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Scattering and the Geometry of the Solution Manifold of $\Box f + \lambda f^3 = 0$

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The wave transforms for the equation $\Box f + \lambda f^3 = 0$ on $\mathbb{R}^4$ can be represented as maps from the Hilbert space $\mathbf{H}(S)$ of finite-energy Cauchy data for the equation $(\Box + 1)\varphi + \lambda\varphi^3 = 0$ on $\mathbb{R} \times S^3$ to Hilbert spaces $\mathbf{H}(C_\pm)$ of finite-energy Goursat data. We show that the wave transforms and associated scattering transform are diffeomorphisms. The action of the group $\tilde{\mathbf{G}} = SO^\sim(2,4)$ on solutions of the latter equation gives rise to actions of $\tilde{\mathbf{G}}$ as diffeomorphisms of $\mathbf{H}(S)$ and $\mathbf{H}(C_\pm)$ which are intertwined by the wave and scattering transforms. Moreover, the spaces $\mathbf{H}(S)$ and $\mathbf{H}(C_\pm)$ have symplectic structures relative to which the group $\tilde{\mathbf{G}}$ and the wave and scattering transforms act as symplectomorphisms. These symplectic structures extend to flat Kähler structures that are invariant under the scaling-extended Poincaré group $\mathbf{P}$. © 1989 Academic Press, Inc.

1. INTRODUCTION

The relation of scattering to the geometry of solution manifolds of relativistic wave equations is of interest not only for its own sake but also for its relevance to quantum field theory [4, 6]. Many basic questions in the subject are unanswered, but for conformally invariant equations the reduction of scattering to a global Goursat problem overcomes a number of difficulties [2, 7].

In the Goursat problem for a wave equation, solutions are determined by data given on a lightcone. Minkowski space, $\mathbf{M}_0 = \mathbb{R}^4$, has an essentially unique conformal embedding in the Einstein universe, $\tilde{\mathbf{M}} = \mathbb{R} \times S^3$, and the boundary of $\mathbf{M}_0$ as embedded is the union of two lightcones, $C_\pm$, representing the limits of spacelike surfaces in $\mathbf{M}_0$ as the Minkowski time approaches $\pm\infty$. Sufficiently regular solutions of a conformally invariant wave equation on $\mathbf{M}_0$ extend to solutions of a closely related equation on $\tilde{\mathbf{M}}$. The scattering transform can then be construed as the map from Goursat data on C_- to Goursat data on C_+.

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The methods of [2] and the present paper apply to equations of the form $\Box f + H'(f) = 0$, where H is a nonnegative homogeneous polynomial of degree four in the multicomponent scalar section f on $\mathbf{M}_0$. For notational simplicity, however, we treat the equation $\Box f + \lambda f^3 = 0$ $(\lambda \geq 0)$ as a representative case. Here the scattering transform is shown to be a diffeomorphism of the Hilbert spaces of finite Einstein energy Goursat data $\mathbf{H}(C_-)$ and $\mathbf{H}(C_+)$. Thus the spaces $\mathbf{H}(C_\pm)$ give distinct but diffeomorphic presentations of the finite-energy solution manifold as a Hilbert space.

We determine symplectic structures for the spaces $\mathbf{H}(C_\pm)$ relative to which the conformal group $\tilde{\mathbf{G}} = SO^\sim(2, 4)$ acts as symplectomorphisms; the scattering transform is then seen to be a symplectomorphism intertwining the actions of $\tilde{\mathbf{G}}$ on $\mathbf{H}(C_-)$ and $\mathbf{H}(C_+)$. The symplectic structures on the spaces $\mathbf{H}(C_\pm)$ extend to flat Kähler structures preserved by the group generated by the Poincaré group and dilations, $\mathbf{P}$, which has $\tilde{\mathbf{P}} \subset \tilde{\mathbf{G}}$. The finite-energy solution manifold thus has $\mathbf{P}$-invariant flat Kähler structures.

2. Space–Time Geometry

In this section we introduce notation concerning the geometry of the generalizations of Minkowski space and the Einstein universe to arbitrary dimension. This material appears in [2, 3] and the references therein.

Let $\mathbf{M}_0$, "Minkowski space," denote $\mathbb{R} \times \mathbb{R}^n$ with the coordinates $(x_0, \mathbf{x})$, where $\mathbf{x} = (x_1, ..., x_n)$, and the "Minkowski" metric $dx_0^2 - d\mathbf{x}^2$, where $d\mathbf{x}^2 = dx_1^2 + \cdots + dx_n^2$. The coordinate x_0 is called the "Minkowski time" and the integer n is the space dimension. In what follows we always assume $n > 1$.

The conformal compactification $\bar{\mathbf{M}}$ of $\mathbf{M}_0$ may be defined as $SO(2, n+1)/SO(1, n)$. $\bar{\mathbf{M}}$ has a unique conformal structure that is preserved by the natural action of $\mathbf{G} = SO(2, n+1)$. As a conformal manifold it is equivalent to $(S^1 \times S^n)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by the product of the antipodal maps on S^1 and S^n, and the usual direct product Lorentzian metric is used on $S^1 \times S^n$.

$\tilde{\mathbf{M}}$ denotes the universal cover of $\bar{\mathbf{M}}$, given the conformal structure lifted up from that of $\bar{\mathbf{M}}$, which is invariant under the natural action of $\tilde{\mathbf{G}}$ on $\tilde{\mathbf{M}}$. If we specify points of $\mathbb{R} \times S^n$ by (τ, u), where $\tau \in \mathbb{R}$ is called the "Einstein time" and $u \in S^n$, and let ds^2 denote the standard Riemannian metric on S^n, then $\tilde{\mathbf{M}}$ is conformally equivalent to the "Einstein universe," $\mathbb{R} \times S^n$ with the conformal structure corresponding to the "Einstein" metric $d\tau^2 - ds^2$.

We specify points of the manifold S^n as images of pairs (ρ, ω), where $\rho \in [0, \pi]$ and $\omega \in S^{n-1}$, under a function defined as follows. We regard S^n as embedded in $\mathbb{R}^{n+1}$ as the unit sphere, so that $\omega = (u_1, ..., u_{n+1})$ where $\Sigma_i u_i^2 = 1$. Regarding S^{n-1} as correspondingly embedded in $\mathbb{R}^n$, we then map $(\rho, \omega) \in [0, \pi] \times S^{n-1}$ into S^n by the map

$$(\rho, \omega) \mapsto (\cos \rho, \sin \rho\ \omega).$$

The map is C^∞, and when restricted to $(0, \pi) \times S^{n-1}$ is a diffeomorphism onto an open subset of S^n denoted by $S^{n,*}$.

If $x \in \mathbf{M}_0$, we let $r = (x_1^2 + \cdots + x_n^2)^{1/2}$, and define $\theta \in S^{n-1} \subset \mathbb{R}^n$ by $\mathbf{x} = r\theta$ for $\mathbf{x} \neq 0$. Then the conformal embedding of $\mathbf{M}_0$ in $\bar{\mathbf{M}}$ lifts to a conformal embedding $\iota: \mathbf{M}_0 \to \tilde{\mathbf{M}}$ given by

$$\sin \tau|_{\iota(x)} = p x_0$$
$$\sin \rho|_{\iota(x)} = p r$$
$$\omega|_{\iota(x)} = \theta,$$

where

$$p = ((1 - (x_0^2 - r^2)/4)^2 + x_0^2)^{-1/2}. \tag{1}$$

We identify $\mathbf{M}_0$ with its image under ι in $\tilde{\mathbf{M}}$,

$$\iota(\mathbf{M}_0) = \{|\tau| + |\rho| < \pi\} \subset \tilde{\mathbf{M}},$$

and extend p to a smooth function on $\tilde{\mathbf{M}}$ by the equation $p = \frac{1}{2}(\cos \tau + \cos \rho)$. The function p is the conformal factor relating the Einstein and Minkowski metrics, i.e., on $\mathbf{M}_0$

$$d\tau^2 - ds^2 = p^2(dx_0^2 - d\mathbf{x}^2).$$

Let $\mathbf{P}$ denote the group of conformal diffeomorphisms of $\mathbf{M}_0$, or the "scale-extended Poincaré group." The action of $\tilde{\mathbf{P}}$ on $\mathbf{M}_0$ extends uniquely to a conformal action on $\tilde{\mathbf{M}}$, and is thus identified with a Lie subgroup of the group $\tilde{\mathbf{G}}$.

The boundary of $\mathbf{M}_0$ as embedded in $\tilde{\mathbf{M}}$ is the union of two lightcones, the "lightcone at future infinity," C_+, and the "lightcone at past infinity," C_-, defined by

$$C_\pm = \{\pm\tau = \pi - \rho\}.$$

We give C_- the coordinates (ρ, ω) obtained by restricting the functions (τ, ρ, ω) on $\tilde{\mathbf{M}}$, thus identifying C_- with S^n; this identification is a homeomorphism, and yields a map from S^n to $\tilde{\mathbf{M}}$ that is smooth on $S^{n,*}$. We use this identification to transfer to C_- the Riemannian metric ds^2 and associated volume form ν on S^n.

3. The Conformal Wave Equation

In this section we describe the conformal wave equation and calculate the conformally invariant inner product of solutions of this equation in terms of Goursat data. Results stated without proof can be found in [2, 3].

We will use $C^k(X)$ to denote the space of all k times continuously differentiable real-valued functions on a manifold X. $L_{p,q}(X)$ will denote the space of all real distributions f on a compact Riemannian manifold X that are in $L_p(X)$ together with their first q derivatives, and denote the norm in this space as $\|\cdot\|_{p,q}$. For $X=C_\pm$ we use the identification of $C_\pm$ with S^n to define $L_{p,q}(C_\pm)$.

Associated with the metric ds^2 on S^n is the (negative) Laplace–Beltrami operator Δ_n, which has a standard realization as a self-adjoint operator on $L_{2,q}(S^n)$. The conformal wave operator on $\tilde{\mathbf{M}}$ is the differential operator $\square + c_n$, where $\square$ denotes the D'Alembertian $\partial_\tau^2 - \Delta_n$ relative to the Einstein metric, and

$$c_n = ((n-1)/2)^2.$$

The conformal wave equation on $\tilde{\mathbf{M}}$ is given by

$$(\square + c_n)\varphi = 0, \tag{2}$$

but it is useful to reformulate it as follows. Let B denote the positive self-adjoint operator $(-\Delta_n + c_n)^{1/2}$ on $L_{2,q}(S^n)$. Then $\mathbf{H}_{-1}(S)$ denotes $L_{2,1/2}(S^n) \oplus L_{2,-1/2}(S^n)$ as a complex Hilbert space with the following complex structure J and inner product $\langle \cdot, \cdot \rangle_{-1}$:

$$J(\Phi_1, \Phi_2) = (-B^{-1}\Phi_2, B\Phi_1)$$

$$\operatorname{Re}\langle(\Phi_1, \Phi_2), (\Psi_1, \Psi_2)\rangle_{-1} = \int_{S^n} (B^{1/2}\Phi_1)(B^{1/2}\Psi_1) + (B^{-1/2}\Phi_2)(B^{-1/2}\Psi_2)$$

$$\operatorname{Im}\langle(\Phi_1, \Phi_2), (\Psi_1, \Psi_2)\rangle_{-1} = \int_{S^n} \Phi_1\Psi_2 - \Phi_2\Psi_1.$$

Then the Cauchy problem for the conformal wave equation is given abstractly as the first-order equation

$$\partial_\tau \Phi(\tau) = A\Phi(\tau), \tag{3}$$

where

$$A(\Phi_1, \Phi_2) = (\Phi_2, -B^2\Phi_1).$$

The operator JA is naturally identifiable with a self-adjoint operator on $\mathbf{H}_{-1}(S)$, which may be defined as the self-adjoint generator of the one-parameter unitary group on $\mathbf{H}_{-1}(S)$ of matrices

$$\begin{pmatrix} \cos tB & B^{-1}\sin tB \\ -B\sin tB & \cos tB \end{pmatrix}.$$

The operator JA is positive, with pure point spectrum bounded away from zero. If $\alpha \in \mathbb{R}$, the domain $D((JA)^\alpha)$ completed with respect to the inner product

$$\langle \Phi, \Psi \rangle_\alpha = \langle \Phi, (JA)^{\alpha+1}\Psi\rangle_{-1}$$

is a complex Hilbert space, which will be denoted as $\mathbf{H}_\alpha(S)$. The common part $\bigcap_\alpha \mathbf{H}_\alpha(S)$ of these domains, with the topology of convergence in each $\mathbf{H}_\alpha(S)$, coincides with $C^\infty(S^n) \oplus C^\infty(S^n)$ in its usual topology, and will be denoted as $\mathbf{H}_\infty(S)$. As a consequence of the spectral theorem, the operators e^{tA} on $\mathbf{H}_{-1}(S)$ extend by continuity to a strongly continuous unitary group on $\mathbf{H}_\alpha(S)$ for any $\alpha \leqslant 0$, and restrict to a strongly continuous unitary group on $\mathbf{H}_\alpha(S)$ for any finite $\alpha \geqslant 0$.

The precise relationship between the conformal wave equation (2) and the first-order equation (3) is then in part as follows. Let $\mathbf{E}$ be the space of smooth solutions of (2) given the $C^\infty(\tilde{\mathbf{M}})$ topology. Suppose that $\Phi \in \mathbf{H}_\infty(S)$. Let $e^{tA}\Phi = (\Phi_1(t), \Phi_2(t))$, and define $\varphi = T\Phi$ by

$$\varphi|_{\tau=t} = \Phi_1(t).$$

Then $\varphi \in \mathbf{E}$; moreover, the map $T\colon \mathbf{H}_\infty(S) \to \mathbf{E}$ thus defined is an isomorphism of the topological vector spaces $\mathbf{H}_\infty(S)$ and $\mathbf{E}$. We use the isomorphism T to transfer to $\mathbf{E}$ the complex structure J and the inner products $\langle \cdot, \cdot \rangle_\alpha$.

The group $\tilde{\mathbf{G}}$ has a representation R on $\mathbf{E}$ given by

$$(R(g^{-1})\varphi)(x) = \|dg_x\|\, \varphi(gx), \qquad g \in \tilde{\mathbf{G}}, \quad x \in \tilde{\mathbf{M}}, \tag{4}$$

where $\|dg_x\|$ denotes the norm of the differential $dg\colon T_x\tilde{\mathbf{M}} \to T_{gx}\tilde{\mathbf{M}}$ relative to the Einstein metric on the tangent spaces $T_x\tilde{\mathbf{M}}$ and $T_{gx}\tilde{\mathbf{M}}$. Since $\tilde{\mathbf{G}}$ acts smoothly on $\tilde{\mathbf{M}}$ and $\|dg_x\|$ depends smoothly on (g, x), the representation $R\colon \tilde{\mathbf{G}} \times \mathbf{E} \to \mathbf{E}$ is continuous. Thus $T^{-1}RT$ is a continuous representation of $\tilde{\mathbf{G}}$ on $\mathbf{H}_\infty(S)$. For any $\alpha \in \mathbb{R}$, $T^{-1}RT$ extends uniquely to a continuous representation of $\tilde{\mathbf{G}}$ on $\mathbf{H}_\alpha(S)$, which we denote by U regardless of the value of α. The value $\alpha = -1$ is special in that the representation U is unitary on the space $\mathbf{H}_{-1}(S)$.

The space $\mathbf{H}_0(S) = \mathbf{H}(S)$ is called the space of "finite-energy Cauchy data," and is important in the theory of nonlinear variants of the conformal wave equation. We write the inner product $\langle \cdot, \cdot \rangle_0$ simply as $\langle \cdot, \cdot \rangle$. Let

$\mathbf{H}(C_\pm)$, the spaces of "finite-energy Goursat data," denote $L_{2,1}(C_\pm)$. Following physics terminology, the spaces $\mathbf{H}(C_-)$ and $\mathbf{H}(C_+)$ may also be called the spaces of "in" and "out" fields, respectively. As shown in [2], there are unique continuous maps $W^\pm: \mathbf{H}(S) \to \mathbf{H}(C_\pm)$ such that

$$W^\pm \Phi = T\Phi \mid C_\pm$$

if $\Phi \in \mathbf{H}_\infty(S)$. The operator $W^\pm$ associates to a finite-energy Cauchy datum the corresponding Goursat datum on the lightcone $C_\pm$. The operators $W^\pm$ are real-linear, one-to-one and onto, and orthogonal in the following sense:

$$\operatorname{Re}\langle \Phi, \Psi \rangle = \int_{C_\pm} (BW^\pm \Phi)(BW^\pm \Psi), \tag{5}$$

where we use the identification of $C_\pm$ with S^n to interpret B as an operator on $\mathbf{H}(C_\pm)$. Thus the spaces $\mathbf{H}(C_\pm)$ have a unique complex structure J and complex inner product $\langle \cdot, \cdot \rangle$ relative to which the operators $W^\pm$ are unitary. Context will serve to distinguish these from the complex structure and inner product on $\mathbf{H}(S)$; in particular, we will use capital Greek letters to denote Gauchy data, and use boldface capital Greek letters to denote Goursat data.

Let $L^\pm = W^\pm A (W^\pm)^{-1}$. The self-adjoint operator $JL^\pm$ on $\mathbf{H}(C_\pm)$ is positive, with pure point spectrum bounded away from zero. If $\alpha \in \mathbb{R}$, the domain of $(L^\pm)^\alpha$ completed with respect to the inner product

$$\langle \mathbf{\Phi}, \mathbf{\Psi} \rangle_\alpha = \langle \mathbf{\Phi}, (L^\pm)^\alpha \mathbf{\Psi} \rangle$$

is a complex Hilbert space, which will be denoted by $\mathbf{H}_\alpha(C_\pm)$. The common part $\bigcap_\alpha \mathbf{H}_\alpha(C_\pm)$ of these domains, with the topology of convergence in each $\mathbf{H}_\alpha(C_\pm)$, will be denoted by $\mathbf{H}_\infty(C_\pm)$. For any $\alpha \in \mathbb{R}$, the operators $W^\pm$ restrict or uniquely extend by continuity to unitary operators from $\mathbf{H}_\alpha(S)$ to $\mathbf{H}_\alpha(C_\pm)$, which will also be denoted by $W^\pm$. We define the continuous representations $U^\pm$ of $\tilde{\mathbf{G}}$ on the spaces $\mathbf{H}_\alpha(C_\pm)$ by

$$U^\pm(g)\mathbf{\Phi} = W^\pm U(g)(W^\pm)^{-1}\mathbf{\Phi}, \qquad g \in \tilde{\mathbf{G}}, \quad \mathbf{\Phi} \in \mathbf{H}_\alpha(C_\pm).$$

In the remainder of this section, we explicitly calculate the complex structure J and inner product $\langle \cdot, \cdot \rangle_{-1}$ in terms of Goursat data.

PROPOSITION 1. *Suppose* $\mathbf{\Phi}, \mathbf{\Psi} \in \mathbf{H}(C_\pm)$. *Then*

$$\operatorname{Im}\langle \mathbf{\Phi}, \mathbf{\Psi} \rangle_{-1} = \int_{C_\pm} (\mathbf{\Phi}\partial_\rho \mathbf{\Psi} - \mathbf{\Psi}\partial_\rho \mathbf{\Phi}). \tag{6}$$

Proof. Note first that since $\mathbf{\Phi}, \mathbf{\Psi} \in \mathbf{H}(C_\pm)$, the function $\mathbf{\Phi}\partial_\rho\mathbf{\Psi} - \mathbf{\Psi}\partial_\rho\mathbf{\Phi}$

is integrable. Let $\mathbf{\Phi} = W^\pm \Phi$, $\mathbf{\Psi} = W^\pm \Psi$. Since $\mathbf{H}_\infty(S)$ is dense in $\mathbf{H}(S)$ and the operators $W^\pm$ are unitary, it suffices that (6) hold if $\Phi, \Psi \in \mathbf{H}_\infty(S)$. This is an immediate consequence of the following lemma, which will be useful in the sequel as well:

LEMMA 2. *Suppose* $\varphi, \psi \in C^2(\tilde{\mathbf{M}})$ *satisfy*

$$(\Box + c_n)\varphi = g\varphi, \qquad (\Box + c_n)\psi = g\psi,$$

where $g \in C(\tilde{\mathbf{M}})$. *Let*

$$\Phi_t = (\varphi, \partial_\tau \varphi)|_{\tau = t}, \qquad \Psi_t = (\psi, \partial_\tau \psi)|_{\tau = t},$$

and let $\mathbf{\Phi} = \varphi \mid C_\pm$ *and* $\mathbf{\Psi} = \psi \mid C_\pm$. *Then* $\operatorname{Im}\langle \Phi_t, \Psi_t \rangle_{-1}$ *is independent of* t, *and the following equation holds*:

$$\operatorname{Im}\langle \Phi_t, \Psi_t \rangle_{-1} = \int_{C_\pm} (\mathbf{\Phi}\partial_\rho \mathbf{\Psi} - \mathbf{\Psi}\partial_\rho \mathbf{\Phi}).$$

Proof. Simple calculations show that the n-form η on $\tilde{\mathbf{M}}$ given by

$$\eta = \varphi \wedge * d\psi - \psi \wedge * d\varphi$$

is closed, and that

$$\int_{\tau = t} \eta = \int_{\tau = t} (\varphi \partial_\tau \psi - \psi \partial_\tau \varphi) = \operatorname{Im}\langle \Phi_t, \Psi_t \rangle_{-1}.$$

Since the submanifolds $\{\tau = t\}$, C_+, and C_- define homologous n-cycles in $\tilde{\mathbf{M}}$,

$$\int_{C_\pm} \eta = \int_{\tau = t} \eta$$

for any value of t. Thus, speaking more carefully, it suffices to show that

$$\int_{C_\pm} i_\pm^* \eta = \int_{C_\pm} (\mathbf{\Phi}\partial_\rho \mathbf{\Psi} - \mathbf{\Psi}\partial_\rho \mathbf{\Phi})\nu \tag{7}$$

holds, where $i_\pm : C_\pm \to \tilde{\mathbf{M}}$ is the inclusion map. For some form μ

$$* d\varphi = (\partial_\tau \varphi \, d\rho + \partial_\rho \varphi \, d\tau) \wedge \sin^{n-1}\rho \, d\omega + \mu \wedge d\tau \wedge d\rho$$

and $i_\pm^* \, d\tau = i_\pm^* \, d\rho$, so

$$\begin{aligned} i_\pm^*(* d\varphi) &= i_\pm^*((\partial_\tau + \partial_\rho)\varphi \sin^{n-1}\rho \, d\rho \wedge d\omega) \\ &= \partial_\rho \mathbf{\Phi} \wedge \nu. \end{aligned}$$

Similarly, $i_\pm^*(*\,d\psi) = \partial_\rho \Psi \wedge v$. Thus

$$i_\pm^* \eta = (\Phi \partial_\rho \Psi - \Psi \partial_\rho \Phi) v,$$

implying that (7) holds. ∎

To describe the symmetric form $\operatorname{Re}\langle\cdot,\cdot\rangle_{-1}$ and the complex structure J on $\mathbf{H}_\infty(C_\pm)$ it is convenient to use an alternate presentation of Goursat data. Let $C_\pm^*$ denote $C_\pm - \{\rho = 0, \pi\}$, and give $C_\pm^*$ the coordinates (s, ω) where

$$s = \mp 2 \cot \rho.$$

These coordinates implement a diffeomorphism by means of which we identify $C_\pm^*$ and $\mathbb{R} \times S^{n-1}$. Given a function $\Phi\colon C_\pm \to \mathbb{R}$, we define the function $\Phi_G\colon \mathbb{R} \times S^{n-1} \to \mathbb{R}$ by

$$\Phi_G(s, \omega) = \sin^{(n-1)/2} \rho\, \Phi(\rho, \omega). \tag{8}$$

THEOREM 3. *Given $\Phi, \Psi \in \mathbf{H}_\infty(C_\pm)$, let the functions Φ_G and Ψ_G on $\mathbb{R} \times S^{n-1}$ be given as in* (8). *Then*

$$\operatorname{Im}\langle \Phi, \Psi \rangle_{-1} = \int_{\mathbb{R} \times S^{n-1}} (\Phi_G\, \partial_s \Psi_G - \Psi_G\, \partial_s \Phi_G)\, ds\, d\omega$$

and

$$\operatorname{Re}\langle \Phi, \Psi \rangle_{-1} = 2 \int_{\mathbb{R} \times S^{n-1}} |\sigma| (F\Phi_G)^- \, (F\Psi_G)\, d\sigma\, d\omega,$$

where F, the Fourier transform in the s variable, is defined as

$$(F\Phi_G)(\sigma, \omega) = (2\pi)^{-1/2} \int e^{-is\sigma}\, \Phi_G(s, \omega)\, ds$$

but may need to be taken in the distributional sense. Moreover, the complex structure J is given as

$$(J\Phi)_G\, (s, \omega) = \pi^{-1} p.v. \int (s - s')^{-1}\, \Phi_G(s', \omega)\, ds'.$$

Proof. First note that since $\partial_s = \frac{1}{2} \sin^2 \rho\, \partial_\rho$ and $ds = 2 \csc^2 \rho\, d\rho$, the equation

$$\operatorname{Im}\langle \Phi, \Psi \rangle_{-1} = \int_{\mathbb{R} \times S^{n-1}} (\Phi_G\, \partial_s \Psi_G - \Psi_G\, \partial_s \Phi_G)\, ds\, d\omega \tag{9}$$

follows from Proposition 1 by a change of variables.

Next we prove that the complex structure J is as stated. As shown in [3], the space $\mathbb{C}\mathbf{E}$ of complex smooth solutions of the conformal wave equation has a complex structure given by the usual multiplication by i, and a $\tilde{\mathbf{G}}$-invariant sesquilinear form given by

$$\langle\langle \varphi, \psi \rangle\rangle = i^{-1} \int_{\tau = 0} \bar{\varphi}\, \partial_\tau \psi - \psi\, \partial_\tau \bar{\varphi}.$$

$\mathbb{C}\mathbf{E}$ is the direct sum of two $\tilde{\mathbf{G}}$-invariant subspaces, $\mathbf{E}_{\text{pos}}$ and $\mathbf{E}_{\text{neg}}$, the positive and negative frequency smooth complex solutions of the conformal wave equation. $\mathbf{E}_{\text{pos}}$ and $\mathbf{E}_{\text{neg}}$ are pre-Hilbert spaces with inner products $\langle\langle\cdot,\cdot\rangle\rangle$ and $-\langle\langle\cdot,\cdot\rangle\rangle$, respectively. The complex structure J on $\mathbf{E}$ extends to a map $J'\colon \mathbb{C}\mathbf{E} \to \mathbb{C}\mathbf{E}$ such that if $\varphi_{\text{pos}} \in \mathbf{E}_{\text{pos}}$ and $\varphi_{\text{neg}} \in \mathbf{E}_{\text{neg}}$,

$$J'(\varphi_{\text{pos}} + \varphi_{\text{neg}}) = -i\varphi_{\text{pos}} + i\varphi_{\text{neg}}.$$

Suppose that $\varphi \in \mathbb{C}\mathbf{E}$. Let $\Phi = \varphi \,|\, C_\pm$ and define Φ_G by (8). Then since the real and imaginary parts of φ are elements of $\mathbf{E}$, (9) implies that

$$\langle\langle \varphi, \varphi \rangle\rangle = i^{-1} \int_{\mathbb{R} \times S^{n-1}} (\bar{\Phi}_G\, \partial_s \Phi_G - \Phi_G\, \partial_s \bar{\Phi}_G)\, ds\, d\omega. \tag{10}$$

Note that

$$\Phi_G(s, \omega) = (1 + \tfrac{1}{4} s^2)^{-(n-1)/4}\, \Phi(\rho, \omega),$$

so Φ_G lies in $C^\infty(\mathbb{R} \times S^{n-1})$ and decays, together with all its derivatives, at least as fast as $(1 + \frac{1}{4}s^2)^{-1/4}$. Thus the Fourier transforms $F\bar{\Phi}_G$ and $F\Phi_G$ are well defined in the distributional sense and (10) implies

$$\langle\langle \varphi, \varphi \rangle\rangle = 2 \int_{\mathbb{R} \times S^{n-1}} \sigma |F\Phi_G(\sigma, \omega)|^2\, d\sigma\, d\omega.$$

As explained in [2], the spaces $\mathbf{E}_{\text{pos}}$ and $\mathbf{E}_{\text{neg}}$ are invariant under the action of the Minkowski time translations $\exp(t\mathbf{T}_0) \in \tilde{\mathbf{G}}$, and

$$(U^\pm(\exp(t\mathbf{T}_0))\Phi)_G(s, \omega) = \Phi_G(s - t, \omega).$$

Thus the space $\{F\Phi_G\colon \varphi \in \mathbf{E}_{\text{pos}}\}$ is invariant under the multiplication operators $e^{it\sigma}$, and by the above

$$\langle\langle \varphi, \varphi \rangle\rangle = 2 \int_{\mathbb{R} \times S^{n-1}} \sigma |F\Phi_G(\sigma, \omega)|^2\, d\sigma\, d\omega \geq 0$$

if $\varphi \in \mathbf{E}_{\text{pos}}$. This implies that given $\varphi \in \mathbb{C}\mathbf{E}$, $\varphi \in \mathbf{E}_{\text{pos}}$ if and only if

$$\operatorname{supp}(F\Phi_G) \subseteq \{(\sigma, \omega)\colon \sigma \geq 0\}.$$

Similarly, given $\varphi \in \mathbb{C}\mathbf{E}$, $\varphi \in \mathbf{E}_{\text{neg}}$ if and only if

$$\operatorname{supp}(F\mathbf{\Phi}_G) \subseteq \{(\sigma, \omega): \sigma \leqslant 0\}.$$

Hence considering $\varphi \in \mathbf{E}$ as an element of $\mathbb{C}\mathbf{E}$, the formula for J' implies that

$$F(J\mathbf{\Phi}_G) = i(\chi_{(-\infty,0)} - \chi_{(0,\infty)}) F\mathbf{\Phi}_G. \tag{11}$$

In other words, up to a sign $J\mathbf{\Phi}_G$ is the Hilbert transform of $\mathbf{\Phi}_G$ in the s variable. Since $\mathbf{\Phi}_G$ has the above smoothness and decay properties the usual formula for the Hilbert transform holds, so that

$$(J\mathbf{\Phi})_G\,(s, \omega) = \pi^{-1} p.v. \int (s - s')^{-1}\, \mathbf{\Phi}_G(s', \omega)\, ds'.$$

Last, note that if $\mathbf{\Phi}, \mathbf{\Psi} \in \mathbf{H}_\infty(C_\pm)$, Eq. (9) implies

$$\begin{aligned}\operatorname{Re}\langle \mathbf{\Phi}, \mathbf{\Psi} \rangle_{-1} &= \operatorname{Im}\langle \mathbf{\Phi}, J\mathbf{\Psi} \rangle_{-1} \\ &= 2i \int_{\mathbb{R} \times S^{n-1}} \sigma (F\mathbf{\Phi}_G)^-\, (FJ\mathbf{\Psi}_G)\, d\sigma\, d\omega,\end{aligned}$$

so Eq. (11) implies that

$$\operatorname{Re}\langle \mathbf{\Phi}, \mathbf{\Psi} \rangle_{-1} = 2 \int_{\mathbb{R} \times S^{n-1}} |\sigma| (F\mathbf{\Phi}_G)^-\, (F\mathbf{\Psi}_G)\, d\sigma\, d\omega. \quad \blacksquare$$

4. Smoothness of the Scattering Transform

If $n = 3$, the nonlinear variant of the conformal wave equation given by

$$(\Box + 1)\varphi + \lambda\varphi^3 = 0, \qquad \lambda \geqslant 0 \tag{12}$$

is also conformally invariant. This equation is equivalent to the (classical) "massless φ^4 theory" equation in Minkowski space, in the following sense. Suppose φ is a smooth solution of (2) on $\tilde{\mathbf{M}}$, and let $f = p\varphi \mid \mathbf{M}_0$, where p is given as in (1). Then f is a smooth solution of

$$\Box f + \lambda f^3 = 0 \tag{13}$$

on $\mathbf{M}_0$, where here $\Box$ denotes the D'Alembertian on $\mathbf{M}_0$; conversely, a sufficiently regular solution of (13) yields a solution of (12). The detailed implications of this equivalence for the scattering theory of the massless φ^4 theory are developed in [2]; in particular, the scattering transform is expressed as a map from $\mathbf{H}(C_-)$ to $\mathbf{H}(C_+)$. In this section we show that the scattering transform is a diffeomorphism.

It is useful to consider an integral equation associated with (12). By nonlinear semigroup theory [5], for any $\Phi \in \mathbf{H}_k(S)$, $k \geqslant 0$ an integer, there is a unique continuous function $\Psi: \mathbb{R} \to \mathbf{H}_k(S)$ such that

$$\Psi(t) = e^{tA}\Phi + \int_0^t e^{(t-s)A} N\Psi(s)\, ds, \tag{14}$$

where

$$N(\Psi_1, \Psi_2) = (0, -\lambda\Psi_1^3).$$

It follows that if $\Phi \in \mathbf{H}_\infty(S)$, there is a unique continuous function $\Psi: \mathbb{R} \to \mathbf{H}_\infty(S)$ such that (14) holds. We will also need the following regularity property:

Proposition 4. *Given $t \in \mathbb{R}$, define $F_t: \mathbf{H}_k(S) \to \mathbf{H}_k(S)$ by $F_t(\Phi) = \Psi(t)$, where $\Psi(t)$ satisfies (14). Then F_t is smooth; that is, for all n and all $\Phi \in \mathbf{H}(S)$ the Frechét derivative $D^n F_t(\Phi)$ exists as a bounded multilinear map from $\mathbf{H}_k(S)^n$ to $\mathbf{H}_k(S)$. Moreover, $\|D^n F_t(\Phi)\|$ is uniformly bounded for t ranging over any bounded interval.*

Proof. One can differentiate the right side of (14) to prove inductively that $D^n F_t(\Phi)$ exists, and satisfies a linear integral equation. The latter, together with the fact that the maps F_t are boundedly Lipschitzian and uniformly so for t ranging over any bounded interval, implies that $\|D^n F_t(\Phi)\|$ is uniformly bounded for t ranging over any bounded interval. $\blacksquare$

The relationship between (12) and the integral equation (14) is in part as follows. Let $\mathbf{E}_\lambda$ be the set of smooth solutions of (12) given the topology of uniform convergence of all derivatives on compact subsets of $\tilde{\mathbf{M}}$. Suppose that $\Phi \in \mathbf{H}_\infty(S)$, and let $\Psi: \mathbb{R} \to \mathbf{H}_\infty(S)$ satisfy (14). Define $\varphi = T_\lambda \Phi$ by

$$\varphi\,|_{\tau = t} = \Psi_1(t).$$

Then $\varphi \in \mathbf{E}_\lambda$; moreover, the map $T_\lambda: \mathbf{H}_\infty(S) \to \mathbf{E}_\lambda$ thus defined is a homeomorphism of $\mathbf{H}_\infty(S)$ and $\mathbf{E}_\lambda$.

In [2] it is shown that there are unique continuous maps, the "wave transforms" $W_\lambda^\pm: \mathbf{H}(S) \to \mathbf{H}(C_\pm)$, such that

$$W_\lambda^\pm \Phi = (T_\lambda \Phi) \mid C_\pm$$

if $\Phi \in \mathbf{H}_\infty(S)$, and that these maps are homeomorphisms. Thus the "scattering transform"

$$S_\lambda = W_\lambda^+ (W_\lambda^-)^{-1}$$

is a homeomorphism from $\mathbf{H}(C_-)$ to $\mathbf{H}(C_+)$. In fact:

THEOREM 5. *The maps* $W_\lambda^\pm : \mathbf{H}(S) \to \mathbf{H}(C_\pm)$ *and the map* $S_\lambda : \mathbf{H}(C_-) \to \mathbf{H}(C_+)$ *are diffeomorphisms.*

Proof. It suffices to show that W_λ^- is a diffeomorphism. First we prove the existence of a wave transform for the inhomogeneous variant of the conformal wave equation. Let R denote the region in $\tilde{\mathbf{M}}$ defined by $\{0 < \tau < \pi - \rho\}$.

LEMMA 6. *For any* $g \in C^\infty(\tilde{\mathbf{M}})$ *and* $\Phi \in \mathbf{H}_\infty(S)$ *there is a unique* $\varphi \in C^\infty(\tilde{\mathbf{M}})$ *such that*

$$(\Box + c_n)\varphi + g = 0$$
$$(\varphi, \partial_\tau \varphi)|_{\tau=0} = \Phi. \tag{15}$$

The restriction $\varphi | C^+$ *is a function of* Φ *and* $g | R$, *and the map* $(g | R, \Phi) \mapsto \varphi | C^+$ *thus defined extends uniquely to a bounded linear operator* $W: L_2(R) \times \mathbf{H}(S) \to \mathbf{H}(C_+)$.

Proof. The existence of $\varphi \in C^\infty(\tilde{\mathbf{M}})$ satisfying (15) is a consequence of the regularity theory of the Cauchy problem, and that $\varphi | C_+$ depends only on Φ and $g | R$ is a consequence of the unit propagation speed associated with the conformal wave equation. Let R' denote the region $[0, \pi] \times S^n \subset \tilde{\mathbf{M}}$. By [2, Lemma 2 and Proposition 3] we have

$$\|\varphi | C_+\|^2 \leqslant \|\Phi\|^2 + \int_R |g\, \partial_\tau \varphi|$$
$$\leqslant \|\Phi\|^2 + \|g | R'\|_2 \, \|\partial_\tau \varphi | R'\|_2$$

and by the formulation of the Cauchy problem as an integral equation,

$$\|\partial_\tau \varphi |_{\tau = t}\|_2 \leqslant \|\Phi\| + \int_0^t \|g|_{\tau = s}\|_2 \, ds$$
$$\leqslant \|\Phi\| + \pi^{1/2} \|g | R'\|_2$$

if $t \in [0, \pi]$. Thus we have

$$\|\varphi | C_+\|^2 \leqslant \|\Phi\|^2 + \|g | R'\|_2 (\|\Phi\| + \pi^{1/2} \|g | R'\|_2)$$

or, since $\varphi | C_+$ depends on g only through $g | R$,

$$\|\varphi | C_+\|^2 \leqslant \|\Phi\|^2 + \|g | R\|_2 (\|\Phi\| + \pi^{1/2} \|g | R\|_2).$$

Since the map $(g | R, \Phi) \mapsto \varphi | C^+$ is linear, the existence of W as a bounded operator from $L_2(R) \times \mathbf{H}(S)$ to $\mathbf{H}(C_+)$ follows. ∎

Now suppose that $\Phi \in \mathbf{H}_\infty(S)$ and let $\varphi = T_\lambda \Phi$. Then $W_\lambda^+(\Phi) = \varphi | C_+$, but by the definition of W in Lemma 6,

$$W(\lambda \varphi^3, \Phi) = \varphi | C_+.$$

Thus we have

$$W_\lambda^+(\Phi) = W(\lambda \varphi^3, \Phi). \tag{16}$$

Since W_λ^+ is continuous and $\mathbf{H}_\infty(S)$ is dense in $\mathbf{H}(S)$, (16) holds for all $\Phi \in \mathbf{H}(S)$ if $\Phi \mapsto W(\lambda \varphi^3, \Phi)$ extends to a smooth function from $\mathbf{H}(S)$ to $\mathbf{H}(C_+)$. To prove the latter, it suffices by Lemma 6 to prove that the function $\Phi \mapsto \varphi^3$ extends to a smooth function from $\mathbf{H}(S)$ to $L_2(R)$, or equivalently that $\Phi \mapsto \varphi$ extends to a smooth function from $\mathbf{H}(S)$ to $L_6(R)$. By Sobolev, this follows from Proposition 4.

Since (16) holds for all $\Phi \in \mathbf{H}(S)$ and $\Phi \mapsto W(\lambda \varphi^3, \Phi)$ is smooth from $\mathbf{H}(S)$ to $\mathbf{H}(C_+)$, the function W_λ^+ is smooth. By [2, Theorem 16], the inverse of W_λ^+ is boundedly Lipschitzian. Thus W_λ^+ is a diffeomorphism. ∎

In connection with this theorem, we note that in [8, lines 10–11] the expression "C^∞" should be replaced by "finite-energy," and the citation of [1] should be replaced by a citation of [2].

5. Geometry of the Solution Manifold

In this section we study the conformally invariant symplectic geometry of the finite-energy solution manifold of Eq. (12), and its relation to the scattering and wave transforms. We also describe Kähler structures on the finite-energy solution manifold that arise from Kähler structures on the spaces on "in" and "out" fields, and are invariant under the Poincaré group and scaling.

The group $\tilde{\mathbf{G}}$ has an action R_λ on $\mathbf{E}_\lambda$ given by

$$(R_\lambda(g^{-1})\varphi)(x) = \|dg_x\| \, \varphi(gx), \qquad g \in \tilde{\mathbf{G}}, \quad x \in \tilde{\mathbf{M}}.$$

Since $\tilde{\mathbf{G}}$ acts smoothly on $\tilde{\mathbf{M}}$ and $\|dg_x\|$ depends smoothly on (g, x), the action $R_\lambda : \tilde{\mathbf{G}} \times \mathbf{E}_\lambda \to \mathbf{E}_\lambda$ is continuous. Thus $T_\lambda^{-1} R_\lambda T_\lambda$ is a continuous action of $\tilde{\mathbf{G}}$ on $\mathbf{H}_\infty(S)$. As shown in [2], this action extends uniquely to a continuous action of $\tilde{\mathbf{G}}$ on $\mathbf{H}(S)$, which we denote by U_λ.

We use the definition of symplectic (resp. Kähler) manifold that involves only "weak" nondegeneracy of the symplectic (resp. Hermitian) form on each tangent space: a continuous bilinear form g on a topological vector space V is "weakly nondegenerate" if for any nonzero $u \in V$ there exists $v \in V$ such that $g(u, v) \neq 0$. Thus the Hilbert space $\mathbf{H}(S)$ becomes a symplectic manifold by identifying any tangent space $T_x\mathbf{H}(S)$ with $\mathbf{H}(S)$ and giving it the symplectic form $\mathrm{Im}\langle \cdot, \cdot \rangle_{-1}$. Similarly, the spaces $\mathbf{H}(C_\pm)$ are symplectic manifolds with symplectic form $\mathrm{Im}\langle \cdot, \cdot \rangle_{-1}$.

For each $g \in \tilde{\mathbf{G}}$ the map $U_\lambda(g)\colon \mathbf{H}(S) \to \mathbf{H}(S)$ is a symplectomorphism, that is, a diffeomorphism such that

$$\mathrm{Im}\langle dU_\lambda(g)u, dU_\lambda(g)v\rangle_{-1} = \mathrm{Im}\langle u, v\rangle_{-1},$$

where $u, v \in T_x\mathbf{H}(S)$. (This is stated in [8], and is also easily derived from Lemma 2 and Proposition 4 above.) We define the group actions $U_\lambda^\pm$ on $\mathbf{H}(C_\pm)$ by

$$U_\lambda^\pm(g) = W_\lambda^\pm U_\lambda(g)(W_\lambda^\pm)^{-1}.$$

THEOREM 7. *The maps $W_\lambda^\pm\colon \mathbf{H}(S) \to \mathbf{H}(C_\pm)$ are symplectomorphisms intertwining the group actions U_λ and $U_\lambda^\pm$. The map $S_\lambda\colon \mathbf{H}(C_-) \to \mathbf{H}(C_+)$ is a symplectomorphism intertwining the group actions U_λ^- and U_λ^+.*

Proof. The only nontrivial point is that the wave transforms $W_\lambda^\pm$ are symplectomorphisms. By continuity it suffices to show that

$$\mathrm{Im}\langle dW_\lambda^+(u), dW_\lambda^+(v)\rangle_{-1} = \mathrm{Im}\langle u, v\rangle_{-1}$$

for $x \in \mathbf{H}_\infty(S)$ and $u, v \in T_x\mathbf{H}(S)$ corresponding to vectors in $\mathbf{H}_\infty(S)$ under the identification of tangent vectors with elements of $\mathbf{H}(S)$. By Proposition 4, $T_\lambda(x+au)(q)$ is a smooth function of $(a, q) \in \mathbb{R} \times \tilde{\mathbf{M}}$. Let

$$\varphi(q) = \partial_a T_\lambda(x+au)(q)|_{a=0}, \qquad q \in \tilde{\mathbf{M}}.$$

Then by Eq. (12) it follows that

$$\begin{aligned}(\Box + 1)\varphi &= \partial_a\{(\Box+1)\, T_\lambda(x+au)\}|_{a=0}\\ &= \partial_a\{-\lambda T_\lambda(x+au)^3\}|_{a=0}\\ &= -3\lambda T_\lambda(x)^2\,\varphi.\end{aligned}$$

Moreover

$$u = (\varphi, \partial_\tau\varphi)|_{\tau=0}$$

and

$$dW_\lambda^+(u) = \partial_a W_\lambda^+(x+au)|_{a=0} = \varphi \mid C_+.$$

Similarly, if we let $\psi = \partial_a T_\lambda(x+av)|_{a=0}$, the function ψ satisfies

$$(\Box + 1)\psi + 3\lambda T_\lambda(x)^2\,\psi = 0,$$
$$(\psi, \partial_\tau\psi)|_{\tau=0} = v,$$

and

$$dW_\lambda^+(v) = \psi \mid C_+.$$

Thus by Theorem 1 and Lemma 2 we have

$$\begin{aligned}\mathrm{Im}\langle dW_\lambda^+(u), dW_\lambda^+(v)\rangle_{-1} &= \mathrm{Im}\langle \varphi \mid C_+, \psi \mid C_+\rangle_{-1}\\ &= \mathrm{Im}\langle(\varphi, \partial_\tau\varphi)|_{\tau=0}, (\psi, \partial_\tau\psi)|_{\tau=0}\rangle_{-1}\\ &= \mathrm{Im}\langle u, v\rangle_{-1}. \quad \blacksquare\end{aligned}$$

THEOREM 8. *The group action $U_\lambda^\pm\colon \tilde{\mathbf{G}} \times \mathbf{H}(C_\pm) \to \mathbf{H}(C_\pm)$ is continuous, and for each $g \in \tilde{\mathbf{G}}$ the map $U_\lambda^\pm(g)\colon \mathbf{H}(C_\pm) \to \mathbf{H}(C_\pm)$ is a symplectomorphism. If $g \in \tilde{\mathbf{P}}$ the map $U_\lambda^\pm(g)\colon \mathbf{H}(C_\pm) \to \mathbf{H}(C_\pm)$ is complex-linear and preserves the continuous sesquilinear form $\langle \cdot, \cdot \rangle_{-1}$ on $\mathbf{H}(C_\pm)$.*

Proof. The only statement that is not an immediate consequence of previous theorems is that for $g \in \tilde{\mathbf{P}}$ the map $U_\lambda^\pm(g)\colon \mathbf{H}(C_\pm) \to \mathbf{H}(C_\pm)$ is complex-linear and preserves $\langle \cdot, \cdot \rangle_{-1}$. The action of the group $\tilde{\mathbf{P}}$ on $\tilde{\mathbf{M}}$ preserves $C_\pm$. Thus if $\mathbf{\Phi} \in \mathbf{H}_\infty(C_\pm)$ and $g \in \tilde{\mathbf{P}}$,

$$(U_\lambda^\pm(g)\,\mathbf{\Phi})(x) = \|dg_x^{-1}\|\;\varphi(g^{-1}x) = (U^\pm(g)\,\mathbf{\Phi})(x)$$

for all $x \in C_\pm$, so $U_\lambda^\pm(g)\mathbf{\Phi} = U^\pm(g)\mathbf{\Phi}$. Since $U_\lambda^\pm(g)$ and $U^\pm(g)$ are continuous on $\mathbf{H}(C_\pm)$ and $\mathbf{H}_\infty(C_\pm)$ is dense in $\mathbf{H}(C_\pm)$, the restriction to $\tilde{\mathbf{P}}$ of $U_\lambda^\pm$ and $U^\pm$ are equal. The statement to be proved then follows from the fact that the representation $U^\pm$ of $\tilde{\mathbf{G}}$ on $\mathbf{H}_{-1}(C_\pm)$ is unitary. $\blacksquare$

A Kähler manifold is "flat" if the associated curvature tensor vanishes. The above theorem implies the following:

COROLLARY 9. *The symplectic structure $\mathrm{Im}\langle \cdot, \cdot \rangle_{-1}$ on $\mathbf{H}(S)$ is the imaginary part of the Kähler structures $g^\pm$ given by*

$$g_\pm(u, v) = \langle dW_\lambda^\pm u, dW_\lambda^\pm v\rangle_{-1}, \qquad u, v \in \mathbf{T}_x\mathbf{H}(S).$$

The Kähler structures $g_\pm$ are flat and are preserved by the action U_λ of the group $\tilde{\mathbf{P}}$ on $\mathbf{H}(S)$.

Proof. This is a straightforward consequence of Theorems 7 and 8. ∎

Note that the action U_λ of $\tilde{\mathbf{P}}$ on $\mathbf{H}(S)$ factors through to an action of $\mathbf{P}$. It is not presently known whether the Kähler structures $g_\pm$ are not only $\mathbf{P}$- but actually $\tilde{\mathbf{G}}$-invariant, or even invariant under the central element $\zeta \in \tilde{\mathbf{G}}$.

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Asymptotics of the Spectral Gap with Applications to the Theory of Simulated Annealing*

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0. Introduction

Let $\mathbf{M}$ be a compact, connected, N-dimensional, Riemmanian manifold with inner product $\langle \cdot, \cdot \rangle$, and let $U: \mathbf{M} \to R^1$ be a given smooth function. For each $\beta > 0$, define $\mathscr{L}_\beta$: $C^\infty(\mathbf{M}) \to C^\infty(\mathbf{M})$ by

$$[\mathscr{L}_\beta \phi] = e^{\beta U} \nabla \cdot (e^{-\beta U} \nabla \phi), \tag{0.1}$$

where we have used ∇ and $\nabla\cdot$, respectively, to denote the gradient and divergence operations corresponding to the Riemmanian structure on $\mathbf{M}$. Next, let μ_0 denote the normalized Reimmanian measure on $\mathbf{M}$, and define the probability measures u_β by

$$\mu_\beta(dx) = \frac{e^{-\beta U(x)}}{Z_\beta} \mu_0(dx) \qquad \text{with} \quad Z_\beta \equiv \int_{\mathbf{M}} e^{-\beta U(x)} \mu_0(dx). \tag{0.2}$$

Using $(\cdot, \cdot)_\beta$ to denote the inner product in $L^2(\mathbf{M}, \mu_\beta)$, one sees that

$$-(\phi, \mathscr{L}_\beta \psi)_\beta = \mathscr{E}_\beta(\phi, \psi) \equiv \int_{\mathbf{M}} \langle \nabla\phi, \nabla\psi \rangle \, d\mu_\beta \tag{0.3}$$

for all ϕ, $\psi \in C^\infty(\mathbf{M})$. In particular, $\mathscr{L}_\beta$ is symmetric and non-positive an operator on $L^2(\mathbf{M}, \mu_\beta)$. In addition, it is well-known and easy to prove that $\mathscr{L}_\beta$ is essentially self-adjoint and that its self-adjoint extensions $\overline{\mathscr{L}_\beta}$ has 0 as a simple, isolated eigenvalue for which the constants are eigenfunctions.

We now define $\lambda(\beta)$ to be the size of the *gap* between 0 and the rest of the spectrum of $\overline{\mathscr{L}_\beta}$. That is,

$$\lambda(\beta) = \inf\{\mathscr{E}_\beta(\phi, \phi) : \phi \in C^\infty(\mathbf{M}) \text{ and } \operatorname{var}_\beta(\phi) = 1\}, \tag{0.4}$$

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