Instructions: Work 2 out of 3 problems in each of the 3 parts for a total of 6 problems.
(1) Let $\Omega$ be an open subset of $\mathbb{R}^d$ and let $C(\Omega) = \{ f : \Omega \to \mathbb{R} \mid f \text{ is continuous} \}$ with the norm,
$$\|f\|_{C(\Omega)} = \sup_{x \in \Omega} |f(x)|.$$ 
Prove that $C(\Omega)$ is a Banach space.

(2) Let $\Omega$ be a bounded domain in $\mathbb{R}^2$, $d \geq 1$, with smooth boundary.
(a) Use the divergence theorem to derive Green’s identity,
$$\int_{\Omega} \Delta u v = -\int_{\Omega} \nabla u \cdot \nabla v + \int_{\partial \Omega} (\nabla u \cdot \mathbf{n}) v,$$
where $u$ and $v$ are smooth scalar-valued functions on $\Omega$, and $\mathbf{n}$ is the outward unit normal vector.
(b) Consider the Cauchy problem,
$$\begin{cases}
\partial_t u = \Delta u + cu & \text{for } (t, x) \in (0, \infty) \times \Omega, \\
u(t, x) = 0 & \text{for } (t, x) \in (0, \infty) \times \partial \Omega, \\
u(0, x) = g(x) & \text{for } x \in \Omega,
\end{cases}$$
on a bounded domain $\Omega \subseteq \mathbb{R}^d$ having a smooth boundary. Here, $c$ is a positive constant. Suppose $u_1$ and $u_2$ are two smooth solutions of the above Cauchy problem with different initial conditions $g_1$ and $g_2$. Show that if $g_1$ and $g_2$ are “close” in $L^2(\Omega)$ then the solutions $u_1$ and $u_2$ are also close in $L^2(\Omega)$ at any later time $t > 0$. Derive an estimate of how close. (Green’s identity and Gronwall’s inequality will be useful here.)

(3) Let $A(t)$ be a continuous function from $t$ in $\mathbb{R}$ to the space of square, real-valued matrices.
(a) Show that for every solution of the (non-autonomous) linear system, $\dot{x} = A(t)x$, we have
$$\|x(t)\| \leq \|x(0)\| e^{\int_0^t \|A(s)\| \, ds},$$
where $\|A(s)\|$ is the operator norm and $\|x(t)\|$ is the usual Euclidean norm.
(b) Show that if $\int_0^t \|A(s)\| \, ds < \infty$ then every solution, $x(t)$, has a finite limit as $t \to \infty$. 

Part 1
PART 2

(1) (a) Find the entropy solution to the Burgers’ equation \( u_t + uu_x = 0 \) with the initial datum

\[
g(x) = \begin{cases} 
1 & \text{if } x \leq 0, \\
1 - x & \text{if } 0 \leq x \leq 1, \\
0 & \text{if } x \geq 1.
\end{cases}
\]

(b) Consider the Burgers’ equation with source term 1 with the initial datum \( x \):

\[
u_t + uu_x = 1, \quad u(t = 0) = x.
\]

Find the equation for the characteristics and also find an explicit formula for the solution of this initial value problem.

(2) Let \( f \in C^2_c(\mathbb{R}^3) \) be given. Define for \( x \in \mathbb{R}^3 \)

\[
u(x) = \int_{\mathbb{R}^3} \Phi(x - y)f(y)dy
\]

where \( \Phi(x) = \frac{1}{4\pi|x|} \). Prove that \(-\Delta u = f\) in \( \mathbb{R}^3 \). You can use the fact \( u \in C^2(\mathbb{R}^3) \) without a proof.

(3) Let \( u \) be a classical solution of the following initial boundary value problem:

\[
u_t = u_{xx}, \quad \text{in } (a, b) \times (0, T)
\]

\[
u(a, t) = \nu(b, t) = 0
\]

\[
u(x, 0) = \nu_0(x)
\]

where \( \nu_0 \) is a continuous function.

(a) Show that the solutions are unique.

(b) Show that there exists a constant \( \alpha > 0 \) such that

\[
\|\nu(\cdot, t)\|_{L^2}^2 \leq e^{-\alpha t}\|\nu_0\|_{L^2}^2.
\]
PART 3

(1) Let $U$ be the open unit ball in $\mathbb{R}^d$.

(a) Let $u(x) = |x|^{-\alpha}$.

For which values of $\alpha > 0$, $d \geq 1$, and $p > 1$ does $u$ belong to $W^{1,p}(U)$?

(b) Show that $u(x) = \log \log \left(1 + |x|^{-1}\right)$

belongs to $W^{1,2}(U)$ but does not belong to $L^\infty(U)$.

(2) Let $U = (0,1)^2$, the unit square in $\mathbb{R}^2$. Can the Lax-Milgram theorem be applied to the bilinear form, $B[u, v] : H^1_0(U) \times H^1_0(U) \to \mathbb{R}$, defined by

$$B[u, v] = \int_0^1 \int_0^1 \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2} - \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} \, dx_1 \, dx_2.$$ 

(3) Suppose $u \in C^2(U) \cap C(\overline{U})$ and let

$$Lu = \sum_{i,j=1}^n a^{ij} u_{x_i x_j},$$

where the coefficient, $a^{ij}$, are continuous and satisfy the uniform ellipticity condition. Prove the weak maximum principle; namely, that if $Lu \leq 0$ then

$$\max_{\overline{U}} u = \max_{\partial U} u.$$
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Part 1

(1) A fundamental solution to the autonomous linear system, \( \dot{x} = Ax \), is a nonsingular matrix-valued function, \( \Phi: \mathbb{R} \rightarrow M^{d \times d} \), with \( \Phi'(t) = A\Phi(t) \).

(a) Show that \( \Psi(t) = e^{At} \) is a fundamental solution satisfying \( \Psi(0) = I \), the identity matrix. (You may use standard facts about \( e^{At} \) without proof.)

(b) Show that \( x(t) = \Phi(t)\Phi(0)^{-1}x_0 \) is a solution to the IVP, \( \dot{x} = Ax, \ x(0) = x_0 \).

(c) Show that any fundamental solution is of the form, \( \Phi(t) = e^{At}M \), for some non-singular matrix \( M \).

(d) Consider the nonhomogeneous linear system, \( \dot{x} = Ax + b(t) \),

where \( b \) is continuous in time. (So \( b \) can vary with time, but \( A \) cannot.) Show that

\[
x(t) = \Phi(t)\Phi(0)^{-1}x_0 + \int_{0}^{t} \Phi(t)\Phi^{-1}(s)b(s)\,ds
\]

is a solution to the IVP, \( \dot{x} = Ax + b(t), \ x(0) = x_0 \).
(2) (a) Consider the linear system of ODEs,
\[ \dot{y}_1 = -y_1, \quad \dot{y}_2 = 2y_2, \]
which has the origin as the only equilibrium point. Determine the explicit solution to this system given the initial condition, \( y(0) = a = (a_1, a_2) \). What are the stable and unstable manifolds for this system? (One or both might be empty.)

(b) Now consider the perturbed, nonlinear system,
\[ \dot{x}_1 = -x_1, \quad \dot{x}_2 = 2x_2 - 5\epsilon x_1^3, \]
which also has the origin as the only equilibrium point. Determine the explicit solution to this system given the initial condition, \( x(0) = a = (a_1, a_2) \). (One method: let \( y_1, y_2 \) be the solution to the linear system in (a) with initial condition, \( (y_1, y_2) = (1, 1) \), assume that \( x_2 = c_1 y_2 + c_2 y_1^3 \), and then determine \( c_1 \) and \( c_2 \).)

(c) What is the stable manifold for the system in (b)?

(3) Consider the system of equations,
\[ \begin{cases} \dot{x}_1 = x_2 - x_1 f(x_1, x_2), \\ \dot{x}_2 = -x_1 - x_2 f(x_1, x_2), \end{cases} \]
where \( f \) lies in \( C^1(\mathbb{R}^2) \).

(a) Show that if \( f \) is positive in some neighborhood of the origin then the origin is an asymptotically stable equilibrium point.

(b) Show that if \( f \) is negative in some neighborhood of the origin then the origin is an unstable equilibrium point.

**Hint** for both parts: Construct a Lyapunov function.
PART 2

(1) Let \( g \) be a bounded, continuous function on \( \mathbb{R}^n \). For \((x,t) \in \mathbb{R}^n \times (0, +\infty)\) define

\[
    u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y, t) g(y) dy,
\]

where \( \Phi \) is the fundamental solution of the heat equation,

\[
    \Phi(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}.
\]

Let \( x_0 \in \mathbb{R}^n \). Prove that

\[
    \lim_{(x,t) \to (x_0, 0)} u(x, t) = g(x_0).
\]

**Hint:** You can use the fact that \( \int_{\mathbb{R}^n} \Phi(x, t) dx = 1 \) for every \( t > 0 \) without proving it. You can also use without proving it the fact that for every \( r_0 > 0 \),

\[
    \lim_{(x,t) \to (x_0, 0)} \int_{|y-x_0| > r_0} \Phi(x-y, t) dy = 0.
\]

In other words, \( \Phi(\cdot, t) \) has mass one and as \( (x, t) \to (x_0, 0) \) all the mass concentrate around the the point \( x_0 \).

(2) Let \( \Omega \subset \mathbb{R}^n \) be a bounded open domain with smooth boundary and define the energy

\[
    E(w) = \frac{1}{2} \int_{\Omega} |\nabla w|^2 dx - \int_{\partial \Omega} hw,
\]

where \( h \) is a smooth function defined on the boundary of \( \Omega \). Suppose \( u \in C^2(\overline{\Omega}) \) satisfies

\[
    E(u) \leq E(w) \text{ for all } w \in C^2(\overline{\Omega}).
\]

What PDE is \( u \) satisfying? What are the boundary conditions? Prove it.

**Hint:** Start by considering perturbation \( u + \epsilon v \) where \( v \in C^2(\Omega) \). This will give you the PDE. Then consider perturbation \( u + \epsilon v \) where \( v \in C^2(\Omega) \) to get the boundary condition.

(3) Let \( u \) and \( v \) belong to \( C^2(U_T) \cap C(\overline{U_T}) \) and satisfy

\[
    u_t = \Delta u + f, \\
    v_t = \Delta v + g.
\]

Show that if \( u \geq v \) on the parabolic boundary \( \Gamma_T \) and \( f \geq g \) in \( U_T \) then \( u \geq v \) in all of \( \overline{U_T} \). This is called a comparison principle.
Part 3

(1) (a) Prove or disprove the following:

Let $U$ be a bounded, open subset of $\mathbb{R}^2$. If $u \in W^{1,2}(U)$, then $u \in L^\infty(U)$ with the estimate

$$\|u\|_{L^\infty(U)} \leq C\|u\|_{W^{1,2}(U)}$$

where $C$ does not depend on $u$.

(b) Let $U$ be a bounded, open set in $\mathbb{R}^n$ with smooth boundary. Show that

$$\|Du\|_{L^2(U)}^2 \leq C\|u\|_{L^2(U)}\|D^2u\|_{L^2(U)}$$

for all $u \in H^1_0(U) \cap H^2(U)$ where $C$ does not depend on $u$.

(2) Consider the following Dirichlet problem

$$-\Delta u + \mu u = f \quad \text{in } U$$

$$u = 0 \quad \text{on } \partial U$$

where $\mu$ is a given constant. $U$ is a bounded, open subset of $\mathbb{R}^n$.

(a) Show the existence of a weak solution $u \in H^1_0(U)$ of the above problem for $\mu > 0$.

(b) Show the existence of a weak solution $u \in H^1_0(U)$ of the above problem for $\mu = 0$.

(c) Discuss the problem when $\mu < 0$.

(3) Consider the Poisson equation with Dirichlet boundary condition:

$$\begin{cases}
-\Delta u = f & \text{in } U \\
u = 0 & \text{on } \partial U
\end{cases}$$

where $U$ is a bounded, open subset of $\mathbb{R}^n$ and $f \in L^2(U)$. We know there exists a weak solution $u \in H^1_0(U)$. Prove that $u \in H^2_{\text{loc}}(U)$. 