Horizontally Homothetic Submersions and Nonnegative Curvature

YE-LIN OU & FREDERICK WILHELM

ABSTRACT. We show that any horizontally homothetic submersion from a compact manifold of nonnegative sectional curvature is a Riemannian submersion.

The lack of examples of manifolds with positive sectional curvature has been a major obstacle to their classification. Apart from $S^n$, every known compact manifold with positive sectional curvature is constructed as the image of a Riemannian submersion of a compact manifold with nonnegative sectional curvature.

Here we study a generalization of Riemannian submersions called “horizontally homothetic” submersions. For this larger class of submersions, the analog of O’Neill’s horizontal curvature equation has exactly one extra term ([22] and [29]). This extra term is always nonnegative and can potentially be positive. So the horizontal curvature equation suggests that a single horizontally homothetic submersion is more likely to have a positively curved image than a given Riemannian submersion. Since horizontally homothetic submersions are (a priori) more abundant, one is lead to believe that they have much more potential for creating positive curvature than Riemannian submersions. Unfortunately, our main result suggests that this is an illusion.

Theorem 1 (Main Theorem). Every horizontally homothetic submersion from a compact Riemannian manifold with nonnegative sectional curvature is a Riemannian submersion (up to a change of scale on the base space).

This generalizes the result in [38] that any horizontally homothetic submersion of a round sphere with 1-dimensional fibers is a Riemannian submersion.

In the special case of maps from $\mathbb{R}^n \to \mathbb{R}$, horizontally homothetic submersions appeared as solutions of the so-called “Infinity Laplace Equation” introduced by Aronsson ([1]) in his study of “optimal” Lipschitz extension of functions in the late 1960s. Nowadays these solutions are called “infinity-harmonic functions,” and have been the subject of a great deal of current research (see e.g. [2], [6],

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[7], [8], [9], [10], [11], [15], [16], [17], [18], [19], [20], [25], [26], [27], [28], [30], [31], [34], and the references therein) including applications in the areas of image processing (see e.g. [13], [40]), mass transfer (see e.g. [19]), and shape metamorphisms (see e.g. [14]).

In full generality, horizontally homothetic submersions arose in the study of p-harmonic morphisms in [3], [5], [12], [22], [23], [29], [32], [36], [37], [38], [42], and [41]. There are many familiar examples of horizontally homothetic submersions from incomplete manifolds with nonnegative curvature that are not Riemannian submersions. Before elaborating, we recall the definition from [3] or [5].

**Definition 2.** A submersion of Riemannian manifolds, \( \pi : M \to B \), is called horizontally homothetic if and only if there is a smooth function \( \lambda : M \to \mathbb{R} \) with vertical gradient so that for all horizontal vectors \( x \) and \( y \)

\[
\lambda^2(x, y)_M = \langle d\pi(x), d\pi(y) \rangle_B.
\]

\( \lambda \) is called the dilation of \( \pi \).

If the condition about the gradient of \( \lambda \) being vertical is dropped, then \( \pi \) is called a horizontally conformal submersion.

The power of Riemannian submersions for creating positive curvature stems from O'Neill's horizontal curvature equation, which implies that a Riemannian submersion \( \pi : M \to N \) does not decrease the curvature of horizontal planes. In fact, the sectional curvature of a horizontal plane spanned by orthonormal vectors \( \{x, y\} \) in \( M \) is related to the curvature of span \( \{d\pi(x), d\pi(y)\} \) by

\[
sec_N(d\pi(x), d\pi(y)) = sec_M(x, y) + 3|A_x y|^2.
\]

Here \( A \) is O'Neill's "integrability" tensor for the horizontal distribution

\[
A_{xy} = \frac{1}{2} [X, Y]_{\text{vert}},
\]

where \( X \) and \( Y \) are arbitrary extensions of \( x \) and \( y \) to horizontal vector fields. Thus if \( M \) has nonnegative curvature, then

\[
sec_N(d\pi(x), d\pi(y)) > 0
\]

if

either \( sec_M(x, y) > 0 \) or \( A_{xy} \neq 0 \).

The generalization of O'Neill's equation for horizontally conformal submersions was discovered independently by Kasue and Washio in [29] and Gudmundsson in [22]. If \( \pi : M \to N \) is horizontally homothetic, then the sectional curvature of a horizontal plane spanned by orthonormal vectors \( \{x, y\} \) in \( M \) is related
to the curvature of span \(d\pi(x), d\pi(y)\) by

\[(2) \quad \lambda^2 \sec_n(d\pi(x), d\pi(y)) = \sec_M(x, y) + 3|A(x, y)|^2 + |\text{grad} \ln \lambda|^2.\]

([22]).

Thus if \(M\) has nonnegative curvature, then \(\sec_n(d\pi(x), d\pi(y)) > 0\) if

either \(\sec_M(x, y) > 0, A_{xy} \neq 0, \) or \(|\text{grad} \lambda|^2 > 0.\)

Thus without our theorem, one would naturally suspect that horizontally homothetic submersions have much more potential for creating positive curvature than Riemannian submersions.

There are many familiar examples of horizontally homothetic submersions of incomplete manifolds with nonnegative sectional curvature that are not Riemannian submersions.

**Example 3.** The radial projection of \(\mathbb{R}^n \setminus \{0\}\) onto \(S^{n-1}\) is horizontally homothetic (see [23] and also [5] for details).

**Example 4.** The metric projection of \(S^n \setminus \{\text{north pole, south pole}\}\) onto the equator is horizontally homothetic (see [23] and also [5] for details).

**Example 5.** View \(S^{p+q+1}\) as the join \(S^p \ast S^q\). Then the metric projection of \(S^{p+q+1} \setminus S^q\) onto \(S^q\) is horizontally homothetic.

For a complete example we need negative curvature.

**Example 6.** Let \(H^n\) be hyperbolic space with the upper half space model,

\[H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}.\]

Then the projection

\[(x_1, \ldots, x_n) \rightarrow (x_1, \ldots, x_{n-1})\]

is a horizontally homothetic submersion onto the Euclidean space \(\mathbb{R}^{n-1}\) (see [23] and also [5] for details).

**Example 7.** While the projection of a warped product onto its first factor is a Riemannian submersion, the projection onto the second factor is horizontally homothetic. All of the preceding examples can be viewed as the projection of a warped product onto its second factor (see [39]). It follows from Theorem 1.3 in [43] that all complete warped products of nonnegative sectional curvature are isometric to Riemannian products. Thus our main theorem can be viewed as a generalization of that result.

There are also examples of horizontally homothetic submersions of incomplete manifolds with nonnegative sectional curvature that are neither Riemannian submersions nor warped products. For these we use projective spaces.
Example 8. There are multiple generalizations of Example 4 and Example 5 that involve $\mathbb{C}P^n$, $\mathbb{H}P^n$, and $\mathbb{C}aP^2$. For example, metric projection of $\mathbb{C}P^n \setminus \{pt\}$ onto the copy of $\mathbb{C}P^{n-1}$ at maximal distance from the point is horizontally homothetic.

In the flat case we have the following example.

Example 9. A horizontally homothetic submersion $u : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the “Infinity Laplace Equation” (see, e.g., [1], [2], [8], [16], and the references therein):

$$\langle \text{grad } u, \text{grad } |\text{grad } u|^2 \rangle = 0.$$ 

The solutions are called infinity-harmonic functions. Aronsson showed that all $C^2$, globally defined infinity harmonic functions $\mathbb{R}^2 \rightarrow \mathbb{R}$ are affine [1]. Although there are nontrivial infinity-harmonic functions on open subsets of $\mathbb{R}^n$, it is not known whether all smooth globally-defined infinity-harmonic functions on $\mathbb{R}^n$ are affine.

Horizontally homothetic submersions have played an important role in the study of $p$-harmonic morphisms. For example, by combining results in [3], [4], [12], and [42] we have the following result.

Theorem 10. Let $m > n \geq 2$ and $\varphi : (M^m, g) \rightarrow (N^n, h)$ be a horizontally conformal submersion.

(I) If $p = n$, then $\varphi$ is a $p$-harmonic map if and only if $\{\varphi^{-1}(y)\}_{y \in N}$ is a minimal foliation of $(M, g)$ of codimension $n$.

(II) If $p \neq n$, then any two of the following conditions imply the third:

(a) $\varphi$ is a $p$-harmonic map,

(b) $\{\varphi^{-1}(y)\}_{y \in N}$ is a minimal foliation of $(M, g)$ of codimension $n$,

(c) $\varphi$ is horizontally homothetic.

For applications of horizontally homothetic submersions in classifying $p$-harmonic morphisms and biharmonic morphisms between certain model spaces see [36] and [37].

Although horizontally homothetic submersions appear to be a much broader class than Riemannian submersions, the presence of a nonconstant smooth function with vertical gradient imposes a fair amount of extra structure. For example, it is easy to show the following result.

Proposition 11. Let $\pi : M \rightarrow B$ be a horizontally homothetic submersion with dilation $\lambda$ and let $r$ be a regular value of $\lambda$ so that $\lambda^{-1}(r)$ is nonempty. Then

$$\pi|_{\lambda^{-1}(r)} : \lambda^{-1}(r) \rightarrow B$$

is a Riemannian submersion with respect to the intrinsic metric on $\lambda^{-1}(r)$. 
The levels of $\lambda$ also force some vanishings of the $A$-tensor of $\pi$, but before we can elaborate we must refine the definition of the $A$-tensor in the horizontally homothetic case.

In the Riemannian case, O'Neill defined the two fundamental tensors,

$$A_ZW = (\nabla_{Z_{\text{horiz}}} W^{\text{vert}})_{\text{horiz}} + (\nabla_{Z_{\text{horiz}}} W^{\text{vert}})_{\text{horiz}}$$

and

$$T_ZW = (\nabla_{Z_{\text{vert}}} W^{\text{horiz}})_{\text{horiz}} + (\nabla_{Z_{\text{vert}}} W^{\text{horiz}})_{\text{vert}}.$$

These tensors are also important for horizontally homothetic submersions, except that it is convenient to modify the definition of the $A$-tensor to

$$A_ZW = \frac{1}{2} [Z_{\text{horiz}}, W_{\text{horiz}}]_{\text{vert}} + (\nabla_{Z_{\text{horiz}}} W^{\text{vert}})_{\text{horiz}, z^\perp},$$

where the last superscript $\text{horiz}, z^\perp$ indicates that we are taking the component that is horizontal and perpendicular to $Z$. In the Riemannian case this definition of the $A$-tensor coincides with O'Neill's since for a Riemannian submersion

$$\frac{1}{2} [Z_{\text{horiz}}, W_{\text{horiz}}]_{\text{vert}} = (\nabla_{Z_{\text{horiz}}} W^{\text{horiz}})_{\text{vert}}$$

and $(\nabla_{Z_{\text{horiz}}} W^{\text{vert}})_{\text{horiz}}$ is already perpendicular to $Z$. On the other hand, we will show that for horizontally homothetic submersions, $\nabla_{Z_{\text{horiz}}} Z_{\text{horiz}}$ has a component that is proportional to $\text{grad} \lambda$ and (dually) $\nabla_{Z_{\text{horiz}}} \text{grad} \lambda$ is proportional to $Z_{\text{horiz}}$.

We assume that the reader has a working knowledge of O'Neill's foundational paper [35]. We use superscripts $\text{horiz}$ and $\text{vert}$ on vectors to denote the horizontal and vertical parts. Similarly $Z_{\text{grad} \lambda}$ stands for the component of $Z$ in the direction of $\text{grad} \lambda$; $Z_{\text{grad} \lambda, \perp}$ stands for the component of $Z$ that is perpendicular to $\text{grad} \lambda$; and more generally, $w_{Z, \perp}$ stands for the component of $w$ that is perpendicular to $Z$. We use

$$D_\nu (f)$$

for the derivative of $f$ in the direction of $\nu$.

For the moment, we study an arbitrary horizontally homothetic submersion

$$\pi : M \to B$$

with dilation $\lambda$. Later we will add the curvature and compactness hypotheses. Let

$$\rho = -\ln \lambda.$$

Then the gradient of $\rho$ is also vertical.
Lemma 12. Let $\gamma$ be a geodesic in $B$, and let $X$ be the horizontal lift of $\gamma$, defined on all of the horizontal lifts of $\gamma$. Then

$$
\begin{bmatrix}
X, \frac{\text{grad } \rho}{|\text{grad } \rho|^2}
\end{bmatrix}
$$

is vertical.

and

$$
\left< \begin{bmatrix}
X, \frac{\text{grad } \rho}{|\text{grad } \rho|^2}
\end{bmatrix}, \text{grad } \rho \right> = 0.
$$

In fact, for any horizontal vector fields $X$ and $Y$,

$$
\langle [X, Y], \text{grad } \rho \rangle = 0.
$$

Moreover, if $V$ is any vertical field that is perpendicular to $\text{grad } \rho$, then

$$
\langle [X, V], \text{grad } \rho \rangle = 0.
$$

Proof: Since $X$ is basic horizontal,

$$
\begin{bmatrix}
X, \frac{\text{grad } \rho}{|\text{grad } \rho|^2}
\end{bmatrix}
$$

is vertical.

For the second statement, note that

$$
\begin{aligned}
\left< \begin{bmatrix}
X, \frac{\text{grad } \rho}{|\text{grad } \rho|^2}
\end{bmatrix}, \text{grad } \rho \right> &= D_{[X, \text{grad } \rho]} \left( \frac{1}{|\text{grad } \rho|^2} \right)(\rho) \\
&= D_X \left( D_{\text{grad } \rho} \frac{1}{|\text{grad } \rho|^2} (\rho) \right) - D_{\text{grad } \rho} \frac{1}{|\text{grad } \rho|^2} (D_X (\rho)) \\
&= D_X (1) - D_{\text{grad } \rho} \frac{1}{|\text{grad } \rho|^2} (0) = 0.
\end{aligned}
$$

In the second term of the third equality we have used the fact that $D_X (\rho) = 0$ for any horizontal vector field $X$, since the dilation $\lambda$ and hence $\rho$ is constant along any horizontal curve.

$\langle [X, Y], \text{grad } \rho \rangle = 0$ and $\langle [X, V], \text{grad } \rho \rangle = 0$ because $X, Y,$ and $V$ are tangent to submanifolds (the levels of $\lambda$) that are perpendicular to $\text{grad } \lambda$.

End Proof.

Lemma 13. For any horizontal vector $X$,

$$
\nabla_X \frac{\text{grad } \rho}{|\text{grad } \rho|^2} = X - \frac{D_X (\text{grad } \rho, \text{grad } \rho)}{2 (\text{grad } \rho, \text{grad } \rho)^2} \text{grad } \rho + \frac{1}{2} \begin{bmatrix}
X, \frac{\text{grad } \rho}{|\text{grad } \rho|^2}
\end{bmatrix}.
$$

In particular, the $A$-tensor satisfies

$$
A \cdot \text{grad } \rho = 0.
$$
Proof. To compute $\langle \nabla_X (\text{grad} \rho \mid \text{grad} \rho \mid^2), X \rangle$ we can now use the Koszul formula to get

\[
2 \left\langle \nabla_X \frac{\text{grad} \rho}{|\text{grad} \rho|^2}, X \right\rangle_M = D_{\text{grad} \rho \mid \text{grad} \rho \mid^2} \langle X, X \rangle_M
\]

\[
= D_{\text{grad} \rho \mid \text{grad} \rho \mid^2} \langle e^{2\rho} (d\pi(X), d\pi(X))_B \rangle
\]

\[
= \frac{\langle d\pi(X), d\pi(X) \rangle_B}{|\text{grad} \rho|^2} D_{\text{grad} \rho} (e^{2\rho})
\]

\[
= \frac{\langle d\pi(X), d\pi(X) \rangle_B}{|\text{grad} \rho|^2} 2e^{2\rho} |\text{grad} \rho|^2
\]

\[
= 2e^{2\rho} \langle d\pi(X), d\pi(X) \rangle_B = 2 \langle X, X \rangle_M.
\]

So

\[
(4) \quad \left\langle \nabla_X \frac{\text{grad} \rho}{|\text{grad} \rho|^2}, X \right\rangle_M = \frac{\langle \nabla_X \frac{\text{grad} \rho}{|\text{grad} \rho|^2}, X \rangle}{|X|^2} = \langle X, X \rangle_M \frac{X}{|X|^2} = X.
\]

Similarly, if $Z$ is any basic horizontal field that is perpendicular to $X$, then the previous lemma and the Koszul formula give

\[
2 \left\langle \nabla_X \frac{\text{grad} \rho}{|\text{grad} \rho|^2}, Z \right\rangle = 0.
\]

If $V$ is any vertical field that is perpendicular to $\text{grad} \rho \mid \text{grad} \rho \mid^2$ and $[X, \text{grad} \rho \mid \text{grad} \rho \mid^2]$, then the Koszul formula gives

\[
2 \left\langle \nabla_X \frac{\text{grad} \rho}{|\text{grad} \rho|^2}, V \right\rangle = 0.
\]

In the direction of $\text{grad} \rho \mid \text{grad} \rho \mid^2$, we have

\[
2 \left\langle \nabla_X \frac{\text{grad} \rho}{|\text{grad} \rho|^2}, \frac{\text{grad} \rho}{|\text{grad} \rho|^2} \right\rangle = D_X \frac{\langle \text{grad} \rho, \text{grad} \rho \rangle}{(\text{grad} \rho, \text{grad} \rho)^2}
\]

\[
= D_X (\text{grad} \rho, \text{grad} \rho)^{-1}
\]

\[
= -\langle \text{grad} \rho, \text{grad} \rho \rangle^{-2} D_X (\text{grad} \rho, \text{grad} \rho)
\]

\[
= -\frac{D_X (\text{grad} \rho, \text{grad} \rho)}{(\text{grad} \rho, \text{grad} \rho)^2}.
\]
So

\[ \left\langle \nabla_X \frac{\text{grad } \rho}{|\text{grad } \rho|^2}, \frac{\text{grad } \rho}{|\text{grad } \rho|} \right\rangle \frac{\text{grad } \rho}{|\text{grad } \rho|} = -\frac{D_X \langle \text{grad } \rho, \text{grad } \rho \rangle}{2 (\text{grad } \rho, \text{grad } \rho)^2} \text{grad } \rho. \]  

In case \([X, \text{grad } \rho / |\text{grad } \rho|^2] = 0\), we are done. If not, take

\[ W = \frac{[X, \text{grad } \rho / |\text{grad } \rho|^2]}{|[X, \text{grad } \rho / |\text{grad } \rho|^2]|}. \]

Then

\[ 2 \left\langle \nabla_X \frac{\text{grad } \rho}{|\text{grad } \rho|^2}, W \right\rangle = \left\langle \left[ X, \frac{\text{grad } \rho}{|\text{grad } \rho|^2} \right], W \right\rangle. \]

From (4), (5) and (6) we obtain equation (3).

The statement about the \(A\)-tensor follows from its (new) definition and the fact that \((\nabla_X \text{grad } \rho / |\text{grad } \rho|^2)_{\text{horiz}} = X\).

Since \(\rho\) is constant along any horizontal geodesic we have the following result.

**Proposition 14.** If the global maximum or minimum of \(\rho\) occurs at \(p\), then it also occurs along the entire length of any horizontal curve passing through \(p\).

In fact, we have the following result.

**Lemma 15.** If \(r\) is a regular point of \(\rho\), then so is the entire length of any horizontal curve passing through \(r\).

**Proof.** Let \(\gamma\) be a curve in the base. Then the horizontal lifts of \(\gamma\) give a family of diffeomorphisms

\[ H_t : \pi^{-1}(\gamma(0)) \rightarrow \pi^{-1}(\gamma(t)), \quad H_t(x) = \tilde{\gamma}(t), \]

where \(\tilde{\gamma}\) is the horizontal lift of \(\gamma\) starting at \(x \in \pi^{-1}(\gamma(0))\). These were studied as early as 1960 in [24] and were called “Holonomy Displacement” maps in [21]. Since a horizontal curve stays in a fixed level of \(\rho\), it follows that for any \(x \in \pi^{-1}(\gamma(0))\), \(\rho \circ H_t(x)\) is independent of \(t\). Thus for a regular point \(r\) of \(\rho\)

\[ d\rho((dH_t)_r(\text{grad } \rho)) \]

is independent of \(t\). Since \(\text{grad } \rho\) is nonzero at \(r\), it follows that

\[ d\rho((dH_t)_r(\text{grad } \rho)) \neq 0 \]

for all \(t\). In particular, \(\text{grad } \rho\) is nonzero for all \(t\). \(\square\)
Generalizing Lemma 2.2 in [38] we have the following result.

**Proposition 16.** Let \( \tilde{y} \) be a horizontal lift of a geodesic \( y \) in \( B \). Then, \( \tilde{y} \) is an intrinsic geodesic of its level set of \( \rho \), and a geodesic of \( M \) only if the level is critical.

**Proof.** Let \( X \) be the horizontal lift of \( y' \), and let \( Z \) be any vertical field that is tangent to a level set of \( \rho \). Then using the Koszul formula

\[
2(\nabla_X X, Z) = -D_Z (X, X) - \langle [X, Z], X \rangle + \langle [Z, X], X \rangle.
\]

All three of these terms are 0. Indeed \( D_Z (X, X) = 0 \), since \( Z \) is tangent to a level of \( \rho \) and \( X \) is basic horizontal and \( \langle [X, Z], X \rangle = 0 \), since \( X \) is basic horizontal.

If \( Z \) is horizontal, then by the Koszul formula

\[
2(\nabla_X X, Z) = 2D_X (X, Z) - D_Z (X, X) - \langle [X, Z], X \rangle + \langle [Z, X], X \rangle,
\]

and the right hand side is 0, since it is equal to a multiple of

\[
2(\nabla_{d\pi (X)} d\pi (X), d\pi (Z)),
\]

which is 0, since \( X \) is a lift of a geodesic field.

It follows that \( \tilde{y} \) is an intrinsic geodesic for its level set of \( \rho \).

On the other hand, since \( X \) is basic horizontal

\[
\left\langle \nabla_X X, \frac{\text{grad } \rho}{| \text{grad } \rho |^2} \right\rangle = -\left\langle X, \nabla_X \frac{\text{grad } \rho}{| \text{grad } \rho |^2} \right\rangle = -\langle X, \nabla_{\text{grad } \rho / | \text{grad } \rho |^2} X \rangle
\]

\[
= -\frac{1}{2} D_{\text{grad } \rho / | \text{grad } \rho |^2} (X, X)
\]

\[
= -\frac{1}{2} D_{\text{grad } \rho / | \text{grad } \rho |^2} (e^{2\rho} \langle d\pi (X), d\pi (X) \rangle_B)
\]

\[
= -\frac{2}{2 | \text{grad } \rho |^2} e^{2\rho} D_{\text{grad } \rho} (\rho)
\]

\[
= -e^{2\rho} \langle d\pi (X), d\pi (X) \rangle_B
\]

\[
= -\langle X, X \rangle_M
\]

and hence \( \nabla_X X \) is nonzero wherever \( \text{grad } \rho \) is nonzero. \( \square \)

**Lemma 17.**

\[
T_{\text{grad } \rho / | \text{grad } \rho |^2} X = -\frac{D_X (\text{grad } \rho, \text{grad } \rho)}{2(\text{grad } \rho, \text{grad } \rho)} \left( \frac{\text{grad } \rho}{| \text{grad } \rho |^2} \right) + \frac{1}{2} | \text{grad } \rho | \left[ \frac{\text{grad } \rho}{| \text{grad } \rho |^2}, X \right].
\]
Proof. Since $\langle \text{grad } \rho \| \text{grad } \rho \rangle^2, X \rangle$ is perpendicular to grad $\rho$, we use Lemma 13 to get

\[
\langle T_{\text{grad } \rho} \| \text{grad } \rho \|, X, \frac{\text{grad } \rho}{\| \text{grad } \rho \|} \rangle \\
= \left\langle \nabla_X \frac{\text{grad } \rho}{\| \text{grad } \rho \|}, \text{grad } \rho \right\rangle \\
= \left\langle X - \frac{D_X \langle \text{grad } \rho, \text{grad } \rho \rangle}{2\langle \text{grad } \rho, \text{grad } \rho \rangle^2} \text{grad } \rho + \frac{1}{2} \left[ X, \frac{\text{grad } \rho}{\| \text{grad } \rho \|^2} \right], \text{grad } \rho \right\rangle \\
= -\frac{D_X \langle \text{grad } \rho, \text{grad } \rho \rangle}{2\langle \text{grad } \rho, \text{grad } \rho \rangle^2} \langle \text{grad } \rho, \text{grad } \rho \rangle \\
= -\frac{D_X \langle \text{grad } \rho, \text{grad } \rho \rangle}{2\langle \text{grad } \rho, \text{grad } \rho \rangle}.
\]

If $V$ is a vertical vector that is perpendicular to grad $\rho$ and $\langle \text{grad } \rho \| \text{grad } \rho \rangle^2, X \rangle$, then the Koszul formula gives

\[
\langle T_{\text{grad } \rho} \| \text{grad } \rho \|, X, V \rangle = \langle \nabla_{\text{grad } \rho} \| \text{grad } \rho \|, X, V \rangle = 0.
\]

Finally, if $W = [\text{grad } \rho \| \text{grad } \rho \rangle^2, X \rangle / [\text{grad } \rho \| \text{grad } \rho \rangle^2, X \rangle]$, then

\[
2\langle T_{\text{grad } \rho} \| \text{grad } \rho \|, X, W \rangle = 2\| \text{grad } \rho \| \langle \nabla_{\text{grad } \rho} \| \text{grad } \rho \rangle^2, X, W \rangle \\
= \| \text{grad } \rho \| \left\langle \left[ \frac{\text{grad } \rho}{\| \text{grad } \rho \|^2}, X \right], W \right\rangle.
\]

The following lemma can be found in [5].

**Lemma 18.** For any basic horizontal field $X$ and any vertical field $V$,

\[
\langle R(X, V) V, X \rangle = \langle (\nabla_X T)_V V, X \rangle - \langle T_V X, T_V X \rangle + \langle A_X V, A_X V \rangle \\
- \langle \nabla_V (\nabla_X V^\text{grad } \rho \text{horiz }), X \rangle + \langle \nabla_X (\nabla_V V^\text{vert }), X \rangle.
\]

**Remark 19.** Note that the first three terms are precisely O'Neill's formula for vertizontal curvature of a Riemannian submersion. In that case, $\langle \nabla_V (\nabla_X V^\text{grad } \rho \text{horiz }), X \rangle$ vanishes since grad $\rho = 0$, and the last term vanishes by the antisymmetry of the A-tensor.
Proof.

\[
\langle R(X, V)V, X \rangle = \langle \nabla_X \nabla_V V, X \rangle - \langle \nabla_V \nabla_X V, X \rangle - \langle \nabla_{[X,V]}V, X \rangle
\]
\[
= \langle \nabla_X (\nabla_V V)_{\text{horiz}}^{\text{grad } \rho}, X \rangle + \langle \nabla_X (\nabla_V V)_{\text{vert}}, X \rangle
- \langle \nabla_V (\nabla_X V)_{\text{horiz}}^{\text{grad } \rho}, X \rangle - \langle \nabla_V (\nabla_X V)_{\text{vert}}, X \rangle
- \langle [X,V]V_{\text{vert}}, X \rangle.
\]

Letting the superscript \(\text{grad } \rho\) denote the component tangent to \(\text{grad } \rho\), and \(\text{grad } \rho_{\perp}\) denote the component perpendicular to \(\text{grad } \rho\), and using the definition of the \(A\)-tensor and Lemma 13, we get that the third term is

\[
- \langle \nabla_V (\nabla_X V)_{\text{horiz}}^\text{grad } \rho, X \rangle
= -\langle \nabla_V (\nabla_X V)_{\text{grad } \rho}^\text{grad } \rho_{\perp}, X \rangle - \langle \nabla_V (\nabla_X V)_{\text{horiz}}, X \rangle
= -\langle \nabla_V (\nabla_X V)_{\text{grad } \rho}^\text{horiz}, X \rangle - \langle \nabla_V (A_X V), X \rangle.
\]

Thus if we also use the definition of the \(T\)-tensor, we get

\[
\langle R(X, V)V, X \rangle = \langle \nabla_X (T_V V), X \rangle - \langle \nabla_V (A_X V), X \rangle - \langle T_V (\nabla_X V)_{\text{vert}}, X \rangle
- \langle \nabla_V (\nabla_X V)_{\text{horiz}}^\text{grad } \rho, X \rangle + \langle \nabla_X (\nabla_V V)_{\text{vert}}, X \rangle
- \langle \nabla_V (\nabla_X V)_{\text{horiz}}^\text{grad } \rho, X \rangle + \langle \nabla_X (\nabla_V V)_{\text{vert}}, X \rangle
= \langle \nabla_X (T_V V), X \rangle - \langle T_V (\nabla_X V)_{\text{vert}}, X \rangle - \langle T_{[X,V]} V_{\text{vert}}, X \rangle
+ \langle T_V (\nabla_X V)_{\text{vert}}, X \rangle + \langle A_X V, A_X V \rangle
- \langle \nabla_V (\nabla_X V)_{\text{horiz}}^\text{grad } \rho, X \rangle + \langle \nabla_X (\nabla_V V)_{\text{vert}}, X \rangle
= \langle (\nabla_X T)_V V, X \rangle - \langle T_V X, T_V X \rangle + \langle A_X V, A_X V \rangle
- \langle \nabla_V (\nabla_X V)_{\text{horiz}}^\text{grad } \rho, X \rangle + \langle \nabla_X (\nabla_V V)_{\text{vert}}, X \rangle.
\]

When \(V = \text{grad } \rho_{\perp}\), the first non-Riemannian term can be simplified as follows.

**Proposition 20.** When \(V = \text{grad } \rho_{\perp}\),

\[
\langle \nabla_V (\nabla_X V)_{\text{horiz}}^\text{grad } \rho, X \rangle = (D_{\text{grad } \rho_{\perp}} | \text{grad } \rho_{\perp} | | \text{grad } \rho_{\perp} |^2) (X, X).
\]
Proof: We use equation (3) to get
\[
\left( \nabla_{\text{grad} \rho \parallel \text{grad} \rho} \left( \nabla \frac{\text{grad} \rho}{|\text{grad} \rho|} \right)^{\text{horiz}} \right) \cdot X
\]
\[
= \left( \nabla_{\text{grad} \rho \parallel \text{grad} \rho} \left( |\text{grad} \rho| \nabla \frac{\text{grad} \rho}{|\text{grad} \rho|^2} \right)^{\text{horiz}} \right) \cdot X
\]
\[
= (\nabla_{\text{grad} \rho \parallel \text{grad} \rho} (|\text{grad} \rho| X)^{\text{horiz}}, X)
\]
\[
= ((D_{\text{grad} \rho \parallel \text{grad} \rho} (|\text{grad} \rho| X) + |\text{grad} \rho| \nabla_{\text{grad} \rho \parallel \text{grad} \rho} |\text{grad} \rho| X, X)
\]
\[
= (D_{\text{grad} \rho \parallel \text{grad} \rho} (|\text{grad} \rho| X, X) + |\text{grad} \rho| |\text{grad} \rho|^2 \langle \nabla_{\text{grad} \rho \parallel \text{grad} \rho} |\text{grad} \rho|^2 X, X \rangle)
\]
\[
= (D_{\text{grad} \rho \parallel \text{grad} \rho} (|\text{grad} \rho| X, X) + |\text{grad} \rho| |\text{grad} \rho|^2 \langle X, X \rangle.
\]

The last non-Riemannian curvature term can also be simplified when \( V = \text{grad} \rho \parallel \text{grad} \rho \).

**Proposition 21.** When \( V = \text{grad} \rho \parallel \text{grad} \rho \),
\[
\langle \nabla_X (\nabla V) \rangle^{\text{vert}}, X \rangle = 0.
\]

Proof: We have
\[
\langle \nabla_X (\nabla V) \rangle^{\text{vert}}, X \rangle = \left( \nabla_X \left( \nabla_{\text{grad} \rho \parallel \text{grad} \rho} \frac{\text{grad} \rho}{|\text{grad} \rho|} \right)^{\text{vert}, \perp} , X \right)
\]
\[
+ \left( \nabla_X \left( \nabla_{\text{grad} \rho \parallel \text{grad} \rho} \frac{\text{grad} \rho}{|\text{grad} \rho|} \right)^{\text{grad} \rho} , X \right)
\]

The first term on the right-hand side of the equation above is 0 by the antisymmetry of the A-tensor. Thus,
\[
\langle \nabla_X (\nabla V) \rangle^{\text{vert}}, X \rangle = \left( \nabla_X \left( \nabla_{\text{grad} \rho \parallel \text{grad} \rho} \frac{\text{grad} \rho}{|\text{grad} \rho|} \right)^{\text{grad} \rho} , X \right).
\]

Since
\[
\left( \nabla_{\text{grad} \rho \parallel \text{grad} \rho} \frac{\text{grad} \rho}{|\text{grad} \rho|}, \frac{\text{grad} \rho}{|\text{grad} \rho|} \right) = \frac{1}{2} D_{\text{grad} \rho \parallel \text{grad} \rho} \left( \frac{\text{grad} \rho}{|\text{grad} \rho|}, \frac{\text{grad} \rho}{|\text{grad} \rho|} \right)
\]
\[
= 0,
\]
the entire last term is 0. \( \Box \)
Thus, when $V = \frac{\text{grad } \rho}{|\text{grad } \rho|}$ we get

$$
\left\langle R \left( X, \frac{\text{grad } \rho}{|\text{grad } \rho|} \right) \frac{\text{grad } \rho}{|\text{grad } \rho|}, X \right\rangle = \left\langle (\nabla_X T)_{\text{grad } \rho/|\text{grad } \rho|} \frac{\text{grad } \rho}{|\text{grad } \rho|}, X \right\rangle
- \left\langle T_{\text{grad } \rho/|\text{grad } \rho|} X, T_{\text{grad } \rho/|\text{grad } \rho|} X \right\rangle
+ \left\langle A_X \frac{\text{grad } \rho}{|\text{grad } \rho|}, A_X \frac{\text{grad } \rho}{|\text{grad } \rho|} \right\rangle
- (D_{\text{grad } \rho/|\text{grad } \rho|} \text{grad } \rho + |\text{grad } \rho|^2)(X, X).
$$

Since the $A$-tensor term vanishes, we get

$$
(7) \quad \left\langle R \left( X, \frac{\text{grad } \rho}{|\text{grad } \rho|} \right) \frac{\text{grad } \rho}{|\text{grad } \rho|}, X \right\rangle
= \left\langle \nabla_X \left( T_{\text{grad } \rho/|\text{grad } \rho|} \frac{\text{grad } \rho}{|\text{grad } \rho|} \right), X \right\rangle
- 2 \left\langle T_{\text{grad } \rho/|\text{grad } \rho|} \left( \nabla_X \frac{\text{grad } \rho}{|\text{grad } \rho|} \right)^{\text{vert}}, X \right\rangle
- (T_{\text{grad } \rho/|\text{grad } \rho|} X, T_{\text{grad } \rho/|\text{grad } \rho|} X)
- (D_{\text{grad } \rho/|\text{grad } \rho|} \text{grad } \rho + |\text{grad } \rho|^2)(X, X).
$$

The second term on the right hand side of the previous display is given by the following result.

**Proposition 22.**

$$
-2 \left\langle T_{\text{grad } \rho/|\text{grad } \rho|} \left( \nabla_X \frac{\text{grad } \rho}{|\text{grad } \rho|} \right)^{\text{vert}}, X \right\rangle = -\frac{1}{2} |\text{grad } \rho|^2 \left\| X, \frac{\text{grad } \rho}{|\text{grad } \rho|^2} \right\|^2.
$$

**Proof.** First note that since $\text{grad } \rho/|\text{grad } \rho|$ has constant length,

$$
\left\langle \nabla_X \frac{\text{grad } \rho}{|\text{grad } \rho|}, \frac{\text{grad } \rho}{|\text{grad } \rho|} \right\rangle = 0.
$$

Thus

$$
\left( \nabla_X \frac{\text{grad } \rho}{|\text{grad } \rho|} \right)^{\text{vert}} = |\text{grad } \rho| \left( \nabla_X \frac{\text{grad } \rho}{|\text{grad } \rho|^2} \right)^{\text{vert, }\perp},
$$
where the superscript \( \text{vert} \) signifies the component that is vertical and perpendicular to \( \text{grad} \rho \). Using equation (3) we obtain
\[
\left( \nabla X \frac{\text{grad} \rho}{|\text{grad} \rho|} \right)^{\text{vert}} = |\text{grad} \rho| \frac{1}{2} \left[ X, \frac{\text{grad} \rho}{|\text{grad} \rho|^2} \right].
\]

Therefore
\[
\left\langle T_{\text{grad} \rho / |\text{grad} \rho|} \left( \nabla X \frac{\text{grad} \rho}{|\text{grad} \rho|} \right)^{\text{vert}}, X \right\rangle = \frac{1}{2} \left\langle T_{\text{grad} \rho} \left[ X, \frac{\text{grad} \rho}{|\text{grad} \rho|^2} \right], X \right\rangle.
\]

If we set \( W = [X, \text{grad} \rho / |\text{grad} \rho|^2] \), then we get
\[
\left\langle T_{\text{grad} \rho / |\text{grad} \rho|} \left( \nabla X \frac{\text{grad} \rho}{|\text{grad} \rho|} \right)^{\text{vert}}, X \right\rangle
= \frac{1}{2} \left\langle T_{\text{grad} \rho} W, X \right\rangle = \frac{1}{2} \left\langle \nabla_{\text{grad} \rho} W, X \right\rangle = \frac{1}{4} \left[ [X, \text{grad} \rho], W \right]
= \frac{1}{4} \left\langle [X, \text{grad} \rho], \left[ X, \frac{\text{grad} \rho}{|\text{grad} \rho|^2} \right] \right\rangle.
\]

Since \([X, \text{grad} \rho / |\text{grad} \rho|^2]\) is perpendicular to \(\text{grad} \rho\) we get
\[
\left\langle T_{\text{grad} \rho / |\text{grad} \rho|} \left( \nabla X \frac{\text{grad} \rho}{|\text{grad} \rho|} \right)^{\text{vert}}, X \right\rangle
= \frac{1}{4} |\text{grad} \rho|^2 \left\langle \left[ X, \frac{\text{grad} \rho}{|\text{grad} \rho|^2} \right], \left[ X, \frac{\text{grad} \rho}{|\text{grad} \rho|^2} \right] \right\rangle,
\]
as desired. \( \square \)

Using Lemma 15, continuity of the second derivatives of \( \rho \), and compactness give us the following result.

**Proposition 23.** Let \( \pi : M \rightarrow B \) be a horizontally homothetic submersion of a compact Riemannian manifold with nonconstant dilation function.

Throughout any horizontal lift of a geodesic in \( B \) that passes through regular points of \( \rho \) and is sufficiently close to the minimum level of \( \rho \),
\[
-(D_{\text{grad} \rho / |\text{grad} \rho|} |\text{grad} \rho| + |\text{grad} \rho|^2) \langle X, X \rangle
\]
is negative and uniformly bounded away from 0 along the given horizontal geodesic.
To control the final vertizontal curvature term
\[
\left\langle \nabla_X \left( T_{\text{grad } \rho / |\text{grad } \rho|} \frac{\text{grad } \rho}{|\text{grad } \rho|} \right), X \right\rangle
\]
we use compactness to get the following result.

**Lemma 24.** For any horizontal lift \( \gamma : (-\infty, \infty) \to M \) of a geodesic in \( B \) that passes through regular points of \( \rho \) and for any \( \varepsilon > 0 \), there is an interval \( [a, b] \) so that
\[
\left| \frac{1}{b - a} \int_a^b \left\langle \nabla_X \left( T_{\text{grad } \rho / |\text{grad } \rho|} \frac{\text{grad } \rho}{|\text{grad } \rho|} \right), X \right\rangle \right| < \varepsilon.
\]

**Proof.** We write
\[
\left\langle \nabla_X \left( T_{\text{grad } \rho / |\text{grad } \rho|} \frac{\text{grad } \rho}{|\text{grad } \rho|} \right), X \right\rangle
= D_X \left\langle T_{\text{grad } \rho / |\text{grad } \rho|} \frac{\text{grad } \rho}{|\text{grad } \rho|}, X \right\rangle - \left\langle T_{\text{grad } \rho / |\text{grad } \rho|} \frac{\text{grad } \rho}{|\text{grad } \rho|}, \nabla_X X \right\rangle,
\]
\[
= D_X \left\langle T_{\text{grad } \rho / |\text{grad } \rho|} \frac{\text{grad } \rho}{|\text{grad } \rho|}, X \right\rangle.
\]
Here
\[
\left\langle T_{\text{grad } \rho / |\text{grad } \rho|} \frac{\text{grad } \rho}{|\text{grad } \rho|}, \nabla_X X \right\rangle = 0
\]
since \( T_{\text{grad } \rho / |\text{grad } \rho|} \frac{\text{grad } \rho}{|\text{grad } \rho|} \) grad \( \rho \) / grad \( \rho \) is horizontal and \( \nabla_X X \) is proportional to grad \( \rho \). If \( \gamma \) were periodic with period \( p \), then
\[
\frac{1}{p} \int_0^p D_X \left\langle T_{\text{grad } \rho / |\text{grad } \rho|} \frac{\text{grad } \rho}{|\text{grad } \rho|}, X \right\rangle = \frac{1}{p} \left\langle T_{\text{grad } \rho / |\text{grad } \rho|} \frac{\text{grad } \rho}{|\text{grad } \rho|}, X \right\rangle \bigg|_0^p = 0.
\]
By compactness of the unit tangent bundle of \( M \), we can find an interval on which \( \gamma \) is almost periodic with the error being any predetermined quantity, yielding the result. \( \square \)

Combining the previous three results with our vertizontal curvature formula (7), we get the following result.

**Proposition 25.** Let \( \pi : M \to B \) be a horizontally homothetic submersion of a compact Riemannian manifold with nonconstant dilation function. Any horizontal lift of a geodesic in \( B \) that passes through regular points and is sufficiently close to the minimum of \( \rho \) passes through some points with some negative vertizontal sectional curvatures.

Our main theorem is a corollary of the previous proposition.
Much of our argument carries through to compact manifolds with nonpositive sectional curvature; however, the two vertizontal terms

\[-2 \left\langle T_{\text{grad } \rho / |\text{grad } \rho|} \left( \nabla_X \frac{\text{grad } \rho}{|\text{grad } \rho|} \right)^{\text{vert}}, X \right\rangle - \langle T_{\text{grad } \rho / |\text{grad } \rho|} X, T_{\text{grad } \rho / |\text{grad } \rho|} X \rangle \]

are nonpositive and one expects them to usually be negative. By assuming them away we get the following theorem.

**Theorem 26.** Every horizontally homothetic submersion of a compact manifold with nonpositive sectional curvature is a Riemannian submersion (up to a change of scale on the base), provided the fibers are totally geodesic.

**Proof.** Since the fibers are totally geodesic and \( \text{grad } \rho / |\text{grad } \rho| \) is in the kernel of the \( A \)-tensor

\[
\left\langle R \left( X, \frac{\text{grad } \rho}{|\text{grad } \rho|} \right) \frac{\text{grad } \rho}{|\text{grad } \rho|}, X \right\rangle = -\langle D_{\text{grad } \rho / |\text{grad } \rho|} |\text{grad } \rho| + |\text{grad } \rho|^2 \rangle \langle X, X \rangle.
\]

If \( \rho \) is nonconstant and we are close enough to a point where it achieves its maximum, then

\[
\left\langle R \left( X, \frac{\text{grad } \rho}{|\text{grad } \rho|} \right) \frac{\text{grad } \rho}{|\text{grad } \rho|}, X \right\rangle = -\langle D_{\text{grad } \rho / |\text{grad } \rho|} |\text{grad } \rho| + |\text{grad } \rho|^2 \rangle \langle X, X \rangle > 0.
\]

So \( \rho \) must be constant if the domain is compact, the sectional curvature is nonpositive, and the fibers are totally geodesic. \( \square \)

**Remark 27.** We remark that by using a Bochner type formula for horizontally homothetic maps with totally geodesic fibers Mustafa shows ([33]) that for \( n \geq 4 \) and \( m \geq n + 2 \), any non-constant submersive harmonic morphism \( \pi : (M^n, g) \rightarrow (N^m, h) \) with totally geodesic fibers from a compact manifold of nonpositive sectional curvature is a Riemannian submersion. When \( n \geq 3 \), Theorem A implies that the class of harmonic submersions with totally geodesic fibers coincides with the class of horizontally homothetic submersions with totally geodesic fibers. So Theorem 26 is equivalent to Mustafa’s result when \( n \geq 4 \) and \( m \geq n + 2 \), and extends Mustafa’s theorem to the cases when either the fibers are 1-dimensional or the base is 2 or 3-dimensional.

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Department of Mathematics
California University at Riverside
Riverside, CA 92521, U. S. A.
E-MAIL: (Ye-Lin Ou) ye.lino@ucr.edu
E-MAIL: (Frederick Wilhelm) fred@ucr.edu

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