 \right) d^3u, \tag{6}$$

where $\widetilde{\nabla}$ denotes the gradient on S^3, and d^3u is the standard volume form on S^3. The explicit formula for this 'Einstein energy' in terms of ϕ:

$$H_e = \int_{x_0=0} \left((1 + \frac{\vec{x}^2}{4})(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\dot{\phi}^2 + \frac{\lambda}{4}\phi^4) - \frac{1}{4}\phi^2 \right) d^3x,$$

shows that having finite Einstein energy imposes a spatial decay condition on the Cauchy data in Minkowski space. One can show that the Einstein energy is always greater than the 'Minkowski energy' given by (4). The key to a simple scattering theory for the massless ϕ^4 theory is to work with the solutions of finite Einstein energy.

For the free theory, the space of Cauchy data $v = (v_1, v_2)$ for which the Einstein energy is finite is a real Hilbert space $\mathbf{H}$ with norm given by

$$\|v\|^2 = \frac{1}{2} \int_{S^3} (\nabla v_1)^2 + v_1^2 + v_2^2.$$

Even for $\lambda \neq 0$, the Einstein energy H_e is finite whenever $(\widetilde{\phi}, \partial_\tau \widetilde{\phi})|_{\tau=0}$ is in $\mathbf{H}$. It easy to prove [6] using nonlinear semigroup theory that for any $v \in \mathbf{H}$ there is a unique solution $\widetilde{\phi}$ of equation (3) with $(\widetilde{\phi}, \partial_\tau \widetilde{\phi})|_{\tau=0} = v$. Thus we will identify finite-Einstein-energy solutions with their $\tau = 0$ Cauchy data.

We now describe analogs of the results of the previous section for the massless ϕ^4 theory. The space $\mathbf{H}$ has a symplectic structure ω given by

$$\omega(u, v) = \int_{S^3} u_1 v_2 - v_1 u_2,$$

where we identify the tangent vectors $u, v \in T_x\mathbf{H}$ with elements of $\mathbf{H}$ by means of parallel translation. The group $\widetilde{\mathbf{G}}$ acts on functions on $\widetilde{\mathbf{M}}$ via

$$g\widetilde{\phi}(y) = \alpha(g^{-1}y)\widetilde{\phi}(g^{-1}y), \qquad g \in \widetilde{\mathbf{G}},\ y \in \widetilde{\mathbf{M}},$$

where α is the conformal factor:

$$(g^{-1})^*(dt^2 - ds^2) = \alpha^2(dt^2 - ds^2).$$

If $\widetilde{\phi}$ is a finite-Einstein-energy solution of (5) then so is $g\widetilde{\phi}$ for all $g \in \widetilde{\mathbf{G}}$. This defines an action of $\widetilde{\mathbf{G}}$ on $\mathbf{H}$, which we denote by V_λ. In fact, the Einstein energy of a solution is just its Minkowski energy plus the Minkowski energy of its transform under conformal inversion. This implies that a solution has finite Einstein energy if and only if all its transforms under elements of $\widetilde{\mathbf{G}}$ have finite Minkowski energy.

We have:

Theorem 4. *For any $g \in \widetilde{\mathbf{G}}$, $V_\lambda(g)\colon \mathbf{H} \to \mathbf{H}$ is a analytic symplectomorphism. For any $v \in \mathbf{H}$, $V_\lambda(g)v$ is continuous as a function of $g \in \widetilde{\mathbf{G}}$.*

Proof - This was shown by Baez, Segal, and Zhou [2, 6, 7]. $\square$

Recall that $\widetilde{\mathbf{P}}$ may be identified with the subgroup of $\widetilde{\mathbf{G}}$ preserving $\mathbf{M}_0$. The action V_λ of $\widetilde{\mathbf{P}}$ on $\mathbf{H}$ factors through $\mathbf{P}$, and we write simply $V_\lambda(t)$ for the action on $\mathbf{H}$ of Minkowski time evolution. There exist wave operators as follows:

Theorem 5. *For any $u \in \mathbf{H}$, there exist $u_+, u_- \in \mathbf{H}$ such that*

$$\lim_{t \to \pm\infty} \|V_\lambda(t)u - V_0(t)u_\pm\| = 0.$$

There exist analytic diffeomorphisms $W_\pm\colon \mathbf{H} \to \mathbf{H}$ such that $u = W_\pm(u_\pm)$ for all $u \in \mathbf{H}$.

Proof - This was proved in a series of papers by Baez, Segal and Zhou in the late 1980's [2, 4, 6, 7, 46]. The key idea is to construct $u_\pm$ in terms of the 'Goursat data' $\widetilde{\phi}|_{C_\pm}$, and to reduce the study of the maps $W_\pm$ to the problem of solving equation (5) given Goursat data of finite Einstein energy. This idea appears formally in the work of Penrose [34, 35], who did not consider the analytical aspects. $\square$

Theorem 6. *The wave operators $W_\pm\colon \mathbf{H} \to \mathbf{H}$ are symplectomorphisms. Moreover,*

$$V_\lambda(g) = W_\pm V_0(g) W_\pm^{-1}$$

for all $g \in \mathbf{P}$.

Proof - These results were proved by Baez [2]. $\square$

As in the massive case, this theorem immediately implies complete integrability:

Corollary 2. *The one-parameter group $V_\lambda(t)$ of analytic symplectomorphisms of $\mathbf{H}$ is completely integrable, with analytic conserved quantities.*

Proof - Baez [4] constructed analytic conserved quantities on $\mathbf{H}$ with vanishing Poisson brackets, generating vector fields spanning a Lagrangian subspace of $T_x\mathbf{H}$ except on a set of first category. Here we sketch the construction, based on earlier work of Baez, Segal and Zhou, of a Hamiltonian group action of an abelian Banach Lie group on $\mathbf{H}$, commuting with the action of time evolution, and implying complete integrability. By Theorem 3 it suffices to consider the free theory. Let

$$C^* = C_- - \{\rho = 0, \pi\}.$$

C^* is a smooth submanifold of $\widetilde{\mathbf{M}}$, diffeomorphic to $\mathbb{R} \times S^2$, and we give it the coordinates (s, ω), where $s = -2\cot\rho$ and $\omega \in S^2 \subset \mathbb{R}^3$ is defined by

$$u = (\cos\rho, (\sin\rho)\omega),$$

where $u \in S^3$ is regarded as a unit vector in $\mathbb{R}^4$. We give C^* the volume form $ds \wedge d^2\omega$, where $d^2\omega$ denotes the standard volume form on S^2. The submanifold C^* is preserved by the action of $\widetilde{\mathbf{P}}$ on $\widetilde{\mathbf{M}}$; the advantage of the coordinates (s, ω) is that Minkowski time translation on C^* corresponds to translation in the s variable:

$$(s, \omega) \mapsto (s + t, \omega).$$

A finite-Einstein-energy solution $\widetilde{\phi}$ of equation (5) need not be continuous, but its restriction to C^* is well-defined almost everywhere. We define the 'in field' $\phi_-\colon C^* \to \mathbb{R}$ by

$$\phi_- = (\sin\rho)\, \widetilde{\phi}|_{C^*}.$$

The factor of $\sin\rho$ makes the action of Minkowski time evolution on Goursat data very simple:

$$(V_\lambda(t)\phi)_-(s, \omega) = \phi_-(s + t, \omega).$$

Let $\mathbf{H}(C^*)$ denote the space of functions on C^* of the form ϕ_- for some finite-energy solution $\widetilde{v}$. The norm on $\mathbf{H}$ transfers to a norm on $\mathbf{H}(C^*)$ given by

$$\|\phi_-\|^2 = \frac{1}{2}\int_{C^*} \left((s^2 + 4)(\partial_s\phi_-)^2 + \phi_-^2 + (\nabla_\omega\phi_-)^2\right) ds d^2\omega$$

where ∇_ω denotes the gradient on S^2. In fact, all functions on C^* for which this norm is finite lie in $\mathbf{H}(C^*)$. Let $\widehat{\phi}_-(\sigma, \omega)$ denote the Fourier transform of ϕ_- in the s variable. For some $c > 0$,

$$\|\phi_-\|^2 \le c\int_{\mathbf{R}\times S^2} (\sigma^2 + 1)(|\partial_\sigma\widehat{\phi}_-|^2 + |\nabla_\omega\widehat{\phi}_-|^2)\, d\sigma d^2\omega.$$

All functions $f(\sigma, \omega)$ with $f(\sigma, \omega) = \overline{f}(-\sigma, \omega)$ and

$$\int_{\mathbf{R}\times S^2} (\sigma^2 + 1)(|\partial_\sigma f|^2 + |\nabla_\omega f|^2)\, d\sigma d^2\omega < \infty$$

are thus of the form $\widehat{\phi}_-(\sigma,\omega)$ for some $\phi_- \in \mathbf{H}(C^*)$. It follows that if $\phi_- \in \mathbf{H}(C^*)$ and $g\colon \mathbb{R}\times S^2 \to U(1)$ is a C^1 function with $g(\sigma,\omega) = \overline{g}(-\sigma,\omega)$, and with derivatives vanishing sufficiently rapidly as $\sigma \to \pm\infty$, then $g\widehat{\phi}_-$ is the Fourier transform of a function in $\mathbf{H}(C^*)$. This allows us to construct the desired Banach Lie group G of C^1 functions $g\colon \mathbb{R}\times S^2 \to U(1)$, with some flexibility as concerns the rate of decay of derivatives of g, acting on $\mathbf{H}(C^*)$ by

$$(\rho(g)\phi_-)\widehat{\ } = g\widehat{\phi}_- \,.$$

Using the fact that the symplectic structure in $\mathbf{H}(C^*)$ is given by

$$\omega(\phi_-,\psi_-) = \int_{C^*} (\phi_-\partial_s\psi_- - \psi_-\partial_s\phi_-)\, ds d^2\omega,$$

one can show G acts as symplectomorphisms. It is easily shown that the action of G commutes with that of Minkowski time evolution, which acts on $\widehat{\phi}_-$ simply as multiplication by a phase, and the proof of complete integrability may be completed along the lines of [4]. Regarding S^3 as a two-point compactification of $\mathbb{R}\times S^2$, the group G may be identified with a subgroup of $C^1(S^3, U(1))$. $\square$

The Yang-Mills equations

Like the massless ϕ^4 theory, the Yang-Mills equations are conformally invariant, but in addition they are gauge invariant. The resulting complications have so far precluded a proof of existence of solutions with arbitary finite-energy Cauchy data, on either Minkowksi space or the Einstein universe. Nonetheless, scattering theory can be formulated for the Yang-Mills equations in terms of the conformal embedding $\iota\colon \mathbf{M}_0 \to \widetilde{\mathbf{M}}$ [3]. As this scattering theory is rather technical, here we just report what is known concerning complete integrability. The possibility that the Yang-Mills equations are completely integrable on a suitable space of solutions may have been neglected due to various results seeming to point to the opposite conclusion. For example, Nikolaevskii and Shchur [32] and Savvidi [38] have considered solutions of Yang-Mills equations that are constant in space at each given time (in temporal gauge, in Minkowski space). By proving that time evolution on this finite-dimensional symplectic manifold is nonintegrable, they conclude that the Yang-Mills equations are not integrable. Saviddi has also shown that radially symmetric solutions near the Wu-Yang static solution (which has a singularity at the origin) form a nonintegrable dynamical system [39]. In neither case, however, are the solutions considered of finite energy, so there would be no contradiction if the Yang-Mills equations were completely integrable on some space of finite-energy solutions. In fact, there is rather large space of finite-energy solutions of the Yang-Mills equations for which there exist infinitely many gauge-invariant conserved quantities.

We begin by considering the Yang-Mills equations on the Einstein universe $\widetilde{\mathbf{M}}$. For simplicity we consider C^2 solutions. Let G be a compact Lie group and $\mathfrak{g}$ its Lie algebra. For simplicity of notation we assume without loss of generality that G is a matrix group. Note that all G-bundles on $\widetilde{\mathbf{M}}$ are trivial, since $\pi_2(G) = 0$ by a theorem of Cartan. We identify the space $\mathbf{A}$ of C^2 $\mathfrak{g}$-valued one-forms on $\widetilde{\mathbf{M}}$ with a space of connections on the trivial G-bundle over $\widetilde{\mathbf{M}}$ by means of the map taking the $\mathfrak{g}$-valued one-form A to the connection $d+A$. We let $\mathcal{G}$ denote the group of C^3 gauge transformations, that is, $C^3(\widetilde{\mathbf{M}}, G)$ with pointwise multiplication. The group $\mathcal{G}$ acts on $\mathbf{A}$ as affine transformations by:

$$gA = \mathrm{Ad}(g)A + (dg)g^{-1}, \qquad g \in \mathcal{G}.$$

The group $\widetilde{\mathbf{G}}$ of conformal transformations acts as linear transformations of $\mathbf{A}$ by:

$$gA = (g^{-1})^* A, \qquad g \in \widetilde{\mathbf{G}}.$$

In particular, let $V(t)$ denote the action of Minkowski time evolution on $\mathbf{A}$. Given $A \in \mathbf{A}$, let $F = dA + \frac{1}{2}[A, A]$. Let $\mathbf{Y}$ be the set of $A \in \mathbf{A}$ satisfying the Yang-Mills equations:

$$d \star F + [A, \star F] = 0.$$

By the invariance of these equations under gauge and conformal transformations, $\mathbf{Y}$ is preserved by the time evolution $V(t)$ and the action of the gauge group $\mathcal{G}$ on $\mathbf{A}$. Moreover, if $A \in \mathbf{Y}$, then $\iota^* A$ is a C^2 solution of the Yang-Mills equations on $\mathbf{M}_0$, where $\iota\colon \mathbf{M}_0 \to \widetilde{\mathbf{M}}$ is the conformal embedding. Thus one may regard $\mathbf{Y}$ as the space of C^2 solutions of the Yang-Mills equations on Minkowski space that extend to C^2 solutions on $\widetilde{\mathbf{M}}$. There are gauge-invariant conserved quantities for such solutions, analogous to the conserved quantities for the massless ϕ^4 theory constructed in Corollary 2:

Theorem 7. *Let $\psi \in C^\infty(\mathfrak{g})$ be an Ad-invariant function with $\psi(0) = 0$. Let $\phi \in C_0(C^*)$ and let u, v be continuous vector fields on C^* invariant under the action of $\mathbb{R}$ as Minkowski time translation. Let $h = \psi(F(u, v))$. Define the function $f\colon \mathbf{A} \to \mathbb{R}$ by:*

$$f(A) = \int_{C^*} \int_{\mathbf{R}} h(s, \omega)\, \phi(t, \omega)\, h(s-t, \omega)\, dt ds d^2\omega,$$

or, more concisely,

$$f(A) = \int_{C^*} h(s, \omega)\, (\phi * h)(s, \omega)\, ds d^2\omega.$$

Then for all $t \in \mathbb{R}$, $V(t)^* f = f$, and for all $g \in \mathcal{G}$, $g^* f = f$.

Proof - This was shown by Baez [5]. Note that the use of convolutions in this theorem is simply a reformulation of use of Fourier transforms in the proof of Corollary 2. $\square$

To obtain conserved quantities on a symplectic manifold of solutions modulo gauge transformations on which time evolution acts as symplectomorphisms, the above theorem will need to be refined. In particular, one probably should work not with C^2 solutions, but with solutions in a fixed gauge having Cauchy data in certain Sobolev spaces. For example, in temporal gauge, Cauchy data for the Yang-Mills equations on $\widetilde{\mathbf{M}}$ are given by $\mathfrak{g}$-valued one-forms A and E on S^3. Let $\mathbf{X}$ denote the space of such pairs for which the components of A lie in the Sobolev space $H^3(S^3)$, the components of E lie in $H^2(S^3)$, and the constraint equation $d \star E + [A, \star E] = 0$ holds. Then Choquet-Bruhat, Paneitz and Segal [14] proved that any Cauchy data $(A, E) \in \mathbf{X}$ determine a unique global solution of the Yang-Mills equations on $\widetilde{\mathbf{M}}$ in temporal gauge. Their proof uses the global existence theorem for the Yang-Mills equation on Minkowski space, proved by Eardley and Moncrief [16], and the conformal embedding of $\mathbf{M}_0$ in $\widetilde{\mathbf{M}}$. (One should note that the theorem of Eardley and Moncrief followed work by Glassey and Strauss [22], Segal [44, 45], Ginibre and Velo [20], and Choquet-Bruhat and Christodoulou [12, 13, 15]. In particular, Christodoulou seems to have been the first to use the embedding of $\mathbf{M}_0$ in $\widetilde{\mathbf{M}}$ in this context. For a review of all this work see Choquet-Bruhat [11].)

The space $\mathbf{X}$ is not the physical phase space for the Yang-Mills equations, however, because there is still left-over gauge freedom corresponding to time-independent gauge transformations. It has been shown by Arms, Marsden and Moncrief [1, 29], and independently by Mitter [28], that one may remove certain singular points from $\mathbf{X}$ and take the quotient by the action of the remaining gauge transformations to obtain a symplectic manifold $\mathbf{Z}$, the nonsingular part of the physical phase space. It is possible that many of the conserved quantities on $\mathbf{Y}$ constructed above extend to functions on $\mathbf{X}$, and that Minkowski time evolution is completely integrable as a one-parameter group of symplectomorphisms of $\mathbb{Z}$. Regardless of whether this turns out to be the case, it would be interesting to understand the significance of these conserved quantities for the Yang-Mills equations.

*

References

[1] J. Arms, J. Marsden and V. Moncrief, Symmetry and bifurcations of momentum mappings, Comm. Math. Phys. **78** (1980/81), 455-478.

[2] J. Baez, Scattering and the geometry of the solution manifold of $\Box f + \lambda f^3 = 0$, J. Funct. Anal. **83** (1989), 317-332.

[3] J. Baez, Scattering for the Yang-Mills equations, Trans. A. M. S. **315** (1989), 823-832.

[4] J. Baez, Scattering and complete integrability in conformally invariant nonlinear theories, J. Math. Phys. **31** (1990), 757-762.

[5] J. Baez, Conserved quantities for the Yang-Mills equations, Adv. Math. **82** (1990), 126-131. (Note that the statement of the present Theorem 7 corrects an error in this paper.)

[6] J. Baez, I. Segal and Z. Zhou, The global Goursat problem and scattering for nonlinear wave equations, J. Funct. Anal. **93** (1990), 239-269.

[7] J. Baez and Z. Zhou, Analyticity of scattering for the ϕ^4 theory, Comm. Math. Phys. **124** (1989), 9-21.

[8] J. Baez and Z. Zhou, Scattering and complete integrability in the massive ϕ^4 theory, Jour. Funct. Anal. **94** (1990), 397-414.

[9] P. Brenner, On scattering and everywhere defined scattering operators for non-linear Klein-Gordon equations, J. Diff. Eq. **56** (1985), 310-344.

[10] F. Calogero, Solution of the one-dimensional N-body problems with quadratic and/or inverse quadratic pair potentials, J. Math. Phys. **12** (1971), 419-436.

[11] Y. Choquet-Bruhat, Global solutions of hyperbolic equations of gauge theories, in *Relativity, Cosmology, Topological Mass, and Supergravity*, ed. C. Aragone, World Scientific, Singapore, 1983.

[12] Y. Choquet-Bruhat and D. Christodoulou, Existence de solutions globales des équations classiques des théories de jauge, C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), 39-42.

[13] Y. Choquet-Bruhat and D. Christodoulou, Existence of global solutions of the Yang-Mills, Higgs and spinor field equations in 3+1 dimensions, Ann. Sci. École Norm. Sup. **14** (1981), 481-506.

[14] Y. Choquet-Bruhat, S. Paneitz, and I. Segal, The Yang-Mills equations on the universal cosmos, J. Funct. Anal. **53** (1983), 112-150.

[15] D. Christodoulou, Solutions globales des équations de Yang et Mills, C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), 139-141.

[16] D. Eardley and V. Moncrief, The global existence of Yang-Mills-Higgs fields on 4-dimensional Minkowski space, Comm. Math. Phys. **83** (1982), 121-212.

[17] M. Flato and J. Simon, Yang-Mills equations are formally linearizable, Lett. Math. Phys. **3** (1979), 279-283.

[18] M. Flato and J. Simon, On a linearization program of non-linear field equations, Phys. Lett. B **94** (1980), 518-522.

[19] D. Freed and K. Uhlenbeck, *Instantons and Four-Manifolds,* Springer-Verlag, New York, 1984.

[20] J. Ginibre and G. Velo, The Cauchy problem for coupled Yang-Mills equations and scalar fields in the temporal gauge, Comm. Math. Phys. **82** (1981), 1-28.

[21] J. Ginibre and G. Velo, Conformal invariance and time decay for nonlinear wave equations, Ann. Inst. H. Poincaré Phys. Théor. **47** (1987), 221-276.

[22] R. Glassey and W. Strauss, Decay of classical Yang-Mills fields, Comm. Math. Phys. **65** (1979), 51-67.

[23] K. Jörgens, Das Anfangswertproblem in Grossen für eine Klasse nichtlinearer Wellengleichungen, Math. Z. **77** (1961), 295-308.

[24] P. Kumlin, Analyticity of the scattering operator for the Klein-Gordon equation with cubic nonlinearity, preprint.

[25] Y. Manin, *Gauge Field Theory and Complex Geometry,* Springer-Verlag, New York, 1988.

[26] B. Marshall, Mixed norm estimates for the Klein-Gordon equation, in *Conference on Harmonic Analysis in Honor of Antoni Zygmund,* Vol. 2, Wadsworth Math. Ser., Wadsworth, Blemont, California, 1983, pp. 638-649.

[27] B. Marshall, W. Strauss and S. Wainger, $L^p - L^q$ estimates for the Klein-Gordon equation, J. Math. Pures et Appl. **59** (1980), 417-440.

[28] P. Mitter, Geometry of the space of gauge orbits and Yang-Mills dynamical system, in *Recent Developments in Gauge Theories*, ed. G. 't Hooft et al, Plenum Press, New York, 1980, pp. 265-292.

[29] V. Moncrief, Reduction of the Yang-Mills equations, in *Differential Geometrical Methods in Mathematical Physics*, eds. P. Garcia, A. Pérez-Rendón, and J. Souriau, Lecture Notes in Mathematics 836, Springer-Verlag, New York, 1980, pp. 276-291.

[30] C. Morawetz and W. Strauss, Decay and scattering of solutions of a nonlinear relativistic wave equation, Comm. Pure Appl. Math. **25** (1972), 1-31.

[31] C. Morawetz and W. Strauss, On a nonlinear scattering operator, Comm. Pure Appl. Math. **26** (1973), 47-54.

[32] E. Nikolaevskii and L. Shchur, The nonintegrability of the classical Yang-Mills equations, Sov. Phys. JETP **58**, 1-7 (1983).

[33] S. Paneitz and I. Segal, Analysis in space-time bundles. I. General considerations and the scalar bundle, J. Funct. Anal. **47** (1982), 78-142.

[34] R. Penrose, Asymptotic properties of fields and space-times, Phys. Rev. Lett. **10**, 66-68 (1963).

[35] R. Penrose and W. Rindler, *Spinors and Space-time*, vols. 1 and 2, New York, Cambridge U. Press, 1984/86.

[36] P. Raczka and W. Strauss, Analyticity properties of the scattering operator in nonlinear relativistic classical and prequantized field theories, Rep. Math. Phys. **16** (1979), 317-327.

[37] M. Reed, *Abstract Nonlinear Wave Equations,* Lecture Notes in Mathematics 507, Springer-Verlag, New York, 1976.

[38] G. Savvidi, Classical and quantum mechanics of non-abelian gauge fields, Nuc. Phys. B **246** (1984), 302-334.

[39] G. Savvidi, S. Matinyan, and E. Prokhorenko, Nonintegrability of time-dependent spherically symmetric Yang-Mills equations, Nuc. Phys. B **298** (1988), 414-428.

[40] I. Segal, Non-linear semi-groups, Ann. Math. **78** (1963), 339-364.

[41] I. Segal, Differential operators in the manifold of solutions of a non-linear differential equation, Jour. Math. Pures et Appl. **44** (1965), 71-132.

[42] I. Segal, Quantization and dispersion for nonlinear relativistic equations, in *Mathematical Theory of Elementary Particles,* ed. R. Goodman and I. Segal, M.I.T. Press, Cambridge, Mass., 1966, pp. 79-108.

[43] I. Segal, Scattering theory as the analysis of elliptic fixed points, in *Global Analysis*, Proc. Symp. Pure Math. **16** (A.M.S., Providence, Rhode Island, 1970), pp. 95-100.

[44] I. Segal, General properties of the Yang-Mills equations in physical space, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), 4638-4639.

[45] I. Segal, The Cauchy problem for the Yang-Mills equations, J. Funct. Anal. **33** (1979), 175-194.

[46] I. Segal, Reduction of scattering to an invariant finite displacement in an ambient space-time, Proc. Nat. Acad. Sci. U.S.A. **81** (1984), 7266-7268.

[47] I. Segal, Nonlinear wave equations, in *Nonlinear Partial Differential Operators and Quantization Procedures,* Lecture Notes in Mathematics 1037, Springer-Verlag, New York, 1983, pp. 115-141.

[48] J. Simon and E. Taflin, Wave operators and analytic solutions for systems of non-linear Klein-Gordon equations and of non-linear Schrödinger equations, Comm. Math. Phys. **99** (1985), 541-562.

[49] W. Strauss, Decay and asymptotics for $\Box u = F(u)$, J. Funct. Anal. **2** (1968), 409-457.

[50] W. Strauss, Everywhere defined wave operators, in *Nonlinear Evolution Equations,* ed. M. Crandall, Academic Press, New York, 1978, pp. 85-101.

[51] W. Strauss, Nonlinear scattering theory at low energy, J. Funct. Anal. **41** (1981), 110-133.

[52] W. Strauss, *Nonlinear Wave Equations,* American Mathematical Society, Providence, Rhode Island, 1989.

[53] R. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions to wave equations, Duke Math. Jour. **44** (1977), 705-714.