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Wick Products of the Free Bose Field

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We consider Wick products of the free Bose field in the abstract context of a complex Hilbert space $\mathbf{H}$ equipped with a self-adjoint operator A satisfying $A \geqslant \varepsilon I$ for some $\varepsilon > 0$. Let $(\mathbf{K}, W, \Gamma, v)$ be the free Bose field over $\mathbf{H}$, and let $H = d\Gamma(A)$. Let $\mathbf{H}_\infty$ and $\mathbf{K}_\infty$ denote $D^\infty(A)$ and $D^\infty(H)$, respectively, given their natural Frechét topologies. Then for any $f_1, ..., f_n \in \mathbf{H}_\infty^*$ the Wick product $:\Phi(f_1) \cdots \Phi(f_n):$ is constructed as a continuous sesquilinear form on $\mathbf{K}_\infty$ characterized by a generalization of the Heisenberg commutation relations. As an application, we treat pointwise products of the free scalar field and its derivatives on $\mathbb{R} \times M$, M an arbitrary complete Riemannian manifold. For example, if $f \in C_0^k(\mathbb{R} \times M)$ for large enough k, then $\int_{\mathbb{R} \times M} f(p) :\varphi(p)^n:$ corresponds to an operator with domain $\mathbf{K}_\infty$. If in addition f is real-valued, $n = 2$, and M is compact, then this operator is essentially self-adjoint.

1. INTRODUCTION

A substantial portion of the theory of quantum fields can be formulated in an abstract Hilbert space context. Traditionally, however, construction of the Wick products of free quantum fields at a point of space-time have relied heavily on symmetries of the space-time, which allow the application of harmonic analysis [6, 9–11]. Here we show how essential aspects of the theory may be developed in the general context of a complex Hilbert space $\mathbf{H}$ equipped with a self-adjoint operator A such that $A \geqslant \varepsilon I$ for some $\varepsilon > 0$. (In physical applications A plays the part of the "single-particle free hamiltonian.") In particular, for any continuous linear functionals $f_1, ..., f_n$ on $D^\infty(A)$ (with its natural Frechét topology) the Wick product $:\Phi(f_1) \cdots \Phi(f_n):$ exists as a sesquilinear form characterized by a generalization of the Heisenberg commutation relations.

This approach allows a unified development of the theory, which may then be applied in a wide variety of contexts. For brevity we consider only the case of Bose fields. We give illustrative applications to the theory of the

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massive neutral scalar field on space-times of the form $\mathbb{R}\times M$, M an arbitrary complete Riemannian manifold.

For a somewhat related study of Wick products, see [8]. Some of the following material first appeared in the author's thesis [1]. The author would like to thank his thesis advisor, Irving Segal, for help and inspiration.

2. Wick Products as Sesquilinear Forms

Given a complex Hilbert space $\mathbf{H}$, there is a unique "free Bose field" $(\mathbf{K}, W, \Gamma, v)$ characterized by the following properties:

(a) $\mathbf{K}$ is a complex Hilbert space.

(b) W is a strongly continuous map from $\mathbf{H}$ to $U(\mathbf{K})$ such that if $f, g\in\mathbf{H}$, then $W(f)\, W(g)=e^{\mathrm{Im}\langle f, g\rangle/2i}W(f+g)$.

(c) Γ is a strongly continuous unitary representation of $U(\mathbf{H})$ on $\mathbf{K}$ such that for any $U\in U(\mathbf{H})$ and $f\in\mathbf{H}$, $\Gamma(U)\, W(f)\, \Gamma(U)^{-1}=W(Uf)$, and for every positive self-adjoint operator A on $\mathbf{H}$, the self-adjoint generator of the group $\Gamma(e^{itA})$ on $\mathbf{K}$ is positive.

(d) The vector $v\in\mathbf{K}$ is cyclic for the action of the $W(f)$ and invariant under the $\Gamma(U)$.

Given $f\in\mathbf{H}$, the "field operator" $\Phi(f)$ is defined to be the self-adjoint generator of the group $W(tf)$, $t\in\mathbb{R}$. Given a self-adjoint operator A on $\mathbf{H}$, $d\Gamma(A)$ is defined to be the self-adjoint generator of the group $\Gamma(e^{itA})$ on $\mathbf{K}$.

In what follows, we will assume that A is a self-adjoint operator on $\mathbf{H}$ with $A\geqslant\varepsilon I$ for some $\varepsilon>0$. Define the Frechét space $\mathbf{H}_\infty$ to be the vector space $D^\infty(A)=\bigcap_{n\geqslant 0}D(A^n)$ with the seminorms $\|A^n\cdot\|$. Let $H=d\Gamma(A)$; by the above we have $H\geqslant 0$. Define the Frechét space $\mathbf{K}_\infty$ to be the vector space $D^\infty(H)=\bigcap_{n\geqslant 0}D(H^n)$ with the seminorms $\|(H+I)^n\cdot\|$. As shown in [9], for any $f\in\mathbf{H}$ the operator $\Phi(f)$ is essentially self-adjoint on $D^\infty(H)$, and if $n\geqslant 0$ is an integer there is a constant c such that

$$\|(H+I)^n\,\Phi(f)x\|\leqslant c\,\|A^nf\|\;\|(H+I)^{n+1/2}\,x\|$$

for all $f\in D(A^n)$ and $x\in D(H^{n+1/2})$. As a consequence, the map $(f, x)\mapsto\Phi(f)x$ is continuous from $\mathbf{H}_\infty\times\mathbf{K}_\infty$ to $\mathbf{K}_\infty$. To treat Wick products of field operators we will use a related continuity result for the "annihilation operator" $a(f)=2^{-1/2}(\Phi(f)+i\Phi(if))$, where $f\in\mathbf{H}$. By the above, the map $(f, x)\mapsto a(f)x$ is continuous from $\mathbf{H}_\infty\times\mathbf{K}_\infty$ to $\mathbf{K}_\infty$, but a stronger statement is true:

PROPOSITION 1. *For every integer $n \geqslant 0$ there exists c such that*

$$\|(H+I)^n a(f)x\| \leqslant c \, \|A^{n-m}f\| \, \|(H+I)^{n+m} x\|$$

for every $f \in D^\infty(A)$, $x \in D^\infty(H)$, and integer $m \geqslant 1$.

Proof. We make use of the following:

LEMMA 2. *If $f \in \mathbf{H}$ and $x \in \mathbf{K}_\infty$, then for all $m \geqslant 1$*

$$\|a(f)x\| \leqslant \|A^{-m}f\| \, \|H^m x\|.$$

Proof. Let P be the self-adjoint projection onto the span of f in $\mathbf{H}$. For all $g \in D^\infty(A)$ we have

$$\|f\|^2 \langle g, Pg\rangle = |\langle f, g\rangle|^2 \leqslant \|A^{-m}f\|^2 \, \|A^m g\|^2 \leqslant \|A^{-m}f\|^2 \langle g, A^{2m} g\rangle$$

hence, as $D^\infty(A)$ is dense in $\mathbf{H}$, $\|f\|^2 P \leqslant \|A^{-m}f\|^2 A^{2m}$. Thus $\|f\|^2 d\Gamma(P) \leqslant \|A^{-m}f\|^2 d\Gamma(A^{2m})$. As is well known, $\|f\|^2 d\Gamma(P) = a(f)^* a(f)$, and $A \geqslant 0$ implies $d\Gamma(A^{2m}) \leqslant d\Gamma(A)^{2m} = H^{2m}$. Thus we have

$$a(f)^* a(f) \leqslant \|A^{-m}f\|^2 H^{2m}.$$

proving the lemma. ∎

Now suppose that $f \in D^\infty(A)$ and $x \in D^\infty(H)$. Differentiating the relation $e^{iHt} W(f) e^{-iHt} x = W(e^{iAt} f)x$ we obtain $[H, a(f)]x = -a(Af)x$, hence

$$H^n a(f)x = \sum_{0 \leqslant k \leqslant n} (-1)^k \binom{n}{k} a(A^k f) H^{n-k} x.$$

By the lemma this implies that for some constant c depending on n and ε, for all $f \in D^\infty(A)$, $x \in D^\infty(H)$, and $m \geqslant 1$

$$\|H^n a(f)x\| \leqslant c \, \|A^{n-m}f\| \, \|(H+I)^{n+m} x\|. \quad ∎$$

The above proposition lets us define annihilation operators $a(f)$ for all $f \in \mathbf{H}_\infty^*$ as follows. For any integer n let $\mathbf{H}_n$ denote the completion of $D(A^n)$ in the norm $\|A^n \cdot\|$. The proposition implies that the function $(f, x) \mapsto a(f)x$ from $\mathbf{H}_\infty \times \mathbf{K}_\infty$ to $\mathbf{K}_\infty$ extends uniquely to a continuous function $a_n \colon \mathbf{H}_n \times \mathbf{K}_\infty \to \mathbf{K}_\infty$. If $n \leqslant m$, we identify $\mathbf{H}_m$ with a subspace of $\mathbf{H}_n$ and identify the restriction of a_n to $\mathbf{H}_m$ with a_m. Moreover we may identify the vector space $\mathbf{H}_\infty^*$ with the union $\bigcup_{-\infty}^{\infty} \mathbf{H}_n$ (in a conjugate-linear manner). Relative to these identifications, the union of the functions a_n is a function from $\mathbf{H}_\infty^* \times \mathbf{K}_\infty$ to $\mathbf{K}_\infty$, which we again write as $(f, x) \mapsto a(f)x$.

We topologize the space $\mathbf{H}_\infty^*$ as the inductive limit of the Banach spaces

$\mathbf{H}_n$. As such, $\mathbf{H}_\infty^*$ is a barrelled space, so the uniform boundedness principle applies [3]. Thus the function $(f, x) \mapsto a(f)x$ is jointly continuous from $\mathbf{H}_\infty^* \times \mathbf{K}_\infty$ to $\mathbf{K}_\infty$.

THEOREM 3. *For each $n \geq 1$, there is a function from $(\mathbf{H}_\infty^*)^n \times \mathbf{K}_\infty^2$ to $\mathbb{C}$,*

$$(f_1, ..., f_n, x, y) \mapsto :\Phi(f_1) \cdots \Phi(f_n): (x, y),$$

such that:

(a) *The function $(f_1, ..., f_n, x, y) \mapsto :\Phi(f_1) \cdots \Phi(f_n): (x, y)$ is jointly continuous, real-linear, and symmetric in the arguments f_i, conjugate-linear in x, and complex-linear in y.*

(b) *For all $f \in \mathbf{H}$ and $x, y \in \mathbf{K}_\infty$, $:\Phi(f): (x, y) = \langle x, \Phi(f)\, y\rangle$.*

(c) *For all $f_1, ..., f_n \in \mathbf{H}_\infty^*$, $:\Phi(f_1) \cdots \Phi(f_n): (v, v) = 0$.*

(d) *Let $:\Phi(f)^n: (x, y)$ stand for $:\Phi(f_1) \cdots \Phi(f_n): (x, y)$ with $f_i = f$ for all $1 \leq i \leq n$. Then for all $f \in \mathbf{H}_\infty^*$, $g \in \mathbf{H}_\infty$, and $x, y \in \mathbf{K}_\infty$,*

$$:\Phi(f)^n: (x, \Phi(g)\, y) - :\Phi(f)^n: (\Phi(g)x, y) = in \operatorname{Im} f(g) :\Phi(f)^{n-1}: (x, y).$$

Moreover, the $:\Phi(f_1) \cdots \Phi(f_n):$ are uniquely characterized by these properties, and satisfy

$$:\Phi(f_1) \cdots \Phi(f_n): (x, y) = 2^{-n/2} \sum_{S \subseteq \{1, ..., n\}} \left\langle \prod_{k \in S} a(f_k)x, \prod_{k \in \{1, ..., n\} - S} a(f_k)\, y \right\rangle. \tag{1}$$

Proof. We prove existence by showing that (1) defines a function from $(\mathbf{H}_\infty^*)^n \times \mathbf{K}_\infty^2$ to $\mathbb{C}$ satisfying (a–d). Note first that by Proposition 1 we have the following extensions of the usual commutativity of annihilation operators: if $f, g \in \mathbf{H}_\infty^*$, $a(f)\, a(g) = a(g)\, a(f)$ as continuous operators from $\mathbf{K}_\infty$ to $\mathbf{K}_\infty$. By Proposition 1 and the fact that the $a(f_k)$ commute, (1) gives a well-defined function from $(\mathbf{H}_\infty^*)^n \times \mathbf{K}_\infty^2$ to $\mathbb{C}$. Property (a) follows from Proposition 1 and the uniform boundedness principle. Property (b) is evident. Property (c) holds because $a(f)v = 0$ for all $f \in \mathbf{H}_\infty^*$. To check (d), note that (1) implies

$$:\Phi(f)^n: (x, y) = 2^{-n/2} \sum_{0 \leq k \leq n} \binom{n}{k} \langle a(f)^k\, x, a(f)^{n-k}\, y\rangle.$$

Extending the usual commutation relations via Proposition 1, if $f \in \mathbf{H}_\infty^*$ and $g \in \mathbf{H}_\infty$,

$$a(f)^p\, \Phi(g) - \Phi(g)\, a(f)^p = 2^{-1/2} p f(g)\, a(f)^{p-1}$$

as operators from $\mathbf{K}_\infty$ to itself, hence

$$
\begin{aligned}
:\Phi(f)^n:(x, \Phi(g)y) &= 2^{-n/2} \sum_{0\leq k\leq n} \binom{n}{k} \langle a(f)^k x, a(f)^{n-k} \Phi(g) y\rangle \\
&= 2^{-n/2} \sum_{0\leq k\leq n} \binom{n}{k} \{\langle a(f)^k \Phi(g)x, a(f)^{n-k} y\rangle \\
&\quad - k2^{-1/2}\overline{f(g)}\langle a(f)^{k-1} x, a(f)^{n-k} y\rangle \\
&\quad + (n-k)\, 2^{-1/2} f(g) \langle a(f)^k x, a(f)^{n-k-1} y\rangle\} \\
&= :\Phi(f)^n:(\Phi(g)x, y) - \frac{1}{2} n\overline{f(g)}\, :\Phi(f)^{n-1}:(x, y) \\
&\quad + \frac{1}{2} n f(g)\, :\Phi(f)^{n-1}:(x, y) \\
&= :\Phi(f)^n:(\Phi(g)x, y) + in \operatorname{Im} f(g)\, :\Phi(f)^{n-1}:(x, y).
\end{aligned}
$$

To prove uniqueness we use the following:

LEMMA 4. *Suppose F is a continuous sesquilinear form on $\mathbf{K}_\infty$ such that $F(x, \Phi(f)y) = F(\Phi(f)x, y)$ for all $f \in \mathbf{H}_\infty$ and $x, y \in \mathbf{K}_\infty$. Then for some $c \in \mathbb{C}$, $F(x, y) = c\langle x, y\rangle$ for all $x, y \in \mathbf{K}_\infty$.*

Proof. Let $N = d\Gamma(I)$, and let P_k be the spectral projection onto the eigenspace of N with eigenvalue k. Via the Fock–Cook representation we identify $P_k\mathbf{K}$ with $S^k\mathbf{H}$, the symmetrized k-fold Hilbert space tensor power of $\mathbf{H}$. Let $\mathbf{D}$ denote the algebraic span of vectors of the form $S(f_1 \otimes \cdots \otimes f_k)$, where $k \geq 0$ is arbitrary and $f_1, ..., f_k \in D^\infty(A)$, and S denotes the symmetrization operator. It is easily seen that $\mathbf{D}$ is dense in $\mathbf{K}_\infty$, and that if $f \in \mathbf{H}_\infty$ the operator $\Phi(f)$ maps $\mathbf{D}$ to itself. As shown in [10], if F is a sesquilinear form on $\mathbf{D}$ with $F(x, \Phi(f)y) = F(\Phi(f)x, y)$ for all $f \in \mathbf{H}_\infty$, all $x, y \in \mathbf{D}$, then F is a constant multiple of $\langle \cdot, \cdot \rangle$. The lemma follows directly from these facts. ∎

Now let

$$(f_1, ..., f_n, x, y) \mapsto :\Psi(f_1) \cdots \Psi(f_n):(x, y)$$

be an alternate set of functions from $(\mathbf{H}_\infty^*)^n \times \mathbf{K}_\infty^2$ to $\mathbb{C}$ satisfying (a)–(d). By Lemma 4 it follows inductively that $:\Psi(f)^n:(x, y) = :\Phi(f)^n:(x, y)$ for all n. In order to conclude that $:\Psi(f_1) \cdots \Psi(f_n):(x, y) = :\Phi(f_1) \cdots \Phi(f_n):(x, y)$, it suffices to note that if V is a real vector space and $F\colon V^n \to \mathbb{R}$ is a symmetric multilinear function with $F(z, ..., z) = 0$ for all $z \in V$, then $F = 0$. This in turn follows from the fact that if V is finite-dimensional, the symmetrized tensor product S^nV is an irreducible representation of GL(V), hence is spanned by elements of the form $z \otimes \cdots \otimes z$, as these span a subrepresentation. ∎

The Wick powers thus defined have the following covariance property:

THEOREM 5. *Let* $U: \mathbf{H} \to \mathbf{H}$ *be a unitary operator that restricts to a continuous linear operator from* $\mathbf{H}_\infty$ *to* $\mathbf{H}_\infty$. *Then* $\Gamma(U)$ *restricts to a continuous linear operator from* $\mathbf{K}_\infty$ *to* $\mathbf{K}_\infty$. *If in addition* $U: \mathbf{H}_\infty \to \mathbf{H}_\infty$ *has a continuous inverse, then for all* $f_1, ..., f_n \in \mathbf{H}_\infty^*$ *and* $x, y \in \mathbf{K}_\infty$ *we have*

$$:\Phi(U^*f_1) \cdots \Phi(U^*f_n): (x, y) = :\Phi(f_1) \cdots \Phi(f_n): (\Gamma(U)x, \Gamma(U)y),$$

where $U^*: \mathbf{H}_\infty^* \to \mathbf{H}_\infty^*$ *denotes the adjoint of* $U: \mathbf{H}_\infty \to \mathbf{H}_\infty$.

Proof. Let $U: \mathbf{H} \to \mathbf{H}$ be a unitary operator which is continuous from $\mathbf{H}_\infty$ to itself. Let the operators N, S, and P_k and the space $\mathbf{D}$ be as in the proof of Lemma 4. As is well known, $\Gamma(U)|_{S^k\mathbf{H}} = U \otimes \cdots \otimes U$ and $H|_{S^k\mathbf{H}} = \sum_{1 \leqslant i \leqslant k} A_{i,k}$, where $A_{i,k}$ is the self-adjoint operator on $S^k\mathbf{H}$ given by $I \otimes \cdots \otimes A \otimes \cdots \otimes I$, with the factor A in the ith place. As a consequence, H and $\Gamma(U)$ map $\mathbf{D}$ to itself.

Writing an arbitrary element $x \in \mathbf{D}$ as a finite sum $\sum_{k \geqslant 0} x_k$ with $x_k \in S^k\mathbf{H}$, we have

$$\|H\Gamma(U)x\| \leqslant \left(\sum_{k \geqslant 0} \left(\sum_{1 \leqslant i \leqslant k} \|A_{i,k}(U \otimes \cdots \otimes U)x_k\| \right)^2 \right)^{1/2}. \tag{2}$$

Since U is continuous from $\mathbf{H}_\infty$ to itself, there exist $m, c \geqslant 0$ such that $\|AUf\| \leqslant c\, \|A^m f\|$ for all $f \in \mathbf{H}_\infty$. This implies that

$$\|A_{i,k}(U \otimes \cdots \otimes U)x_k\| \leqslant c\, \|(I \otimes \cdots \otimes A^m \otimes \cdots \otimes I)x_k\|, \tag{3}$$

where the factor of A^m occurs in the ith place. As may be checked by formula for H above,

$$I \otimes \cdots \otimes A^m \otimes \cdots \otimes I \leqslant H^m|_{S^k\mathbf{H}}.$$

Since these two operators commute on a common domain of essential self-adjointness, namely $S^k\mathbf{H} \cap \mathbf{D}$, we have

$$\|(I \otimes \cdots \otimes A^m \otimes \cdots \otimes I)x_k\| \leqslant \|H^m x_k\|. \tag{4}$$

By (2), (3), and (4) we have

$$\|H\Gamma(U)x\| \leqslant c \left(\sum_{k \geqslant 0} k\, \|H^m x_k\|^2 \right)^{1/2}. \tag{5}$$

Since $A \geqslant \varepsilon I$, we have $H \geqslant \varepsilon N$, and these operators commute, so $\|Hu\| \geqslant \varepsilon \|Nu\|$ for all $u \in \mathbf{D}$. Since $Nx_k = kx_k$, (5) implies

$$\|H\Gamma(U)x\| \leqslant c\varepsilon^{-1} \left(\sum_{k \geqslant 0} \|H^{m+1}x_k\|^2 \right)^{1/2} = c\varepsilon^{-1} \|H^{m+1}x\|.$$

It follows that for any $n \geqslant 0$, $x \in \mathbf{D}$, we have

$$\|H^n\Gamma(U)x\| \leqslant (c\varepsilon^{-1})^n \|H^{n(m+1)}x\|.$$

Since $\mathbf{D}$ is dense in $\mathbf{K}_\infty$ this inequality is also valid for any $x \in \mathbf{K}_\infty$, so $\Gamma(U)$ is continuous from $\mathbf{K}_\infty$ to itself, as was to be shown.

Next suppose U and U^{-1} are continuous from $\mathbf{H}_\infty$ to itself. Then $\Gamma(U)$ and $\Gamma(U^{-1})$ are continuous from $\mathbf{K}_\infty$ to itself, so

$$\begin{aligned} i\Phi(U^*f)x &= \partial_t W(tU^*f)x|_{t=0} = \partial_t \Gamma(U^{-1})\, W(tf)\, \Gamma(U)x|_{t=0} \\ &= \Gamma(U^{-1})\, \partial_t W(tf)\, \Gamma(U)x|_{t=0} = i\Gamma(U^{-1})\, \Phi(f)\, \Gamma(U)x \end{aligned}$$

for all $f \in \mathbf{H}_\infty$ and $x \in \mathbf{K}_\infty$. Similarly, $\Phi(U^*if)x = \Gamma(U^{-1})\, \Phi(if)\, \Gamma(U)x$, so $a(U^*f) = \Gamma(U^{-1})\, a(f)\, \Gamma(U)$ as continuous operators from $\mathbf{H}_\infty$ to itself. It follows from formula (1) that if $f_1, ..., f_n \in \mathbf{H}_\infty$ and $x, y \in \mathbf{K}_\infty$, then

$$:\Phi(U^*f_1) \cdots \Phi(U^*f_n):(x, y) = :\Phi(f_1) \cdots \Phi(f_n):(\Gamma(U)x, \Gamma(U)y).$$

By the continuity stated in part (a) of Theorem 3, this equation holds for all $f_1, ..., f_n \in \mathbf{H}_\infty^*$. ∎

3. Free Bose Fields on Manifolds

We now apply the general theory to the case of free Bose fields on manifolds. In these applications $\mathbf{H}_\infty$ is a subspace of the space of sections of a vector bundle over a manifold. In what follows, manifolds will always be assumed paracompact and C^∞.

Proposition 6. *Let $C^\infty(X, E)$ be the space of C^∞ sections of a C^∞ real vector bundle E over a manifold X. Let* $\mathbf{H}$ *be a complex Hilbert space, let A be a self-adjoint operator on* $\mathbf{H}$ *such that $A \geqslant \varepsilon I$ for some $\varepsilon > 0$, and suppose that there is a continuous real-linear embedding $T: \mathbf{H}_\infty \to C^\infty(X, E)$. Then given $\mu \in C^\infty(X, E)^*$ there is a unique element $T^\# \mu \in \mathbf{H}_\infty^*$ such that for all $f \in \mathbf{H}_\infty$, $\mathrm{Re}(T^\# \mu(f)) = \mu(Tf)$. The map $T^\#: C^\infty(X, E)^* \to \mathbf{H}_\infty^*$ is real-linear. Define*

$$:\varphi(\mu_1) \cdots \varphi(\mu_n):(x, y) = :\Phi(T^\# \mu_1) \cdots \Phi(T^\# \mu_n):(x, y)$$

for $\mu_1, ..., \mu_n \in C^\infty(X, E)^*$, $x, y \in \mathbf{K}_\infty$. *Then the function* $(\mu_1, ..., \mu_n, x, y) \mapsto\ :\varphi(\mu_1) \cdots \varphi(\mu_n): (x, y)$ *from* $(C^\infty(X, E)^*)^n \times \mathbf{K}_\infty^2$ *to* $\mathbb{C}$ *is real-linear in each argument* μ_i, *complex-linear and continuous in* y, *and conjugate-linear and continuous in* x.

Proof. The only point that is not a direct consequence of Theorem 3 is the existence of a unique function $T^\#$ with the required properties. Here note that given a continuous real functional μ on $\mathbf{V}$, $T^\#\mu$ must be a complex-linear functional on $\mathbf{H}_\infty$ with $\mathrm{Re}(T^\#\mu(f)) = \mu(Tf)$. This implies that $T^\#\mu(f) = \mu(Tf) + i^{-1}\mu(T(if))$. With this definition it is easy to see that $T^\#$ has the required properties. ∎

The hypotheses of the above proposition are applicable to Minkowskian and Euclidean free quantum fields, as well as to "light-cone" and "infinite momentum frame" quantization (in mathematical terms, the Goursat problem [2]) and white noise on complete Riemannian manifolds [11]. Note that $C^\infty(X, E)^*$ contains functionals f of the form

$$f(g) = \eta(Dg(p)), \qquad g \in \mathbf{V},$$

where $p \in X$, $\eta \in (E_p)^*$, and D is any linear differential operator on E with C^∞ coefficients. This permits the definition of pointwise Wick products of fields and their derivatives.

As a concrete and notationally simple example, consider the "free neutral scalar field of mass m" on $\mathbb{R} \times M$, where M is a complete Riemannian manifold. Here the Hilbert space $\mathbf{H}$ is taken to be a space of Cauchy data for real solutions of the Klein–Gordon equation

$$(\Box + m^2)\psi = 0$$

on $\mathbb{R} \times M$.

To be precise, suppose $m > 0$, and for real α let $H^\alpha(M)$ denote the completion of the space of real-valued functions $f \in C_0^\infty(M)$ with respect to the norm

$$\|f\|_\alpha = \|(\Delta + m^2)^{\alpha/2} f\|_{L^2(M)},$$

where Δ denotes the unique extension to a self-adjoint operator on $L^2(M)$ of the (nonnegative) Laplace–Beltrami operator on $C_0^\infty(M)$ [4]. Let $H^\infty(M)$ denote the intersection of the spaces $H^\alpha(M)$, a Frechét space with seminorms $\|\cdot\|_\alpha$.

Let $\mathbf{H} = H^{1/2}(M) \oplus H^{-1/2}(M)$. Let $B = (\Delta + m^2)^{1/2}$, and give $\mathbf{H}$ the structure of a complex Hilbert space with complex structure

$$J = \begin{pmatrix} 0 & -B^{-1} \\ B & 0 \end{pmatrix}$$

and inner product

$$\langle (f_1, f_2), (g_1, g_2)\rangle$$
$$= \langle B^{1/2}f_1, B^{1/2}g_1\rangle + \langle B^{-1/2}f_2, B^{-1/2}g_2\rangle + i(\langle f_1, g_2\rangle - \langle f_2, g_1\rangle), \quad (6)$$

where all the inner products on the right side are those of $L^2(M)$. Let

$$A = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

realized as a self-adjoint operator on $\mathbf{H}$. Note that with this choice of A, $\mathbf{H}_\infty$ is isomorphic to $H^\infty(M) \oplus H^\infty(M)$, hence continuously embedded in $C^\infty(M) \oplus C^\infty(M)$.

There is a continuous map $T: \mathbf{H}_\infty \to C^\infty(\mathbb{R} \times M)$ given by

$$T(f)(t, q) = (e^{JAt}f)_1\,(q), \qquad (t, q) \in \mathbb{R} \times M,$$

where $(e^{JAt}f)_1$ denotes the first component of the pair $e^{JAt}f \in \mathbf{H}_\infty$. Given $\mu_1, ..., \mu_n \in C^\infty(\mathbb{R} \times M)^*$, we may define the function $(\mu_1, ..., \mu_n, x, y) \mapsto {:}\varphi(\mu_1) \cdots \varphi(\mu_n){:}\,(x, y)$ from $(C^\infty(\mathbb{R} \times M)^*)^n \times \mathbf{K}^2_\infty$ to $\mathbb{C}$ as in Proposition 6.

Given $p_1, ..., p_n \in \mathbb{R} \times M$, define ${:}\varphi(p_1) \cdots \varphi(p_n){:}$ to be ${:}\varphi(\delta_{p_1}) \cdots \varphi(\delta_{p_n}){:}$, where δ_{p_i} is the Dirac delta measure at p_i, an element of $C_\infty(\mathbb{R} \times M)^*$. Then we have:

THEOREM 7. *Suppose M is a complete Riemannian manifold. The function*

$$(p_1, ..., p_n, x, y) \mapsto {:}\varphi(p_1) \cdots \varphi(p_n){:}\,(x, y),$$

defined as above, is continuous from $(\mathbb{R} \times M)^n \times \mathbf{K}^2_\infty$ to $\mathbb{C}$, linear in y, conjugate-linear in x, and C^∞ as a function of $(p_1, ..., p_n) \in (\mathbb{R} \times M)^n$.

Proof. This is a consequence of Theorem 3, Proposition 6, and the fact that the map $p \mapsto T^\#(\delta_p)$ is C^∞ (in the Frechét sense) from $\mathbb{R} \times M$ to $\mathbf{H}^*_\infty$ with its inductive limit topology. ∎

The Wick power ${:}\varphi(p)^n{:}$ is defined to be the sesquilinear form ${:}\varphi(p_1) \cdots \varphi(p_n){:}$ with $p_1, ..., p_n = p \in \mathbb{R} \times M$. When integrated against sufficiently smooth functions on $\mathbb{R} \times M$, the Wick powers give rise to densely defined operators on $\mathbf{K}$. The following result illustrates this.

THEOREM 8. *Suppose M is a complete Riemannian manifold, and suppose $K \subseteq \mathbb{R} \times M$ is compact. Then for any $n, \alpha \geqslant 0$ there exist $c, k, \beta \geqslant 0$ such that*

$$\left| \int_{\mathbb{R} \times M} f(p)\, {:}\varphi(p)^n{:}\, ((H+I)^\alpha x, y) \right| \leqslant c\, \|f\|_{C^k}\, \|x\|\, \|(H+I)^\beta y\|$$

for all $f \in C_0^k(K)$ and $x, y \in \mathbf{K}_\infty$. Thus for any $f \in C_0^k(\mathbb{R} \times M)$, the sesquilinear form $\int_{\mathbb{R} \times M} f(p) :\varphi(p)^n:$ corresponds to an operator on $\mathbf{K}$ with invariant domain $D^\infty(H)$.

Proof. Given a compact subset $K \subseteq \mathbb{R} \times M$ choose a bounded interval $I \subseteq \mathbb{R}$ and a compact set $S \subseteq M$ such that $K \subseteq I \times S$. Given $x, y \in \mathbf{K}_\infty$, let $x_l = (P_{[l,l+1)}H)x$ and $y_m = (P_{[m,m+1)}H)y$. By Theorem 7, for some $\gamma, c_1 > 0$,

$$\|:\varphi(0, q)^n: ((H+I)^\alpha x, y)\|_{L^\infty(S)} \leqslant c_1 \|(H+I)^\gamma x\| \|(H+I)^\gamma y\|$$

for all $x, y \in \mathbf{K}_\infty$. Thus by Theorem 5 we have

$$\|:\varphi(t, q)^n: ((H+I)^\alpha x_l, y_m)\|_{L^\infty(\mathbb{R} \times S)} \leqslant c_1(l+1)^\gamma (m+1)^\gamma \|x_l\| \|y_m\|. \quad (7)$$

Given $f \in C_0^k(K)$, let $f = \sum_{j=-\infty}^{\infty} f_j$, with each $\hat{f}_j$ supported in $[j, j+1) \times S$, where ^ denotes the Fourier transform in the time variable. Each function f_j is supported in $\mathbb{R} \times S$. Choosing k large enough we have for some $c_2 > 0$

$$\|f_j\|_{L^1} \leqslant c_2(|j|+1)^{-\gamma-1} \|f\|_{C^k} \quad (8)$$

for all $f \in C_0^k(K)$.

Since the sequences $\|x_l\|$ and $\|y_l\|$ decrease more rapidly than the reciprocal of any polynomial in l, by (7) and (8) we have an absolutely convergent sum

$$\int_{\mathbb{R} \times M} f(p) :\varphi(p)^n: (x, y) = \sum_{j=-\infty}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \int_{\mathbb{R} \times S} f_j(t, q) :\varphi(t, q)^n: (x_l, y_m)\, dt\, dq$$

and since for each $q \in S$ the Fourier transform of the integrand is supported in $(j+m-l-1, j+m-l+2)$, the integral vanishes unless j equals $l-m-1$ or $l-m$. Thus we have

$$\left| \int_{\mathbb{R} \times M} f(p) :\varphi(p)^n: (x, y) \right| \leqslant \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{d=0,1} \int_{\mathbb{R} \times S} |f_{l-m-d}(t, q) :\varphi(t, q)^n: (x_l, y_m)|\, dt\, dq.$$

By (7) and (8) it then follows that for some $c_3 > 0$

$$\left|\int_{\mathbb{R}\times S} f(p) :\varphi(p)^n: ((H+I)^\alpha x, y)\right|$$

$$\leqslant c_1 c_2 \|f\|_{C^k} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{d=0,1} (|l-m-d|+1)^{-\gamma-1}$$

$$\times (l+1)^\gamma (m+1)^\gamma \|x_l\| \|y_m\|$$

$$\leqslant c_3 \|f\|_{C^k} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} (|l-m|+1)^{-\gamma-1} (l+1)^\gamma (m+1)^\gamma \|x_l\| \|y_m\|.$$

Since $l+1 \leqslant (|l-m|+1)(m+1)$ for l, m in the indicated range, the above is less than or equal to

$$c_3 \|f\|_{C^k} \sum_{l,m} (l+1)^{-1} (m+1)^{2\gamma+1} \|x_l\| \|y_m\|.$$

By Cauchy–Schwarz we conclude that for some $c > 0$ this is less than or equal to

$$c \|f\|_{C^k} \|x\| \|(H+I)^{2\gamma+2} y\|,$$

from which the theorem follows. ∎

An examination of the proof of this theorem makes it clear that it could be sharpened in a number of directions; in particular, the estimate (8) could be replaced by one which only required a certain number of derivatives of $f(t, q)$ with respect to t to lie in L^1.

Suppose that $f \in C_0^\infty(\mathbb{R} \times M)$ is real-valued, so that $\int_{\mathbb{R}\times M} f(p) :\varphi(p)^n:$ as defined in Theorem 8 is a densely defined symmetric operator. It is of some interest to find conditions under which this operator admits self-adjoint extensions, and when such extensions are unique. Existing results along these lines include the following. If $n=1$, the operator is essentially self-adjoint on $D^\infty(H)$. By the technique of [11], if f is even with respect to t the operator admits a self-adjoint extension, not known to be unique. An essential self-adjointness result for arbitrary n, applicable to special choices of f and M, appears in [7].

To conclude, we prove the following result for $n=2$:

THEOREM 9. *Suppose M is a compact Riemannian manifold. Then if k is sufficiently large and $f \in C_0^k(\mathbb{R} \times M)$, the operator $R = \int_{\mathbb{R}\times M} f(p) :\varphi(p)^2:$ is essentially self-adjoint on $D^\infty(H)$.*

Proof. Suppose $g \in D^\infty(A)$ and $x, y \in D^\infty(H)$. Let $\eta(p)$ denote the element of $\mathbf{H}_\infty^*$ corresponding to $\delta_p \in C^\infty(\mathbb{R} \times M)^*$. By Theorem 3,

$$\langle x, [R, \Phi(g)]\, y\rangle = 2i\Phi\left(\int_{\mathbb{R}\times M} f(p)\, \mathrm{Im}(\eta(p)\, g)\, \eta(p)\right)(x, y).$$

Identifying $\mathbf{H}_\infty^*$ with $H^\infty(M)^* \oplus H^\infty(M)^*$ by means of the pairing (6), and writing $p = (t, q) \in \mathbb{R} \times M$, we have $\eta(0, q) = (B^{-1}\delta_q, 0)$, hence

$$\eta(t, q) = (B^{-1} \cos tB\, \delta_q, \sin tB\, \delta_q).$$

Thus

$$\int_{\mathbb{R}\times M} f(p)\, \mathrm{Im}(\eta(p)\, g)\, \eta(p)$$
$$= \int_{\mathbb{R}\times M} f(t, q)(B^{-1} \cos tBg_2 - \sin tBg_1)(q)(B^{-1} \cos tB\, \delta_q, \sin tB\, \delta_q).$$

For $h \in C_0^k(M)$ and bounded measurable $F\colon \mathbb{R} \to \mathbb{R}$,

$$\int_M h(q)\, F(B)\, \delta_q\, dq = F(B) h$$

the integrand being a compactly supported continuous $H^\infty(M)^*$-valued function. Thus

$$\int_{\mathbb{R}\times M} f(p)\, \mathrm{Im}(\eta(p)\, g)\, \eta(p)$$
$$= \int_{\mathbb{R}\times M} f(t, q)(B^{-2} \cos^2 tBg_2 - B^{-1} \cos tB \sin tBg_1,$$
$$B^{-1} \cos tB \sin tBg_2 - \sin^2 tBg_1).$$

It follows that

$$\langle x, [R, \Phi(g)]\, y\rangle = i\langle x, \Phi(Tg)\, y\rangle, \tag{9}$$

where $T\colon D^\infty(H) \to D^\infty(H)$ is given by

$$T = 2\int_{\mathbb{R}\times M} f(t, q)\, T(t)\, dt\, dq$$

and

$$T(t) = \begin{pmatrix} -B^{-1}\cos tB \sin tB & B^{-2}\cos^2 tB \\ -\sin^2 tB & B^{-1}\cos tB \sin tB \end{pmatrix}.$$

Define the unitary equivalence $U: \mathbf{H} \to L^2(M)$ (the complex L^2 space) by $U(g_1, g_2) = B^{1/2}g_1 + iB^{-1/2}g_2$. Then

$$\begin{aligned} &UT(t)\, U^{-1}(g_1 + ig_2) \\ &\qquad = B^{-1}\{\cos tB \sin tB(ig_2 - g_1) + \cos^2 tBg_2 - i\sin^2 tBg_1\}. \end{aligned}$$

It follows that while T is not complex-linear, it is a member of $sp(\mathbf{H})$, i.e., it is a bounded real-linear operator such that $\operatorname{Im}\langle x, Ty\rangle + \operatorname{Im}\langle Tx, y\rangle = 0$ for all $x, y \in \mathbf{H}$. Moreover, if T^* denotes the adjoint with respect to $\operatorname{Re}\langle\cdot,\cdot\rangle$, a calculation using (10) shows that

$$U(T + T^*)\, U^{-1}(g_1 + ig_2) = 2i\int_{\mathbb{R}\times M} f(t, q)\, B^{-1}e^{2itB}\bar{g}\, dt\, dq.$$

This operator is the product of $g \mapsto 2iB^{-1}\bar{g}$, which is a bounded transformation of $L^2(M)$, with the operator $\int f(t, q)e^{2itB}\, dt\, dq$ (the integral taken in the strong operator topology). Let $F(t) = \int_M f(t, q)\, dq$; we have $F \in C_0^k(\mathbb{R})$, and

$$\int_{\mathbb{R}\times M} f(t, q)e^{2itB}\, dt\, dq = \int_{\mathbb{R}} \hat{F}(\lambda)e^{2i\lambda t}\, dE(\lambda),$$

where $dE(\lambda)$ is the spectral projection-valued measure corresponding to B. If k is large enough, this operator is Hilbert–Schmidt, hence $T + T^*$ is Hilbert–Schmidt. We may then make use of the following:

LEMMA 10 (Klein). *Suppose $T \in sp(\mathbf{H})$ and $T + T^*$ is Hilbert–Schmidt. Then there is a self-adjoint operator S on* $\mathbf{K}$, *essentially self-adjoint on* $\mathbf{D}$ *(as defined in Lemma* 4) *such that*

$$e^{-itS}\Phi(g)e^{itS} = \Phi(e^{tT}g)$$

for all $g \in \mathbf{H}$.

Proof. This is Proposition 2 of [5]. ∎

Note that if S satisfies the conclusions of this Lemma, so does $S + cI$ for any $c \in \mathbb{R}$. We will suppose that

$$\langle v, Sv\rangle = \langle v, Rv\rangle. \tag{11}$$

Now we show that $R|\mathbf{D} = S|\mathbf{D}$. First we note:

LEMMA 11. *If $x \in \mathbf{D}$ and $g \in D^\infty(A)$, then x is in the domain of $[S, \Phi(g)]$ and $[S, \Phi(g)]x = i\Phi(Tg)x$.*

Proof. Let $x \in \mathbf{D}$ and $g \in D^\infty(A)$. Then $\Phi(g)x \in \mathbf{D} \subseteq D(S)$ so x is in the domain of $S\Phi(g)$. Thus we need to show that Sx is in the domain of $\Phi(g)$ and that $\Phi(g)Sx = (S\Phi(g) - i\Phi(Tg))x$. Since $\Phi(g) = \Phi(g)^*$ it is enough to show that for all $y \in D(\Phi(g))$,

$$\langle \Phi(g)\, y, Sx \rangle = \langle y, (S\Phi(g) - i\Phi(Tg))x \rangle. \tag{12}$$

Since $\mathbf{D}$ is dense in $D(\Phi(g))$ with its graph norm topology, it suffices to prove (11) for all $y \in \mathbf{D}$. When $y \in \mathbf{D}$, by Lemma 11 we have

$$\begin{aligned} \langle \Phi(g)\, y, Sx \rangle &= -i\,\partial_t \langle e^{-itS}\Phi(g)\, y, x\rangle|_{t=0} \\ &= -i\,\partial_t \langle \Phi(e^{tT}g)e^{-itS}y, x\rangle|_{t=0} \\ &= -i\,\partial_t \langle e^{-itS}y, \Phi(e^{tT}g)x\rangle|_{t=0}. \end{aligned}$$

Since $x \in D(\Phi(Tg))$, and $y \in D(S)$ by Lemma 10, this implies (12). ∎

It follows from this lemma, formula (9), and remarks in the proof of Lemma 4 that $R|\mathbf{D}$ and $S|\mathbf{D}$ differ by a multiple of the identity operator. By (11) it follows that $R|\mathbf{D} = S|\mathbf{D}$.

Since S is essentially self-adjoint on $\mathbf{D}$, $\bar{R} \supseteq (\overline{R|\mathbf{D}}) = (\overline{S|\mathbf{D}}) = S$, so $\bar{R}$ is a symmetric extension of S. Since S is self-adjoint this implies $\bar{R} = S$. Thus R is essentially self-adjoint, as was to be shown. ∎

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