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Scattering and Complete Integrability in the Massive φ^4 Theory

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The wave and scattering operators for the equation $(\square + m^2)\varphi + \lambda\varphi^3 = 0$ with $m, \lambda > 0$ on four-dimensional Minkowski space are analytic on a neighborhood of the space of finite-energy Cauchy data, $L^{2,1}(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. This allows the construction of infinitely many independent conserved quantities with vanishing Poisson brackets and implies that the massive φ^4 theory is completely integrable at low energy.

1. INTRODUCTION

We begin by recalling some basic facts about scattering for the massive φ^4 theory equation,

$$(\square + m^2)\varphi + \lambda\varphi^3 = 0, \qquad m > 0, \qquad \lambda \geqslant 0, \tag{1}$$

where φ is a real function on Minkowski space $\mathbf{M}_0$, which we identify with $\mathbb{R}^4$ given the coordinates $(x_0, x_1, x_2, x_3) = (x_0, \mathbf{x})$. Let $L^{q,r}(\mathbb{R}^n)$ denote the Sobolev space of functions on $\mathbb{R}^n$ with r derivatives in L^q, where here and in what follows all function spaces are of *real* functions unless otherwise specified. The space of finite-energy Cauchy data, $\mathbf{H}$, is defined as the Hilbert space $L^{2,1}(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ with norm given by

$$\|(u_1, u_2)\|^2 = \frac{1}{2}\int_{\mathbb{R}^3} (\nabla u_1)^2 + m^2 u_1^2 + u_2^2.$$

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Given $u \in \mathbf{H}$ there is a unique distributional solution φ of (1) with $u = (\varphi, \dot\varphi)|_{x_0=0}$. Let $V(t)\,u = (\varphi, \dot\varphi)|_{x_0=t}$. Then V is a strongly continuous one-parameter group of diffeomorphisms of $\mathbf{H}$. In the special case $\lambda = 0$, (1) reduces to the linear Klein–Gordon equation, and we denote the corresponding one-parameter group by U. For all $t \in \mathbb{R}$, $U(t)$ is an orthogonal linear operator on $\mathbf{H}$. V may be defined in terms of U via an integral equation: if N denotes the function from $\mathbf{H}$ to $\mathbf{H}$ given by $N(u_1, u_2) = (0, -\lambda u_1^3)$, then

$$V(t)\,u = U(t)\,u + \int_0^t U(t-s)\,N(V(s)\,u)\,ds.$$

For any $u \in \mathbf{H}$, there exist $u_+, u_- \in \mathbf{H}$ such that

$$\lim_{t \to \pm\infty} \|U(t)\,u_\pm - V(t)\,u\| = 0,$$

or, equivalently,

$$u_\pm = u + \lim_{t \to \pm\infty} \int_0^t U(-s)\,N(V(s)\,u)\,ds.$$

In fact there are homeomorphisms, the wave operators $W_\pm : \mathbf{H} \to \mathbf{H}$, such that $W_\pm(u_\pm) = u$. The scattering operator $S: \mathbf{H} \to \mathbf{H}$ may thus be defined by $S = (W_+)^{-1}\,W_-$.

The most natural way to prove the existence of the limit above is to prove absolute convergence of the integral $\int_{-\infty}^{+\infty} U(-s)\,N(V(s)\,u)\,ds$. By the Sobolev inequalities it suffices to show that

$$\int_{-\infty}^{+\infty} \left(\int_{\mathbb{R}^3} \varphi(t,x)^6\,dt\right)^{1/2} dt < \infty.$$

In other words, it suffices to show that $\varphi \in L^3(L^6)$, where we write $L^p(L^q)$ for $L^p(\mathbb{R}, L^q(\mathbb{R}^3))$. Though this has been shown for u satisfying various decay conditions [8, 11], it has never been shown for all $u \in \mathbf{H}$. Thus Strauss took an indirect approach, whereby for all sufficiently small $u \in \mathbf{H}$ he proved the existence of the limit without proving absolute convergence of the integral [12, 13]. By a substantial elaboration of Strauss' method, Brenner [4] was able to construct an everywhere-defined scattering operator $S: \mathbf{H} \to \mathbf{H}$.

In fact, $\varphi \in L^3(L^6)$ for all $u \in \mathbf{H}$. Let $L^p(L^{q,r})$ denote $L^p(\mathbb{R}, L^{q,r}(\mathbb{R}^3))$ and let $\mathbf{X} = L^\infty(L^{2,1}) \cap L^3(L^6)$, a Banach space with the norm

$$\|\varphi\|_{\mathbf{X}} = \|\varphi\|_{L^\infty(L^{2,1})} + \|\varphi\|_{L^3(L^6)}.$$

Then we have:

THEOREM 1. *If $u \in \mathbf{H}$ and φ is the unique distributional solution of* (1) *with $u = (\varphi, \dot\varphi)|_{x_0=0}$, then $\varphi \in \mathbf{X}$. Moreover, for some neighborhood Ω of the origin of $\mathbf{H}$ the map $u \mapsto \varphi$ is analytic from Ω to $\mathbf{X}$.*

By "analytic" we mean real-analytic, that is, infinitely Fréchet-differentiable with a locally norm-convergent Taylor series. Our proof of Theorem 1 makes use of Brenner's work and recent work of Marshall, who essentially proved that $u \in \mathbf{H}$ implies $\varphi \in \mathbf{X}$ in the linear case [6].

Theorem 1 has as a consequence:

THEOREM 2. *The operators $W_\pm, S: \mathbf{H} \to \mathbf{H}$ are analytic in a neighborhood Ω of the origin of $\mathbf{H}$.*

Theorem 2 strengthens earlier results of Raczka and Strauss [10] and the authors [3]. The analyticity of the wave operators in a neighborhood Ω of the origin allows the construction of infinitely many analytic functions $F_i: \Omega \to \mathbb{R}$ that are conserved quantities for the nonlinear time evolution $V(t)$. In fact, the massive φ^4 theory is completely integrable at low energy.

To make this precise, recall that the space $\mathbf{H}$ has a symplectic structure ω given by

$$\omega(u, v) = \int_{x_0=t} (u_1 v_2 - u_2 v_1)\,d^3x,$$

where $u, v \in T_x\mathbf{H}$ are tangent vectors identified with vectors in $\mathbf{H}$. The following propositions extend the work of Morawetz and Strauss [9]:

PROPOSITION 1. *The operators $W_\pm, S: \mathbf{H} \to \mathbf{H}$ are symplectomorphisms on a neighborhood Ω of the origin of $\mathbf{H}$.*

PROPOSITION 2. *For all $t \in \mathbb{R}$, $V(t) = W_\pm\,U(t)\,W_\pm^{-1}$.*

Using the above propositions we obtain the following:

THEOREM 3. *There exists a neighborhood Ω of the origin of $\mathbf{H}$ that is invariant under $V(t)$ for all $t \in \mathbb{R}$, and there exist analytic functions $F_j: \Omega \to \mathbb{R}$ such that for all j and all $t \in \mathbb{R}$, $V(t)^*\,F_j = F_j$. There exist vector fields v_j on Ω such that*

$$dF_j = \omega(v_j, \cdot).$$

Moreover: (1) *For all j and k, $\omega(v_j, v_k) = 0$.* (2) *Generically, i.e., except for u in a set of first category in Ω, the subspace $L = \{v \in T_u\mathbf{H} : \forall j\, dF_j(v) = 0\}$ is isotropic in $T_u\mathbf{H}$. (That is, $\omega(v, w) = 0$ for all $v, w \in L$.)*

In other words, the functions F_i are a complete set of integrals of motion for the massive φ^4 theory at low energy, with pairwise vanishing Poisson brackets. (For a discussion of complete integrability see, for example, [1].) A similar result has been obtained for the *massless* φ^4 theory without any such "small data" condition [2]. If one could show that the map $u \mapsto \varphi$ is analytic from all of $\mathbf{H}$ to $\mathbf{X}$, then the "small data" condition in Proposition 1 and Theorems 2 and 3 could be omitted.

2. PROOF OF ANALYTICITY

First we note that Theorem 1 holds for the linear Klein–Gordon equation. Given $u \in \mathbf{H}$, let φ_0 denote the unique distributional solution of the linear Klein–Gordon equation $(\Box + m^2)\,\varphi_0 = 0$ with $(\varphi_0, \dot{\varphi}_0)|_{x_0=0} = u$.

LEMMA 1 (B. Marshall). $\|\varphi_0\|_{\mathbf{X}} \leqslant c\,\|u\|_{\mathbf{H}}$.

Proof. Marshall proved this in the case where $\varphi_0|_{x_0=0} = 0$; we defer the proof to the Appendix. ∎

We write the solution to (1) with initial data $u \in \mathbf{H}$ as

$$\varphi(t) = \varphi_0(t) - \lambda \int_0^t K(t-s)\,\varphi(s)^3\,ds,$$

where $K(t) = B^{-1} \sin tB$ and $B = \sqrt{\Delta + m^2}$. We will use the following estimates on $K(t)$:

LEMMA 2. *Suppose that* $1/q + 1/q' = 1$, $1 < q' \leqslant q < \infty$, $\delta = 1/2 - 1/q$, *and* $5\delta = r' - r + 1$. *Then*

$$\|K(t)\,g\|_{q,r} \leqslant k(t)\,\|g\|_{q',r'},$$

where

$$k(t) \leqslant \begin{cases} ct^{-\delta} & \text{if} \quad t \in (0, 1]; \\ ct^{-3\delta}(\ln(1+t))^{2\delta} & \text{if} \quad t \in [1, \infty). \end{cases}$$

Proof. This follows from Brenner's estimate A [4], but we present a simplified proof in the Appendix. ∎

LEMMA 3. *Given* $\varphi_1, \varphi_2, \varphi_3 \in \mathbf{X}$, *let*

$$\psi(t) = \int_0^t K(t-s)\,\varphi_1(s)\,\varphi_2(s)\,\varphi_3(s)\,ds.$$

Then $\psi \in \mathbf{X}$ *and the map* $(\varphi_1, \varphi_2, \varphi_3) \mapsto \psi$ *is analytic from* $\mathbf{X}^3$ *to* $\mathbf{X}$.

Proof. Since the map $(\varphi_1, \varphi_2, \varphi_3) \mapsto \psi$ is multilinear it suffices to show that

$$\|\psi\|_{\mathbf{X}} \leqslant c\,\|\varphi_1\|_{\mathbf{X}}\,\|\varphi_2\|_{\mathbf{X}}\,\|\varphi_3\|_{\mathbf{X}}.$$

Let $F(t, s) = K(t-s)\,\varphi_1(s)\,\varphi_2(s)\,\varphi_3(s)$. By Hölder's inequality

$$\|\psi(t)\|_6 \leqslant \left\| \int_0^t F(t, s)\,ds \right\|_2^{1/9} \left\| \int_0^t F(t, s)\,ds \right\|_8^{8/9}. \tag{2}$$

We estimate the first factor using Hölder's inequality as follows:

$$\begin{aligned} \left\| \int_0^t F(t, s)\,ds \right\|_2 &\leqslant c \left\| \int_0^t K(t-s)\,\varphi_1(s)\,\varphi_2(s)\,\varphi_3(s)\,ds \right\|_2 \\ &\leqslant c \int_{-\infty}^{\infty} \|\varphi_1(s)\,\varphi_2(s)\,\varphi_3(s)\|_2\,ds \\ &\leqslant c' \int_{-\infty}^{\infty} \|\varphi_1(s)\|_6\,\|\varphi_2(s)\|_6\,\|\varphi_3(s)\|_6\,ds \\ &\leqslant c'\,\|\varphi_1\|_{\mathbf{X}}\,\|\varphi_2\|_{\mathbf{X}}\,\|\varphi_3\|_{\mathbf{X}}. \end{aligned} \tag{3}$$

To treat the second factor, we first note that by Lemma 2,

$$\|F(t, s)\|_8 \leqslant k(t-s)\,\|\varphi_1(s)\,\varphi_2(s)\,\varphi_3(s)\|_{8/7, 1},$$

where $k \in L^1(\mathbb{R})$. It follows from Young's inequality that for some constant M,

$$\begin{aligned} \left\| \left(\left\| \int_0^t F(t, s)\,ds \right\|_8 \right)^{8/9} \right\|_{L^3(\mathbb{R})} &\leqslant \left\| \left(\int_0^t \|F(t, s)\|_8\,ds \right) \right\|_{L^{8/3}(\mathbb{R})}^{8/9} \\ &\leqslant M\,\|\varphi_1 \varphi_2 \varphi_3\|_{L^{8/3}(L^{8/7,1})}^{8/9}. \end{aligned} \tag{4}$$

Writing Φ_i for $\varphi_i(t)$, by Hölder's inequality we have

$$\begin{aligned} &\|\Phi_1 \Phi_2 \Phi_3\|_{8/7,1} \\ &\quad \leqslant \|\Phi_1\|_{24/7}\,\|\Phi_2\|_{24/7}\,\|\Phi_3\|_{24/7} + (\|\Phi_1\|_{2,1}\,\|\Phi_2\|_{24/5}\,\|\Phi_3\|_6 + \cdots), \end{aligned}$$

where the dots indicate two terms obtained from the first by cyclic permutations of 1, 2, 3. Using Hölder's inequality again we obtain

$$\begin{aligned} \|\varphi_1 \varphi_2 \varphi_3\|_{L^{8/3}(L^{8/7,1})} &\leqslant \|\varphi_1\|_{L^8(L^{24/7})}\,\|\varphi_2\|_{L^8(L^{24/7})}\,\|\varphi_3\|_{L^8(L^{24/7})} \\ &\quad + (\|\varphi_1\|_{L^\infty(L^{2,1})}\,\|\varphi_2\|_{L^{24}(L^{24/5})}\,\|\varphi_3\|_{L^3(L^6)} + \cdots). \end{aligned}$$

By Lemma 4 below, for some constant c,

$$\|\varphi\|_{L^8(L^{24/7})},\ \|\varphi\|_{L^{24}(L^{24/5})} \leqslant c\, \|\varphi\|_{\mathbf{X}}.$$

It follows that for some constant c,

$$\|\varphi_1 \varphi_2 \varphi_3\|_{L^{8/3}(L^{8/7,1})} \leqslant c\, \|\varphi_1\|_{\mathbf{X}}\, \|\varphi_2\|_{\mathbf{X}}\, \|\varphi_3\|_{\mathbf{X}}.$$

Thus (4) implies that for some constant M,

$$\left\| \left(\left\| \int_0^t F(t,s)\, ds \right\|_8 \right)^{8/9} \right\|_{L^3(\mathbb{R})} \leqslant M(\|\varphi_1\|_{\mathbf{X}}\, \|\varphi_2\|_{\mathbf{X}}\, \|\varphi_3\|_{\mathbf{X}})^{8/9}. \tag{5}$$

By (2), (3), and (5) we have for some constant C,

$$\|\psi\|_{L^3(L^6)} \leqslant C\, \|\varphi_1\|_{\mathbf{X}}\, \|\varphi_2\|_{\mathbf{X}}\, \|\varphi_3\|_{\mathbf{X}}.$$

To complete the proof it thus suffices to prove that

$$\|\psi\|_{L^\infty(L^{2,1})} \leqslant C'\, \|\varphi_1\|_{\mathbf{X}}\, \|\varphi_2\|_{\mathbf{X}}\, \|\varphi_3\|_{\mathbf{X}}$$

for some constant C'. This may be seen as follows: for all $t \in \mathbb{R}$,

$$\begin{aligned} \|\psi(t)\|_{2,1} &\leqslant c \left\| \int_0^t K(t-s)\, \varphi_1(s)\, \varphi_2(s)\, \varphi_3(s)\, ds \right\|_{2,1} \\ &\leqslant c \int_{-\infty}^{\infty} \|\varphi_1(s)\, \varphi_2(s)\, \varphi_3(s)\|_2\, ds \\ &\leqslant c' \int_{-\infty}^{\infty} \|\varphi_1(s)\|_6\, \|\varphi_2(s)\|_6\, \|\varphi_3(s)\|_6\, ds \\ &\leqslant c'\, \|\varphi_1\|_{\mathbf{X}}\, \|\varphi_2\|_{\mathbf{X}}\, \|\varphi_3\|_{\mathbf{X}} \end{aligned}$$

by Hölder's inequality. ∎

Lemma 4. *For some constant c, for all $\varphi \in \mathbf{X}$,*

$$\|\varphi\|_{L^8(L^{24/7})},\ \|\varphi\|_{L^{24}(L^{24/5})} \leqslant c\, \|\varphi\|_{\mathbf{X}}.$$

Proof. It follows from the Sobolev inequalities that if $r \in [2, 6]$,

$$\|\varphi\|_{L^\infty(L^r)},\ \|\varphi\|_{L^3(L^6)} \leqslant C\, \|\varphi\|_{\mathbf{X}}.$$

Therefore by the Riesz–Thorin theorem $\|\varphi\|_{L^p(L^q)} \leqslant c\, \|\varphi\|_{\mathbf{X}}$ if $1/p = \theta/3$ and $1/q = \theta/6 + (1-\theta)/r$, where $0 \leqslant \theta \leqslant 1$. If we choose $\theta = \frac{3}{8}$, $r = \frac{30}{11} \in [2, 6]$, then

$$\frac{1}{p} = \frac{1}{8}, \qquad \frac{1}{q} = \frac{1}{16} + \frac{11}{48} = \frac{7}{24};$$

hence $\|\psi\|_{L^8(L^{24/7})} \leqslant c\, \|\varphi\|_{\mathbf{X}}$. If we choose $\theta = \frac{1}{8}$, $r = \frac{14}{3} \in [2, 6]$, then

$$\frac{1}{p} = \frac{1}{24}, \qquad \frac{1}{q} = \frac{1}{48} + \frac{9}{48} = \frac{5}{24};$$

hence $\|\psi\|_{L^{24}(L^{24/5})} \leqslant c\, \|v\|$. ∎

Proof of Theorem 1. Given $u \in \mathbf{H}$, let φ be the unique distributional solution of (1) with $(\varphi, \dot{\varphi})|_{x_0 = 0} = u$. It follows from conservation of energy that $\varphi \in L^\infty(L^{2,1})$; so to show that $\varphi \in \mathbf{X}$, it suffices to prove $\varphi \in L^3(L^6)$. The proof resembles that of Lemma 3. Writing

$$\varphi(t) = \varphi_0(t) - \lambda \int_0^t K(t-s)\, \varphi(s)^3\, ds,$$

by Lemma 1 it suffices to show that $\psi \in L^3(L^6)$, where

$$\psi(t) = -\lambda \int_0^t K(t-s)\, \varphi(s)^3\, ds.$$

By Hölder's inequality,

$$\|\psi(t)\|_6 \leqslant \|\psi(t)\|_2^{1/9}\, \|\psi(t)\|_8^{8/9}. \tag{6}$$

By conservation of energy, the first factor is bounded by a constant independent of t:

$$\|\psi(t)\|_2 \leqslant \|\varphi_0(t)\|_2 + \|\varphi(t)\|_2 \leqslant c. \tag{7}$$

To treat the second factor, we note that by Lemma 2,

$$\|\psi(t)\|_8 \leqslant \lambda \int_0^t k(t-s)\, \|\varphi(s)^3\|_{8/7,1}\, ds.$$

By Lemma 5 below, for some constant c,

$$\|\varphi(s)^3\|_{8/7,1} \leqslant c\, \|\varphi(s)\|_{2,1}^{3/2}\, \|\varphi(s)\|_8^{3/2};$$

so by conservation of energy, for some constant C,

$$\|\psi(t)\|_8 \leqslant C \int_0^t k(t-s)\, \|\varphi(s)\|_8^{3/2}\, ds \tag{8}$$

for all t.

It follows from (6), (7), and (8) that for some constant c,

$$\|\psi(t)\|_6 \leqslant c \left(\int_0^t k(t-s)\, \|\varphi(s)\|_8^{3/2}\, ds \right)^{8/9}. \tag{9}$$

Brenner's theorem 1 in [4] implies that $\varphi \in L^4(L^8)$. It follows that $\|\varphi(s)\|_8^{3/2} \in L^{8/3}$. Since $k \in L^1(\mathbb{R})$, it follows from Young's inequality that

$$\int_0^t k(t-s)\, \|\varphi(s)\|_8^{3/2}\, ds \in L^{8/3}(\mathbb{R});$$

so by (9), $\psi \in L^3(L^6)$ as desired.

We prove that $u \mapsto \varphi$ is analytic for small u using the contractive mapping technique. This will also provide a different proof that $\varphi \in \mathbf{X}$ for small $u \in \mathbf{H}$. By Lemmas 1 and 3, there is an analytic map $R: \mathbf{H} \times \mathbf{X} \to \mathbf{X}$ given by

$$R(u, \psi)(t) = \varphi_0(t) - \lambda \int_0^t K(t-s)\, \psi(s)^3\, ds,$$

where φ_0 is as above.

By Lemmas 1 and 3 we can choose c such that

$$\lambda \left\| \int_0^t K(t-s)\, \varphi_1(s)\, \varphi_2(s)\, \varphi_3(s)\, ds \right\|_{\mathbf{X}} \leqslant c\, \|\varphi_1\|_{\mathbf{X}}\, \|\varphi_2\|_{\mathbf{X}}\, \|\varphi_2\|_{\mathbf{X}}$$

and

$$\|\varphi_0\|_{\mathbf{X}} \leqslant c\, \|u\|_{\mathbf{H}}.$$

Choose $r > 0$ such that $16cr^2 < 1$. Suppose that $c\, \|u\|_{\mathbf{H}} \leqslant r$, so that $\|\varphi_0\|_{\mathbf{X}} \leqslant r$. Then we claim that the map $R(u, \cdot): \mathbf{X} \to \mathbf{X}$ preserves the open set

$$E = \{\psi \in \mathbf{X} : \|\psi - \varphi_0\|_{\mathbf{X}} < r\},$$

and is a contradiction on this set. If $\varphi \in E$, then

$$\|R(u, \psi) - \varphi_0\|_{\mathbf{X}} \leqslant c\, \|\psi\|_{\mathbf{X}}^3 \leqslant c(\|\psi - \varphi_0\|_{\mathbf{X}} + \|\varphi_0\|_{\mathbf{X}})^3 < r,$$

so $R(u, \psi) \in E$. Moreover, if $\psi_1, \psi_2 \in E$, then

$$\begin{aligned}
\|R(u, \psi_1) - R(u, \psi_2)\|_{\mathbf{X}} &= \lambda \left\| \int_0^t K(t-s)(\psi_1(s)^3 - \psi_2(s)^3)\, ds \right\|_{\mathbf{X}} \\
&\leqslant c(\|\psi_1\|_{\mathbf{X}} + \|\psi_2\|_{\mathbf{X}})^2\, \|\psi_1 - \psi_2\|_{\mathbf{X}} \\
&\leqslant c(2\, \|\varphi_0\|_{\mathbf{X}} + \|\psi_1 - \varphi_0\|_{\mathbf{X}} + \|\psi_2 - \varphi_0\|_{\mathbf{X}})^2\, \|\psi_1 - \psi_2\|_{\mathbf{X}} \\
&\leqslant 16cr^2\, \|\psi_1 - \psi_2\|_{\mathbf{X}} < \|\psi_1 - \psi_2\|_{\mathbf{X}}.
\end{aligned}$$

Given $u \in \mathbf{H}$ there is a unique $\varphi \in \mathbf{X}$ such that $R(u, \varphi) = \varphi$. Since $R(u, \cdot)$ is a contraction of the open set E if $c\, \|u\|_{\mathbf{H}} < r$, it follows from the analytic implicit function theorem [3] that the function $u \mapsto \varphi$ is analytic from $\{u : c\, \|u\|_{\mathbf{H}} < r\}$ to $\mathbf{X}$. ∎

LEMMA 5. *For some constant c,*

$$\|f^3\|_{8/7,1} \leqslant c\, \|f\|_{2,1}^{3/2}\, \|f\|_8^{3/2}$$

for all $f \in L^{2,1}(\mathbb{R}^3) \cap L^8(\mathbb{R}^3)$.

Proof. Since

$$\|f^3\|_{8/7,1} \leqslant \|f^3\|_{8/7} + 3\, \|f^2\, \nabla f\|_{8/7},$$

it suffices to estimate each of the terms on the right side. For the first term, Hölder's inequality implies

$$\|f^3\|_{8/7} \leqslant \|f\|_8^{3/2}\, \|f\|_{24/11}^{3/2},$$

and since $24/11 \in [2, 6]$, the Sobolev inequalities imply

$$\|f^3\|_{8/7} \leqslant c\, \|f\|_8^{3/2}\, \|f\|_{2,1}^{3/2}.$$

For the second term, Hölder's inequality implies

$$\|f^2\, \nabla f\|_{8/7} \leqslant \|f^2\|_{8/3}\, \|\nabla f\|_2 \leqslant \|f\|_8^{3/2}\, \|f\|_{8/3}^{1/2}\, \|\nabla f\|_2,$$

and since $\frac{8}{3} \in [2, 6]$, the Sobolev inequalities imply

$$\|f^2\, \nabla f\|_{8/7} \leqslant c\, \|f\|_8^{3/2}\, \|f\|_{2,1}^{3/2}. \quad ∎$$

Proof of Theorem 2. The map $u \mapsto \varphi$ given by Theorem 1 is analytic from a neighborhood Ω of the origin of $\mathbf{H}$ to $L^3(L^6)$. It follows that for $u \in \Omega$ the inverse wave operators are given by

$$W_{\pm}^{-1} u = u + \int_0^{\pm\infty} U(-s)(0, -\lambda\varphi(s)^3)\, ds,$$

the integral converging absolutely in $\mathbf{H}$, and that $W_{\pm}^{-1}: \Omega \to \mathbf{H}$ are analytic. It is easily seen that the derivative of the maps $W_{\pm}^{-1}$ at the origin is the identity operator on $\mathbf{H}$. It follows from the analytic inverse function theorem [3] that the maps $W_{\pm}$ are also analytic on a neighborhood of the origin of $\mathbf{H}$. Since $S = (W_+)^{-1}\, W_-$, the map S is analytic on a neighborhood of the origin of $\mathbf{H}$ as well. ∎

3. Proof of Complete Integrability

Proof of Proposition 1. Morawetz and Strauss [9] have shown that

$$\omega(dW_{\pm} v, dW_{\pm} w) = \omega(v, w) \tag{10}$$

if $v, w \in T_u\mathscr{F}$ and $u \in \mathscr{F}$, where $\mathscr{F}$ is a certain dense subspace of $\mathbf{H}$ such that $W_\pm : \mathscr{F} \to \mathscr{F}$ is a diffeomorphism. By Theorem 2, the maps $W_\pm$ are real-analytic on a neighborhood Ω of the origin of $\mathbf{H}$, so (10) holds for all $v, w \in T_u\mathbf{H}$ if $u \in \Omega$. ∎

Proof of Proposition 2. This follows directly from the definition of the maps $W_\pm$. ∎

Proof of Theorem 3. Note that by conservation of energy there exist arbitrarily small neighborhoods of the origin in $\mathbf{H}$ that are invariant under $V(t)$ for all t, namely those of the form

$$\Omega = \left\{ u \in \mathbf{H} : \int \frac{1}{2}\left((\nabla u_1)^2 + m^2 u_1^2 + u_2^2\right) + \frac{\lambda}{4} u_1^4 < \varepsilon \right\}.$$

Choosing Ω small enough, by Theorem 2 and Propositions 1 and 2, W_-^{-1} is an analytic symplectomorphism from Ω to a $U(t)$-invariant neighborhood Ω' of the origin of $\mathbf{H}$, and W_-^{-1} intertwines the action of $V(t)$ on Ω with the action of $U(t)$ on Ω'. Thus it suffices to prove the theorem for the action of $U(t)$ on Ω', that is, in the free case.

Given $\mathbf{k} \in \mathbb{R}^3$, let $k_0 = (\mathbf{k}^2 + m^2)^{1/2}$. There is an isomorphism $R: \mathbf{H} \to L^2(\mathbb{R}^3, \mathbb{C})$ such that:

$$\varphi_0(x) = (2\pi)^{-3/2} \operatorname{Re} \int (Ru)(\mathbf{k})\, e^{i(k_0 x_0 - \mathbf{k}\cdot\mathbf{x})} k_0^{-1}\, d^3k.$$

Given a real-valued function $h \in C_0^\infty(\mathbb{R}^3)$, define the function $F_h : \mathbf{H} \to \mathbb{R}$ by

$$F_h(u) = \int h(\mathbf{k})\, |(Ru)(\mathbf{k})|^2\, k_0^{-1}\, d^3k.$$

Note that F_h is analytic on $\mathbf{H}$. Furthermore, given $t \in \mathbb{R}$ and $u \in \mathbf{H}$, $U(t)^* F_h = F_h$ because $RU(t)\, u = e^{ik_0 t} Ru$.

We define the functions F_j and corresponding vector fields v_j as follows. Let $\{h_j\}$ be a sequence of real functions in $C_0^\infty(\mathbb{R}^3)$ spanning $L^1(\mathbb{R}^3, \mathbb{C})^*$ in its weak-* topology, and let $F_j = F_{h_j}$. Clearly the map $u \mapsto R^{-1}(2ih_j Ru)$ is continuous and linear from $\mathbf{H}$ to $\mathbf{H}$ and can be identified with an analytic vector field on $\mathbf{H}$, which we denote by v_j.

It is well known that if $v, w \in T_u\mathbf{H}$ are identified with vectors in $\mathbf{H}$ by translation:

$$\omega(v, w) = \operatorname{Im} \int Rv\overline{Rw} k_0^{-1}\, d^3k.$$

Thus if $w \in T_u\mathbf{H}$,

$$dF_j(w) = 2 \operatorname{Re} \int h_j Ru\overline{Rw} k_0^{-1}\, d^3k = \omega(v_j, w),$$

as claimed.

Statement (1) is proved as follows:

$$\omega(v_i, v_j) = \operatorname{Im} \int Rv_i\overline{Rv_j} k_0^{-1}\, d^3k = 4 \operatorname{Im} \int h_i h_j\, |Ru|^2\, k_0^{-1}\, d^3k = 0.$$

To prove statement (2), we show that the space $L \subset T_u\mathbf{H}$ is isotropic for all $u \in D$, where $D \subset \mathbf{H}$ is the set of u such that Ru is a.e. nonzero. An argument as in [2] shows that the complement of D is of first category in $\mathbf{H}$. Suppose that $u \in D$ and $v \in T_u\mathbf{H}$. Then if $v \in L$,

$$2 \operatorname{Re} \int h_i Ru\overline{Rv} k_0^{-1}\, d^3k = 0$$

for all i, which implies that $Ru\overline{Rv}$ is a.e. imaginary, since $\{h_i\}$ spans $L^1(\mathbb{R}^3, \mathbb{C})^*$. Thus given v and w in L, $Rv\overline{Rw}\, |Ru|^2$ is a.e. real, so by the definition of D, $Rv\overline{Rw}$ is a.e. real. It follows that

$$\omega(v, w) = \operatorname{Im} \int Rv\overline{Rw} k_0^{-1}\, d^3k = 0$$

as was to be shown. ∎

Appendix

Proof of Lemma 1. We follow the arguments of Marshall, Strauss, and Wainger [6, 7]. For brevity we use φ rather than φ_0 to denote the unique distributional solution of $(\Box + m^2)\, \varphi = 0$ with $(\varphi, \dot\varphi)|_{x_0 = 0} = u$. By conservation of energy, $\|\varphi\|_{L^\infty(L^{2,1})} \leqslant C\, \|u\|_{\mathbf{H}}$, so it suffices to show that $\|\varphi\|_{L^3(L^6)} \leqslant C\, \|u\|_{\mathbf{H}}$.

Note that

$$B^{-b}\varphi(t, \cdot) = B^{-b} \cos(tB)\, u_1 + B^{-(b+1)} \sin(tB)\, u_2.$$

Let dV denote the measure $dV = k_0^{-(b+1)}\, d^3k$ on the surface $\{k_0 = (\mathbf{k}^2 + m^2)^{1/2}\}$ in $\mathbf{M}_0$. Then

$$B^{-b}\varphi(t, \mathbf{x}) = (2\pi)^{-3/2} \operatorname{Re} \int F(\mathbf{k})\, e^{i(k_0 t - \mathbf{k}\cdot\mathbf{x})}\, dV,$$

where

$$F(\mathbf{k}) = (2\pi)^{-3/2} \int (Bu_1 - iu_2)(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3x.$$

In other words, $B^{-b}\varphi = \operatorname{Re} \widehat{F\,dV}$, where $\widehat{\ }$ denotes the usual Poincaré-invariant inverse Fourier transform on $\mathbf{M}_0$, up to a constant. Suppose that

b, p, and q satisfy the hypotheses of Lemma 6 below. Then using that lemma and the Plancherel theorem, it follows that

$$\|B^{-b}\varphi\|_{L^p(L^q)} \leqslant \|\widehat{F\,dV}\|_{L^p(L^q)} \leqslant C\|F\|_{L^2(dV)} = C\|B^{-(b+1)/2}(Bu_1 - iu_2)\|_2,$$

hence

$$\|B^{(1-b)/2}\varphi\|_{L^p(L^q)} \leqslant c\ \|Bu_1 - iu_2\|_2.$$

In particular, we may choose $b = \frac{1}{2}$, $p = 3$, and $q = 5$. The Sobolev inequalities then imply

$$\|\varphi\|_{L^3(L^6)} \leqslant c\ \|B^{1/4}\varphi\|_{L^3(L^5)} \leqslant c'\ \|Bu_1 - iu_2\|_2 = C\ \|u\|_{\mathbf{H}},$$

as was to be shown. ∎

LEMMA 6. *If $q \geqslant 2$ and the following hold,*

$$b = 5\left(\frac{1}{2} - \frac{1}{q}\right) - 1, \tag{11}$$

$$\frac{2}{3} < p\left(\frac{1}{2} - \frac{1}{q}\right) < 1, \tag{12}$$

then

$$\|\widehat{F\,dV}\|_{L^p(L^q)} \leqslant C\ \|F\|_{L^2(dV)}.$$

Proof of Lemma 6. By duality and an argument given by P. Tomas [14], it is enough to show that

$$\|\hat{F} * \widehat{dV}\|_{L^p(L^q)} \leqslant C\ \|\hat{F}\|_{L^{p'}(L^{q'})},$$

where $1/p + 1/p' = 1$, $1/q + 1/q' = 1$.

Note that if $f(s) = \|\hat{F}(s, \cdot)\|_{q'}$, and $k(t)$ is such that $\|h * \widehat{dV}(t, \cdot)\|_q \leqslant k(t)\ \|h\|_{q'}$ for all $h \in L^{q'}(\mathbb{R}^3)$, then

$$\|\hat{F} * \widehat{dV}\|_{L^p(L^q)} \leqslant \|k * f\|_{L^p(\mathbb{R})}.$$

Young's inequality implies that

$$\|k * f\|_p \leqslant \|k\|_{p/2}\ \|f\|_{p'} = \|k\|_{p/2}\ \|\hat{F}\|_{L^{p'}(L^{q'})}.$$

Thus to prove the lemma it suffices to show that $k \in L^{p/2}(\mathbb{R})$ for some k such that

$$\|h * \widehat{dV}(t, \cdot)\|_q \leqslant k(t)\ \|h\|_{q'}.$$

Note that

$$h * \widehat{dV}(t, \cdot) = B^{-(b+1)}e^{itB}h.$$

Given $t \in \mathbb{R}$ and $z \in \mathbb{C}$, let

$$P(z, t) = B^{-(z+1)}e^{itB}.$$

We will show by interpolation that $\|P(b, \cdot)\|_{L^{q'}, L^q} \in L^{p/2}(\mathbb{R})$.

Note, first, that if $\mathrm{Re}(z) = -1$, the Plancherel theorem implies $\|P(z, t)\|_{L^2, L^2} \leqslant 1$. Second, as we show below,

$$\|P(3/2, t)\|_{L^1, L^\infty} \leqslant ct^{-1}(1+t)^{-1/2}\ln(2+t) \tag{13}$$

for all $t > 0$. Moreover, [5], for all $y \in \mathbb{R}$, $\|B^{iy}\|_{L^\infty, BMO} \leqslant c(1 + |y|^2)$. Composing these operators we conclude that

$$\|P(z, t)\|_{L^1, BMO} \leqslant c(1 + |\mathrm{Im}(z)|)^2\, t^{-1}(1+t)^{-1/2}\ln(2+t)$$

if $\mathrm{Re}(z) = \frac{3}{2}$. The space BMO plays the role of L^∞ in complex interpolation theory [5]. Therefore by the Stein interpolation theorem, $P(b, t)$ maps $L^{q'}(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ and

$$\|P(b, t)\|_{L^{q'}, L^q} \leqslant ct^{-\varepsilon}(1+t)^{-\varepsilon/2}\ln(2+t)^\varepsilon,$$

where $\varepsilon \in [0, 1]$, $b = 5\varepsilon/2 - 1$, and $1/q = (1-\varepsilon)/2$; hence $\varepsilon = 1 - 2/q$. It follows that $\|P(b, t)\|_{L^{q'}, L^q} \in L^{p/2}$ if $q > 2$ and (11) and (12) hold.

Now let us show that (13) holds. By the definition of $P(z, t)$,

$$P(\tfrac{3}{2}, t)f = B^{-5/2}e^{itB}f = G(t, \cdot) * f,$$

where

$$G(t, \mathbf{x}) = \int k_0^{-5/2} e^{i(k_0 t - \mathbf{k}\cdot\mathbf{x})}\, d^3k.$$

Thus it is enough to show that for some constant C,

$$|G(t, \mathbf{x})| \leqslant Ct^{-1}(1+t)^{-1/2}\ln(2+t)$$

for every $\mathbf{x} \in \mathbb{R}^3$. Let $R = \|\mathbf{x}\|$ and $r = \|\mathbf{k}\|$. Using spherical coordinates, it follows that

$$\begin{aligned} G(t, \mathbf{x}) &= 4\pi \int_0^\infty \int_0^\pi k_0^{-5/2} e^{ik_0 t} e^{irR\cos\varphi} r^2 \sin\varphi\, d\varphi\, dr \\ &= \frac{8\pi}{R}\int_0^\infty k_0^{-5/2} e^{ik_0 t} r \sin(rR)\, dr. \\ &= \frac{8\pi}{itR}\int_0^\infty k_0^{-3/2}\frac{d}{dr}(e^{ik_0 t})\sin(rR)\, dr, \end{aligned}$$

and integrating by parts gives

$$G(t, \mathbf{x}) = \frac{8\pi i}{t} \int_0^\infty e^{ik_0 t} \left(\frac{\cos(rR)}{k_0^{3/2}} - \frac{3r \sin(rR)}{2Rk_0^{7/2}} \right) dr.$$

It follows that $|G(t, \mathbf{x})| \leqslant ct^{-1}$ for $0 < t < 1$. For $t \geqslant 1$ we have

$$\begin{aligned} G(t, \mathbf{x}) &= \frac{8\pi i}{t} \int_0^\infty e^{ik_0 t} \frac{\cos(rR)}{k_0^{3/2}}\, dr - \frac{8\pi}{t^2} \int_0^\infty \frac{d}{dr}(e^{ik_0 t}) \frac{3 \sin(rR)}{2Rk_0^{5/2}}\, dr \\ &= \frac{8\pi i}{t} \int_0^\infty e^{ik_0 t} \frac{\cos(rR)}{k_0^{3/2}}\, dr + \frac{8\pi}{t^2} \int_0^\infty e^{ik_0 t} \frac{3 \cos(rR)}{2k_0^{5/2}}\, dr \\ &\quad - \frac{8\pi}{t^2} \int_0^\infty \cos(k_0 t) \frac{15r \sin(rR)}{4Rk_0^{9/2}}\, dr. \end{aligned}$$

Denote the three terms in the last expression by I_1, I_2, and I_3, respectively. It is easy to see that $|I_2|, |I_3| \leqslant Ct^{-3/2}$, since

$$\left| e^{ik_0 t} \frac{\cos(rR)}{k_0^{5/2}} \right|, \left| e^{ik_0 t} \frac{r \sin(rR)}{Rk_0^{9/2}} \right| \leqslant ck_0^{-5/2}.$$

Thus we only need to prove that $|I_1| \leqslant Ct^{-3/2} \ln(1+t)$ or

$$J = \left| \int_0^\infty k_0^{-3/2} e^{ik_0 t} \cos(rR)\, dr \right| \leqslant ct^{-1/2} \ln(1+t),$$

for $t \geqslant 1$. J can be estimated as follows:

$$\begin{aligned} J &\leqslant \left| \int_0^t e^{ik_0 t} \frac{\cos(rR)}{k_0^{3/2}}\, dr \right| + \left| \int_t^\infty e^{ik_0 t} \frac{\cos(rR)}{k_0^{3/2}}\, dr \right| \\ &\leqslant \left| \int_0^t e^{ik_0 t} \frac{\cos(rR)}{k_0^{3/2}}\, dr \right| + c \int_t^\infty r^{-3/2}\, dr \\ &\leqslant \left| \int_0^t e^{ik_0 t} \frac{\cos(rR)}{k_0^{3/2}}\, dr \right| + 2ct^{-1/2}. \end{aligned}$$

To estimate the integral

$$I = \int_0^t e^{ik_0 t} \frac{\cos(rR)}{k_0^{3/2}}\, dr,$$

we use the following:

LEMMA 7. *(1) If h is a function on an interval $[a, b]$ such that $h'' \geqslant \lambda > 0$, then $|\int_a^b e^{ih(x)}\, dx| \leqslant 8\lambda^{-1/2}$. (2) If h is a function on an interval $[a, b]$ such that $h'' \geqslant 0$ and $h' \geqslant \beta > 0$, then $|\int_a^b e^{ih(x)}\, dx| \leqslant 4\beta^{-1}$.*

Proof. This is due to Van der Corput; see [15]. ∎

Let

$$H(s, t) = \int_0^s e^{ik_0 t} \cos(rR)\, dr = \int_0^s \frac{1}{2} (e^{i(k_0 t + rR)} + e^{i(k_0 t - rR)})\, dr.$$

We have

$$I = H(t, t)(t^2 + m^2)^{-3/4} + \int_0^t H(r, t)\, rk_0^{-7/2}\, dr.$$

From Lemma 7 and $(d^2/dr^2)(k_0 t \pm rR) = tm^2 k_0^{-3}$, we have

$$|H(r, t)| \leqslant ct^{-1/2} k_0^{3/2}.$$

Therefore

$$|H(t, t)(t^2 + m^2)^{-3/4}| \leqslant ct^{-1/2}$$

and

$$\begin{aligned} \left| \int_0^t H(r, t)\, rk_0^{-7/2}\, dr \right| &\leqslant ct^{-1/2} \int_0^t k_0^{-2} r\, dr \\ &\leqslant c' t^{-1/2} \ln(1+t); \end{aligned}$$

so for some constant c, $|I| \leqslant ct^{-1/2} \ln(1+t)$. ∎

Proof of Lemma 2. Recall that $K(t)\, g = B^{-1} \sin(tB)\, g$. It is easy to see that if $\mathrm{Re}(z) = -1$,

$$\|B^{-z} K(t)\, g\|_2 \leqslant \|g\|_2.$$

As we show below,

$$\|B^{-3/2} K(t)\, g\|_\infty \leqslant m(t)\, \|g\|_1, \tag{14}$$

where

$$m(t) \leqslant \begin{cases} ct^{-1/2} & \text{if } \ t \in (0, 1/m]; \\ ct^{-3/2} \ln(1+t) & \text{if } \ t \in [1/m, \infty). \end{cases}$$

As in the proof of Lemma 1, the Stein interpolation theorem implies that

$$\|B^{-b} K(t)\, g\|_q \leqslant k(t)\, \|g\|_{q'},$$

where $\varepsilon \in [0, 1]$, $b = 5\varepsilon/2 - 1$, $1/q = (1-\varepsilon)/2$, and $k(t)$ satisfies

$$k(t) \leqslant \begin{cases} ct^{-\varepsilon/2} & \text{if } t \in (0, 1/m]; \\ ct^{-3\varepsilon/2}(\ln(1+t))^{\varepsilon} & \text{if } t \in [1/m, \infty). \end{cases}$$

Thus,

$$\|B^{-b}K(t)\, B^{r'}g\|_q \leqslant k(t)\, \|B^{r'}g\|_{q'},$$

which implies that

$$\|K(t)\, g\|_{q,r} \leqslant k(t)\, \|g\|_{q',r'},$$

where $r = r' - b = r' + 1 - 5\varepsilon/2$. Taking $\delta = \varepsilon/2$, we obtain the desired estimate.

To prove (14) it suffices to show that

$$\|I(t,\cdot) * g\|_\infty \leqslant \|I(t,\cdot)\|_\infty\, \|g\|_1 \leqslant m(t)\, \|g\|_1,$$

where

$$I(t, \mathbf{x}) = \int_0^\infty \frac{\sin(k_0 t)}{k_0^{5/2}} \frac{r \sin(rR)}{R}\, dr. \tag{15}$$

From the proof of Lemma 1 we already know that $|I(t,\mathbf{x})| \leqslant ct^{-3/2}\ln(1+t)$ if $t \geqslant 1/m$. We only need to show that $|I(t,\mathbf{x})| \leqslant ct^{-1/2}$ for $t \in [0, 1/m]$. Letting $s = R/t$ and $u = k_0 t$, we can rewrite (15) as

$$I(t, x) = t^{-1/2}J(t, \mathbf{x}) = t^{-1/2} \int_{mt}^\infty \frac{\sin u}{u^{1/2}} \frac{\sin(s\sqrt{u^2 - m^2t^2})}{su}\, du.$$

If $s \geqslant \frac{1}{8}$ then

$$|J(t,\mathbf{x})| \leqslant \int_0^1 u^{-1/2}\, du + \int_1^\infty s^{-1}u^{-3/2}\, du \leqslant 2 + 16.$$

In the case of $s \leqslant \frac{1}{8}$, we divide the integral for $J(t,\mathbf{x})$ into three parts J_1, J_2, and J_3, given by

$$J_1 = \int_{s^{-2/3}}^\infty s^{-1}u^{-3/2} \sin u \sin v\, du,$$

$$J_2 = \int_2^{s^{-2/3}} s^{-1}u^{-3/2} \sin u \sin v\, du,$$

$$J_3 = \int_{mt}^2 s^{-1}u^{-3/2} \sin u \sin v\, du,$$

where $v = s\sqrt{u^2 - m^2t^2}$.

Since $u^{-3/2}$ is a decreasing function for $u \in [s^{-2/3}, \infty)$, and $(d/du)(u \pm v) \geqslant \frac{1}{2}$, the second intermediate value theorem and Lemma 7 imply that

$$|J_1| \leqslant \sup_w \left| \int_{s^{-2/3}}^w e^{i(u+v)}\, du \right| + \sup_w \left| \int_{s^{-2/3}}^w e^{i(u-v)}\, du \right| \leqslant 16.$$

For $u \in [2, s^{-2/3}]$ the function $s^{-1}u^{-3/2} \sin v$ is decreasing since

$$\frac{d}{du} \frac{\sin v}{su^{3/2}} = \left(v \cot v \frac{u^2}{u^2 - m^2t^2} - \frac{3}{2} \right) \frac{\sin v}{u^{3/2}} \leqslant \left(\frac{4}{3} - \frac{3}{2} \right) \frac{\sin v}{u^{5/2}} \leqslant 0,$$

noting that $v \cot v \leqslant 1$, where $v \in [0, \frac{1}{2}]$. By the second intermediate value theorem, this implies

$$|J_2| = \left| \int_2^{s^{-2/3}} s^{-1}u^{-3/2} \sin u \sin v\, du \right| \leqslant 2.$$

Finally, it is trivial that

$$|J_3| = \left| \int_{mt}^2 \frac{\sin u}{u^{1/2}} \frac{\sin s\sqrt{u^2 - m^2t^2}}{su}\, du \right| \leqslant \int_0^2 u^{-1/2}\, du = 4.$$

Combining these estimates gives $|I(t,\mathbf{x})| \leqslant ct^{-1/2}$ when $t \in (0, 1/m]$. ∎

As Marshall, Strauss, and Wainger have shown [7], with further work one can remove the logarithmic term from the statement of Lemma 2 and the proofs of Lemmas 1 and 2.

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